Example 6.6 (The Helmholtz Equation in Three Dimensions) In three dimensions and in **rectangular coordinates** $\boldsymbol{r} = (x, y, z)$, the function $f(x, y, z) = X(x)Y(y)Z(z)$ is a solution of the ODE $-\triangle f = k^2 f$ as long as X, Y, and Z satisfy $-X_a'' = a^2 X_a$, $-Y_b'' = b^2 Y_b$, and $-Z_c'' = c^2 Z_c$ with $a^2 + b^2 + c^2 = k^2$. We set $X_a(x) = \alpha \sin ax + \beta \cos ax$ and so forth. Arbitrary linear combinations of the products $X_a Y_b Z_c$ also are solutions of Helmholtz's equation $-\triangle f = k^2 f$ as long as $a^2 + b^2 + c^2 = k^2$.

In **cylindrical coordinates** (ρ, ϕ, z), the laplacian (6.34) is

$$\nabla \cdot \nabla f = \triangle f = \frac{1}{\rho}\left[\left(\rho\, f_{,\rho}\right)_{,\rho} + \frac{1}{\rho}\, f_{,\phi\phi} + \rho\, f_{,zz}\right] \tag{6.49}$$

and so if we substitute $f(\rho, \phi, z) = \mathrm{P}(\rho)\, \Phi(\phi)\, Z(z)$ into Helmholtz's equation $-\triangle f = \alpha^2 f$ and multiply both sides by $-\rho^2/\mathrm{P}\,\Phi\, Z$, then we get

$$\frac{\rho^2}{f}\triangle f = \frac{\rho^2 \mathrm{P}'' + \rho \mathrm{P}'}{\mathrm{P}} + \frac{\Phi''}{\Phi} + \rho^2 \frac{Z''}{Z} = -\,\alpha^2 \rho^2. \tag{6.50}$$

If we set $Z_k(z) = e^{kz}$, then this equation becomes (6.46) with k^2 replaced by $\alpha^2 + k^2$. Its solution then is

$$f(\rho, \phi, z) = J_n(\sqrt{\alpha^2 + k^2}\rho)\, e^{in\phi}\, e^{kz} \tag{6.51}$$

in which n must be an integer if the solution is to apply to the full range of ϕ from 0 to 2π. The case in which $\alpha = 0$ corresponds to Laplace's equation with solution $f(\rho, \phi, z) = J_n(k\rho)e^{in\phi}e^{kz}$. We could have required Z to satisfy $Z'' = -k^2 Z$. The solution (6.51) then would be

$$f(\rho, \phi, z) = J_n(\sqrt{\alpha^2 - k^2}\,\rho)\, e^{in\phi}\, e^{ikz}. \tag{6.52}$$

But if $\alpha^2 - k^2 < 0$, we write this solution in terms of the **modified Bessel function** $I_n(x) = i^{-n} J_n(ix)$ (section 9.3) as

$$f(\rho, \phi, z) = I_n(\sqrt{k^2 - \alpha^2}\,\rho)\, e^{in\phi}\, e^{ikz}. \tag{6.53}$$

In **spherical coordinates**, the laplacian (6.35) is

$$\triangle f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 f}{\partial \phi^2} \tag{6.54}$$

in which the first term is $r^{-1}(rf)_{,rr}$. If we set $f(r, \theta, \phi) = R(r)\,\Theta(\theta)\,\Phi_m(\phi)$ where $\Phi_m = e^{im\phi}$ and multiply both sides of the Helmholtz equation $-\triangle f = k^2 f$ by $-r^2/R\Theta\Phi$, then we get

$$\frac{\left(r^2 R'\right)'}{R} + \frac{(\sin\theta\,\Theta')'}{\sin\theta\,\Theta} - \frac{m^2}{\sin^2\theta} = -k^2\, r^2. \tag{6.55}$$

The first term is a function of r, the next two terms are functions of θ, and the last term is a constant. So we set the r-dependent terms equal to a constant $\ell(\ell+1) - k^2$ and the θ-dependent terms equal to $-\ell(\ell+1)$, and we require the **associated Legendre function** $\Theta_{\ell,m}(\theta)$ to satisfy (8.91)

$$\left(\sin\theta\, \Theta'_{\ell,m}\right)' / \sin\theta + \left[\ell(\ell+1) - m^2/\sin^2\theta\right] \Theta_{\ell,m} = 0. \tag{6.56}$$

If $\Phi(\phi) = e^{im\phi}$ is to be single valued for $0 \le \phi \le 2\pi$, then the parameter m must be an integer. The constant ℓ also must be an integer with $-\ell \le m \le \ell$ (example 6.29, section 8.12) if $\Theta_{\ell,m}(\theta)$ is to be single valued and finite for $0 \le \theta \le \pi$. The product $f = R\,\Theta\,\Phi$ then will obey Helmholtz's equation $-\triangle f = k^2 f$ if the radial function $R_{k,\ell}(r) = j_\ell(kr)$ satisfies

$$\left(r^2 R'_{k,\ell}\right)' + \left[k^2 r^2 - \ell(\ell+1)\right] R_{k,\ell} = 0 \tag{6.57}$$

which it does because the **spherical Bessel function** $j_\ell(x)$ obeys Bessel's equation (9.63)

$$\left(x^2\, j'_\ell\right)' + [x^2 - \ell(\ell+1)]\, j_\ell = 0. \tag{6.58}$$

In three dimensions, Helmholtz's equation separates in 11 standard coordinate systems (Morse and Feshbach, 1953, pp. 655–664). □

6.6 Wave Equations

You can easily solve some of the linear homogeneous partial differential equations of electrodynamics (exercise 6.6) and quantum field theory.

Example 6.7 (The Klein-Gordon Equation) In Minkowski space, the analog of the laplacian in natural units ($\hbar = c = 1$) is (summing over a from 0 to 3)

$$\Box = \partial_a \partial^a = \triangle - \frac{\partial^2}{\partial x^{02}} = \triangle - \frac{\partial^2}{\partial t^2} \tag{6.59}$$

and the Klein-Gordon wave equation is

$$\left(\Box - m^2\right) A(x) = \left(\triangle - \frac{\partial^2}{\partial t^2} - m^2\right) A(x) = 0. \tag{6.60}$$

If we set $A(x) = B(px)$ where $px = p_a x^a = \boldsymbol{p} \cdot \boldsymbol{x} - p^0 x^0$, then the kth partial derivative of A is p_k times the first derivative of B

$$\frac{\partial}{\partial x^k} A(x) = \frac{\partial}{\partial x^k} B(px) = p_k B'(px) \tag{6.61}$$

and so the Klein-Gordon equation (6.60) becomes

$$\left(\Box - m^2\right) A = (\boldsymbol{p}^2 - (p^0)^2)B'' - m^2 B = \; p^2 B'' - m^2 B = 0 \qquad (6.62)$$

in which $p^2 = \boldsymbol{p}^2 - (p^0)^2$. Thus if $B(p \cdot x) = \exp(ip \cdot x)$ so that $B'' = -B$, and if the energy-momentum 4-vector $(p^0, \boldsymbol{p})$ satisfies $p^2 + m^2 = 0$, then $A(x)$ will satisfy the Klein-Gordon equation. The condition $p^2 + m^2 = 0$ relates the energy $p^0 = \sqrt{\boldsymbol{p}^2 + m^2}$ to the momentum $\boldsymbol{p}$ for a particle of mass m. □

Example 6.8 (Field of a Spinless Boson) The quantum field

$$\phi(x) = \int \frac{d^3p}{\sqrt{2p^0(2\pi)^3}} \left[a(\boldsymbol{p})e^{ipx} + a^\dagger(\boldsymbol{p})e^{-ipx}\right] \qquad (6.63)$$

describes spinless bosons of mass m. It satisfies the Klein-Gordon equation $\left(\Box - m^2\right)\phi(x) = 0$ because $p^0 = \sqrt{\boldsymbol{p}^2 + m^2}$. The operators $a(\boldsymbol{p})$ and $a^\dagger(\boldsymbol{p})$ respectively represent the annihilation and creation of the bosons and obey the commutation relations

$$[a(\boldsymbol{p}), a^\dagger(\boldsymbol{p}')] = \delta^3(\boldsymbol{p} - \boldsymbol{p}') \;\text{ and }\; [a(\boldsymbol{p}), a(\boldsymbol{p}')] = [a^\dagger(\boldsymbol{p}), a^\dagger(\boldsymbol{p}')] = 0 \qquad (6.64)$$

in units with $\hbar = c = 1$. These relations make the field $\phi(x)$ and its time derivative $\dot{\phi}(y)$ satisfy **the canonical equal-time commutation relations**

$$[\phi(\boldsymbol{x}, t), \dot{\phi}(\boldsymbol{y}, t)] = i\,\delta^3(\boldsymbol{x} - \boldsymbol{y}) \;\text{ and }\; [\phi(\boldsymbol{x}, t), \phi(\boldsymbol{y}, t)] = [\dot{\phi}(\boldsymbol{x}, t), \dot{\phi}(\boldsymbol{y}, t)] = 0 \qquad (6.65)$$

in which the dot means time derivative. □

Example 6.9 (Field of the Photon) The electromagnetic field has four components, but in the Coulomb or radiation gauge $\nabla \cdot \boldsymbol{A}(x) = 0$, the component A_0 is a function of the charge density, and the vector potential $\boldsymbol{A}$ in the absence of charges and currents satisfies the wave equation $\Box \boldsymbol{A}(x) = 0$ for a spin-one massless particle. We write it as

$$\boldsymbol{A}(x) = \sum_{s=1}^{2} \int \frac{d^3p}{\sqrt{2p^0(2\pi)^3}} \left[\boldsymbol{e}(\boldsymbol{p}, s)\, a(\boldsymbol{p}, \boldsymbol{s})\, e^{ipx} + \boldsymbol{e}^*(\boldsymbol{p}, s)\, a^\dagger(\boldsymbol{p}, \boldsymbol{s})\, e^{-ipx}\right] \qquad (6.66)$$

in which the sum is over the two possible polarizations s. The energy p^0 is equal to the modulus $|\boldsymbol{p}|$ of the momentum because the photon is massless, $p^2 = 0$. The dot-product of the polarization vectors $\boldsymbol{e}(\boldsymbol{p}, s)$ with the momentum vanishes $\boldsymbol{p} \cdot \boldsymbol{e}(\boldsymbol{p}, s) = 0$ so as to respect the gauge condition