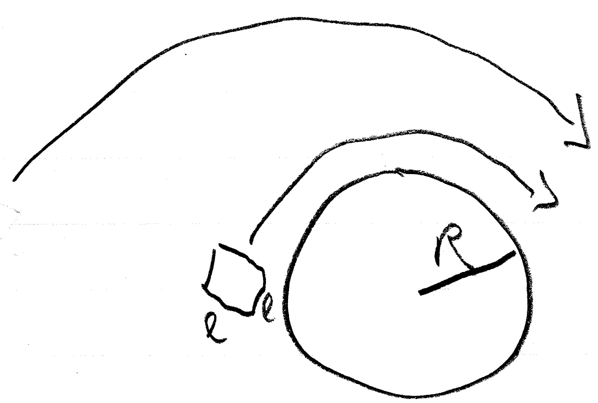
Something interesting happens for $\rho \approx \rho_c$. Look at Fig. 5.3 on page 166. A fluid element has volume l^3 and mass ρl^3 and so it experiences an inertial force

$$f_{tot} = \frac{m v^2}{R} = \frac{l^3 \rho v^2}{R}$$

as it accelerates around the sphere of radius R.

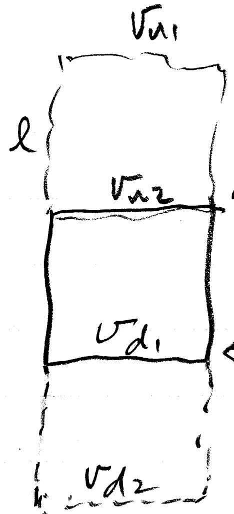


The total force on the fluid element is

$$f_p + f_f = f_{tot} = m a = \text{mass} \times \text{acceleration}.$$

Here f_p is force due to pressure; f_f that of friction

To estimate f_f, we look at picture:



$$f_u = \eta \ell^2 \frac{(v_{u1} - v_{u2})}{\ell} \approx \eta \ell^2 \frac{dv_u}{dr}$$

$$f_d = \eta \ell^2 \frac{(v_{d2} - v_{d1})}{\ell} \approx -\eta \ell^2 \frac{dv_d}{dr}$$

So the net frictional force on the solid cube of volume ℓ^3 is

$$f_u + f_d = \eta \ell^2 \left(\frac{dv_u}{dr} - \frac{dv_d}{dr} \right) \approx \eta \ell^3 \frac{d^2 v}{dr^2}.$$

Now we guestimate

$$\frac{d^2 v}{dr^2} \approx \frac{v}{R^2}.$$

The frictional force f_f then is

$$f_f = \eta \ell^3 \frac{v}{R^2}.$$

And $f_{tot} = \dfrac{m v^2}{R} = \rho \dfrac{\ell^3 v^2}{R}$.

The ratio of these two forces

$$\mathcal{R} = \frac{f_{tot}}{f_f} = \frac{\ell^3 \rho v^2}{R} \frac{1}{(\eta \ell^3 v / R^2)}$$

is the Reynolds number

$$\mathcal{R} = \frac{\rho v R}{\eta}.$$

When $\mathcal{R}$ is small, friction dominates.
When $\mathcal{R}$ is big, pressure dominates.
For $\mathcal{R} \gtrsim 1000$, turbulence rules.

So we have four forces here:

$f_c = \eta^2/\rho$ is the force scale of the fluid; it is the critical viscous force.

$f_{tot} = ma$ is the total force; it is the sum $f_{tot} = f_p + f_f$ of the force f_p due to pressure and the force f_f of friction.

The Reynolds number $\mathcal{R} = f_{tot} / f_f$ is the ratio of the total force to the frictional force.

Ex. If $\mathcal{R} \ll 1$, friction dominates because

$$\frac{f_p}{f_f} = \frac{f_{tot} - f_f}{f_f} = \mathcal{R} - 1 \approx -1$$

and friction nearly cancels pressure, so the fluid does not accelerate, $f_{tot} \approx 0$.

So to hold a ball of radius R in place in a fluid when Reynolds's number $\mathcal{R} \ll 1$, one must apply a pressure force $f_p = -f_f$.

Taking $f_f = \dfrac{\eta \, \ell^3 v}{R^2}$ with $\ell = R$, we get

$$f_f = \eta R v.$$

So $$\frac{|f_p|}{f_c} = \frac{f_f}{f_c} = \frac{\eta R v}{\eta^2/\rho} = \rho \frac{R v}{\eta} = \mathcal{R} \ll 1.$$

At low $\mathcal{R}$, the force f_p needed to hold the ball fixed is much less than f_c the critical force $f_c = \eta^2/\rho$ of the fluid.

$\boxed{5E}$ Suppose $\mathcal{R} >> 1$. What is f_p/f_c?

Now $$\mathcal{R} = \frac{f_{tot}}{f_f} = \frac{f_f + f_p}{f_f} = 1 + \frac{f_p}{f_f} >> 1.$$

So $f_p >> f_f$, and $f_p \approx f_{tot}$. So

$$\frac{f_p}{f_c} \approx \frac{f_{tot}}{f_c} = \frac{\ell^3 \rho v^2}{R \quad \eta^2/\rho} = \frac{\ell^3 \rho^2 v^2}{R \eta^2}.$$

Setting $\ell = R$, we get

$$\frac{f_p}{f_c} \approx \frac{R^3 \rho^2 v^2}{R \eta^2} = \frac{R^2 \rho^2 v^2}{\eta^2} = \left(\frac{R\rho v}{\eta}\right)^2 = \mathcal{R}^2 >> 1.$$

Numbers: A 30m whale swimming at $v = 10\,\mathrm{ms}^{-1}$ in water with $\eta = 10^{-3}\,\mathrm{kg\,m^{-1}s^{-1}}$ has

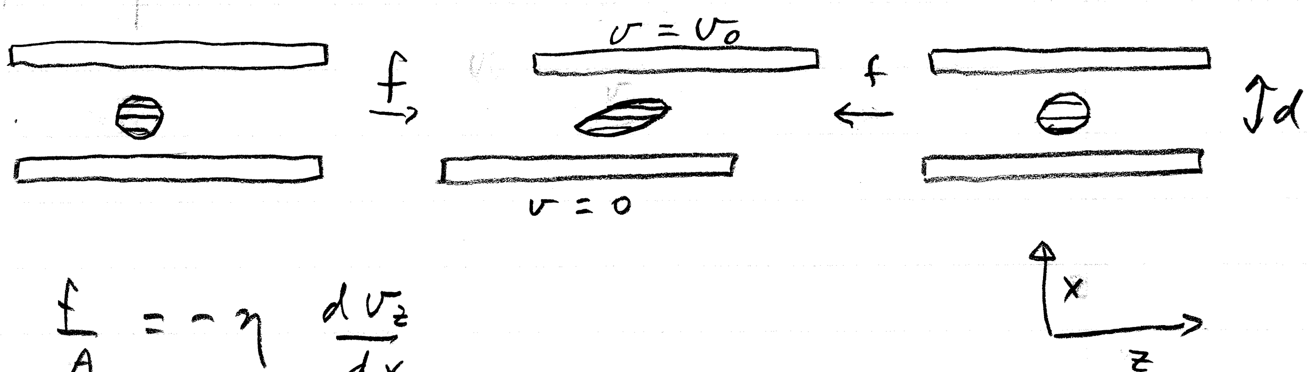
$$\mathcal{R} = \frac{30 \;\, 10^3 \;\, 10}{10^{-3}} = 3 \times 10^8 \qquad \text{since } \rho_{H_2O} = 10^3\,\mathrm{kg\,m^{-3}}.$$

But a 1μm bacterium swimming at $v = 30\ \mu m\,s^{-1}$ in water has

$$\mathcal{R} = \frac{10^{-6}\ 30\times10^{-6}\times10^{3}}{10^{-3}} = 3\times10^{-5}.$$

In the world of microbiology, $\mathcal{R} \ll 1$.

Imagine a viscous fluid between two plates of area A separated by a distance d



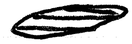
$$\frac{f}{A} = -\eta\,\frac{dv_z}{dx}$$

At $f \simeq 0$ and $\mathcal{R} \simeq 0$, there is very little acceleration in the fluid and

$$\frac{d^2 v_z}{dx^2} = 0$$

So $$v_z(x) = \frac{x}{d}\,v_0\,.$$

A volume element ⊜ at (x_0, z_0) initially moves to $(x_0, z_0 + (x/d)\,t\,v_0)$ at time t.

If we reverse the motion, pulling the top plate backwards at $v = -v_0$ for the same period t of time, then the blob  returns to  at time $2t$.

As long as the force applied to the plate is well below the critical viscous force $f_c = \eta^2/\rho$, the process  $\to$  will be reversed  $\to$  even if some jerky motions occur as long as

$$f(t+t') = -f(t-t') \quad \text{for } 0<t'<t$$

holds approximately.

Mechanics in vacuum is different. Here, e.g., $x(t) = v_0 t - \frac{1}{2} g t^2$ for a rock thrown up at $t=0$ with speed v_0. It obeys

$$-\frac{dU}{dx} = m \frac{d^2 x}{dt^2}$$

and so $\tilde{x}(t) = x(-t)$ is also a solution

$$-\frac{dU}{d\tilde{x}} = -\frac{dU}{dx} = m\frac{d^2x}{dt^2} = m\frac{d^2\tilde{x}}{dt^2}.$$

The sign of the force is the same

$$-\frac{dU}{d\tilde{x}} = -\frac{dU}{dx}$$

in vacuum mechanics.

But when friction dominates mechanics, we have an effective equation of motion

$$\frac{d\langle x\rangle}{dt} = \frac{f(t)}{\eta}$$

and so to get $\langle\tilde{x}(t)\rangle = \langle x(-t)\rangle$ we need

$$\frac{d\langle\tilde{x}(t)\rangle}{dt} = \frac{d\langle x(-t)\rangle}{dt} = -\frac{f(-t)}{\eta}$$

or $f(t) = -f(-t)$.

Swimming: Pushing paddles down and then raising them back up does not work at low Reynolds number.

paddle $\zeta_1(v-u) = \zeta_0 u$ drag on body

$$\zeta_1 v = (\zeta_0 + \zeta_1) u$$

forward stroke

$$u = \frac{\zeta_1 v}{\zeta_0 + \zeta_1}$$

$$\Delta x = t u$$

Similarly $\zeta_1 v' = (\zeta_0 + \zeta_1) u'$

backward stroke

$$u' = \frac{\zeta_1 v'}{\zeta_0 + \zeta_1}$$

Now $t'v' = tv$ if paddles return to original places

So $\Delta x' = -t'u' = -\dfrac{t'\zeta_1 v'}{\zeta_0 + \zeta_1} = -\dfrac{t\zeta_1 v}{\zeta_0 + \zeta_1} = -\Delta x$

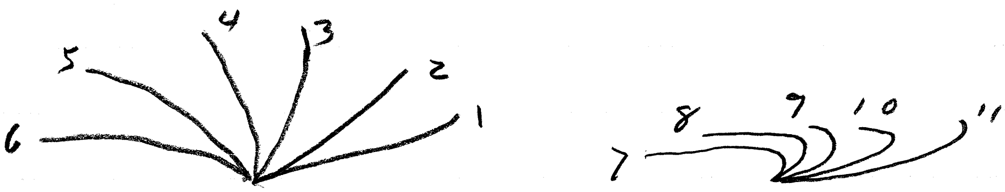
Thus

$$\Delta x' = -\Delta x$$

and we go nowhere.

Two solutions:

A cilium's stroke



This clearly works because the drag on a cilium that ~~moves~~ $\perp$ thru the fluid is greater than on one that moves $\parallel$ to fluid. Cilia in lungs work this way and sweep junk out of our lungs.

A flagellum is a corkscrew or a helix that rotates.

$$\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= p\theta \end{aligned}\right\} \quad \vec{r} = (x, y, z) \text{ is a point on a helix}$$

So as θ increases, z does too. And the bacterium moves. In the limit of infinite viscosity, and vanishing R, we can picture the flagellum as a corkscrew turning in a cork — as the screw turns, the corkscrew advances into the cork.

To swim or not to swim at speed v? It goes $d = tv$ but diffusion goes $d \sim \sqrt{6Dt}$, so swimming makes sense if

$$\frac{d}{v} < \frac{d^2}{6D} \qquad \left(D = \frac{L^2}{2\Delta t}\right)$$

or

$$v > \frac{6D}{d} \quad \text{or} \quad \frac{vd}{D} > 6.$$

The dimensionless quantity vd/D is

the Peclet number. For a cilium with $d = 1\mu m$ and $D = (1\mu m)^2 ms^{-1}$

$$\frac{D}{d} = 1\mu m\ ms^{-1} = 10^3 \mu m\ s^{-1}.$$

Bacteria don't swim this fast. But for protozoa, both d and v are bigger and swimming to get food works. Bacteria with defective mutant flagella do well in nutricious media.

But foraging is different from swimming: Bacteria use flagella to get to food. E. coli swims in a random direction and then stops and swims in a different random direction. If the concentration of food rises during a swim, then the E. coli swims longer; if decreasing, it stops sooner. This biased random walk works.

Now $v \simeq 30\ \mu m\ s^{-1}$, and so to beat diffusion, they must swim a distance

$$d > \frac{D}{v} \simeq \frac{1\ \mu m^2\ ms^{-1}}{30\ \mu m\ s^{-1}}$$

or

$$d > \frac{1000}{30}\ \mu m \simeq 30\ \mu m$$

or 30 body lengths, which is what they do.

Attack & Escape: A swimmer at low $\mathcal{R}$ and size R disturbs the fluid out to a distance $\simeq R$. So its prey can sense its approach.

So medium-small creatures briefly accelerate to a much higher $\mathcal{R}$.
The Cyclops crustacean strikes at accelerations up to $12\ m s^{-2}$, briefly hitting $\mathcal{R} = 500$. And the sessile (attached) protozoan Vorticella, when fearful, retracts its stalk from $0.2 \sim 0.3$ mm down to $0.1 \sim 0.15$ mm at speeds of $80\ mm s^{-1}$!

Vascular networks bring oxygen and food to cells and remove their waste; they also transport lymph, which is the fluid of the immune system.
So how does a fluid flow thru a pipe

of length L and radius R if its viscosity is η?

$df_1 = 2\pi r p \, dr$ is pressure force on shell from r to $r+dr$. This shell is dragged forward by the faster fluid at $r' < r$

$$df_3 = -\eta(2\pi r L)\frac{dv(r)}{dr}$$

and dragged backward by the slower fluid at $r' > r$

$$df_2 = \eta\left(2\pi(r+dr)L\right)\left.\frac{dv(r')}{dr'}\right|_{r'=r+dr}$$

Now

$$\left.\frac{dv'(r')}{dr'}\right|_{r'=r+dr} = \frac{dv}{dr} + dr\,\frac{d^2v}{dr^2}$$

and we require that $0 = df_1 + df_2 + df_3$.

$$0 = 2\pi r p\, dr + \eta 2\pi (r + dr) L \left(\frac{dv}{dr} + dr \frac{d^2 v}{dr^2} \right)$$

$$- \eta\, 2\pi r L \frac{dv}{dr} \qquad \text{or}$$

$$0 = 2\pi r p\, dr + \eta 2\pi L\, dr \frac{dv}{dr} + \eta 2\pi L r\, dr \frac{d^2 v}{dr^2}$$

or

$$0 = \frac{rp}{L\eta} + \frac{dv}{dr} + r \frac{d^2 v}{dr^2} .$$

By setting $f = \frac{dv}{dr}$, we see

that this equation really is a first-order inhomogeneous differential equation in disguise

$$0 = \frac{rp}{L\eta} + f + r \frac{df}{dr} .$$

The standard way to solve such

an equation is to set

$$\alpha(r) = \exp \int^r \frac{dr'}{r'} = e^{\ln r} = r$$

and then to set

$$f(r) = \frac{1}{\alpha(r)} \left[\int^r \alpha(r') \left(-\frac{p}{L\eta} \right) dr' + c \right]$$

$$= \frac{1}{r} \left[\int^r \left(-\frac{p}{L\eta} \right) r' dr' + c \right]$$

$$= \frac{1}{r} \left[-\frac{pr^2}{2L\eta} + c \right]$$

So

$$\frac{dv}{dr} = -\frac{pr}{2L\eta} + \frac{c}{r}$$

$$v(r) = A - \frac{pr^2}{4L\eta} + c \ln r.$$

Since $v(0)$ is finite, $C = 0$

Since $v(R) = 0$, $A = \dfrac{pR^2}{4\eta L}$.

So the velocity profile is

$$v(r) = \frac{(R^2 - r^2)}{4L\eta} p .$$

The total rate of flow then is

$$Q = \int_0^R 2\pi r \, v(r) \, dr$$

$$= \frac{p}{4L\eta} 2\pi \int_0^R r \, (R^2 - r^2) \, dr$$

$$= \frac{\pi p}{2L\eta} \left[R^2 \frac{r^2}{2} - \frac{r^4}{4} \right]_0^R$$

$$= \frac{\pi p}{2L\eta} \left(\frac{R^4}{2} - \frac{R^4}{4} \right) \quad \text{or}$$

$$Q = \frac{\pi R^4}{8L\eta} p . \qquad (5.18)$$

This Hagen-Poiseuille rule for laminar flow says the rate goes as R^4.