Notes for Chapter Four

In 1828, Robert Brown, a botanist observed 1 μm diameter pollen grains thru a microscope and noticed that they danced about incessantly. He hoped that these colloidal particles were a new life form, but reluctantly concluded otherwise.

By the 1860's, some people inferred that this brownian motion was due to the collisions of the colloidal particles with water molecules since the dancing was more vigorous at higher temperatures at which the kinetic energy $\frac{3}{2}kT$ of the water molecules was higher.

But there were two puzzles:

How can a tiny water molecule move a grain of pollen enough to be seen in a microscope?

Water molecules at room temperature move at 10^3 m/s. They are perhaps nanometers apart. So their collision rate is

$$\frac{10^3 \ m s^{-1}}{10^{-9} \ m} = 10^{12} \ s^{-1}$$

10^{12} collisions per second! This is much too fast to see and much faster than the dancing of pollen grains.

These puzzles attracted Einstein's interest in 1905.

Random Walks

First, one dimension: Flip a coin every second and move one step to the right on heads, one to the left on tails. After two tosses, the possible outcomes are HH, HT, TH, & TT. So the chance of going nowhere is $P_0 = \frac{1}{2} = 0.5$.

After 10^4 seconds, the probability of staying put is very small. It's the probability of throwing 5000 heads and 5000 tails. The total population of possible outcomes is 2^{10^4}. So now P_0 is the number of outcomes with 5000 heads divided by 2^{10^4}.

First, let's do the case of 4 steps.

How many sequences of integers ≤ 4 have 2 heads? There are 4 choices for n_1, 3 for n_2. So

$$N_{S4}(2H) = 4 \cdot 3 = 12.$$

But a sequence of tosses has $n_1 < n_2$ etc. So the number of real sequences in time is

$$N_4(2) = \frac{4 \cdot 3}{2} = 6.$$

For big numbers, we use factorial notation: $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$, $2! = 2 \cdot 1 = 2$, so

$$N_4(2) = \frac{4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1 \times 2 \cdot 1} = \frac{4!}{2!\,2!} = 6.$$

How many sequences of 10000 integers have 5000 heads? There are 10000 choices for n_1, 9999 for n_2, 9998 for n_3, etc. So the number of such sequences is

$$N_S(5000H) = 10000 \times 9999 \times \cdots 5001$$

$$= \frac{1000\;!}{5000\;!}$$

But the number of sequences of tosses with $n_1 < n_2 < n_3$ etc. is

$$N_{10000}(5000) = \frac{1000!}{5000!\;5000!}.$$

Stirling's formula for $M!$ is

$$\ln M! \approx M \ln M - M + \frac{1}{2}\ln(2\pi M).$$

So the probability of staying put after 10000 tosses is

$$P_0 = \frac{N_{1000}(5000)}{2^{10^4}} .$$

$\boxed{4A}$ Let's use Stirling's formula to estimate P_0.

$$\ln P_0 = \ln\left(\frac{10000!}{5000!\,5000!}\,\frac{1}{2^{10^4}}\right)$$

$$= \ln 10000! - 2\ln 5000! - 10000\ln 2$$

$$= 10000(\ln 2\cdot 5000 - 1) + \tfrac{1}{2}\ln 4\pi 5000$$

$$-2\cdot\left[5000(\ln 5000 - 1) + \tfrac{1}{2}\ln 2\pi 5000\right]$$

$$- 10000\ln 2$$

$$\ln P_0 = 10000 \ln 2 + 10000 \ln 5000 - 10000$$

$$+ \frac{1}{2} \ln 4\pi + \frac{1}{2} \ln 5000$$

$$- 10000 \ln 5000 + 10000$$

$$- \ln 2\pi - \ln 5000 - 1000 \ln 2$$

$$= \frac{1}{2} \ln 4\pi + \frac{1}{2} \ln 5000 - \ln 2\pi - \ln 5000$$

$$= \ln \left(\frac{\sqrt{2\pi \times 10000}}{2\pi \times 5000} \right).$$

So

$$P_0 = \exp \ln P_0$$

$$= \frac{\sqrt{2\pi \times 10000}}{2\pi \times 5000} = \frac{\sqrt{4\pi}}{2\pi \sqrt{5000}}$$

$$= \frac{1}{\sqrt{\pi \times 5000}} = \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{10000}}$$

$$= \frac{1}{100} \sqrt{\frac{2}{\pi}} \simeq 0.008.$$

The number of ways of choosing a set of M objects out of a set of N objects is

$$\frac{N!}{M!\,(N-M)!} = \binom{N}{M}$$

pronounced "N choose M."

We found that the probability of returning to the origin after 5000 steps was $P_0 = 0.008$. This is small, but it's huge compared to the probability of moving to the right 10000 times, which is

$$P_{10000} = \frac{1}{2}^{10000} = \left(e^{\ln\frac{1}{2}}\right)^{10000}$$

$$= e^{-10000 \times (0.6931)} \cong \left(10^{\log e}\right)^{-6931}$$

What's $\log e$?

$$10^{x} = e = e^{x \ln 10}$$

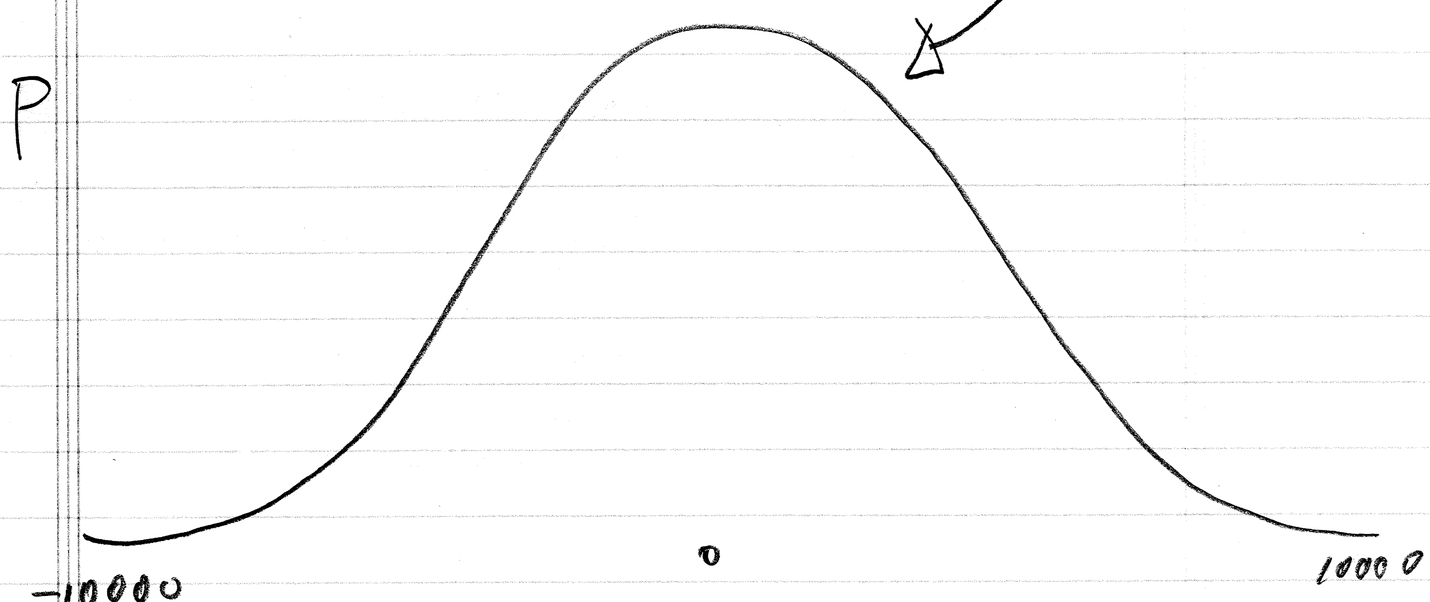
So $x = \log e = 1/\ln 10 = 0.4343.$

So

$$P_{5000} \cong 10^{-6931 \times 0.4343}$$

$$\approx 10^{-1505} \approx 10^{-3010} << 0.008 = P_0.$$

One is more likely to wind up somewhere in the middle.



Say each step is of length L.

Displacement j is $k_j L$ where $k_j = \pm 1$. After j steps, one is at x_j. Say $x_0 = 0$. Then $x_1 = k_1 L$, and $x_j = x_{j-1} + k_j L$.

Walks of 3 steps:

			x_3
+	+	+	3
+	+	−	1
−	+	+	1
+	−	+	1
+	−	−	−1
−	+	−	−1
−	−	+	−1
−	−	−	−3

We compute

$$\langle x_3^2 \rangle = \frac{9 + 6 \cdot 1 + 9}{8} = \frac{24}{8} = 3.$$

$\boxed{4B}$ Random walks of four steps, $+ = R$, $- = L$:

				x_4	x_4^2
+	+	+	+	$4L$	$16L^2$
+	+	+	−	$2L$	$4L^2$
+	+	−	+	$2L$	$4L^2$
+	−	+	+	$2L$	$4L^2$
−	+	+	+	$2L$	$4L^2$
+	+	−	−	0	0
+	−	+	−	0	0
+	−	−	+	0	0
−	−	+	+	0	0
−	+	−	+	0	0
−	+	+	−	0	0
−	−	−	+	$-2L$	$4L^2$
−	−	+	−	$-2L$	$4L^2$
−	+	−	−	$-2L$	$4L^2$
+	−	−	−	$-2L$	$4L^2$
−	−	−	−	$-4L$	$16L^2$

So $$\langle x_4^2 \rangle = \frac{(16 + 4\cdot 4 + 4\cdot 4 + 16)L^2}{16}$$

$$= 4L^2$$

$$\langle x_4 \rangle = 0$$

So the variance$(x_4) = \langle x_4^2 \rangle - \langle x_4 \rangle^2 = 4L^2$.

How about N steps?

Let step j be $k_j L$ where

$k_j = \pm 1$, and $+1$ is as likely as -1.

Then $x_j = x_{j-1} + k_j L$ and

$$< x_N^2 > = < (x_{N-1} + k_N \ L)^2 >$$

$$= < x_{N-1}^2 > + 2L < x_{N-1} k_N > + L^2 < k_{N-1}^2 >.$$

Now

$$< k_{N-1}^2 > = \frac{1^2 + (-1)^2}{2} = 1.$$

Also, x_{N-1} and k_{N-1} are totally uncorrelated, and $< k_N > = 0$ (which implies $< x_N > = 0$), so

$$< x_{N-1} k_N > = < x_{N-1} > < k_N > = 0 \cdot 0 = 0$$

So

$$< x_N^2 > = < x_{N-1}^2 > + L^2$$

$$< x_1^2 > = L^2$$

$$< x_2^2 > = 2L^2$$

$$< x_N^2 > = N L^2.$$