6.18 In example 6.29, show that the solutions associated with the roots $r = 0$ and $r = 1$ are the same.

6.19 For a hydrogen atom, we set $V(r) = -e^2 / 4\pi\varepsilon_0 r \equiv -q^2 / r$ in (6.439) and get $(r^2 R''_{n,\ell})' + [(2m/h^2)(E_{n,\ell} + Zq^2 / r)] R_{n,\ell} = 0$. So at big $r$, $R''_{n,\ell} \approx -2mE_{n,\ell}R_{n,\ell}/h^2$ and $R_{n,\ell} \sim \exp(-\sqrt{2mE_{n,\ell}r/h})$. At tiny $r$, $(r^2 R''_{n,\ell})' \approx \ell(\ell + 1)R_{n,\ell}$ and $R_{n,\ell}(r) \sim r^\ell$. Set $R_{n,\ell}(r) = r^\ell \exp(-\sqrt{2mE_{n,\ell}r/h})P_n(\ell)$ and apply the method of Frobenius to find the values of $E_{n,\ell}$ for which $R_{n,\ell}$ is suitably normalizable.

6.20 Show that as long as the matrix $\mathcal{Y}_{kj} = y_k^{(\ell_j)}(x_j)$ is nonsingular, the $n$ boundary conditions

$$b_j = y^{(\ell_j)}(x_j) = \sum_{k=1}^n c_k y_k^{(\ell_j)}(x_j)$$

(6.440)

determine the $n$ coefficients $c_k$ of the expansion (6.222) to be

$$C^T = B^T \mathcal{Y}^{-1} \quad \text{or} \quad C_k = \sum_{j=1}^n b_j \mathcal{Y}^{-1}_{jk}.$$  

(6.441)

6.21 Show that if the real and imaginary parts $u_1$, $u_2$, $v_1$, and $v_2$ of $\psi$ and $\chi$ satisfy boundary conditions at $x = a$ and $x = b$ that make the boundary term (6.240) vanish, then its complex analog (6.242) also vanishes.

6.22 Show that if the real and imaginary parts $u_1$, $u_2$, $v_1$, and $v_2$ of $\psi$ and $\chi$ satisfy boundary conditions at $x = a$ and $x = b$ that make the boundary term (6.240) vanish, and if the differential operator $L$ is real and self adjoint, then (6.238) implies (6.243).

6.23 Show that if $D$ is the set of all twice-differentiable functions $u(x)$ on $[a, b]$ that satisfy Dirichlet’s boundary conditions (6.245) and if the function $p(x)$ is continuous and positive on $[a, b]$, then the adjoint set $D^*$ defined as the set of all twice-differentiable functions $v(x)$ that make the boundary term (6.247) vanish for all functions $u \in D$ is $D$ itself.

6.24 Same as exercise (6.23) but for Neumann boundary conditions (6.246).

6.25 Use Bessel’s equation (6.307) and the boundary conditions $u(0) = 0$ for $n > 0$ and $u(1) = 0$ to show that the eigenvalues $\lambda$ are all positive.

6.26 Show that after the change of variables $u(x) = J_n(kx) = J_n(\rho)$, the self-adjoint differential equation (6.307) becomes Bessel’s equation (6.308).

6.27 Derive Bessel’s inequality (6.378) from the inequality (6.377).

6.28 Repeat example 6.41 using $J_1$’s instead of $J_0$’s. Hint: the Mathematica command Do[Print[N[BesselJZero[1, k], 10]], {k, 1, 100, 1}] gives the first 100 zeros $z_{1,k}$ of the Bessel function $J_1(x)$ to 10 significant figures.