real. Thus the Fourier series (2.12) for a real function $f(x)$ is

$$\begin{aligned} f(x) &= d_0 + \sum_{n=1}^{\infty} d_n\, e^{inx} + \sum_{n=-\infty}^{-1} d_n\, e^{inx} \\ &= d_0 + \sum_{n=1}^{\infty} \left[d_n\, e^{inx} + d_{-n}\, e^{-inx} \right] = d_0 + \sum_{n=1}^{\infty} \left[d_n\, e^{inx} + d_n^*\, e^{-inx} \right] \\ &= d_0 + \sum_{n=1}^{\infty} d_n\,(\cos nx + i \sin nx) + d_n^*\,(\cos nx - i \sin nx) \\ &= d_0 + \sum_{n=1}^{\infty} (d_n + d_n^*)\, \cos nx + i(d_n - d_n^*)\, \sin nx. \end{aligned} \tag{2.18}$$

Let's write d_n as

$$d_n = \frac{1}{2}\,(a_n - ib_n), \quad \text{so that} \quad a_n = d_n + d_n^* \quad \text{and} \quad b_n = i(d_n - d_n^*). \tag{2.19}$$

Then the Fourier series (2.18) for a real function $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\, \cos nx + b_n\, \sin nx. \tag{2.20}$$

What are the formulas for a_n and b_n? By (2.19 & 2.12), the coefficient a_n is

$$a_n = \int_0^{2\pi} \left[e^{-inx}\, f(x) + e^{inx}\, f^*(x) \right] \frac{dx}{2\pi} = \int_0^{2\pi} \frac{\left(e^{-inx} + e^{inx}\right)}{2}\, f(x)\, \frac{dx}{\pi} \tag{2.21}$$

since the function $f(x)$ is real. So the coefficient a_n of $\cos nx$ in (2.20) is the cosine integral of $f(x)$

$$a_n = \int_0^{2\pi} \cos nx\, f(x)\, \frac{dx}{\pi} = \int_{-\pi}^{\pi} \cos nx\, f(x)\, \frac{dx}{\pi}. \tag{2.22}$$

Similarly, equations (2.19 & 2.12) and the reality of $f(x)$ imply that the coefficient b_n is the sine integral of $f(x)$

$$b_n = \int_0^{2\pi} i \frac{\left(e^{-inx} - e^{inx}\right)}{2} f(x)\, \frac{dx}{\pi} = \int_0^{2\pi} \sin nx\, f(x)\, \frac{dx}{\pi} = \int_{-\pi}^{\pi} \sin nx\, f(x)\, \frac{dx}{\pi}. \tag{2.23}$$

The real Fourier series (2.20) and the cosine (2.22) and sine (2.23) integrals