α. The probability $P(n)$ of finding n quanta in the state $|\alpha\rangle$ is the square of the absolute value of the inner product $\langle n|\alpha\rangle$

$$P(n) = |\langle n|\alpha\rangle|^2 = \frac{|\alpha|^{2n}}{n!}\, e^{-|\alpha|^2} \tag{13.64}$$

which is a Poisson distribution $P(n) = P_P(n, |\alpha|^2)$ with mean and variance $\mu = \langle n \rangle = V(\alpha) = |\alpha|^2$. □

13.5 The Gaussian Distribution

Gauss considered the binomial distribution in the limit $N \to \infty$ with the probability p fixed. In this limit, the binomial probability

$$P_B(n, p, N) = \frac{N!}{n!\,(N-n)!}\, p^n\, q^{N-n} \tag{13.65}$$

is very tiny unless n is near pN which means that $n \approx pN$ and $N - n \approx (1-p)N = qN$ are comparable. So the limit $N \to \infty$ effectively is one in which n and $N - n$ also tend to infinity. The approximation (13.54)

$$P_B(n, p, N) \approx \sqrt{\frac{N}{2\pi n(N-n)}} \left(\frac{pN}{n}\right)^n \left(\frac{qN}{N-n}\right)^{N-n} R_3(n, N) \tag{13.66}$$

applies in which $R_3(n, N) \to 1$ as N, $N-n$, and n all increase without limit.

Because the probability $P_B(n, p, N)$ is negligible unless $n \approx pN$, we set $y = n - pN$ and treat y/n as small. Since $n = pN + y$ and $N - n = (1-p)N + pN - n = qN - y$, we may write the square-root as

$$\begin{aligned} \sqrt{\frac{N}{2\pi\, n\,(N-n)}} &= \frac{1}{\sqrt{2\pi\, N\, [(pN+y)/N]\, [(qN-y)/N]}} \\ &= \frac{1}{\sqrt{2\pi\, pqN\, (1 + y/pN)\,(1 - y/qN)}}. \end{aligned} \tag{13.67}$$

Since y remains finite as $N \to \infty$, we get in this limit

$$\lim_{N\to\infty} \sqrt{\frac{N}{2\pi\, n\,(N-n)}} = \frac{1}{\sqrt{2\pi\, pqN}}. \tag{13.68}$$