and its adjoint $A^\dagger$ is the operator $IA^\dagger I$

$$A^\dagger = \sum_{n,\ell=1}^{N} |n\rangle\langle n|A^\dagger|\ell\rangle = \sum_{n,\ell=1}^{N} |\ell\rangle\langle n|A|\ell\rangle^*|n\rangle = \sum_{n,\ell=1}^{N} |n\rangle\langle \ell|A^*|n\rangle^*|\ell\rangle$$

in which we used the definition (1.147) of the adjoint of an outer product and then interchanged $\ell$ and $n$. It follows that $\langle n|A^\dagger|\ell\rangle = \langle \ell|A^*|n\rangle^*$ so that the matrix $A^\dagger_{n\ell}$ that represents $A^\dagger$ in this basis is

$$A^\dagger_{n\ell} = \langle n|A^\dagger|\ell\rangle = \langle \ell|A^*|n\rangle^* = A^*_{\ell n} = A^*_{n\ell}$$

(1.149)

in agreement with our definiton (1.28) of the adjoint of a matrix as the transpose of its complex conjugate, $A^\dagger = A^{*T}$. We also have

$$\langle g|A^\dagger f\rangle = \langle g|A^\dagger f\rangle = \langle f|A^*g\rangle^* = \langle A^*g|f\rangle.$$

(1.150)

Taking the adjoint of the adjoint is by (1.147)

$$\left[ (z |n\rangle\langle \ell|)^\dagger \right]^\dagger = [z^* |\ell\rangle\langle n|] = z |n\rangle\langle \ell|$$

(1.151)

the same as doing nothing at all. This also follows from the matrix formula (1.149) because both $(A^*)^* = A$ and $(A^{*T})^T = A$, and so

$$\left( A^\dagger \right)^\dagger = \left( A^{*T} \right)^* = A$$

(1.152)

the adjoint of the adjoint of a matrix is the original matrix.

Before Dirac, the adjoint $A^\dagger$ of a linear operator $A$ was defined by

$$(g, A^\dagger f) = (A g, f) = (f, A^*g)^*.$$  

(1.153)

This definition also implies that $A^{\dagger\dagger} = A$ since

$$(g, A^{\dagger\dagger} f) = (A^\dagger g, f) = (f, A^\dagger g)^* = (A^*g, f)^* = (g, A f).$$

(1.154)

We also have $(g, A f) = (g, A^{\dagger\dagger} f) = (A^\dagger g, f)$.

### 1.14 Self-Adjoint or Hermitian Linear Operators

An operator $A$ that is equal to its adjoint $A^\dagger = A$ is **self adjoint** or **hermitian**. In view of (1.149), the matrix elements of a self-adjoint linear operator $A$ satisfy $\langle n|A^\dagger|\ell\rangle = \langle \ell|A|n\rangle^* = \langle n|A|\ell\rangle$ in any orthonormal basis. So a matrix that represents a hermitian operator is equal to the transpose of its complex conjugate

$$A_{n\ell} = \langle n|A|\ell\rangle = \langle n|A^\dagger|\ell\rangle = \langle \ell|A|n\rangle^* = A^*_{\ell n} = A^\dagger_{n\ell}.$$

(1.155)