Example 6.6 (The Helmholtz Equation in Three Dimensions) In three dimensions and in **rectangular coordinates** $\boldsymbol{r} = (x, y, z)$, the function $f(x, y, z) = X(x)Y(y)Z(z)$ is a solution of the ODE $-\triangle f = k^2 f$ as long as X, Y, and Z satisfy $-X_a'' = a^2 X_a$, $-Y_b'' = b^2 Y_b$, and $-Z_c'' = c^2 Z_c$ with $a^2 + b^2 + c^2 = k^2$. We set $X_a(x) = \alpha \sin ax + \beta \cos ax$ and so forth. Arbitrary linear combinations of the products $X_a\, Y_b\, Z_c$ also are solutions of Helmholtz's equation $-\triangle f = k^2 f$ as long as $a^2 + b^2 + c^2 = k^2$.

In **cylindrical coordinates** (ρ, ϕ, z), the laplacian (6.34) is

$$\nabla \cdot \nabla f = \triangle f = \frac{1}{\rho}\left[\left(\rho\, f_{,\rho}\right)_{,\rho} + \frac{1}{\rho}\, f_{,\phi\phi} + \rho\, f_{,zz} \right] \tag{6.49}$$

and so if we substitute $f(\rho, \phi, z) = \mathrm{P}(\rho)\,\Phi(\phi)\,Z(z)$ into Helmholtz's equation $-\triangle f = \alpha^2 f$ and multiply both sides by $-\rho^2/\mathrm{P}\,\Phi\,Z$, then we get

$$\frac{\rho^2}{f}\triangle f = \frac{\rho^2 \mathrm{P}'' + \rho \mathrm{P}'}{\mathrm{P}} + \frac{\Phi''}{\Phi} + \rho^2 \frac{Z''}{Z} = -\,\alpha^2 \rho^2. \tag{6.50}$$

If we set $Z_k(z) = e^{kz}$, then this equation becomes (6.46) with k^2 replaced by $\alpha^2 + k^2$. Its solution then is

$$f(\rho, \phi, z) = J_n(\sqrt{\alpha^2 + k^2}\rho)\, e^{in\phi}\, e^{kz} \tag{6.51}$$

in which n must be an integer if the solution is to apply to the full range of ϕ from 0 to 2π. The case in which $\alpha = 0$ corresponds to Laplace's equation with solution $f(\rho, \phi, z) = J_n(k\rho)e^{in\phi}e^{kz}$. We could have required Z to satisfy $Z'' = -k^2 Z$. The solution (6.51) then would be

$$f(\rho, \phi, z) = J_n(\sqrt{\alpha^2 - k^2}\,\rho)\, e^{in\phi}\, e^{ikz}. \tag{6.52}$$

But if $\alpha^2 - k^2 < 0$, we write this solution in terms of the **modified Bessel function** $I_n(x) = i^{-n} J_n(ix)$ (section 9.3) as

$$f(\rho, \phi, z) = I_n(\sqrt{k^2 - \alpha^2}\,\rho)\, e^{in\phi}\, e^{ikz}. \tag{6.53}$$

In **spherical coordinates**, the laplacian (6.35) is

$$\triangle f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 f}{\partial \phi^2} \tag{6.54}$$

in which the first term is $r^{-1}(rf)_{,rr}$. If we set $f(r, \theta, \phi) = R(r)\,\Theta(\theta)\,\Phi_m(\phi)$ where $\Phi_m = e^{im\phi}$ and multiply both sides of the Helmholtz equation $-\triangle f = k^2 f$ by $-r^2/R\Theta\Phi$, then we get

$$\frac{\left(r^2 R'\right)'}{R} + \frac{(\sin\theta\,\Theta')'}{\sin\theta\,\Theta} - \frac{m^2}{\sin^2\theta} = -k^2\, r^2. \tag{6.55}$$