the interval $[-1,1]$. Let's promote x to a complex variable z, put the cut on the negative real axis, and write the square root as $\sqrt{1-x^2} = -i\sqrt{x^2-1} = -i\sqrt{(z-1)(z+1)}$ so that for $x = z = i\epsilon$ we have as $\epsilon \to 0$ that $1 = \sqrt{1-x^2} = -i\sqrt{-1} = -ii = 1$. Now the function $f(z) = 1/[(z-k)(-i)\sqrt{(z-1)(z+1)}]$ is analytic everywhere except along a cut on the interval $[-1,1]$ and at $z = k$. The circle $z = Re^{i\theta}$ for $0 \le \theta \le 2\pi$ is a ghost contour as $R \to \infty$. If we shrink-wrap this ccw contour around the pole at $z = k$ and the interval $[-1,1]$, then we get $0 = -2I + 2\pi i/[(-i)\sqrt{k-1}\sqrt{k+1}]$ or

$$I = -\frac{\pi}{\sqrt{k-1}\sqrt{k+1}}. \tag{5.193}$$

So if $k = -2$, then $I = \pi/\sqrt{3}$, while if $k = 2$, then $I = -\pi/\sqrt{3}$. □

Example 5.31 (Contour Integral with a Cut) Let's compute the integral

$$I = \int_0^\infty \frac{x^a}{(x+1)^2}\, dx \tag{5.194}$$

for $-1 < a < 1$. We promote x to a complex variable z and put the cut on the positive real axis. Since

$$\lim_{|z|\to\infty} \frac{|z|^{a+1}}{|z+1|^2} = 0, \tag{5.195}$$

the integrand vanishes faster than $1/|z|$, and we may add two ghost contours, $\mathcal{G}_+$ counter-clockwise around the upper half-plane and $\mathcal{G}_-$ counter-clockwise around the lower half-plane, as shown in Fig. 5.8.

We add a contour $\mathcal{C}_-$ that runs from $-\infty$ to the double pole at $z = -1$, loops around that pole, and then runs back to $-\infty$; the two long contours along the negative real axis cancel because the cut in θ lies on the positive real axis. So the contour integral along $\mathcal{C}_-$ is just the clockwise integral around the double pole which by Cauchy's integral formula (5.34) is

$$\oint_{\mathcal{C}_-} \frac{z^a}{(z-(-1))^2}\, dz = -2\pi i \left.\frac{dz^a}{dz}\right|_{z=-1} = 2\pi i\, a\, e^{\pi a i}. \tag{5.196}$$

We also add the integral I_- from ∞ to 0 just below the real axis

$$I_- = \int_\infty^0 \frac{(x-i\epsilon)^a}{(x-i\epsilon+1)^2}\, dx = \int_\infty^0 \frac{\exp(a(\ln(x)+2\pi i))}{(x+1)^2}\, dx \tag{5.197}$$

which is

$$I_- = -e^{2\pi a i} \int_0^\infty \frac{x^a}{(x+1)^2}\, dx = -e^{2\pi a i}\, I. \tag{5.198}$$