Notes for Chapter 5

If a particle of mass m and volume V is at height z in a gravitational field of acceleration g and in a fluid of mass density ρ, then its gravitational potential energy is

$$U(z) = gz(m - V\rho).$$

At equilibrium at temperature T, the concentration of such particles is

$$c(z) = c(0)\, e^{-gz(m-V\rho)/(kT)}.$$

In this formula for sedimentation equilibrium, the combination $m - V\rho$ is the net mass m_{net} identified long ago by Archimedes.

The globular protein myoglobin has $m \approx 17\,000$ g mole^{-1} and $m_{net} \approx 0.25\,m$.

The <u>scale height</u>

$$z_* = \frac{kT}{m_{net}\,g} = \frac{4.1 \times 10^{-21}\,\mathrm{J}}{m_{net}\,g}$$

in this case for $T = T_r$ is $z_* \approx 59$ m.

We expect $c(z) = c(0)\,e^{-z/z_*}$. So in a 4 cm test tube held vertical

$$c(0.04) = c(0)\,e^{-\frac{.04}{59}} \cong .999\,c(0).$$

The suspension of myoglobin never settles to the bottom. It is a <u>colloidal suspension</u> or a <u>colloid</u>. DNA & pollen grains in water are colloids. But grains of sand in water do settle — waves flow over not into a beach.

A suspension settles on a height scale of h if $m_{net} g h$ is bigger than kT. In terms of m_{net}, the suspension's concentration $C(z)$ is

$$C(z) = C(0)\, e^{-gz\, m_{net}/kT} \quad . \qquad (5.1)$$

$\boxed{5A}$ Milk fat has mass density

$$\rho_{m,f} = 0.91 \text{ g cm}^{-3}.$$

Water has $\rho_{m,w} = 1.00 \text{ g cm}^{-3}$.

The droplets have volume

$$\frac{4}{3}\pi R^3 = \frac{4}{3}\pi\left(\frac{10^{-6}\,m}{2}\right)^3 = \frac{\pi}{6}\, 10^{-18}\, m^3.$$

So

$$m_{net} = (\rho_{m,f} - \rho_{m,w})V = -0.09 g \frac{\pi}{6} 10^{-18} \frac{m^3}{cm^3}$$

$$= -0.09 \frac{\pi}{6} 10^{-12} \text{ kg } \frac{g}{kg}$$

$$= -\frac{\pi}{6}(0.09) \times 10^{-15} \text{ kg} = -1.5\pi \times 10^{-17} \text{ kg}$$

$$gz = 9.8 \frac{m}{s^2} \, 0.25\, m = \frac{9.8}{4} \, m^2 s^{-2}$$

$$kT = 4.1 \times 10^{-21} \, J$$

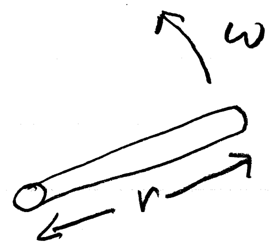
$$c(h) = c(0)\, e^{-gz\, m_{net}/kT}$$

$$\frac{c(h)}{c(0)} = e^{+\frac{9.8}{4} \frac{1.5\pi \times 10^{-17}}{4.1 \times 10^{-21}}}$$

$$= e^{2.8 \times 10^4} = 2.6 \cdot 10^{12145}.$$

So eventually, homogenized milk settles out with the fat on top.

Einstein's insight into gravity started from his assertion that all accelerations are physically equivalent. The acceleration in a centrifuge of angular velocity ω at a distance r from the axis of rotation



is

$$g' = r\omega^2.$$

$\boxed{5B}$ $$[g'] = \mathcal{L}\mathcal{T}^{-2}$$

The length is r and $[\omega] = s^{-1}$, so we guess that $g' = r\omega^2$.

Let's look at a particle of m_{net} a distance r from the axis of a centrifuge of angular velocity ω. In the rotating frame of the tube, the particle feels a force

$$f = r\omega^2 m_{net}.$$

This force results in a drift velocity

$$v = \frac{m_{net}\, r \omega^2}{\zeta}$$

where ζ is the viscous-friction coefficient.

By Einstein's relation

$$\zeta D = kT$$

we have

$$v = \frac{m_{net}\, r \omega^2 D}{kT}$$

which implies a drift flux

$$j_v = c v = \frac{c\, m_{net}\, r \omega^2 D}{kT}$$

where c is the concentration.

The diffusive flux by Fick's Law

is

$$j_d = -D \frac{dc}{dr}.$$

So the total flux is

$$j = j_v + j_d = D\left(-\frac{dc}{dr} + \frac{r\omega^2 m_{net}\, c}{kT}\right).$$

At equilibrium, $j = 0$ and so

$$\frac{dc}{dr} = \left(\frac{r\omega^2 m_{net}}{kT}\right) c$$

or

$$c(r) = c(0)\, e^{\frac{r^2\omega^2 m_{net}}{2kT}} = c(0)\, e^{-\frac{U(r)}{kT}}$$

which is the Boltzmann formula with $U(r) = -\frac{1}{2} r^2 \omega^2 m_{n}$.

To avoid explicit reference to the acceleration g', one defines the sedimentation coefficient

$$s = \frac{v_d}{g'} = \frac{m_{net}}{\zeta} = \frac{m_{net}\, D}{kT}.$$

This s has units $[s] = T$, units of time. It is roughly the time for a particle to reach its terminal speed.

$\boxed{5C}$ (a) $\zeta = 6\pi\eta R$

So

$$[\eta] = \frac{[\zeta]}{[R]} = \mathcal{M}\mathcal{T}^{-1}\mathcal{L}^{-1}$$

Note

$$\eta_{H_2O} = 10^{-3}\ \mathrm{kg\,m^{-1}\,s^{-1}} = 10^{-3}\ \mathrm{Pa\,s}.$$

Pascal = pressure

(b)

$$f = m_{net}\, g$$

$$v = f/\zeta \qquad\qquad \zeta = 6\pi\eta R$$

$$m_{net} = 90\ \mathrm{kg\,m^{-3}}\ \tfrac{4}{3}\pi R^3 \qquad\qquad vt = h = \tfrac{1}{4}\,\mathrm{m}$$

So the time for a milk-fat globule of radius R to go a distance h in skim milk in a gravitational field g is

$$t = \frac{h}{v} = \frac{h}{f/\zeta} = \frac{h\zeta}{f} = \frac{h\,6\pi\eta R}{m_{net}\,g}$$

$$= \frac{6\pi h\eta R}{\frac{4\pi}{3}R^3\, 90 g}\ \mathrm{kg^{-1}m^3} = \frac{18}{4}\frac{h\eta}{R^2\, 90 g} = \frac{h\eta}{20 R^2 g}$$

For homogenized milk, $R = \frac{1}{2} 10^{-6}$ m, and

$$t_h = \frac{1}{4} \frac{10^{-3}}{20 (5 \cdot 10^{-7})^2 10} \frac{m^2 kg\, m^{-1} s^{-1}}{kg\, m\, s^{-2}}$$

$$= 5 \times 10^6 s = 58 \text{ days}.$$

But for raw milk, $R = 2.5 \times 10^{-6}$ m, and so since R is 5 times bigger

$$t_r = \frac{5}{25} \times 10^6 s = \frac{1}{5} 10^6 = 2 \times 10^5 s$$

$$= \underline{56 \text{ hours}} = 2.3 \text{ days}.$$

Recall, again, the viscosity η of room-temperature water is $\eta = 10^{-3}$ kg m^{-1}s^{-1} = 10^{-3} Pa s.

Polymer coils, again: $R \propto m^p \propto L^p$.

Random-walk theory (p. 123) gave $p = \frac{1}{2}$.

$$s = \frac{v_d}{g} = \frac{m_{net}\, g}{g\, \eta} = \frac{m_{net}}{\eta} = \frac{(m - \rho V)}{6\pi \eta R}$$

Now if $V = v m$, then since $R = \alpha m^p$

$$s = \frac{m(1 - v\rho)}{6\pi \eta R} = \frac{m(1-v\rho)}{6\pi \eta \alpha m^p}$$

we get

$$s \propto m^{1-p}.$$

Fig. 4.7b of page 123 shows $p \approx \frac{1}{2}$,

that is, $s \propto m^{.44}$. And Fig. 4.7a

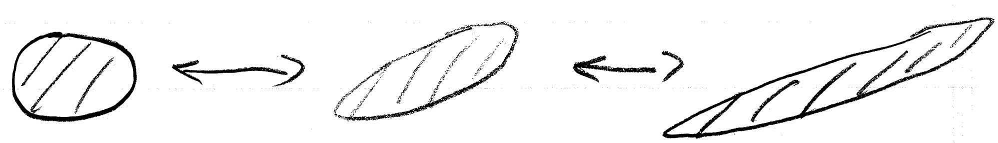
gives $p = 0.57$, so our model gives

$$s \propto m^{1-0.57} = m^{0.43}$$

which is very close to $m^{0.44}$.

Recall how the red drop of corn syrup in the wide beaker of corn syrup spread out as I moved the stirring rod four times in clockwise circles —

and then moved back to its original drop-like shape when I reversed the motion of the stirring rod in four counter-clockwise circles. This demo illustrated <u>laminar</u> <u>flow</u>: the corn syrup moved slowly in an organized way in which adjacent layers of fluid slide past each other



The flow of the syrup is laminar because the viscosity of corn syrup is $\eta = 5 \text{ kg m}^{-1}\text{s}^{-1}$, which is 5000 times that of water $\eta = 10^{-3} \text{ kg m}^{-1}\text{s}^{-1}$. If one did this demo using milk in tea, the flow would have been turbulent, and the four counter-clockwise movements would have mixed the milk further; they would not have restored the milk to its original shape.

Laminar flow happens when a quantity called the Reynolds number

$$\mathcal{R} = \frac{v R \rho}{\eta}$$

in which v is speed, R is size, ρ is density, and η viscosity. $[\mathcal{R}] = 0$, the Reynolds number is dimensionless. Laminar flow occurs when $\mathcal{R}$ is tiny. Turbulence occurs when $\mathcal{R}$ is big.

Let's try to understand the Reynolds number $\mathcal{R}$. Consider two plates of area A separated by a distance d with a fluid of viscosity η between them. If one slowly drags one plate at speed v, then the shearing viscous force f on the other plate obeys the empirical law

$$f = \frac{\eta v A}{d}, \qquad [\zeta] = \frac{F}{V} = \frac{ML}{T^2}\frac{T}{L} = \frac{M}{T}$$

$$[\eta] = \frac{[\zeta]}{L} = \frac{M}{LT}$$

Such a fluid is said to be newtonian.

$\boxed{5D}$ Check the units:

$[\eta] = \mathcal{M}\mathcal{L}^{-1}\mathcal{T}^{-1}$, $[A] = \mathcal{L}^2$

$[d] = \mathcal{L}$, $[v] = \mathcal{L}\mathcal{T}^{-1}$, so

$$\frac{[\eta][v][A]}{[d]} = \frac{\mathcal{M}\mathcal{L}^{-1}\mathcal{T}^{-1}\mathcal{L}\mathcal{T}^{-1}\mathcal{L}^2}{\mathcal{L}} = \mathcal{M}\mathcal{L}\mathcal{T}^{-2}$$

$= [ma] = [f]$. So $f = \frac{\eta v A}{d}$ works.

(We'll discuss isotropic newtonian fluids — ones that are the same in all directions.)

An isotropic, newtonian fluid is completely characterized by its density ρ and its viscosity η.

Since $[\eta] = \mathcal{M}\mathcal{L}^{-1}\mathcal{T}^{-1}$, the ratio η^2/ρ has the dimensions of force

$$\left[\frac{\eta^2}{\rho}\right] = \frac{\mathcal{M}^2\mathcal{L}^{-2}\mathcal{T}^{-2}}{\mathcal{M}\mathcal{L}^{-3}} = \mathcal{M}\mathcal{L}\mathcal{T}^{-2}.$$

This ratio

$$f_c = \frac{\eta^2}{\rho}$$

is called the <u>viscous critical force</u>. Flow is viscous or laminar when the dimensionless ratio f/f_c is small.

So forces small compared to f_c result in viscous or laminar flow. When the force f is big compared to f_c, then inertia becomes important, and the flow is turbulent.

A marble in corn syrup pushed by a force less than 0.03 N results in motion dominated by friction

$$f_c = \frac{\eta^2}{\rho} = \frac{25}{1000} = 3 \times 10^{-2}\ N$$

So if $f < 0.03\ N$, then

$$\frac{f}{f_c} < \frac{.03}{.03} = 1 .$$

But in water,

$$f_c = \frac{(10^{-3})^2}{10^3} = 10^{-9}\ N$$

and so even $f = 10^{-6}\ N$ gives $f/f_c = 10^3$.

But for tiny forces — those of biological interest in cells — $f/f_c \ll 1$ even for $f_c = 10^{-9} N$ because such forces f are in the piconewton range.

$[\rho] = \mathcal{M} \mathcal{L}^{-3}$ and $[\eta] = \mathcal{M} \mathcal{L}^{-1} \mathcal{T}^{-1}$.

The ratio η^2/ρ is the critical force, but no combination of ρ and η has the dimensions of length.

$$[\rho^\alpha][\eta^\beta] = \mathcal{M}^\alpha \mathcal{L}^{-3\alpha} \mathcal{M}^\beta \mathcal{L}^{-\beta} \mathcal{T}^{-\beta}$$

$$= \mathcal{M}^{\alpha+\beta} \mathcal{L}^{-3\alpha-\beta} \mathcal{T}^{-\beta} \stackrel{?}{=} \mathcal{L} \quad ?$$

We'd need $\alpha = -\beta$ and $\beta = 0$, and so both $\alpha = \beta = 0$, but that is a number not a length. A newtonian fluid has no length scale.