PROBLEMS

5.1 *Friction versus dissipation*

Gilbert says: You say that friction and dissipation are two manifestations of the same thing. So high viscosity must be a very dissipative situation. Then why do I get beautifully ordered, laminar motion only in the *high*-viscosity case? Why does my ink blob miraculously reassemble itself only in this case?

Sullivan: Um, uh ...

Help Sullivan out.

5.2 *Density profile*

Finish the derivation of particle density in an equilibrium colloidal suspension (begun in Section 5.1.1) by finding the constant prefactor in Equation 5.1. That is, find a formula for the equilibrium number density $c(x)$ of particles with net weight $m_{\mathrm{net}}g$ as a function of the height x. The total number of particles is N; the height of the test tube is h and its cross sectional area is A.

5.3 *Archibald method*

Sedimentation is a key analytical tool in the lab for the study of big molecules. Consider a particle of mass m and volume V in a fluid of mass density ρ_{m} and viscosity η.

a. Suppose a test tube is spun in the plane of a wheel, pointing along one of the "spokes." The artificial gravity field in the centrifuge is not uniform; rather, it is stronger at one end of the tube than the other. Hence the sedimentation rate will not be uniform either. Suppose that one end lies a distance r_1 from the center, and the other end is at $r_2 = r_1 + \ell$. The centrifuge is spun at angular frequency ω. Adapt the formula $v_{\mathrm{drift}} = gs$ (Equation 5.3 on page 160) to find an analogous formula for the drift speed in terms of s in the centrifuge case.

Eventually, sedimentation will stop and an equilibrium profile will emerge. It may take quite a long time for the whole test tube to reach its equilibrium distribution. In that case, Equation 5.2 on page 160 is not the most convenient way to measure the mass parameter m_{net}. The **Archibald method** uses the fact that the *ends* of the test tube equilibrate rapidly, as follows.

b. There can be no flux of material through the ends of the tube. Thus, the Fick-law flux must cancel the flux you found in (a). Write down two equations expressing this statement at the two ends of the tube.

c. Derive the following expression for the mass parameter in terms of the concentration and its gradient at one end of the tube:

$$m_{\mathrm{net}} = (\text{stuff}) \times \left.\frac{\mathrm{d}c}{\mathrm{d}r}\right|_{r=r_1},$$

and a similar formula for the other end, where (stuff) is some factors that you are to find. The concentration and its gradient can be measured photometrically in the lab, thus allowing a measurement of m_{net} long before the whole test tube has come to equilibrium.

5.4 *Coasting at low Reynolds*

The chapter asserted that tiny objects stop moving at once when we stop pushing them. Let's see.

a. Consider a bacterium, idealized as a sphere of radius $1\,\mu\text{m}$, propelling itself at $1\,\mu\text{m}\,\text{s}^{-1}$. At time zero, the bacterium suddenly stops swimming and coasts to a stop, following Newton's Law of motion with the Stokes drag force. How far does it travel before it stops? Comment.

b. Our discussion of Brownian motion assumed that each random step was independent of the previous one; thus, for example, we neglected the possibility of a residual drift speed left over from the previous step. In the light of (a), would you say that this assumption is justified for a bacterium?

5.5 *Blood flow*

Your heart pumps blood into your aorta. The maximum flow rate into the aorta is about $500\,\text{cm}^3\,\text{s}^{-1}$. Assume that the aorta has diameter 2.5 cm, that the flow is laminar (not very accurate), and that blood is a Newtonian fluid with viscosity roughly equal to that of water.

a. Find the pressure drop per unit length along the aorta. Express your answer in SI units. Compare the pressure drop along a 10 cm section of aorta with atmospheric pressure (10^5 Pa).

b. How much power does the heart expend just pushing blood along a 10 cm section of aorta? Compare your answer with your basal metabolic rate, about 100 W, and comment.

c. The fluid velocity in laminar pipe flow is zero at the walls of the pipe and maximum at the center. Sketch the velocity as a function of distance r from the center. Find the velocity at the center. [*Hint:* The total volume flow rate, which you are given, equals $\int v(r) 2\pi r \mathrm{d}r$.]

5.6 T2 *Kinematic viscosity*

a. Although the kinematic viscosity ν has the same dimensions $\mathbb{L}^2/\mathbb{T}$ as any other diffusion constant, its physical meaning is quite different from that of D, and its numerical value for water is quite different from the value of D for self-diffusion of water molecules. Find the value of ν from η and compare with D.

b. Still, these values are related. Show, by combining Einstein's relation and the Stokes formula, that taking the radius R of a water molecule to be about 0.2 nm leads to a satisfactory order-of-magnitude prediction of ν from D, R, and the mass density of water.

5.7 T2 *No going back*

Section 5.2.3 argued that the motion of a gently sheared, flat layer would retrace its history if we reverse the applied force. When the force is large, so that we cannot ignore the inertial term in Newton's Law of motion, where exactly does the argument fail?

5.8 $\boxed{T_2}$ *Intrinsic viscosity of a polymer in solution*

Section 4.3.2 argued that a long polymer chain in solution would be found in a random-walk conformation at any instant of time.[4] This claim is not easy to verify directly, so let's approach the question indirectly, by examining the *viscosity* of a polymer solution.

Figure 5.2b on page 163 shows two parallel plates separated by distance d, with the space filled with water of viscosity η. If one plate slides sideways at speed v, then both plates feel viscous force $\eta v/d$ per unit area. Suppose now that a small fraction ϕ of the volume between plates is filled with *solid objects*, taking up space previously taken by water. Then, at speed v, the shear strain rate in the remaining fluid must be greater than before, and the viscous force will be greater, too.

a. To estimate the shear strain rate, imagine that all the rigid objects are lying in a solid layer of thickness ϕd attached to the bottom plane, effectively reducing the gap between the plates. Then what is the viscous force per area?
b. We can express the result by saying that the suspension has an "effective viscosity" η' bigger than η. (Your result for the speed of milk separation in Your Turn 5C on page 161 was actually a bit too high, in part because of this effect.) Write an expression[5] for the relative change $(\eta' - \eta)/\eta$. Use $\phi \ll 1$ to simplify your answer.
c. We want to explore the proposition that a polymer N segments long behaves like a sphere with radius $\alpha L N^p$ for some power p. (Here L is the segment length and α is a constant of proportionality; we won't need the exact values of these parameters.) What do we expect p to be? What then is the volume fraction ϕ of a suspension of c such spheres per volume? Express your answer in terms of the total mass M of a polymer, the mass m per monomer, the concentration of polymer c, L, and α.
d. Discuss the experimental data in Figure 5.13 in the light of your analysis. Each set of points joined by a line represents measurements taken on a family of polymers with various numbers N of identical monomers; each monomer has the same mass m. The total mass $M = Nm$ of each polymer is on the x-axis. The quantity $[\eta]_\Theta$ on the vertical axis is called the polymer's intrinsic viscosity; it is defined as $(\eta' - \eta)/(\eta \rho_{\rm m,p})$, where $\rho_{\rm m,p}$ is the mass of dissolved polymer per volume of solvent. [*Hint:* Recall $\rho_{\rm m,p}$ is small. Write everything in terms of the fixed segment length L, the fixed monomer mass m, and the variable total mass M.]
e. What combination of L and m could we measure from the data? (Don't actually calculate it.)

5.9 $\boxed{T_2}$ *Friction as diffusion*

Section 5.2.1′ on page 187 claimed that viscous friction can be interpreted as the diffusive transport of momentum. The argument was that, in the planar geometry, when the flux of momentum given by Equation 5.21 leaves the top plate, it exerts a resisting drag force. When this momentum arrives at the bottom plate, it exerts an entraining force. So far, the argument is quite correct.

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[4] This problem concerns a polymer under "theta conditions" (see Section 4.3.1′ on page 148).

[5] The expression you'll get is not quite complete, because of some effects we left out, but its scaling is right when ϕ is small. Einstein obtained the full formula in his doctoral dissertation. (Then he fixed a computational error six years later!)

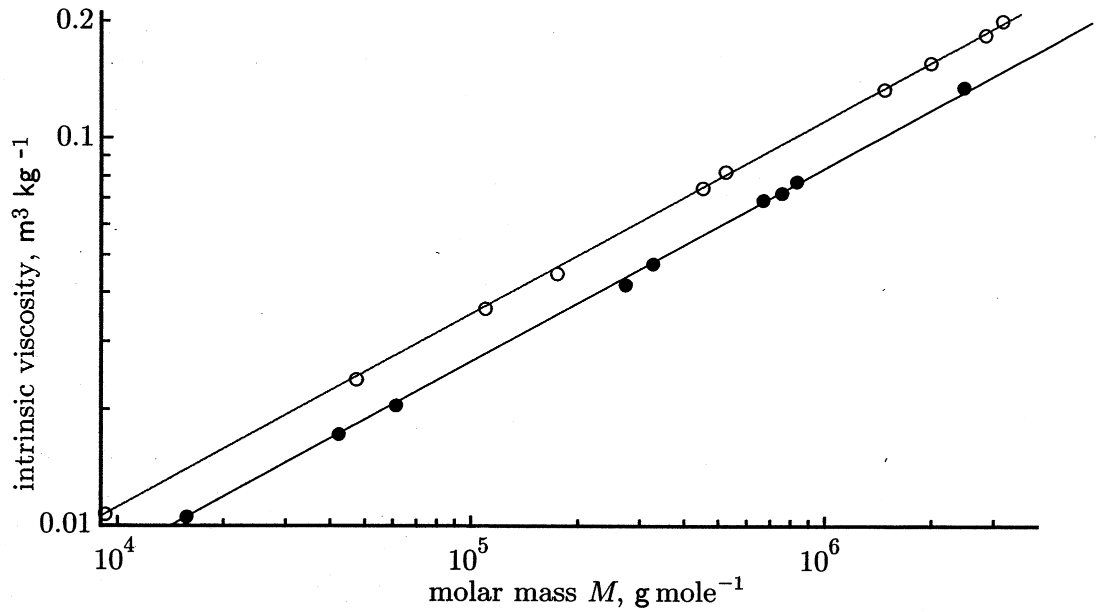

Figure 5.13: (Experimental data.) Log-log plot of the intrinsic viscosity $[\eta]_\Theta$ for polymers with different values for the molar mass M. The two data sets shown represent different combinations of polymer type, solvent type, and temperature, both corresponding to "theta solvent" conditions. *Open circles:* Polyisobutylene in benzene at 24°C. *Solid circles:* Polystyrene in cyclohexane at 34°C. The two *lines* each have logarithmic slope $\frac{1}{2}$. [Data from Flory, 1953.]

Viscous friction is more complicated than ordinary diffusion, however, because momentum is a vector quantity, whereas number density is a scalar. For example, Section 5.2.2 noted that the viscous force law (Equation 5.9 on page 168) needs to be modified for situations other than planar geometry. The required modification really matters if we want to get the correct answer for the spinning-rod problem (Figure 5.11b on page 180).

We consider a long cylinder of radius R with its axis along the $\hat{\mathbf{z}}$ direction and centered at $x = y = 0$. Some substance surrounds the cylinder. First suppose that this substance is *solid ice.* When we crank the cylinder, everything rotates as a rigid object with some angular frequency ω. The velocity at position $\mathbf{r}$ is then $\mathbf{v}(\mathbf{r}) = (-\omega y, +\omega x, 0)$. Certainly nothing is rubbing against anything, and there should be no dissipative friction—the frictional transport of momentum had better be zero. And yet, if we examine the point $\mathbf{r}_0 = (r_0, 0, z)$, we find a nonzero gradient

$$\left.\frac{dv_y}{dx}\right|_{\mathbf{r}=\mathbf{r}_0} = \omega.$$

Evidently, our formula for the flux of momentum in planar geometry (Equation 5.21 on page 187) needs some modification for the nonplanar case.

We want a modified form of Equation 5.21 that applies to cylindrically symmetrical flows and vanishes when the flow is rigid rotation. Letting $r \equiv \|\mathbf{r}\| = \sqrt{x^2 + y^2}$, we can write a cylindrically symmetrical flow as

$$\mathbf{v}(\mathbf{r}) = \big(-yg(r), xg(r), 0\big).$$

The case of rigid rotation corresponds to the choice of a constant angular velocity $g(r)$. You are about to find $g(r)$ for a different situation, namely, fluid flow. We can think of this flow as a set of nested cylinders, each with a *different* value of $g(r)$.

Near any point, say, $\mathbf{r}_0$, let $\mathbf{u}(\mathbf{r}) = (-yg(r_0), xg(r_0))$ be the rigidly rotating vector field that agrees with $\mathbf{v}(\mathbf{r})$ at $\mathbf{r}_0$. We then replace Equation 5.21 by

$$(j_{p_y})_x(\mathbf{r}_0) = -\eta \left(\left.\frac{dv_y}{dx}\right|_{\mathbf{r}=\mathbf{r}_0} - \left.\frac{du_y}{dx}\right|_{\mathbf{r}=\mathbf{r}_0} \right). \qquad \text{(cylindrical geometry)} \qquad (5.22)$$

In this formula, $\eta \equiv \nu\rho_m$, the ordinary viscosity. Equation 5.22 is the proposed modification of the momentum-transport rule. It says that we compute dv_y/dx and *subtract off* the corresponding quantity with $\mathbf{u}$, to ensure that rigid rotation incurs no frictional resistance.

a. Each cylindrical shell of fluid exerts a torque on the next one and feels a torque from the previous one. These torques must balance. Show that, as a result, the tangential force per area across the surface at fixed r is $(\tau/L)/(2\pi r^2)$, where τ is the external torque on the central cylinder and L is the cylinder's length.
b. Set your result from (a) equal to Equation 5.22 and solve for the function $g(r)$.
c. Find τ/L as a constant times ω. Hence, find the constant C in Equation 5.19 on page 183.

5.10 $\boxed{T_2}$ *Pause and tumble*

In between straight-line runs, *E. coli* pauses. If it just turned off its flagellar motors during the pauses, eventually the bacterium would find itself pointing in a new, randomly chosen direction, as a result of rotational Brownian motion.

If you haven't done Problem 4.9, do it now and compare your answer to part (d) with the measured pause time of 0.14 s. Do you think the bacterium just shuts down its flagellar motors and waits during the pauses? Explain your reasoning.