which satisfies the condition (4.112) that defines an asymptotic series

$$\begin{aligned}\lim_{x\to\infty} x^n R_n(x) &= \lim_{x\to\infty} (-1)^n \frac{n!\, e^{-x}}{x} \int_0^\infty e^{-u} \frac{du}{\left(1+\frac{u}{x}\right)^{n+1}} \\ &= \lim_{x\to\infty} (-1)^n \frac{n!\, e^{-x}}{x} \int_0^\infty e^{-u}\, du \\ &= \lim_{x\to\infty} (-1)^n \frac{n!\, e^{-x}}{x} = 0 \end{aligned} \tag{4.123}$$

for fixed n. □

Asymptotic series often occur in physics. In such physical problems, a small parameter λ usually plays the role of $1/x$. A perturbative series

$$S_n(\lambda) = \sum_{k=0}^{n} a_k\, \lambda^k \tag{4.124}$$

is an asymptotic expansion of the physical quantity $S(\lambda)$ if the remainder

$$R_n(\lambda) = S(\lambda) - S_n(\lambda) \tag{4.125}$$

satisfies for fixed n

$$\lim_{\lambda\to 0} \lambda^{-n}\, R_n(\lambda) = 0. \tag{4.126}$$

The WKB approximation and the Dyson series for quantum electrodynamics are asymptotic expansions in this sense.

4.13 Some Electrostatic Problems

Gauss's law $\boldsymbol{\nabla} \cdot \boldsymbol{D} = \rho$ equates the divergence of the **electric displacement** $\boldsymbol{D}$ to the density ρ of **free charges** (charges that are free to move in or out of the dielectric medium—as opposed to those that are part of the medium and bound to it by molecular forces). In electrostatic problems, Maxwell's equations reduce to Gauss's law and the static form $\boldsymbol{\nabla} \times \mathbf{E} = 0$ of Faraday's law which implies that the electric field $\boldsymbol{E}$ is the gradient of an electrostatic potential $\boldsymbol{E} = -\boldsymbol{\nabla} V$. (James Maxwell 1831–1879, Michael Faraday 1791–1867)

Across an interface with normal vector $\hat{\boldsymbol{n}}$ between two dielectrics, the tangential electric field is continuous while the normal electric displacement jumps by the surface density of free charge σ

$$\hat{\boldsymbol{n}} \times (\boldsymbol{E}_2 - \boldsymbol{E}_1) = 0 \quad \text{and} \quad \sigma = \hat{\boldsymbol{n}} \cdot (\boldsymbol{D}_2 - \boldsymbol{D}_1)\,. \tag{4.127}$$

In a **linear isotropic dielectric**, the electric displacement $\boldsymbol{D}$ is proportional to the electric field $\boldsymbol{D} = \epsilon_m \boldsymbol{E}$, where the **permittivity** $\epsilon_m = \epsilon_0 + \chi_m = K_m \epsilon_0$ of the material differs from that of the vacuum ϵ_0 by the **electric susceptibility** χ_m and by the **relative permittivity** K_m. The permittivity of the vacuum is the **electric constant** $\epsilon_0 = 8.85418782 \times 10^{-12}$ F/m.

An electric field $\boldsymbol{E}$ exerts on a charge q a **force** $\boldsymbol{F} = q\boldsymbol{E}$ even in a dielectric medium. The electrostatic energy W of a system of linear dielectrics is the volume integral

$$W = \frac{1}{2} \int \boldsymbol{D} \cdot \boldsymbol{E} \, d^3 r. \tag{4.128}$$

Example 4.15 (Field of a Charge Near an Interface) Consider two semi-infinite dielectrics of permittivities ϵ_1 and ϵ_2 separated by an infinite horizontal x-y-plane. What is the electrostatic potential due to a charge q in region 1 at a height h above the interface?

The easy way to solve this problem is to put an image charge q' at the same distance from the interface in region 2 so that the potential in region 1 is

$$V_1(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_1} \left(\frac{q}{\sqrt{x^2 + y^2 + (z-h)^2}} + \frac{q'}{\sqrt{x^2 + y^2 + (z+h)^2}} \right). \tag{4.129}$$

This potential satisfies Gauss's law $\boldsymbol{\nabla} \cdot \boldsymbol{D} = \rho$ in region 1. In region 2, the potential

$$V_2(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_2} \, \frac{q''}{\sqrt{x^2 + y^2 + (z-h)^2}} \tag{4.130}$$

also satisfies Gauss's law. The continuity (4.127) of the tangential component of $\boldsymbol{E}$ tells us that the partial derivatives of V_1 and V_2 in the x (or y) direction must be the same at $z = 0$

$$\frac{\partial V_1(x, y, 0)}{\partial x} = \frac{\partial V_2(x, y, 0)}{\partial x}. \tag{4.131}$$

The discontinuity equation (4.127) for the electric displacement says that at the interface at $z = 0$ with no surface charge

$$\epsilon_1 \frac{\partial V_1(x, y, 0)}{\partial z} = \epsilon_2 \frac{\partial V_2(x, y, 0)}{\partial z}. \tag{4.132}$$

These two equations (4.131 & 4.132) allow one to solve for q' and q''

$$q' = \frac{\epsilon_1 - \epsilon_2}{\epsilon_1 + \epsilon_2} \, q \quad \text{and} \quad q'' = \frac{2\epsilon_2}{\epsilon_1 + \epsilon_2} \, q. \tag{4.133}$$