We can relate this to a Bessel function by setting $\lambda = (|x|/2m)\exp(-\alpha)$

$$\begin{aligned} G(x) &= \frac{1}{(4\pi)^{n/2}} \left(\frac{2m}{x}\right)^{(n/2-1)} \int_{-\infty}^{\infty} e^{-mx\cosh\alpha + (n/2-1)\alpha}\, d\alpha \\ &= \frac{2}{(4\pi)^{n/2}} \left(\frac{2m}{x}\right)^{(n/2-1)} \int_{0}^{\infty} e^{-mx\cosh\alpha} \cosh(n/2-1)\alpha\, d\alpha \\ &= \frac{2}{(4\pi)^{n/2}} \left(\frac{2m}{x}\right)^{(n/2-1)} K_{n/2-1}(mx) \end{aligned} \tag{5.151}$$

where $x = |x| = \sqrt{x^2}$ and K is a modified Bessel function of the second kind (9.98). If $n = 3$, this is (exercise 5.27) the Yukawa potential (5.141). □

Example 5.24 (A Fourier Transform) As another example, let's consider the integral

$$J(x) = \int_{-\infty}^{\infty} \frac{e^{ikx}}{(k^2+m^2)^2}\, dk. \tag{5.152}$$

We may add ghost contours as in the preceding example, but now the integrand has double poles at $k = \pm im$, and so we must use Cauchy's integral formula (5.36) for the case of $n = 1$, which is Eq.(5.34). For $x > 0$, we add a ghost contour in the UHP and find

$$\begin{aligned} J(x) &= \oint \frac{e^{ikx}}{(k+im)^2(k-im)^2}\, dk = 2\pi i \frac{d}{dk} \frac{e^{ikx}}{(k+im)^2}\bigg|_{k=im} \\ &= \frac{\pi}{2m^2}\left(x + \frac{1}{m}\right) e^{-mx}. \end{aligned} \tag{5.153}$$

If $x < 0$, then we add a ghost contour in the LHP and find

$$\begin{aligned} J(x) &= \oint \frac{e^{ikx}}{(k+im)^2(k-im)^2}\, dk = -2\pi i \frac{d}{dk} \frac{e^{ikx}}{(k-im)^2}\bigg|_{k=-im} \\ &= \frac{\pi}{2m^2}\left(-x + \frac{1}{m}\right) e^{mx}. \end{aligned} \tag{5.154}$$

Putting the two together, we get

$$J(x) = \int_{-\infty}^{\infty} \frac{e^{ikx}}{(k^2+m^2)^2} dk = \frac{\pi}{2m^2}\left(|x| + \frac{1}{m}\right) e^{-m|x|}. \tag{5.155}$$

as the Fourier transform of $1/(k^2+m^2)^2$. □

Example 5.25 (Integral of a Complex Gaussian) As another example of the use of ghost contours, let us use one to do the integral

$$I = \int_{-\infty}^{\infty} e^{wx^2}\, dx \tag{5.156}$$