Something interesting happens for \( P > P_c \). Look at Fig. 5.3 on page 166. A fluid element has volume \( l^3 \) and mass \( \rho l^3 \) and so it experiences an inertial force

\[
f_{\text{tot}} = \frac{mv^2}{R} = \frac{l^3 \rho v^2}{R}
\]

as it accelerates around the sphere of radius \( R \).

The total force on the fluid element is

\[
f_p + f_f = f_{\text{tot}} = m \alpha = \text{mass} \times \text{acceleration}
\]

Here \( f_p \) is force due to pressure; \( f_f \), that of friction.
To estimate $f_x$, we look at a picture:

\[ f_u = \eta l^2 \frac{(v_{a_1} - v_{a_2})}{e} \sim \eta l^2 \frac{dv_a}{dr} \]

\[ f_d = \eta l^2 (v_{d_2} - v_{d_1}) \sim -\eta l^2 \frac{dv_d}{dr} \]

So the net frictional force on the solid cube of volume $l^3$ is

\[ f_u + f_d = \eta l^2 \left( \frac{dv_a}{dr} - \frac{dv_d}{dr} \right) \sim \eta l^3 \frac{d^2 v}{dr^2} . \]

Now we estimate

\[ \frac{d^2 v}{dr^2} \sim \frac{v}{R^2} . \]

The frictional force $f_f$ then is

\[ f_f = \eta l^3 \frac{v}{R^2} . \]

And $f_{\text{tot}} = \frac{m v^2}{R} = \rho l^3 \frac{v^2}{R}$. 
The ratio of these two forces

\[ R = \frac{f_{tot}}{f_f} = \frac{\frac{e^3 p r^2}{R}}{\gamma \frac{e^3 v}{R^2}} \]

is the Reynolds number

\[ R = \frac{\rho u r}{\eta} \]

When \( R \) is small, friction dominates.

When \( R \) is big, pressure dominates.

For \( R \geq 1000 \), turbulence rules.

So we have four forces here:

- \( f_c = \frac{\eta^2}{\rho} \) is the force scale of the fluid;
- it is the critical viscous force.

- \( f_{tot} = m a \) is the total force; it is the sum \( f_{tot} = f_p + f_f \) of the force \( f_p \) due to pressure and the force \( f_f \) of friction.
The Reynolds number $R = \frac{f_{tot}}{f_f}$ is the ratio of the total force to the frictional force.

Ex. If $R << 1$, friction dominates because

$$\frac{f_p}{f_f} = \frac{f_{tot} - f_f}{f_f} = R - 1 \approx -1$$

and friction nearly cancels pressure, so the fluid does not accelerate, $f_{tot} \approx 0$. So to hold a ball of radius $R$ in place in a fluid when Reynolds's number $R << 1$, one must apply a pressure force $f_p = -f_f$.

Taking $f_f = \eta \frac{v^2}{R^2}$ with $l = R$, we get

$$f_f = \eta R v.$$ 

So

$$\frac{|f_p|}{f_c} = \frac{f_f}{f_c} = \frac{\eta R v}{\eta^2 / \rho} = \frac{p R v}{\eta} = R << 1.$$
At low \( R \), the force \( f_p \) needed to hold the ball fixed is much less than \( f_c \) the critical force \( f_c = \frac{\eta^2}{\rho} \) of the fluid.

**SE** Suppose \( R \gg 1 \). What is \( \frac{f_p}{f_c} \)?

Now \( R = \frac{f_{\text{tot}}}{f_f} = \frac{f_f + f_p}{f_f} = 1 + \frac{f_p}{f_f} \gg 1 \).

So \( f_p \gg f_f \), and \( f_p \approx f_{\text{tot}} \). So

\[
\frac{f_p}{f_c} \approx \frac{f_{\text{tot}}}{f_c} = \frac{\frac{\rho}{\rho}v^2}{R \eta^2 / \rho} = \frac{\rho^2 v^2}{R \eta^2}.
\]

Setting \( \rho = R \), we get

\[
\frac{f_p}{f_c} = \frac{R^3 \rho^2 v^2}{R \eta^2} = \frac{R^2 \rho^2 v^2}{\eta^2} = \left( \frac{R \rho v}{\eta} \right)^2 = R^2 \gg 1.
\]

**Number:** A 30m whale swimming at \( v = 10 \text{ m s}^{-1} \) in water with \( \eta = 10^{-3} \text{ kg m}^{-1} \text{s}^{-1} \) has

\[
R = \frac{30 \times 10^3}{10^{-3}} = 3 \times 10^8 \quad \text{since} \quad \rho_{\text{H}_{2} \text{O}} = 10^3 \text{ kg m}^{-3}.
\]
But a lam bacterium swimming at $v = 30 \mu m s^{-1}$ in water has

$$R = 10^{-6} \frac{30 \times 10^{-6} \times 10^3}{10^{-3}} = 3 \times 10^{-5}.$$  

In the world of microbiology, $R << 1$.

Imagine a viscous fluid between two plates of area $A$ separated by a distance $d$.

At $t = 0$ and $R \ll 0$, there is very little acceleration in the fluid and

$$\frac{d^2 V_x}{dx^2} = 0$$

So

$$V_x(x) = \frac{x}{d} V_0.$$  

A volume element $d\Theta$ at $(x_0, z_0)$ initially moves to $(x_0, z_0 + (x/d) V_0)$ at time $t$. 
If we reverse the motion, pulling the top plate backwards at \( v = -v_0 \) to the same period \( t \) of time, then the blob returns to \( \Theta \) at time \( 2t \).

As long as the force applied to the plate is well below the critical viscous force \( f_c = \eta^2/\rho \), the process \( 0 \rightarrow 0 \) will be reversed \( 0 \rightarrow 0 \) even if some jerky motions occur, as long as
\[
f(t + t') = -f(t - t') \quad \text{for} \quad 0 < t' < t
\]
holds approximately.

Mechanics in vacuum is different.
Here, e.g., \( x(t) = v_0 t - \frac{1}{2}gt^2 \) for a rock thrown up at \( t = 0 \) with speed \( v_0 \). It obeys
\[
-\frac{dU}{dx} = m \frac{d^2x}{dt^2}
\]
and so \( \dot{x}(t) = x(-t) \) is also a solution

\[
-\frac{dU}{dx} = -\frac{dV}{dx} = m \frac{d^2 x}{dt^2} = m \frac{d^2 \tilde{x}}{dt^2}
\]

The sign of the force is the same

\[
-\frac{dV}{dx} = -\frac{dV}{dx}
\]

in vacuum mechanics.

But when friction dominates mechanics, we have an effective equation of motion

\[
\frac{d\langle x \rangle}{dt} = -\frac{f(t)}{m}
\]

and so to get \( \langle \dot{x}(t) \rangle = \langle x(-t) \rangle \)

we need

\[
\frac{d\langle \dot{x}(t) \rangle}{dt} = \frac{d\langle x(-t) \rangle}{dt} = -\frac{f(-t)}{m}
\]

or \( f(t) = -f(-t) \).
Swimming: Pushing paddles down and then raising them back up does not work at low Reynolds number.

\[ y_1(u - u) = \text{drag on body} \]

\[ y_1, v = (y_0 + y_1)u \]

\[ u = \frac{y_1, v}{y_0 + y_1} \quad \text{forward stroke} \]

\[ \Delta x = t u \]

Similarly,

\[ y_1, v' = (y_0 + y_1)u' \]

\[ u' = \frac{y_1, v'}{y_0 + y_1} \quad \text{backward stroke} \]

Now \( t'v' = tv \) if paddles return to original place.

So \( \Delta x' = -t'v' = -t' y'_1, u' = -t y'_1, u' = -\Delta x \]

Thus \( \Delta x' = -\Delta x \)

and we go nowhere.
Two solutions:

A cilium's stroke

This clearly works because the drag on a cilium that moves I thin the fluid is greater than on one that moves II to fluid. Cilia in lungs work this way and sweep junk out of our lungs.

A flagellum is a corkscrew or a helix that rotates.

\[
\begin{align*}
    x &= r \cos \theta \\
    y &= r \sin \theta \\
    z &= p \theta
\end{align*}
\]

\[r^2 = (x, y, z) \text{ is a point on a helix}\]
So as $\theta$ increases, $z$ does too. And the bacterium moves. In the limit of infinite viscosity, and vanishing $R$, we can picture the flagellum as a corkscrew turning in a cork — as the screw turns, the corkscrew advances into the cork.

To swim or not to swim at speed $v$? It goes $x = tv$ but diffusion goes $\sqrt{4Dt}$, so swimming makes sense if

$$\frac{d}{v} < \frac{d^2}{6D}$$

or

$$v > \frac{6D}{d} \quad \text{or} \quad \frac{vd}{D} > 6.$$ 

The dimensionless quantity $vd/D$ is
the Peclet number. For a cilium with $d = 1 \mu m$ and $D = (1 \mu m)^2 m s^{-1}$

$$\frac{D}{d} = 1 \mu m m s^{-1} < 10^3 \mu m s^{-1}.$$ Bacteria don't swim this fast. But for protozoa, both $d$ and $v$ are bigger and swimming to get food works.

Bacteria with defective mutant flagella do well in nutritious media.

But foraging is different from swimming. Bacteria use flagella to get to food. E. coli swims in a random direction and then stops and swims in a different random direction. If the concentration of food rises during
a swim, then the E. coli swim's longer; if decreasing, it stops sooner. This bacterial random walk works.

Now \( v = 30 \text{ mm s}^{-1} \), and so to beat diffusion, they must swim a distance

\[
d > \frac{D}{v} = \frac{1 \text{ mm}^2 \text{ ms}^{-1}}{30 \text{ mm s}^{-1}}
\]

or

\[
d > \frac{1000}{30} \text{ mm} \approx 30 \text{ mm}
\]

or 30 body lengths, which is what they do.

**Attack & Escape**: A swimmer at low \( R \) and size \( R \) disturbs the fluid out to a distance \( 2R \). So its prey can sense its approach.
So medium-small creatures briefly accelerate to a much higher \( R \).

The Culex crassicauda strikes at accelerations up to \( 12 \, \text{m/s}^2 \), briefly hitting \( R = 500 \). And the sessile (attached) protozoan Ventricella, when fearful, retracts its stolons from \( 0.2 \sim 0.3 \, \text{mm} \) down to \( 0.1 \sim 0.15 \, \text{mm} \) at speeds of \( 80 \, \text{mm/s} \).

Vascular networks bring oxygen and food to cells and remove their wastes; they also transport lymph, which is the fluid of the immune system.

So how does a fluid flow thru a pipe?
of length $L$ and radius $R$ if its viscosity is $\eta$?

\[ df_1 = 2\pi r p dr \] is pressure force on shell from $r$ to $r + dr$. This shell is dragged forward by the faster fluid at $r < r'$

\[ df_3 = -\eta (2\pi r L) \frac{dv(r)}{dr} \]

and dragged backward by the slower fluid at $r' > r$

\[ df_2 = \eta (2\pi (r + dr) L) \frac{dv(r')}{dr'} \bigg|_{r' = r + dr} \]

Now

\[ \frac{dv'(r')}{dr'} \bigg|_{r' = r + dr} = \frac{dv}{dr} + dr \frac{d^2v}{dr^2} \]

and we require that $0 = df_1 + df_2 + df_3$. 
\[ 0 = 2\pi \nu p dr + \eta 2\pi (r + dr)(dr + dr) \left( \frac{dr}{dr} + dr \frac{d^2r}{dr^2} \right) \]

\[ + \eta 2\pi r L \frac{dr}{dr} \]

\[ 0 = 2\pi \nu p dr + \eta 2\pi L dr \frac{dr}{dr} + \eta 2\pi L r dr \frac{d^2r}{dr^2} \]

\[ 0 = \nu p \frac{dr}{L^2} + \frac{dr}{dr} + r \frac{d^2r}{dr^2} \]

By setting \( f = \frac{dr}{dr} \), we see that this equation really is a first-order inhomogeneous differential equation in disguise

\[ 0 = \frac{\nu p}{L^2} + f + r \frac{df}{dr} \]

The standard way to solve such
an equation is to set
\[ \alpha(n) = \exp \int_0^n \frac{dv'}{v'} = e^{\ln n} = n \]
and then to set
\[
f(n) = \frac{1}{\alpha(n)} \left[ \int^n \alpha(h) \left(-\frac{P}{L\eta}\right) dh' + C \right]
\]
\[= \frac{1}{n} \left[ \int^n \left(-\frac{P}{L\eta}\right) dh' + C \right]
\]
\[= \frac{1}{n} \left[ -\frac{Pr^2}{2L\eta} + C \right]
\]
So
\[
\frac{dv}{dr} = \frac{1}{n} \frac{Pr}{2L\eta} + C
\]
\[v(n) = A - \frac{Pr^2}{4L\eta} + C \ln n.
\]
Since \( v(0) \) is finite, \( C = 0 \)
Since \( v(R) = 0 \), \( A = \frac{PR^2}{4\eta L} \).
So the velocity profile is

\[ v(r) = \frac{\left(R^2 - r^2\right)p}{4\pi \eta} \]

The total rate of flow then is

\[ Q = \int_0^\infty \frac{2\pi r v(r)}{4\pi \eta} \, dr \]

\[ = \frac{p}{4\pi \eta} \int_0^R \left[R^2 - r^2\right] \, dr \]

\[ = \frac{\pi p}{2\pi \eta} \left[ \frac{R^2}{2} - \frac{r^4}{4} \right]_0^R \]

\[ = \frac{\pi p}{2\pi \eta} \left( \frac{R^4}{2} - \frac{R^4}{4} \right) = \frac{\pi p R^4}{8\pi \eta} \]

\[ Q = \frac{\pi p R^4}{8\pi \eta} \quad \text{(5.18)} \]

This Hagen-Poiseuille rule for laminar flow says the rate goes as \( R^4 \).