3.10 Derivatives and Integrals of Laplace Transforms

The derivatives of a Laplace transform $f(s)$ are by its definition (3.125)

$$\frac{d^n f(s)}{ds^n} = \int_0^\infty dt\,(-t)^n\,e^{-st}\,F(t). \tag{3.139}$$

They usually are well defined if $f(s)$ is well defined. For instance, if we differentiate the Laplace transform $f(s) = 1/s$ of the function $F(t) = 1$ as given by (3.126), then we find

$$(-1)^n\,\frac{d^n s^{-1}}{ds^n} = \frac{n!}{s^{n+1}} = \int_0^\infty dt\,e^{-st}\,t^n \tag{3.140}$$

which tells us that the Laplace transform of t^n is $n!/s^{n+1}$.

The result of differentiating the function $F(t)$ also has a simple form. Integrating by parts, we find for the Laplace transform of $F'(t)$

$$\begin{aligned}\int_0^\infty dt\,e^{-st}\,F'(t) &= \int_0^\infty dt\,\left\{\frac{d}{dt}\left[e^{-st}\,F(t)\right] - F(t)\frac{d}{dt}\,e^{-st}\right\}\\ &= -\,F(0) + \int_0^\infty dt\,F(t)\,s\,e^{-st}\\ &= -\,F(0) + s\,f(s).\end{aligned} \tag{3.141}$$

The indefinite integral of the Laplace transform (3.125) is

$${}^{(1)}f(s) \equiv \int ds_1\,f(s_1) = \int_0^\infty dt\,\frac{e^{-st}}{(-t)}\,F(t) \tag{3.142}$$

and its nth indefinite integral is

$${}^{(n)}f(s) \equiv \int ds_n \ldots \int ds_1\,f(s_1) = \int_0^\infty dt\,\frac{e^{-st}}{(-t)^n}\,F(t). \tag{3.143}$$

If $f(s)$ is a well-behaved function, then these indefinite integrals usually are well defined for $s > 0$ as long as $F(t) \to 0$ suitably as $t \to 0$.

3.11 Laplace Transforms and Differential Equations

Suppose we wish to solve the differential equation

$$P(d/ds)\,f(s) = j(s). \tag{3.144}$$

By writing $f(s)$ and $j(s)$ as Laplace transforms

$$f(s) = \int_0^\infty e^{-st}\,F(t)\,dt \quad \text{and} \quad j(s) = \int_0^\infty e^{-st}\,J(t)\,dt. \tag{3.145}$$