5.31 The Bessel function $J_n(x)$ is given by the integral

$$J_n(x) = \frac{1}{2\pi i} \oint_C e^{(x/2)(z-1/z)} \frac{dz}{z^{n+1}} \tag{5.352}$$

along a counter-clockwise contour about the origin. Find the generating function for these Bessel functions, that is, the function $G(x,z)$ whose Laurent series has the $J_n(x)$'s as coefficients

$$G(x,z) = \sum_{n=-\infty}^{\infty} J_n(x)\, z^n. \tag{5.353}$$

5.32 Show that the Heaviside function $\theta(y) = (y+|y|)/(2|y|)$ is given by the integral

$$\theta(y) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{iyx} \frac{dx}{x - i\epsilon} \tag{5.354}$$

in which ϵ is an infinitesimal positive number.

5.33 Show that the integral of $\exp(ik)/k$ along the contour from $k = L$ to $k = L + iH$ and then to $k = -L + iH$ and then down to $k = -L$ vanishes in the double limit $L \to \infty$ and $H \to \infty$.

5.34 Use a ghost contour and a cut to evaluate the integral

$$I = \int_{-1}^{1} \frac{dx}{(x^2+1)\sqrt{1-x^2}} \tag{5.355}$$

by imitating example 5.30. Be careful when picking up the poles at $z = \pm i$. If necessary, use the explicit square-root formulas (5.188) and (5.189).

5.35 Redo the previous exercise (5.34) by defining the square roots so that the cuts run from $-\infty$ to -1 and from 1 to ∞. Take advantage of the evenness of the integrand and integrate on a contour that is slightly above the whole real axis. Then add a ghost contour around the upper half plane.

5.36 Show that if u is even and v is odd, then the Hilbert transforms (5.265) imply (5.267).

5.37 Show why the principal-part identity (5.224) lets one write the Kramers-Kronig integral (5.287) for the index of refraction in the regularized form (5.292).

5.38 Use the formula (5.283) for the group velocity and the regularized expression (5.292) for the real part of the index of refraction $n_r(\omega)$ to derive formula (5.293) for the group velocity.

5.39 Show that the quarter-circles of the Abel-Plana contours $\mathcal{C}_\pm$ contribute $\frac{1}{2}(f(n_1) + f(n_2))$ to the sum S in the formula $T_x = I_x + S$.