cancel leaving

$$\sum_{\ell=1}^{K} c_\ell \left(\lambda_\ell - \lambda_{K+1}\right) V^{(\ell)} = 0 \tag{1.251}$$

in which all the factors $(\lambda_\ell - \lambda_{K+1})$ are different from zero since by assumption all the eigenvalues are different. But this last equation says that the first K eigenvectors are linearly dependent, which contradicts our assumption that they were linearly in dependent. This contradiction tells us that **if all N eigenvectors of an $N \times N$ square matrix have different eigenvalues, then they are linearly independent**. Similarly, if any $n < N$ eigenvectors of an $N \times N$ square matrix have different eigenvalues, then they are linearly independent.

An eigenvalue λ that is a single root of the characteristic equation (1.245) is associated with a single eigenvector; it is called a **simple eigenvalue**. An eigenvalue λ that is an nth root of the characteristic equation is associated with n eigenvectors; it is said to be an ***n*-fold degenerate eigenvalue** or to have **algebraic multiplicity** n. Its **geometric multiplicity** is the number $n' \le n$ of linearly independent eigenvectors with eigenvalue λ. A matrix with $n' < n$ for any eigenvalue λ is **defective**. Thus an $N \times N$ matrix with fewer than N linearly independent eigenvectors is defective.

Example 1.35 (A Defective 2×2 Matrix) Each of the 2×2 matrices

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{1.252}$$

has only one linearly independent eigenvector and so is defective. □

Suppose A is an $N \times N$ matrix that is not defective. We may use its N linearly independent eigenvectors $V^{(\ell)} = |\ell\rangle$ to define the columns of an $N \times N$ matrix S as $S_{k\ell} = V_k^{(\ell)}$. In terms of S, the eigenvalue equation (1.242) takes the form

$$\sum_{k=1}^{N} A_{ik} S_{k\ell} = \lambda_\ell S_{i\ell}. \tag{1.253}$$

Since the columns of S are linearly independent, the determinant of S does not vanish—the matrix S is **nonsingular**—and so its inverse S^{-1} is well-defined by (1.197). So we may multiply this equation by S^{-1} and get

$$\sum_{i,k=1}^{N} \left(S^{-1}\right)_{ni} A_{ik} S_{k\ell} = \sum_{i=1}^{N} \lambda_\ell \left(S^{-1}\right)_{ni} S_{i\ell} = \lambda_n \delta_{n\ell} \tag{1.254}$$