i with the column index k held fixed

$$\det A = \sum_{i=1}^{N} A_{ik} C_{ik} = \sum_{i=1}^{N} A_{ki} C_{ki} \tag{1.195}$$

in order to prove that

$$\delta_{k\ell} \det A = \sum_{i=1}^{N} A_{ik} C_{i\ell} = \sum_{i=1}^{N} A_{ki} C_{\ell i}. \tag{1.196}$$

For $k = \ell$, this formula just repeats Laplace's expansion (1.195). But for $k \neq \ell$, it is Laplace's expansion for the determinant of a matrix A' that is the same as A but with its ℓth column replaced by its kth one. Since the matrix A' has two identical columns, its determinant vanishes, which explains (1.196) for $k \neq \ell$.

This rule (1.196) provides a formula for the inverse of a matrix A whose determinant does not vanish. Such matrices are said to be **nonsingular**. The inverse A^{-1} of an $N \times N$ nonsingular matrix A is the transpose of the matrix of cofactors divided by $\det A$

$$\left(A^{-1}\right)_{\ell i} = \frac{C_{i\ell}}{\det A} \quad \text{or} \quad A^{-1} = \frac{C^{\mathsf{T}}}{\det A}. \tag{1.197}$$

To verify this formula, we use it for A^{-1} in the product $A^{-1}A$ and note that by (1.196) the ℓkth entry of the product $A^{-1}A$ is just $\delta_{\ell k}$

$$\left(A^{-1}A\right)_{\ell k} = \sum_{i=1}^{N} \left(A^{-1}\right)_{\ell i} A_{ik} = \sum_{i=1}^{N} \frac{C_{i\ell}}{\det A} A_{ik} = \delta_{\ell k}. \tag{1.198}$$

Example 1.26 (Inverting a 2×2 Matrix) Let's apply our formula (1.197) to find the inverse of the general 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{1.199}$$

We find then

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \tag{1.200}$$

which is the correct inverse as long as $ad \neq bc$. □

The simple example of matrix multiplication

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} a & xa + b & ya + zb + c \\ d & xd + e & yd + ze + f \\ g & xg + h & yg + zh + i \end{pmatrix} \tag{1.201}$$