To see why a normal matrix can be diagonalized by a unitary transformation, let us consider an \( N \times N \) normal matrix \( V \) which (since it is square (section 1.25)) has \( N \) eigenvectors \( |n\rangle \) with eigenvalues \( v_n \)

\[
(V - v_n I) |n\rangle = 0. 
\]  
(1.311)

The square of the norm (1.80) of this vector must vanish

\[
\| (V - v_n I) |n\rangle \|^2 = \langle n | (V - v_n I) \dagger (V - v_n I) |n\rangle = 0. 
\]  
(1.312)

But since \( V \) is normal, we also have

\[
\langle n | (V - v_n I) \dagger (V - v_n I) |n\rangle = \langle n | (V - v_n I) \dagger (V - v_n I) \dagger |n\rangle. 
\]  
(1.313)

So the square of the norm of the vector \( (V - v_n I) |n\rangle \) also vanishes \( \| (V - v_n I) |n\rangle \|^2 = 0 \) which tells us that \( |n\rangle \) also is an eigenvector of \( V \) with eigenvalue \( v_n \)

\[
V |n\rangle = v_n |n\rangle \quad \text{and so} \quad \langle n | V = v_n \langle n |. 
\]  
(1.314)

If now \( |m\rangle \) is an eigenvector of \( V \) with eigenvalue \( v_m \)

\[
V |m\rangle = v_m |m\rangle 
\]  
(1.315)

then we have

\[
\langle n | V |m\rangle = v_m \langle n |m\rangle 
\]  
(1.316)

and from (1.314)

\[
\langle n | V |m\rangle = v_n \langle n |m\rangle. 
\]  
(1.317)

Subtracting (1.316) from (1.317), we get

\[
(v_n - v_m) \langle n |m\rangle = 0 
\]  
(1.318)

which shows that any two eigenvectors of a normal matrix \( V \) with different eigenvalues are orthogonal.

Usually, all \( N \) eigenvalues of an \( N \times N \) normal matrix are different. In this case, all the eigenvectors are orthogonal and may be individually normalized. But even when a set \( D \) of eigenvectors has the same (degenerate) eigenvalue, one may use the argument (1.291–1.297) to find a suitable set of orthonormal eigenvectors with that eigenvalue. Thus every \( N \times N \) normal matrix has \( N \) orthonormal eigenvectors. It follows then from the argument of equations (1.300–1.303) that every \( N \times N \) normal matrix \( V \) can be diagonalized by an \( N \times N \) unitary matrix \( U \)

\[
V = U V^{(d)} U^\dagger 
\]  
(1.319)