A system that is measured to be in a state $|n\rangle$ cannot simultaneously be measured to be in an orthogonal state $|m\rangle$. The probabilities sum to unity because the system must be in some state.

Since the density operator ρ is hermitian, it has a complete, orthonormal set of eigenvectors $|k\rangle$ all of which have nonnegative eigenvalues ρ_k

$$\rho|k\rangle = \rho_k|k\rangle. \tag{1.428}$$

They afford for it the expansion

$$\rho = \sum_{k=1}^{N} \rho_k |k\rangle\langle k| \tag{1.429}$$

in which the eigenvalue ρ_k is the probability that the system is in the state $|k\rangle$.

1.37 Correlation Functions

We can define two Schwarz inner products for a density matrix ρ. If $|f\rangle$ and $|g\rangle$ are two states, then the inner product

$$(f, g) \equiv \langle f|\rho|g\rangle \tag{1.430}$$

for $g = f$ is nonnegative, $(f, f) = \langle f|\rho|f\rangle \geq 0$, and satisfies the other conditions (1.73, 1.74, & 1.76) for a Schwarz inner product.

The second Schwarz inner product applies to operators A and B and is defined (Titulaer and Glauber, 1965) as

$$(A, B) = \mathrm{Tr}\left(\rho A^\dagger B\right) = \mathrm{Tr}\left(B\rho A^\dagger\right) = \mathrm{Tr}\left(A^\dagger B \rho\right). \tag{1.431}$$

This inner product is nonnegative when $A = B$ and obeys the other rules (1.73, 1.74, & 1.76) for a Schwarz inner product.

These two degenerate inner products are not inner products in the strict sense of (1.73–1.79), but they are Schwarz inner products, and so (1.92–1.93) they satisfy the Schwarz inequality (1.93)

$$(f, f)(g, g) \geq |(f, g)|^2. \tag{1.432}$$

Applied to the first, vector, Schwarz inner product (1.430), the Schwarz inequality gives

$$\langle f|\rho|f\rangle\langle g|\rho|g\rangle \geq |\langle f|\rho|g\rangle|^2 \tag{1.433}$$

which is a useful property of density matrices. Application of the Schwarz inequality to the second, operator, Schwarz inner product (1.431) gives (Titulaer and Glauber, 1965)

$$\mathrm{Tr}\left(\rho A^\dagger A\right)\mathrm{Tr}\left(\rho B^\dagger B\right) \geq \left|\mathrm{Tr}\left(\rho A^\dagger B\right)\right|^2. \tag{1.434}$$

The operator $E_i(x)$ that represents the ith component of the electric field at the point x is the hermitian sum of the "positive-frequency" part $E_i^{(+)}(x)$ and its adjoint $E_i^{(-)}(x) = (E_i^{(+)}(x))^\dagger$

$$E_i(x) = E_i^{(+)}(x) + E_i^{(-)}(x). \tag{1.435}$$

Glauber has defined the first-order correlation function $G_{ij}^{(1)}(x,y)$ as (Glauber, 1963b)

$$G_{ij}^{(1)}(x,y) = \mathrm{Tr}\left(\rho E_i^{(-)}(x)E_j^{(+)}(y)\right) \tag{1.436}$$

or in terms of the operator inner product (1.431) as

$$G_{ij}^{(1)}(x,y) = \left(E_i^{(+)}(x), E_j^{(+)}(y)\right). \tag{1.437}$$

By setting $A = E_i^{(+)}(x)$, *etc.*, it follows then from the Schwarz inequality (1.434) that the correlation function $G_{ij}^{(1)}(x,y)$ is bounded by (Titulaer and Glauber, 1965)

$$\left|G_{ij}^{(1)}(x,y)\right|^2 \leq G_{ii}^{(1)}(x,x)G_{jj}^{(1)}(y,y). \tag{1.438}$$

Interference fringes are sharpest when this inequality is saturated

$$\left|G_{ij}^{(1)}(x,y)\right|^2 = G_{ii}^{(1)}(x,x)G_{jj}^{(1)}(y,y) \tag{1.439}$$

which can occur only if the correlation function $G_{ij}^{(1)}(x,y)$ factorizes (Titulaer and Glauber, 1965)

$$G_{ij}^{(1)}(x,y) = \mathcal{E}_i^*(x)\mathcal{E}_j(y) \tag{1.440}$$

as it does when the density operator is an outer product of coherent states

$$\rho = |\{\alpha_k\}\rangle\langle\{\alpha_k\}| \tag{1.441}$$

which are eigenstates of $E_i^{(+)}(x)$ with eigenvalue $\mathcal{E}_i(x)$ (Glauber, 1963b,a)

$$E_i^{(+)}(x)|\{\alpha_k\}\rangle = \mathcal{E}_i(x)|\{\alpha_k\}\rangle. \tag{1.442}$$