Fourier Series

and (2.3) as

\[
    f(x) = \sum_{n=-\infty}^{\infty} d_n e^{inx} \quad \text{and} \quad d_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} \, dx.
\]

(2.12)

One also may use the rules

\[
    f(x) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} c_n e^{inx} \quad \text{and} \quad c_n = \int_{-\pi}^{\pi} f(x) e^{-inx} \, dx.
\]

(2.13)

**Example 2.3** (Fourier Series for \(\exp(-2|x|)\)) Let’s compute the Fourier series for the real function \(f(x) = \exp(-2|x|)\) on the interval \((-\pi, \pi)\). Using Eq.(2.10) for the shifted interval and the \(2\pi\)-placement convention (2.12),
we find for the coefficient $d_n$

$$d_n = \int_{-\pi}^{\pi} \frac{dx}{2\pi} e^{-inx} e^{-m|x|}$$

(2.14)

which we may split into the two pieces

$$d_n = \int_{0}^{\pi} \frac{dx}{2\pi} e^{(m-in)x} + \int_{0}^{\pi} \frac{dx}{2\pi} e^{-(m+in)x}.$$  (2.15)

After doing the integrals, we find

$$d_n = \frac{1}{\pi} \frac{m}{m^2 + n^2} \left[ 1 - (-1)^n e^{-\pi m} \right].$$  (2.16)

Here since $m$ is real, $d_n = d^*_n$, but also $d_n = d_{-n}$. So the coefficients $d_n$ satisfy the condition (2.7) that holds when the function $f(x)$ is real, $d_n = d^*_{-n}$. The Fourier series for $\exp(-m|x|)$ with $d_n$ given by (2.16) is

$$e^{-m|x|} = \sum_{n=-\infty}^{\infty} d_n e^{inx} = \sum_{n=-\infty}^{\infty} \frac{1}{\pi} \frac{m}{m^2 + n^2} \left[ 1 - (-1)^n e^{-\pi m} \right] e^{inx}$$

$$= \frac{(1 - e^{-\pi m})}{m\pi} + \sum_{n=1}^{\infty} \frac{2}{\pi} \frac{m}{m^2 + n^2} \left[ 1 - (-1)^n e^{-\pi m} \right] \cos(nx).$$  (2.17)

In Fig. 2.2, the 10-term (dashes) Fourier series for $m = 2$ is plotted from $x = -2\pi$ to $x = 2\pi$. The function $\exp(-2|x|)$ itself is represented by a solid line. Although it is not periodic, its Fourier series is periodic with period $2\pi$. The 10-term Fourier series represents the function $\exp(-2|x|)$ quite well within the interval $[-\pi, \pi]$.

In what follows, we usually won’t bother to use different letters to distinguish between the symmetric (2.2 & 2.3) and asymmetric (2.12 & 2.13) conventions on the placement of the $2\pi$’s.

\[\square\]

2.4 Real Fourier Series for Real Functions

The rules (2.1–2.3 and 2.10–2.13) for Fourier series are simple and apply to functions that are continuous and periodic — whether complex or real. If the function $f(x)$ is real, then by (2.7) $d_{-n} = d^*_n$, whence $d_0 = d^*_0$, so $d_0$ is