hermitian matrix has N orthogonal eigenvectors. To do this, we'll first show that the space of vectors orthogonal to an eigenvector $|n\rangle$ of a hermitian operator A

$$A|n\rangle = \lambda|n\rangle \tag{1.298}$$

is **invariant** under the action of A—that is, $\langle n|y\rangle = 0$ implies $\langle n|A|y\rangle = 0$. We'll use successively the definition of $A^\dagger$, the hermiticity of A, the eigenvector equation (1.298), the definition of the inner product, and the reality of the eigenvalues of a hermitian matrix:

$$\langle n|A|y\rangle = \langle A^\dagger n|y\rangle = \langle An|y\rangle = \langle \lambda n|y\rangle = \bar{\lambda}\langle n|y\rangle = 0. \tag{1.299}$$

Thus the space of vectors orthogonal to an eigenvector of a hermitian operator A is invariant under the action of that operator.

Now a hermitian operator A acting on an N-dimensional vector space S is represented by an $N \times N$ hermitian matrix, and so it has at least one eigenvector $|1\rangle$. The subspace of S consisting of all vectors orthogonal to $|1\rangle$ is an $(N-1)$-dimensional vector space S_{N-1} that is invariant under the action of A. On this space S_{N-1}, the operator A is represented by an $(N-1)\times(N-1)$ hermitian matrix A_{N-1}. This matrix has at least one eigenvector $|2\rangle$. The subspace of S_{N-1} consisting of all vectors orthogonal to $|2\rangle$ is an $(N-2)$-dimensional vector space S_{N-2} that is invariant under the action of A. On S_{N-2}, the operator A is represented by an $(N-2)\times(N-2)$ hermitian matrix A_{N-2} which has at least one eigenvector $|3\rangle$. By construction, the vectors $|1\rangle$, $|2\rangle$, and $|3\rangle$ are mutually orthogonal. Continuing in this way, we see that A **has** N**orthogonal eigenvectors** $|k\rangle$ **for** $k = 1, 2, \ldots, N$. Thus no hermitian matrix is defective.

The N orthogonal eigenvectors $|k\rangle$ of an $N \times N$ matrix A can be normalized and used to write the $N \times N$ identity operator I as

$$I = \sum_{k=1}^{N} |k\rangle\langle k|. \tag{1.300}$$

On multiplying from the left by the matrix A, we find

$$A = AI = A\sum_{k=1}^{N} |k\rangle\langle k| = \sum_{k=1}^{N} a_k|k\rangle\langle k| \tag{1.301}$$

which is the diagonal form of the hermitian matrix A. This expansion of A as a sum over outer products of its eigenstates multiplied by their eigenvalues exhibits the possible values a_k of the physical quantity represented by the matrix A when selective, nondestructive measurements $|k\rangle\langle k|$ of the quantity A are done.