and the formula (1.118) for the kth orthonormal linear combination of the vectors $|V_\ell\rangle$ is

$$|U_k\rangle = \frac{|u_k\rangle}{\sqrt{\langle u_k|u_k\rangle}}. \tag{1.139}$$

The vectors $|U_k\rangle$ are not unique; they vary with the order of the $|V_k\rangle$. □

Vectors and linear operators are abstract. The numbers we compute with are inner products like $\langle g|f\rangle$ and $\langle g|A|f\rangle$. In terms of N orthonormal basis vectors $|n\rangle$ with $f_n = \langle n|f\rangle$ and $g_n^* = \langle g|n\rangle$, we can use the expansion (1.131) to write these inner products as

$$\begin{aligned}\langle g|f\rangle &= \langle g|I|f\rangle = \sum_{n=1}^{N}\langle g|n\rangle\langle n|f\rangle = \sum_{n=1}^{N} g_n^* f_n \\ \langle g|A|f\rangle &= \langle g|IAI|f\rangle = \sum_{n,\ell=1}^{N}\langle g|n\rangle\langle n|A|\ell\rangle\langle \ell|f\rangle = \sum_{n,\ell=1}^{N} g_n^*\, A_{n\ell}\, f_\ell\end{aligned} \tag{1.140}$$

in which $A_{n\ell} = \langle n|A|\ell\rangle$. We often gather the inner products $f_\ell = \langle \ell|f\rangle$ into a column vector f with components $f_\ell = \langle \ell|f\rangle$

$$f = \begin{pmatrix} \langle 1|f\rangle \\ \langle 2|f\rangle \\ \vdots \\ \langle N|f\rangle \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_3 \end{pmatrix} \tag{1.141}$$

and the $\langle n|A|\ell\rangle$ into a matrix A with matrix elements $A_{n\ell} = \langle n|A|\ell\rangle$. If we also line up the inner products $\langle g|n\rangle = \langle n|g\rangle^*$ in a row vector that is the transpose of the complex conjugate of the column vector g

$$g^\dagger = (\langle 1|g\rangle^*,\ \langle 2|g\rangle^*, \ldots, \langle N|g\rangle^*) = (g_1^*, g_2^*, \ldots, g_N^*) \tag{1.142}$$

then we can write inner products in matrix notation as $\langle g|f\rangle = g^\dagger f$ and as $\langle g|A|f\rangle = g^\dagger A f$.

If we switch to a different basis, say from $|n\rangle$'s to $|\alpha_n\rangle$'s, then the components of the column vectors change from $f_n = \langle n|f\rangle$ to $f'_n = \langle \alpha_n|f\rangle$, and similarly those of the row vectors $g^\dagger$ and of the matrix A change, but the bras, the kets, the linear operators, and the inner products $\langle g|f\rangle$ and $\langle g|A|f\rangle$