electric field $\boldsymbol{D} = \epsilon_m \boldsymbol{E}$, where the **permittivity** $\epsilon_m = \epsilon_0 + \chi_m = K_m \epsilon_0$ of the material differs from that of the vacuum ϵ_0 by the **electric susceptibility** χ_m and by the **relative permittivity** K_m. The permittivity of the vacuum is the **electric constant** ϵ_0.

An electric field $\boldsymbol{E}$ exerts on a charge q a **force** $\boldsymbol{F} = q\boldsymbol{E}$ even in a dielectric medium. The electrostatic energy W of a system of linear dielectrics is the volume integral

$$W = \frac{1}{2} \int \boldsymbol{D} \cdot \boldsymbol{E} \, d^3 r. \tag{4.128}$$

Example 4.15 (Field of a Charge Near an Interface) Consider two semi-infinite dielectrics of permittivities ϵ_1 and ϵ_2 separated by an infinite horizontal x-y-plane. What is the electrostatic potential due to a charge q in region 1 at a height h above the interface?

The easy way to solve this problem is to put an image charge q' at the same distance from the interface in region 2 so that the potential in region 1 is

$$V_1(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_1} \left(\frac{q}{\sqrt{x^2 + y^2 + (z-h)^2}} + \frac{q'}{\sqrt{x^2 + y^2 + (z+h)^2}} \right). \tag{4.129}$$

This potential satisfies Gauss's law $\boldsymbol{\nabla} \cdot \boldsymbol{D} = \rho$ in region 1. In region 2, the potential

$$V_2(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_2} \frac{q''}{\sqrt{x^2 + y^2 + (z-h)^2}} \tag{4.130}$$

also satisfies Gauss's law. The continuity (4.127) of the tangential component of $\boldsymbol{E}$ tells us that the partial derivatives of V_1 and V_2 in the x (or y) direction must be the same at $z = 0$

$$\frac{\partial V_1(x, y, 0)}{\partial x} = \frac{\partial V_2(x, y, 0)}{\partial x}. \tag{4.131}$$

The discontinuity equation (4.127) for the electric displacement says that at the interface at $z = 0$ with no surface charge

$$\epsilon_1 \frac{\partial V_1(x, y, 0)}{\partial z} = \epsilon_2 \frac{\partial V_2(x, y, 0)}{\partial z}. \tag{4.132}$$

These two equations (4.131 & 4.132) allow one to solve for q' and q''

$$q' = \frac{\epsilon_1 - \epsilon_2}{\epsilon_1 + \epsilon_2} q \quad \text{and} \quad q'' = \frac{2\epsilon_2}{\epsilon_1 + \epsilon_2} q. \tag{4.133}$$

In the limit $h \to 0$, the potential in region 1 becomes

$$V_1(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_1} \frac{q}{\sqrt{x^2+y^2+z^2}} \left(1 + \frac{\epsilon_1 - \epsilon_2}{\epsilon_1 + \epsilon_2}\right) = \frac{q}{4\pi\bar{\epsilon}r} \tag{4.134}$$

in which $\bar{\epsilon}r$ is the mean permittivity $\bar{\epsilon} = (\epsilon_1 + \epsilon_2)/2$. Similarly in region 2, the potential is

$$V_2(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_2} \frac{q}{\sqrt{x^2+y^2+z^2}} \frac{2\epsilon_2}{\epsilon_1 + \epsilon_2} = \frac{q}{4\pi\bar{\epsilon}r} \tag{4.135}$$

in the limit $h \to 0$. □

Example 4.16 (A Charge Near a Plasma Membrane) A eukaryotic cell (the kind with a nucleus) is surrounded by a plasma membrane, which is a phospholipid bilayer about 5 nm thick. Both sides of the plasma membrane are in contact with salty water. The permittivity of the water is $\epsilon_w \approx 80\epsilon_0$ while that of the membrane considered as a simple lipid slab is $\epsilon_\ell \approx 2\epsilon_0$.

Let's think about the potential felt by an ion in the water outside a cell but near its membrane, and let us for simplicity imagine the membrane to be infinitely thick so that we can use the simple formulas we've derived. The potential due to the ion, if its charge is q, is then given by equation (4.129) with $\epsilon_1 = \epsilon_w$ and $\epsilon_2 = \epsilon_\ell$. The image-charge term in $V_1(\boldsymbol{r})$ is the potential due to the polarization of the membrane and the water by the ion. It is the potential felt by the ion. Since the image charge by (4.133) is $q' \approx q$, the potential the ion feels is $V_i(z) \approx q/8\pi e_w z$. The force on the ion then is

$$F = -qV_i'(z) = \frac{q^2}{8\pi e_w z}. \tag{4.136}$$

It always is positive no matter what the sign of the charge is. A lipid slab in water repels ions. Similarly, a charge in a lipid slab is attracted to the water outside the slab.

Now imagine an electric dipole in water near a lipid slab. Now there are two equal and opposite charges and two equal and opposite mirror charges. The net effect is that the slab repels the dipole. So lipids repel water molecules; they are said to be **hydrophobic**. This is one of the reasons why folding proteins move their hydrophobic amino acids inside and their polar or **hydrophilic** ones outside.

With some effort, one may use the method of images to compute the electric potential of a charge in or near a plasma membrane taken to be a lipid slab of finite thickness.

The electric potential in the lipid bilayer $V_\ell(\rho, z)$ of thickness t due to a