Example 1.46 Suppose A is the 3×2 matrix

$$A = \begin{pmatrix} r_1 & p_1 \\ r_2 & p_2 \\ r_3 & p_3 \end{pmatrix} \tag{1.385}$$

and the vector $|y\rangle$ is the cross-product $|y\rangle = \boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$. Then no solution $|x\rangle$ exists to the equation $A|x\rangle = |y\rangle$ (unless $\boldsymbol{r}$ and $\boldsymbol{p}$ are parallel) because $A|x\rangle$ is a linear combination of the vectors $\boldsymbol{r}$ and $\boldsymbol{p}$ while $|y\rangle = \boldsymbol{L}$ is perpendicular to both $\boldsymbol{r}$ and $\boldsymbol{p}$. □

Even when the matrix A is square, the equation (1.381) sometimes has no solutions. For instance, if A is a square matrix that vanishes, $A = 0$, then (1.381) has no solutions whenever $|y\rangle \neq 0$. And when $N > M$, as in for instance

$$\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \tag{1.386}$$

the solution (1.384) is never unique, for we may add to it any linear combination of the vectors $|n\rangle$ that A annihilates for $M < n \leq N$

$$|x\rangle = \sum_{n=1}^{\min(M,N)} \frac{\langle m_n|y\rangle}{S_n} |n\rangle + \sum_{n=M+1}^{N} x_n |n\rangle. \tag{1.387}$$

These are the vectors $|n\rangle$ for $M < n \leq N$ that A maps to zero since they do not occur in the sum (1.362) which stops at $n = \min(M, N) < N$.

Example 1.47 (The CKM Matrix) In the standard model, the mass matrices of the u, c, t and d, s, b quarks are 3×3 complex matrices M_u and M_d with singular-value decompositions $M_u = U_u \Sigma_u V_u^\dagger$ and $M_d = U_d \Sigma_d V_d^\dagger$ whose singular-values are the quark masses. The unitary CKM matrix $U_u^\dagger U_d$ (Cabibbo, Kobayashi, Maskawa) describes transitions among the quarks mediated by the $W^\pm$ gauge bosons. By redefining the quark fields, one may make the CKM matrix real, apart from a phase that violates charge-conjugation-parity (CP) symmetry. □

The adjoint of a complex symmetric matrix M is its complex conjugate, $M^\dagger = M^*$. So by (1.351), its right singular vectors $|n\rangle$ are the eigenstates of $M^* M$

$$M^* M|n\rangle = S_n^2 |n\rangle \tag{1.388}$$

and by (1.366) its left singular vectors $|m_n\rangle$ are the eigenstates of MM^*

$$MM^*|m_n\rangle = (M^*M)^* \, |m_n\rangle = S_n^2|m_n\rangle. \tag{1.389}$$

Thus its left singular vectors are the complex conjugates of its right singular vectors, $|m_n\rangle = |n\rangle^*$. So the unitary matrix V is the complex conjugate of the unitary matrix U, and the SVD of M is (Autonne, 1915)

$$M = U\Sigma U^{\mathsf{T}}. \tag{1.390}$$

1.32 The Moore-Penrose Pseudoinverse

Although a matrix A has an inverse A^{-1} if and only if it is square and has a nonzero determinant, one may use the singular-value decomposition to make a pseudoinverse A^+ for an arbitrary $M \times N$ matrix A. If the singular-value decomposition of the matrix A is

$$A = U \, \Sigma \, V^\dagger \tag{1.391}$$

then the Moore-Penrose pseudoinverse (Eliakim H. Moore 1862–1932, Roger Penrose 1931–) is

$$A^+ = V \, \Sigma^+ \, U^\dagger \tag{1.392}$$

in which Σ^+ is the transpose of the matrix Σ with every nonzero entry replaced by its inverse (and the zeros left as they are). One may show that the pseudoinverse A^+ satisfies the four relations

$$\begin{aligned} A\,A^+\,A = A \quad &\text{and} \quad A^+\,A\,A^+ = A^+ \\ \left(A\,A^+\right)^\dagger = A\,A^+ \quad &\text{and} \quad \left(A^+\,A\right)^\dagger = A^+\,A. \end{aligned} \tag{1.393}$$

and that it is the only matrix that does so.

Suppose that all the singular values of the $M \times N$ matrix A are positive. In this case, if A has more rows than columns, so that $M > N$, then the product AA^+ is the $N \times N$ identity matrix I_N

$$A^+A = V^\dagger \Sigma^+ \Sigma V = V^\dagger I_N V = I_N \tag{1.394}$$

and AA^+ is an $M \times M$ matrix that is not the identity matrix I_M. If instead A has more columns than rows, so that $N > M$, then AA^+ is the $M \times M$ identity matrix I_M

$$AA^+ = U\Sigma\Sigma^+U^\dagger = UI_MU^\dagger = I_M \tag{1.395}$$

but A^+A is an $N \times N$ matrix that is not the identity matrix I_N. If the