we write $G(x-y)$ in terms of the complete set of eigenfunctions u_k as

$$G(x-y) = \sum_{k=1}^{\infty} \frac{u_k(x) u_k(y)}{\lambda_k} \tag{6.410}$$

so that the action $Lu_k = \lambda_k \rho u_k$ turns G into

$$L\, G(x-y) = \sum_{k=1}^{\infty} \frac{L\, u_k(x) u_k(y)}{\lambda_k} = \sum_{k=1}^{\infty} \rho(x)\, u_k(x)\, u_k(y) = \delta(x-y) \tag{6.411}$$

our $\alpha = 1$ series expansion (6.374) of the delta function.

6.39 Green's Functions in One Dimension

In one dimension, we can explicitly solve the inhomogeneous ordinary differential equation $L\, f(x) = g(x)$ in which

$$L = -\frac{d}{dx}\left(p(x)\frac{d}{dx}\right) + q(x) \tag{6.412}$$

is formally self adjoint. We'll build a Green's function from two solutions u and v of the homogeneous equation $L\, u(x) = L\, v(x) = 0$ as

$$G(x,y) = \frac{1}{A}\left[\theta(x-y)u(y)v(x) + \theta(y-x)u(x)v(y)\right] \tag{6.413}$$

in which $\theta(x) = (x + |x|)/(2|x|)$ is **the Heaviside step function** (Oliver Heaviside 1850–1925), and A is a constant which we'll presently identify. We'll show that the expression

$$f(x) = \int_a^b G(x,y)\, g(y)\, dy = \frac{v(x)}{A}\int_a^x u(y)\, g(y)\, dy + \frac{u(x)}{A}\int_x^b v(y)\, g(y)\, dy$$

solves our inhomogeneous equation. Differentiating, we find after a cancellation

$$f'(x) = \frac{v'(x)}{A}\int_a^x u(y)\, g(y)\, dy + \frac{u'(x)}{A}\int_x^b v(y)\, g(y)\, dy. \tag{6.414}$$

Differentiating again, we have

$$\begin{aligned} f''(x) &= \frac{v''(x)}{A} \int_a^x u(y)\, g(y)\, dy + \frac{u''(x)}{A} \int_x^b v(y)\, g(y)\, dy \\ &\quad + \frac{v'(x)u(x)g(x)}{A} - \frac{u'(x)v(x)g(x)}{A} \\ &= \frac{v''(x)}{A} \int_a^x u(y)\, g(y)\, dy + \frac{u''(x)}{A} \int_x^b v(y)\, g(y)\, dy \\ &\quad + \frac{W(x)}{A}\, g(x) \end{aligned} \tag{6.415}$$

in which $W(x)$ is the wronskian $W(x) = u(x)v'(x) - u'(x)v(x)$. The result (6.266) for the wronskian of two linearly independent solutions of a self-adjoint homogeneous ODE gives us $W(x) = W(x_0)\, p(x_0)/p(x)$. We set the constant $A = -W(x_0)p(x_0)$ so that the last term in (6.415) is $-g(x)/p(x)$. It follows that

$$Lf(x) = \frac{[Lv(x)]}{A} \int_a^x u(y)\, g(y)\, dy + \frac{[Lu(x)]}{A} \int_x^b v(y)\, g(y)\, dy + g(x) = g(x). \tag{6.416}$$

But $Lu(x) = Lv(x) = 0$, so we see that f satisfies our inhomogeneous equation $Lf(x) = g(x)$.

Example 6.45 (Green's functions with boundary conditions) To use the Green's function (6.413) with $A = W(x_0)p(x_0)$ to solve the ODE (6.412) subject to the Dirichelet boundary conditions $f(a) = 0 = f(b)$, we choose solutions $u(x)$ and $v(x)$ of the homogeneous equations $Lu(x) = 0 = Lv(x)$ that obey these boundary conditions, $u(a) = 0 = v(b)$. For then our formula $f(x) = \int_a^b G(x,y)g(y)dy$ gives

$$f(a) = \frac{u(a)}{A} \int_a^b v(y)\, g(y)\, dy = 0 = f(b) = \frac{v(b)}{A} \int_a^b u(y)\, g(y)\, dy. \tag{6.417}$$

Similarly, to impose the Neumann boundary conditions $f'(a) = 0 = f'(b)$, we choose solutions $u(x)$ and $v(x)$ of the homogeneous equations $Lu(x) = 0 = Lv(x)$ that obey these boundary conditions, $u'(a) = 0 = v'(b)$, so that our formula (6.414) for $f'(x)$ gives

$$f'(a) = \frac{u'(a)}{A} \int_a^b v(y)\, g(y)\, dy = 0 = f'(b) = \frac{v'(b)}{A} \int_a^b u(y)\, g(y)\, dy. \tag{6.418}$$

For instance, to solve the equation $-f''(x) - f(x) = \exp x$, with the mixed boundary conditions $f(-\pi) = 0$ and $f'(\pi) = 0$, we choose from among the solutions $\alpha \cos x + \beta \sin x$ of the homogeneous equation $-f'' - f = 0$, the functions $u(x) = \sin x$ and $v(x) = \cos x$. Substituting them into the formula (6.413) and setting $p(x) = 1$ and $A = -W(x_0) = \sin^2(x_0) + \cos^2(x_0) = 1$, we find as the Green's function

$$G(x, y) = \theta(x - y) \sin y \cos x + \theta(y - x) \sin x \cos y. \tag{6.419}$$

The solution then is

$$\begin{aligned} f(x) &= \int_a^b G(x, y)\, e^y\, dy \\ &= \int_{-\pi}^{\pi} [\theta(x - y) \sin y \cos x + \theta(y - x) \sin x \cos y]\, e^y\, dy \\ &= \cos x \int_{-\pi}^{x} e^y\ \sin y\, dy + \sin x \int_{x}^{\pi} e^y\ \cos y\, dy \\ &= -\frac{1}{2} \left(e^{-\pi} \cos x + e^{\pi} \sin x + e^x \right). \end{aligned} \tag{6.420}$$

□

6.40 Nonlinear Differential Equations

The field of nonlinear differential equations is too vast to cover here, but we may hint at some of its features by considering some examples from cosmology and particle physics.

The Friedmann equations of general relativity (11.413 & 11.415) for the dimensionless scale factor $a(t)$ of a homogeneous, isotropic universe are (in natural units, $c = 1$)

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\,(\rho + 3p) \quad \text{and} \quad \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\,\rho - \frac{k}{a^2} \tag{6.421}$$

in which k respectively is 1, 0, and -1 for closed, flat, and open geometries. (The scale factor $a(t)$ tells how much space has expanded or contracted by the time t.) These equations become more tractable when the energy density ρ is due to a single constituent whose pressure p is related to it by an equation of state $p = w\rho$. Conservation of energy $\dot{\rho} = -3\,\dot{a}(\rho + p)/a$ (11.429–11.434) then ensures (exercise 6.30) that the product $\rho\, a^{3(1+w)}$ is independent of time. The constant w respectively is 1/3, 0, and -1 for