To see why a normal matrix can be diagonalized by a unitary transformation, let us consider an $N \times N$ normal matrix V which (since it is square (section 1.25)) has N eigenvectors $|n\rangle$ with eigenvalues v_n

$$(V - v_n I)\,|n\rangle = 0. \tag{1.311}$$

The square of the norm (1.80) of this vector must vanish

$$\|\,(V - v_n I)\,|n\rangle\,\|^2 = \langle n|\,(V - v_n I)^\dagger\,(V - v_n I)\,|n\rangle = 0. \tag{1.312}$$

But since V is normal, we also have

$$\langle n|\,(V - v_n I)^\dagger\,(V - v_n I)\,|n\rangle = \langle n|\,(V - v_n I)\,(V - v_n I)^\dagger\,|n\rangle. \tag{1.313}$$

So the square of the norm of the vector $\left(V^\dagger - v_n^* I\right)|n\rangle = (V - v_n I)^\dagger\,|n\rangle$ also vanishes $\|\,\left(V^\dagger - v_n^* I\right)|n\rangle\,\|^2 = 0$ which tells us that $|n\rangle$ also is an eigenvector of $V^\dagger$ with eigenvalue v_n^*

$$V^\dagger|n\rangle = v_n^*|n\rangle \quad \text{and so} \quad \langle n|V = v_n\langle n|. \tag{1.314}$$

If now $|m\rangle$ is an eigenvector of V with eigenvalue v_m

$$V|m\rangle = v_m|m\rangle \tag{1.315}$$

then we have

$$\langle n|V|m\rangle = v_m\langle n|m\rangle \tag{1.316}$$

and from (1.314)

$$\langle n|V|m\rangle = v_n\langle n|m\rangle. \tag{1.317}$$

Subtracting (1.316) from (1.317), we get

$$(v_n - v_m)\,\langle n|m\rangle = 0 \tag{1.318}$$

which shows that **any two eigenvectors of a normal matrix V with different eigenvalues are orthogonal**.

Usually, all N eigenvalues of an $N \times N$ normal matrix are different. In this case, all the eigenvectors are orthogonal and may be individually normalized. But even when a set D of eigenvectors has the same (degenerate) eigenvalue, one may use the argument (1.291–1.297) to find a suitable set of orthonormal eigenvectors with that eigenvalue. Thus **every $N \times N$ normal matrix has N orthonormal eigenvectors**. It follows then from the argument of equations (1.300–1.303) that every $N \times N$ normal matrix V can be diagonalized by an $N \times N$ unitary matrix U

$$V = UV^{(d)}U^\dagger \tag{1.319}$$