is the Lorentz-invariant function $\Delta_+(x-y)$

$$
\begin{aligned}
[\phi^+(x), \phi^-(y)] &= \int \frac{d^3p\, d^3q}{(2\pi)^3 2\sqrt{q^0 p^0}}\, e^{ipx - iqy}\, [a(\boldsymbol{p}), a^\dagger(\boldsymbol{q})] \\
&= \int \frac{d^3p}{(2\pi)^3 2p^0}\, e^{ip(x-y)} = \Delta_+(x-y)
\end{aligned} \tag{5.247}
$$

in which $p(x-y) = \boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y}) - p^0(x^0-y^0)$.

At points x that are space-like, that is, for which $x^2 = \boldsymbol{x}^2 - (x^0)^2 \equiv r^2 > 0$, the Lorentz-invariant function $\Delta_+(x)$ depends only upon $r = +\sqrt{x^2}$ and has the value (Weinberg, 1995, p. 202)

$$
\Delta_+(x) = \frac{m}{4\pi^2 r}\, K_1(mr) \tag{5.248}
$$

in which the Hankel function K_1 is

$$
K_1(z) = -\frac{\pi}{2}\, [J_1(iz) + iN_1(iz)] = \frac{1}{z} + \frac{z}{2}\left[\ln\left(\frac{z}{2}\right) + \gamma - \frac{1}{2}\right] + \ldots \tag{5.249}
$$

where J_1 is the first Bessel function, N_1 is the first Neumann function, and $\gamma = 0.57721\ldots$ is the Euler-Mascheroni constant.

The Feynman propagator arises most simply as the mean value in the vacuum of the **time-ordered product** of the fields $\phi(x)$ and $\phi(y)$

$$
\mathcal{T}\{\phi(x)\phi(y)\} \equiv \theta(x^0-y^0)\phi(x)\phi(y) + \theta(y^0-x^0)\phi(y)\phi(x). \tag{5.250}
$$

The operators $a(\boldsymbol{p})$ and $a^\dagger(\boldsymbol{p})$ respectively annihilate the vacuum ket $a(\boldsymbol{p})|0\rangle = 0$ and bra $\langle 0|a^\dagger(\boldsymbol{p}) = 0$, and so by (5.245 & 5.246) do the positive- and negative-frequency parts of the field $\phi^+(z)|0\rangle = 0$ and $\langle 0|\phi^-(z) = 0$. Thus the mean value in the vacuum of the time-ordered product is

$$
\begin{aligned}
\langle 0|\mathcal{T}\{\phi(x)\phi(y)\}\,|0\rangle &= \langle 0|\theta(x^0-y^0)\phi(x)\phi(y) + \theta(y^0-x^0)\phi(y)\phi(x)|0\rangle \\
&= \langle 0|\theta(x^0-y^0)\phi^+(x)\phi^-(y) + \theta(y^0-x^0)\phi^+(y)\phi^-(x)|0\rangle \\
&= \langle 0|\theta(x^0-y^0)[\phi^+(x),\phi^-(y)] \\
&\qquad + \theta(y^0-x^0)[\phi^+(y),\phi^-(x)]|0\rangle.
\end{aligned} \tag{5.251}
$$

But by (5.247), these commutators are $\Delta_+(x-y)$ and $\Delta_+(y-x)$. Thus the mean value in the vacuum of the time-ordered product

$$
\begin{aligned}
\langle 0|\mathcal{T}\{\phi(x)\phi(y)\}\,|0\rangle &= \theta(x^0-y^0)\Delta_+(x-y) + \theta(y^0-x^0)\Delta_+(y-x) \\
&= -i\Delta_F(x-y)
\end{aligned} \tag{5.252}
$$

is the Feynman propagator (5.241) multiplied by $-i$. □