PROBLEMS

6.1 *Tall tale*

The mythical lumberjack Paul Bunyan usually cut down trees, but on one occasion he attempted to diversify and run his own sawmill. As the historians tell it, "Instead of turning out lumber the mill began to take in piles of sawdust and turn it back into logs. They soon found out the trouble: A technician had connected everything up backwards."

Can we reject this story on the basis of the Second Law?

6.2 *Entropy change upon equilibration*

Consider two boxes of ideal gas. The boxes are thermally isolated from the world and, initially, from each other as well. Each box holds N molecules in volume V. Box *1* starts with temperature $T_{i,1}$, whereas box *2* starts with $T_{i,2}$. (The subscript "i" means "initial," and "f" will mean "final.") So the initial total energies are $E_{i,1} = N\frac{3}{2}k_B T_{i,1}$ and $E_{i,2} = N\frac{3}{2}k_B T_{i,2}$.

Now we put the boxes into thermal contact with each other but still isolated from the rest of the world. We know they'll eventually come to the same temperature, as argued in Equation 6.10.

a. What is this temperature?

b. Show that the change of total entropy S_{tot} is then

$$k_B \frac{3}{2} N \ln \frac{(T_{i,1} + T_{i,2})^2}{4T_{i,1}T_{i,2}}.$$

c. Show that this change is always ≥ 0. [*Hint:* Let $X = \frac{T_{i,1}}{T_{i,2}}$ and express the change of entropy in terms of X. Plot the resulting function of X.]

d. Under a special circumstance, the change in S_{tot} will be zero: When? Why?

6.3 *Bobble Bird*

The Bobble Bird toy dips its beak into a cup of water, rocks back until the water has evaporated, then dips forward and repeats the cycle. All you need to know about the internal mechanism is that after each cycle, it returns to its original state: There is no spring winding down and no internal fuel getting consumed. You could even attach a little ratchet to the toy and extract a little mechanical work from it, maybe lifting a small weight.

a. Where does the energy to do this work come from?

b. Your answer in (a) may at first seem to contradict the Second Law. Explain why it does not. [*Hint:* What system discussed in Chapter 1 does this device resemble?]

6.4 *Efficient energy storage*

Section 6.5.3 discussed an energy-transduction machine. We can see some similar lessons from a simpler system, an energy-*storage* device. Any such device in the cellular world will inevitably lose energy, as a result of viscous drag, so we imagine pushing a ball through a viscous fluid. We push with constant external force f; as the ball moves, it compresses a spring (Figure 6.11). According to the Hooke relation, the

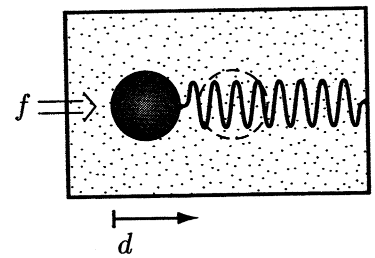

Figure 6.11: A simple energy-storage device. A tank filled with a viscous fluid contains an elastic element (spring) and a bead, whose motion is opposed by viscous drag.

spring resists compression with an elastic force $f = kd$, where k is the spring constant.[8] When this force balances the external force, the ball stops moving, at $d = f/k$.

Throughout the process, the applied force was fixed, so by this point we've done work $fd = f^2/k$. But integrating the Hooke relation shows that our spring has stored only $\int_0^d f(x)\mathrm{d}x = \frac{1}{2}kd^2$, or $\frac{1}{2}f^2/k$. The rest of the work we did went to generate heat. Indeed, at every position x along the way from 0 to d, some of the applied force compresses the spring while the rest goes to overcome viscous friction.

Nor can we get back all the stored energy, $\frac{1}{2}f^2/k$, because we lose even more to friction as the spring relaxes. Suppose that we suddenly reduce the external force to a value f_1 that is smaller than f.

a. Find how far the ball moves and how much work it does against the external force. We'll call the latter quantity the "useful work" recovered from the storage device.
b. For what constant value of f_1 will the useful work be maximal? Show that even with this optimal choice, the useful work output is only half of what was stored in the spring, or $\frac{1}{4}f^2/k$.
c. How could we make this process more efficient? [*Hint:* Keep in mind Idea 6.20.]

6.5 *Atomic polarization*

Suppose that we have a lot of noninteracting atoms (a gas) in an external magnetic field. You may take as given the fact that each atom can be in one of two states, whose energies differ by an amount $\Delta E = 2\mu B$, depending on the strength of the magnetic field B. Here μ is some positive constant, and B is also positive. Each atom's magnetization is taken to be $+1$ if it's in the lower energy state or -1 if it's in the higher state.

a. Find the *average* magnetization of the entire sample as a function of the applied magnetic field B. [*Remark:* Your answer can be expressed in terms of ΔE by using a hyperbolic trigonometric function; if you know these, then write it this way.]
b. Discuss how your solution behaves when $B \to \infty$ and when $B \to 0$, and why your results make sense.

6.6 *Polymer mesh*

D. Discher studied the mechanical character of the red blood cell cytoskeleton, a polymer network attached to its inner membrane. Discher attached a bead of diameter

[8] Another Hooke relation appeared in Chapter 5, where the force resisting a shear deformation was proportional to the size of the deformation (Equation 5.14 on page 172).

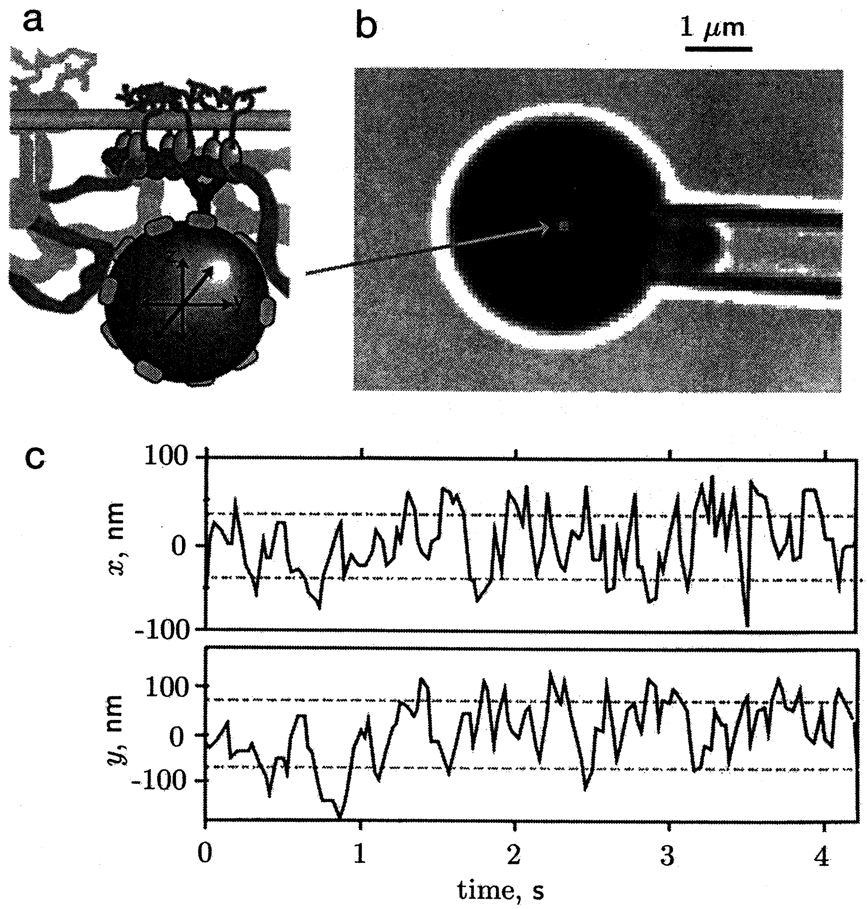

Figure 6.12: (Schematic; optical micrograph; experimental data.) (**a**) Attachment of a single fluorescent nanoparticle to actin in the red blood cell cortex. (**b**) The red cell, with attached particle, is immobilized by partially sucking it into a micropipette (*right*) of diameter 1 μm. (**c**) Tracking of the thermal motion of the nanoparticle gives information about the elastic properties of the cortex. [Digital image kindly supplied by D. Discher; see Discher, 2000.]

40 nm to this network (Figure 6.12a). The network acts as a spring, constraining the free motion of the bead. He then asked, "What is the stiffness (spring constant) of this spring?"

In the macroworld, we'd answer this question by applying a known force to the bead, measuring the displacement Δx in the x direction, and using $f = k\Delta x$. But it's not easy to apply a known force to such a tiny object. Instead, Discher just passively observed the thermal motion of the bead (Figure 6.12c). He found the bead's root-mean-square deviation from its equilibrium position, at room temperature, to be $\sqrt{\langle(\Delta x)^2\rangle} = 35$ nm; from this, he computed the spring constant k. What value did he find?

6.7 *Inner ear*

A. J. Hudspeth and coauthors found a surprising phenomenon while studying signal transduction by the inner ear. Figure 6.13a shows a bundle of stiff fibers (called stereocilia) projecting from a sensory cell. The fibers sway when the surrounding inner-ear

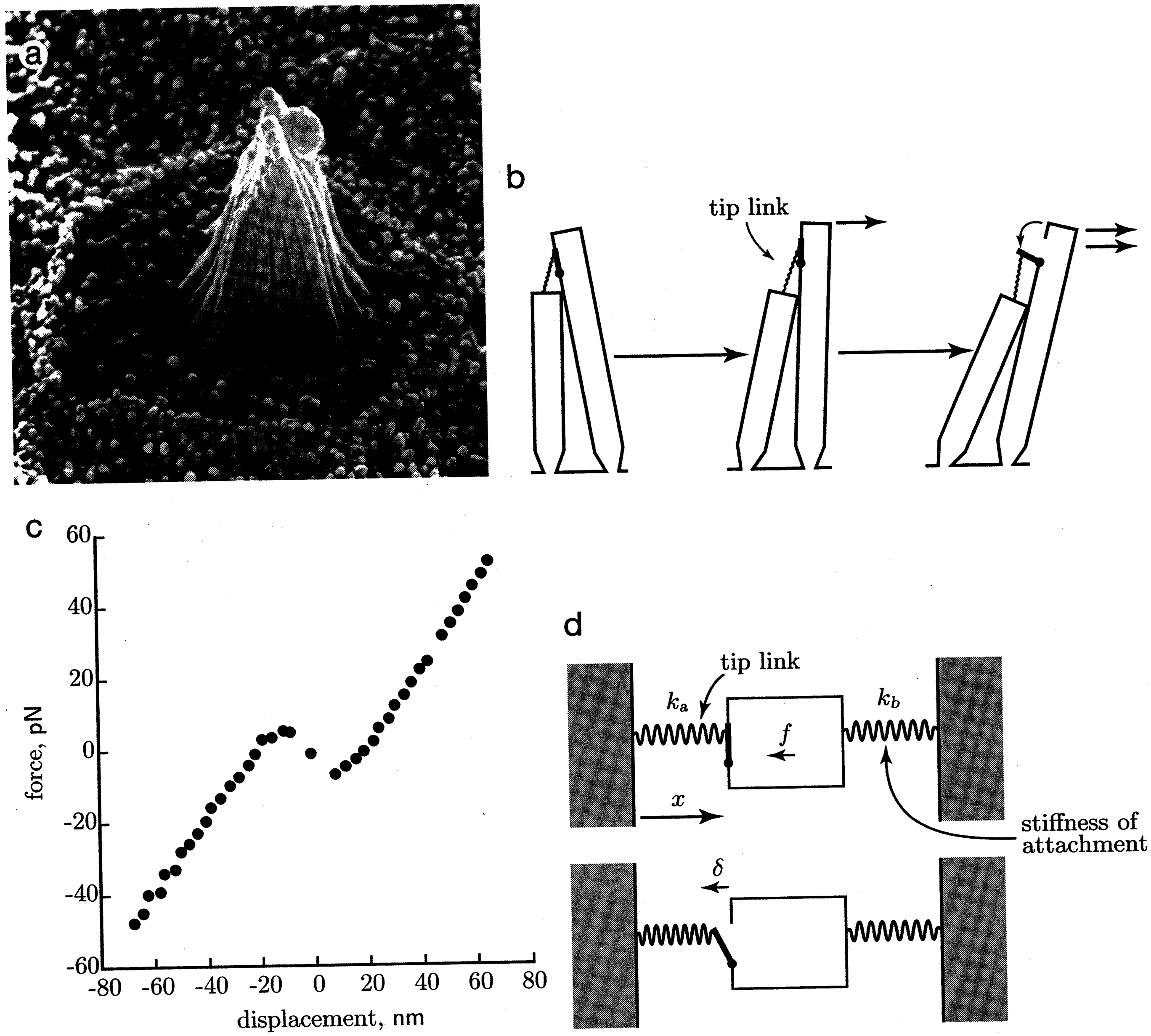

Figure 6.13: (Scanning electron micrograph; diagram; experimental data; diagram) (a) Bundle of stereocilia projecting from an auditory hair cell. (b) Pushing the bundle to the right causes a relative motion between two neighboring stereocilia in the bundle, stretching the tip link, a thin filament joining them. At large enough displacement, the tension in the tip link can open a "trap door." (c) Force exerted by the hair bundle in response to imposed displacements. Positive values of f correspond to forces directed to the left in (b); positive values of x correspond to displacements to the right. (d) Mechanical model for stereocilia. The *left spring* represents the tip link. The spring on the *right* represents the stiffness of the attachment point where the stereocilium joins the main body of the hair cell. The two springs exert a combined force f. The model envisions N of these units in parallel. [(a) Digital image kindly supplied by A. J. Hudspeth; (c) data from Martin et al., 2000.]

fluid moves. Other micrographs (not shown) revealed thin, flexible filaments (called tip links) joining each fiber in the bundle to its neighbor (wiggly line in the sketch, Figure 6.13b).

The experimenters measured the force-displacement relation for the bundle by using a tiny glass fiber to poke it. A feedback circuit maintained a fixed displacement for the bundle's tip and reported back the force needed to maintain this displacement. The surprise is that the experiments gave the complex curve shown in panel (c). A simple spring has a stiffness $k = \frac{df}{dx}$ that is constant (independent of x). The diagram shows that the bundle of stereocilia behaves like a simple spring at large deflections; but in the middle, it has a region of *negative* stiffness!

To explain their observations, the experimenters hypothesized a trap door at one end of the tip link (top right of the wiggly line in Figure 6.13b), and proposed that the trap door was effectively a two-state system.

a. Explain qualitatively how this hypothesis helps us to understand the data.

b. In particular, explain why the bump in the curve is rounded, not sharp.

c. In its actual operation, the hair bundle is not clamped; its displacement can wander at will, subject to applied forces from motion of the surrounding fluid. At zero applied force, the curve shows *three* possible displacements, at about −20, 0, and +20 nm. But really, we will never observe one of these three values. Which one? Why?

6.8 $\boxed{T_2}$ *Energy fluctuations*

Figure 6.2 implies that the relative fluctuations of energy between two macroscopic subsystems in thermal contact will be very small in equilibrium. Confirm this statement by calculating the root-mean-square deviation of E_A as a fraction of its mean value. [*Hints:* Suppose the two subsystems are identical, as assumed in the figure. Work out the probability $P(E_A)$ that the joint system will be in a microstate with E_A on one side and $E_{\mathrm{tot}} - E_A$ on the other side. Approximate $\ln P(E_A)$ near its peak by a suitable quadratic function, $A - B(E_A - \frac{1}{2}E_{\mathrm{tot}})^2$. Use this approximate form to estimate the RMS deviation.]

6.9 $\boxed{T_2}$ *The Langevin function*

Repeat Problem 6.5 for a slightly different situation: Instead of having just two discrete allowed values, our system has a continuous, unit-vector variable $\hat{\mathbf{n}}$ that can point in any direction in space. Its energy is a constant plus $-a\hat{\mathbf{n}} \cdot \hat{\mathbf{z}}$, or $-a\hat{\mathbf{n}}_z = -a\cos\theta$. Here a is a positive constant with units of energy and θ is the polar angle of $\hat{\mathbf{n}}$.

a. Find the probability distribution $P(\theta, \varphi)\mathrm{d}\theta\mathrm{d}\varphi$ for the directions that $\hat{\mathbf{n}}$ may point.

b. Compute the partition function $Z(a)$ and the free energy $F(a)$ for this system. Then compute the quantity $\langle\hat{\mathbf{n}}_z\rangle$. (Your answer is sometimes called the **Langevin function.**) Find the limiting behavior at high temperature and make sure your answer is reasonable.

6.10 $\boxed{T_2}$ *Gating compliance*

(Continuation of Problem 6.7.) We can model the system in Figure 6.13 quantitatively as follows. We think of the bundle of stereocilia as a collection of N elastic units in parallel. Each element has two springs: One, with spring constant k_a and equilibrium position x_a, represents the elasticity of the tip link filament. The other spring, characterized by k_b and x_b, represents the stiffness of the stereocilium's attachment point (provided by a bundle of actin filaments). See panel (d) of the figure.

The first spring attaches via a hinged element (the "trap door"). When the hinge is in its open state, the attachment point is a distance δ to the left of its closed state relative to the body of the stereocilium. The trap door is itself a two-state system with a free energy change ΔF_0 to jump to its open state.

a. Derive the formula $f_{\text{closed}}(x) = k_a(x - x_a) + k_b(x - x_b)$ for the net force on the stereocilium in the closed state. Rewrite this in the more compact form $f_{\text{closed}} = k(x - x_1)$ and find the effective parameters k and x_1 in terms of the earlier quantities. Then find the analogous formula for the state in which the trap door is open.

b. The total force f_{tot} is the sum of N terms. In $P_{\text{open}}N$ of these terms, the trap door is open; in the remaining $(1-P_{\text{open}})N$, it is closed. To find the open probability using Equation 6.34 on page 225, we need the free energy difference $\Delta F(x)$ between the system's two states (at fixed x). This difference is a constant, ΔF_0, plus a term involving the energy stored in spring a. Get a formula for $\Delta F(x)$.

c. Assemble the pieces of your answer to get the force $f_{\text{tot}}(x)$ in terms of the unknown parameters N, k_a, k_b, x_a, x_b, δ, and ΔF_1, where $\Delta F_1 \equiv \Delta F_0 + \frac{1}{2}k_a\delta^2$. That's a lot of parameters, but some of them enter only in fixed combination. Show that your answer can be expressed as

$$f_{\text{tot}}(x) = K_{\text{tot}}x + f_0 - \frac{Nz}{1 + e^{-z(x-x_0)/k_BT}},$$

and find the quantities K_{tot}, f_0, z, and x_0 in terms of the earlier parameters.

d. Hudspeth and coauthors fit this model to their data and to other known facts. They found $N = 65$, $K_{\text{tot}} = 1.1\,\text{pN}\,\text{nm}^{-1}$, $x_0 = -2.2\,\text{nm}$, and $f_0 = 25\,\text{pN}$. Graph the formula in (c), using these values. Use various trial values for z, starting from zero and moving upward. What value of z gives a curve resembling the data?

e. The authors also estimated that $k_a = 2 \cdot 10^{-4}\,\text{N}\,\text{m}^{-1}$. Use this value and your answer from (d) to find δ. Is this a reasonable value?