Example 6.6 (The Helmholtz Equation in Three Dimensions) In three dimensions and in **rectangular coordinates** \( r = (x, y, z) \), the function
\[
f(x, y, z) = X(x)Y(y)Z(z)
\]
is a solution of the ODE \(- \Delta f = k^2 f\) as long as \( X, Y, \) and \( Z \) satisfy \(-X'' = a^2 X, -Y'' = b^2 Y, \) and \(-Z'' = c^2 Z\) with \( a^2 + b^2 + c^2 = k^2 \). We set \( X_a(x) = \alpha \sin ax + \beta \cos ax \) and so forth. Arbitrary linear combinations of the products \( X_a Y_b Z_c \) also are solutions of Helmholtz’s equation \(- \Delta f = k^2 f\) as long as \( a^2 + b^2 + c^2 = k^2 \).

In **cylindrical coordinates** \((\rho, \phi, z)\), the laplacian (6.34) is
\[
\nabla \cdot \nabla f = \Delta f = \frac{1}{\rho} \left( (\rho f_{\rho})_{\rho} + (\rho f_{\phi})_{\phi} + \rho f_{zz} \right)
\]
and so if we substitute \( f(\rho, \phi, z) = P(\rho) \Phi(\phi) Z(z) \) into Helmholtz’s equation \(- \Delta f = \alpha^2 f\) and multiply both sides by \(-\rho^2 / P \Phi Z\), then we get
\[
\frac{\rho^2}{f} \Delta f = \frac{\rho^2 P'' + \rho P'}{P} + \frac{\Phi''}{\Phi} + \frac{k^2}{\rho^2} Z = -\alpha^2 \rho^2.
\]
If we set \( Z_k(z) = e^{kz} \), then this equation becomes (6.46) with \( k^2 \) replaced by \( \alpha^2 + k^2 \). Its solution then is
\[
f(\rho, \phi, z) = J_n(\sqrt{\alpha^2 + k^2 \rho}) e^{in\phi} e^{kz}
\]
in which \( n \) must be an integer if the solution is to apply to the full range of \( \phi \) from 0 to \( 2\pi \). The case in which \( \alpha = 0 \) corresponds to Laplace’s equation with solution \( f(\rho, \phi, z) = J_n(k\rho) e^{im \phi} e^{kz} \). We could have required \( Z \) to satisfy \( Z'' = -k^2 Z \). The solution (6.51) then would be
\[
f(\rho, \phi, z) = J_n(\sqrt{\alpha^2 - k^2 \rho}) e^{im \phi} e^{kz}.
\]
But if \( \alpha^2 - k^2 < 0 \), we write this solution in terms of the **modified Bessel function** \( I_n(x) = i^{-n} J_{n}(ix) \) (section 9.3) as
\[
f(\rho, \phi, z) = I_n(\sqrt{k^2 - \alpha^2 \rho}) e^{im \phi} e^{kz}.
\]

In **spherical coordinates**, the laplacian (6.35) is
\[
\Delta f = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 f}{\partial \phi^2}
\]
in which the first term is \( r^{-1}(rf)_{rr} \). If we set \( f(r, \theta, \phi) = R(r) \Theta(\theta) \Phi_m(\phi) \) where \( \Phi_m = e^{im \phi} \) and multiply both sides of the Helmholtz equation \(- \Delta f = k^2 f\) by \(-r^{-2} R \Theta \Phi\), then we get
\[
\frac{\left(\frac{r^2 R'}{R}\right)'}{r^2} + \frac{(\sin \theta \Theta')'}{\sin \theta \Theta} - \frac{m^2}{\sin^2 \theta} = -k^2 r^{-2}.
\]