We use them to represent rotations, translations, Lorentz transformations, and internal-symmetry transformations.

### 1.20 Determinants

The **determinant** of a $2 \times 2$ matrix $A$ is

$$
\det A = |A| = A_{11}A_{22} - A_{21}A_{12}.
$$

(1.175)

In terms of the $2 \times 2$ antisymmetric $(e_{ij} = -e_{ji})$ matrix $e_{12} = 1 = -e_{21}$ with $e_{11} = e_{22} = 0$, this determinant is

$$
\det A = \sum_{i=1}^{2} \sum_{j=1}^{2} e_{ij}A_{i1}A_{j2}.
$$

(1.176)

It’s also true that

$$
e_{k\ell} \det A = \sum_{i=1}^{2} \sum_{j=1}^{2} e_{ij}A_{ik}A_{j\ell}.
$$

(1.177)

These definitions and results extend to any square matrix. If $A$ is a $3 \times 3$ matrix, then its determinant is

$$
\det A = \sum_{ijk=1}^{3} e_{ijk}A_{i1}A_{j2}A_{k3}
$$

(1.178)

in which $e_{ijk}$ is totally antisymmetric with $e_{123} = 1$, and the sums over $i, j,$ & $k$ run from 1 to 3. More explicitly, this determinant is

$$
\det A = \sum_{ijk=1}^{3} e_{ijk}A_{i1}A_{j2}A_{k3}
\quad = \sum_{i=1}^{3} A_{i1} \sum_{j=1}^{3} e_{ijk}A_{j2}A_{k3}
\quad = A_{11} (A_{22}A_{33} - A_{32}A_{23}) + A_{21} (A_{32}A_{13} - A_{12}A_{33})
\quad \quad + A_{31} (A_{12}A_{23} - A_{22}A_{13}).
$$

(1.179)

The **minor** $M_{i\ell}$ of the matrix $A$ is the $2 \times 2$ determinant of the matrix $A$ without row $i$ and column $\ell$, and the **cofactor** $C_{i\ell}$ is the minor $M_{i\ell}$ multiplied by $(-1)^{i+\ell}$. Thus $\det A$ is the sum

$$
\det A = A_{11}(-1)^2 (A_{22}A_{33} - A_{32}A_{23}) + A_{21}(-1)^3 (A_{12}A_{33} - A_{32}A_{13})
\quad + A_{31}(-1)^4 (A_{12}A_{23} - A_{22}A_{13})
\quad = A_{11}C_{11} + A_{21}C_{21} + A_{31}C_{31}
$$

(1.180)
of the products $A_{i1}C_{i1} = A_{i1}(-1)^{i+1}M_{i1}$ where

$$
C_{11} = (-1)^2 M_{11} = A_{22}A_{33} - A_{23}A_{32}
$$

$$
C_{21} = (-1)^3 M_{21} = A_{32}A_{13} - A_{12}A_{33}
$$

$$
C_{31} = (-1)^4 M_{31} = A_{12}A_{23} - A_{22}A_{13}.
$$

(1.181)

**Example 1.25 (Determinant of a $3 \times 3$ Matrix)**  The determinant of a $3 \times 3$ matrix is the dot product of the vector of its first row with the cross-product of the vectors of its second and third rows:

$$
\begin{vmatrix}
U_1 & U_2 & U_3 \\
V_1 & V_2 & V_3 \\
W_1 & W_2 & W_3
\end{vmatrix} = \sum_{ijk=1}^{3} e_{ijk} U_i V_j W_k = \sum_{i=1}^{3} U_i (V \times W)_i = U \cdot (V \times W)
$$

which is called the scalar triple product.

Laplace used the totally antisymmetric symbol $e_{i_1i_2...i_N}$ with $N$ indices and with $e_{123...N} = 1$ to define the determinant of an $N \times N$ matrix $A$ as

$$
\det A = \sum_{i_1i_2...i_N=1}^{N} e_{i_1i_2...i_N} A_{i_11}A_{i_22}...A_{i_NN}
$$

(1.182)

in which the sums over $i_1...i_N$ run from 1 to $N$. In terms of cofactors, two forms of his expansion of this determinant are

$$
\det A = \sum_{i=1}^{N} A_{ik}C_{ik} = \sum_{k=1}^{N} A_{ik}C_{ik}
$$

(1.183)

in which the first sum is over the row index $i$ but not the (arbitrary) column index $k$, and the second sum is over the column index $k$ but not the (arbitrary) row index $i$. The cofactor $C_{ik}$ is $(-1)^{i+k}M_{ik}$ in which the minor $M_{ik}$ is the determinant of the $(N-1) \times (N-1)$ matrix $A$ without its $i$th row and $k$th column. It’s also true that

$$
\sum_{i_1i_2...i_N=1}^{N} e_{i_1i_2...i_N} A_{i_1k_1}A_{i_2k_2}...A_{i_Nk_N}.
$$

(1.184)

The key feature of a determinant is that it is an antisymmetric combination of products of the elements $A_{ik}$ of a matrix $A$. One implication of this antisymmetry is that the interchange of any two rows or any two columns changes the sign of the determinant. Another is that if one adds a multiple of one column to another column, for example a multiple $xA_{i2}$ of column 2