We use them to represent rotations, translations, Lorentz transformations, and internal-symmetry transformations.

1.20 Determinants

The **determinant** of a 2×2 matrix A is

$$\det A = |A| = A_{11}A_{22} - A_{21}A_{12}. \tag{1.175}$$

In terms of the 2×2 antisymmetric ($e_{ij} = -e_{ji}$) matrix $e_{12} = 1 = -e_{21}$ with $e_{11} = e_{22} = 0$, this determinant is

$$\det A = \sum_{i=1}^{2}\sum_{j=1}^{2} e_{ij}A_{i1}A_{j2}. \tag{1.176}$$

It's also true that

$$e_{k\ell}\det A = \sum_{i=1}^{2}\sum_{j=1}^{2} e_{ij}A_{ik}A_{j\ell}. \tag{1.177}$$

These definitions and results extend to any square matrix. If A is a 3×3 matrix, then its determinant is

$$\det A = \sum_{ijk=1}^{3} e_{ijk}A_{i1}A_{j2}A_{k3} \tag{1.178}$$

in which e_{ijk} is totally antisymmetric with $e_{123} = 1$, and the sums over $i, j,$ & k run from 1 to 3. More explicitly, this determinant is

$$\begin{aligned}\det A &= \sum_{ijk=1}^{3} e_{ijk}A_{i1}A_{j2}A_{k3}\\ &= \sum_{i=1}^{3} A_{i1}\sum_{jk=1}^{3} e_{ijk}A_{j2}A_{k3}\\ &= A_{11}\left(A_{22}A_{33} - A_{32}A_{23}\right) + A_{21}\left(A_{32}A_{13} - A_{12}A_{33}\right)\\ &\qquad + A_{31}\left(A_{12}A_{23} - A_{22}A_{13}\right). \end{aligned}\tag{1.179}$$

The **minor** $M_{i\ell}$ of the matrix A is the 2×2 determinant of the matrix A without row i and column ℓ, and the **cofactor** $C_{i\ell}$ is the minor $M_{i\ell}$ multiplied by $(-1)^{i+\ell}$. Thus $\det A$ is the sum

$$\begin{aligned}\det A &= A_{11}(-1)^2\left(A_{22}A_{33} - A_{32}A_{23}\right) + A_{21}(-1)^3\left(A_{12}A_{33} - A_{32}A_{13}\right)\\ &\qquad + A_{31}(-1)^4\left(A_{12}A_{23} - A_{22}A_{13}\right)\\ &= A_{11}C_{11} + A_{21}C_{21} + A_{31}C_{31}\end{aligned}\tag{1.180}$$

of the products $A_{i1}C_{i1} = A_{i1}(-1)^{i+1}M_{i1}$ where

$$\begin{aligned} C_{11} &= (-1)^2 M_{11} = A_{22}A_{33} - A_{23}A_{32} \\ C_{21} &= (-1)^3 M_{21} = A_{32}A_{13} - A_{12}A_{33} \\ C_{31} &= (-1)^4 M_{31} = A_{12}A_{23} - A_{22}A_{13}. \end{aligned} \tag{1.181}$$

Example 1.25 (Determinant of a 3×3 Matrix) The determinant of a 3×3 matrix is the dot product of the vector of its first row with the cross-product of the vectors of its second and third rows:

$$\begin{vmatrix} U_1 & U_2 & U_3 \\ V_1 & V_2 & V_3 \\ W_1 & W_2 & W_3 \end{vmatrix} = \sum_{ijk=1}^{3} e_{ijk} U_i V_j W_k = \sum_{i=1}^{3} U_i (\boldsymbol{V} \times \boldsymbol{W})_i = \boldsymbol{U} \cdot (\boldsymbol{V} \times \boldsymbol{W})$$

which is called the scalar triple product. □

Laplace used the totally antisymmetric symbol $e_{i_1 i_2 \dots i_N}$ with N indices and with $e_{123\dots N} = 1$ to define the determinant of an $N \times N$ matrix A as

$$\det A = \sum_{i_1 i_2 \dots i_N = 1}^{N} e_{i_1 i_2 \dots i_N} A_{i_1 1} A_{i_2 2} \dots A_{i_N N} \tag{1.182}$$

in which the sums over $i_1 \dots i_N$ run from 1 to N. In terms of cofactors, two forms of his expansion of this determinant are

$$\det A = \sum_{i=1}^{N} A_{ik} C_{ik} = \sum_{k=1}^{N} A_{ik} C_{ik} \tag{1.183}$$

in which the first sum is over the row index i but not the (arbitrary) column index k, and the second sum is over the column index k but not the (arbitrary) row index i. The cofactor C_{ik} is $(-1)^{i+k} M_{ik}$ in which the minor M_{ik} is the determinant of the $(N-1) \times (N-1)$ matrix A without its ith row and kth column. It's also true that

$$e_{k_1 k_2 \dots k_N} \det A = \sum_{i_1 i_2 \dots i_N = 1}^{N} e_{i_1 i_2 \dots i_N} A_{i_1 k_1} A_{i_2 k_2} \dots A_{i_N k_N}. \tag{1.184}$$

The key feature of a determinant is that it is an *antisymmetric* combination of products of the elements A_{ik} of a matrix A. One implication of this antisymmetry is that the interchange of any two rows or any two columns changes the sign of the determinant. Another is that if one adds a multiple of one column to another column, for example a multiple xA_{i2} of column 2