real. Thus the Fourier series (2.12) for a real function \( f(x) \) is

\[
f(x) = d_0 + \sum_{n=1}^{\infty} d_n e^{inx} + \sum_{n=-\infty}^{-1} d_n e^{inx}
\]

\[
= d_0 + \sum_{n=1}^{\infty} [d_n e^{inx} + d_{-n} e^{-inx}] = d_0 + \sum_{n=1}^{\infty} [d_n e^{inx} + d_n^* e^{-inx}]
\]

\[
= d_0 + \sum_{n=1}^{\infty} d_n (\cos nx + i \sin nx) + d_n^* (\cos nx - i \sin nx)
\]

\[
= d_0 + \sum_{n=1}^{\infty} (d_n + d_n^*) \cos nx + i(d_n - d_n^*) \sin nx.
\] (2.18)

Let’s write \( d_n \) as

\[
d_n = \frac{1}{2} (a_n - ib_n), \text{ so that } a_n = d_n + d_n^* \text{ and } b_n = i(d_n - d_n^*). \quad (2.19)
\]

Then the Fourier series (2.18) for a real function \( f(x) \) is

\[
f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx. \quad (2.20)
\]

What are the formulas for \( a_n \) and \( b_n \)? By (2.19 & 2.12), the coefficient \( a_n \) is

\[
a_n = \int_0^{2\pi} [e^{-inx} f(x) + e^{inx} f^*(x)] \frac{dx}{2\pi} = \int_0^{2\pi} \frac{e^{-inx} + e^{inx}}{2} f(x) \frac{dx}{\pi}
\]

since the function \( f(x) \) is real. So the coefficient \( a_n \) of \( \cos nx \) in (2.20) is the cosine integral of \( f(x) \)

\[
a_n = \int_0^{2\pi} \cos nx f(x) \frac{dx}{\pi} = \int_{-\pi}^{\pi} \cos nx f(x) \frac{dx}{\pi}. \quad (2.22)
\]

Similarly, equations (2.19 & 2.12) and the reality of \( f(x) \) imply that the coefficient \( b_n \) is the sine integral of \( f(x) \)

\[
b_n = \int_0^{2\pi} i \frac{e^{-inx} - e^{inx}}{2} f(x) \frac{dx}{\pi} = \int_0^{2\pi} \sin nx f(x) \frac{dx}{\pi} = \int_{-\pi}^{\pi} \sin nx f(x) \frac{dx}{\pi}. \quad (2.23)
\]

The real Fourier series (2.20) and the cosine (2.22) and sine (2.23) integrals