by setting its derivatives with respect to ρ_n, λ_1, and λ_2 equal to zero

$$\frac{\partial L}{\partial \rho_n} = -k\left(\ln \rho_n + 1\right) - \lambda_1 - \lambda_2 E_n = 0 \tag{1.339}$$

$$\frac{\partial L}{\partial \lambda_1} = \sum_n \rho_n - 1 = 0 \tag{1.340}$$

$$\frac{\partial L}{\partial \lambda_2} = \sum_n \rho_n E_n - E = 0. \tag{1.341}$$

The first (1.339) of these conditions implies that

$$\rho_n = \exp\left[-(\lambda_1 + \lambda_2 E_n + k)/k\right] \tag{1.342}$$

We satisfy the second condition (1.340) by choosing λ_1 so that

$$\rho_n = \frac{\exp(-\lambda_2 E_n/k)}{\sum_n \exp(-\lambda_2 E_n/k)}. \tag{1.343}$$

Setting $\lambda_2 = 1/T$, we define the temperature T so that ρ satisfies the third condition (1.341). Its eigenvalue ρ_n then is

$$\rho_n = \frac{\exp(-E_n/kT)}{\sum_n \exp(-E_n/kT)}. \tag{1.344}$$

In terms of the inverse temperature $\beta \equiv 1/(kT)$, the density operator is

$$\rho = \frac{e^{-\beta H}}{\mathrm{Tr}\left(e^{-\beta H}\right)} \tag{1.345}$$

which is the **Boltzmann distribution**. □

1.31 The Singular-Value Decomposition

Every complex $M \times N$ rectangular matrix A is the product of an $M \times M$ unitary matrix U, an $M \times N$ rectangular matrix Σ that is zero except on its main diagonal which consists of A's nonnegative singular values S_k, and an $N \times N$ unitary matrix $V^\dagger$

$$A = U\,\Sigma\,V^\dagger. \tag{1.346}$$

This singular-value decomposition is a key theorem of matrix algebra.

Suppose A is a linear operator that maps vectors in an N-dimensional vector space V_N into vectors in an M-dimensional vector space V_M. The spaces V_N and V_M will have infinitely many orthonormal bases $\{|n,a\rangle \in V_N\}$ and $\{|m,b\rangle \in V_M\}$ labeled by continuous parameters a and b. Each pair of