for $n \geq -1$ and zero otherwise. So the Laurent series for $f(z)$ is

$$f(z) = \frac{1}{z(z+1)} = \sum_{n=-1}^{\infty} (-1)^{n+1} z^n. \tag{5.92}$$

The series starts at $n = -1$, not at $n = -\infty$, because $f(z)$ is meromorphic with only a simple pole at $z = 0$. □

Example 5.10 (The Argument Principle) Consider the counter-clockwise integral

$$\frac{1}{2\pi i} \oint_{\mathcal{C}} f(z) \frac{g'(z)}{g(z)} dz \tag{5.93}$$

along a contour $\mathcal{C}$ that lies inside a simply connected region R in which $f(z)$ is analytic and $g(z)$ meromorphic. If the function $g(z)$ has a zero or a pole of order n at $w \in R$ and no other singularity in R

$$g(z) = a_n(w)(z-w)^n \tag{5.94}$$

then the ratio g'/g is

$$\frac{g'(z)}{g(z)} = \frac{n(z-w)^{n-1}}{(z-w)^n} = \frac{n}{z-w} \tag{5.95}$$

and the integral is

$$\frac{1}{2\pi i} \oint_{\mathcal{C}} f(z) \frac{g'(z)}{g(z)} dz = \frac{1}{2\pi i} \oint_{\mathcal{C}} f(z) \frac{n}{z-w} dz = n\, f(w). \tag{5.96}$$

Any function $g(z)$ meromorphic in R will possess a Laurent series

$$g(z) = \sum_{k=n}^{\infty} a_k(w)(z-w)^k \tag{5.97}$$

about each point $w \in R$. One may show (exercise 5.18) that as $z \to w$ the ratio g'/g again approaches (5.95). It follows that the integral (5.93) is a sum of $n_\ell f(w_\ell)$ at the zeros and poles of $g(z)$ that lie within the contour $\mathcal{C}$

$$\frac{1}{2\pi i} \oint_{\mathcal{C}} f(z) \frac{g'(z)}{g(z)} dz = \sum_\ell \frac{1}{2\pi i} \oint_{\mathcal{C}} f(z) \frac{n_\ell}{z-w_\ell} = \sum_\ell n_\ell\, f(w_\ell) \tag{5.98}$$

in which $|n_\ell|$ is the multiplicity of the ℓth zero or pole. □