The generating function $g(t,x)$ is even under the reflection of both independent variables, so

$$g(t,x) = \sum_{n=0}^{\infty} t^n \, P_n(x) = \sum_{n=0}^{\infty} (-t)^n \, P_n(-x) = g(-t,-x) \tag{8.51}$$

which implies that

$$P_n(-x) = (-1)^n \, P_n(x) \quad \text{whence} \quad P_{2n+1}(0) = 0. \tag{8.52}$$

With more effort, one can show that

$$P_{2n}(0) = (-1)^n \, \frac{(2n-1)!!}{(2n)!!} \quad \text{and that} \quad |P_n(x)| \le 1. \tag{8.53}$$

8.7 Schlaefli's Integral

Schlaefli used Cauchy's integral formula (5.36) and Rodrigues's formula

$$P_n(x) = \frac{1}{2^n \, n!} \left(\frac{d}{dx} \right)^n (x^2 - 1)^n \tag{8.54}$$

to express $P_n(z)$ as a counterclockwise contour integral around the point z

$$P_n(z) = \frac{1}{2^n \, 2\pi i} \oint \frac{(z'^2 - 1)^n}{(z' - z)^{n+1}} \, dz'. \tag{8.55}$$

8.8 Orthogonal Polynomials

Rodrigues's formula (8.8) generates other families of orthogonal polynomials. The n-th order polynomials R_n in which the e_n are constants

$$R_n(x) = \frac{1}{e_n w(x)} \, \frac{d^n}{dx^n} \left[w(x) \, Q^n(x) \right] \tag{8.56}$$

are orthogonal on the interval from a to b with weight function $w(x)$

$$\int_a^b R_n(x) \, R_k(x) \, w(x) \, dx = N_n \, \delta_{nk} \tag{8.57}$$

as long as $Q(x)$ vanishes at a and b (exercise 8.8)

$$Q(a) = Q(b) = 0. \tag{8.58}$$