Notes on Gaussian Integration

$$\int_{-\infty}^{\infty} dx\, e^{-ax^2} = \sqrt{\int_{-\infty}^{\infty} dx\,dy\, e^{-a(x^2+y^2)}}$$

$$= \sqrt{\int_0^\infty 2\pi r\,dr\, e^{-ar^2}} = \sqrt{\int_0^\infty \frac{2\pi r}{a} dr\, e^{-r^2}}$$

$$= \sqrt{\frac{\pi}{a}\int_0^\infty dz\, e^{-z}} = \sqrt{\frac{\pi}{a}}$$

$$\int_{-\infty}^{\infty} dx\, e^{-\frac{a}{2}x^2 - bx - c} = \int dx\, e^{-\frac{a}{2}(x+\alpha)^2 + \beta}$$

$$-a\alpha = -b \qquad \alpha = b/a$$

$$-a\frac{\alpha^2}{2} + \beta = -\frac{a}{2}\frac{b^2}{a^2} + \beta = -c \qquad \beta = -c + \frac{b^2}{2a}$$

So

$$\int dx\, e^{-\frac{a}{2}x^2 - bx - c} = e^{\frac{b^2}{2a} - c} \int dx\, e^{-\frac{a}{2}(x + b/a)^2}$$

$$= e^{+\frac{b^2}{2a} - c} \int dx\, e^{-\frac{a}{2}x^2} = \sqrt{\frac{2\pi}{a}}\, e^{\frac{b^2}{2a} - c}$$

Now let A be a positive symmetric matrix

$$\mathcal{J} = \int \prod_{i=1}^{n} dx_i \; e^{-\frac{1}{2} x_i A_{ij} x_j - b_\ell x_\ell - c}$$

$$A = S^T a S \qquad a_{ij} = a_i \delta_{ij}$$

and $S^T S = S S^T = 1$, $S^* = S$, S is orthogonal.

So

$$\mathcal{J} = \int \prod dx_i \; e^{-\frac{1}{2} x_i (S^T a S)_{ij} x_j - b_\ell x_\ell - c}$$

$$= \int \prod dx_i \; e^{-\frac{1}{2} x_i S_{ki} a_k S_{kj} x_j - b_\ell x_\ell - c}$$

let $y_k = S_{kj} x_j$ The Jacobian

$$\det \frac{\partial y_k}{\partial x_j} = \det S = 1 \qquad x_\ell = S^T_{\ell k} y_k = S_{k\ell} y_k$$

$$\mathcal{J} = \int \prod dy_i \, e^{-\frac{1}{2} a_k y_k^2 - y_k S_{k\ell} b_\ell - c}$$

$$= \prod \sqrt{\frac{2\pi}{a_k}} \; e^{\frac{(S_{k\ell} b_\ell)^2}{2a_k} - c}$$

$$A = S^T a S$$
$$A^{-1} = S^{-1} a^{-1} S = S^T a^{-1} S$$

$$\prod a_k = \det A = \det S^T a S$$

$$\frac{(S_{k\ell} b_\ell)^2}{2 a_k} = \frac{1}{2} S_{k\ell} \frac{b_\ell}{a_k} S_{km} b_m = \frac{1}{2} b_\ell S^T_{\ell k} a^{-1}_k S_{km} b_m$$
$$= \frac{1}{2} b_\ell A^{-1}_{\ell m} b_m$$

So

$$\mathcal{I} = \int \prod_i^n dx_i \; e^{-\frac{1}{2} x_i A_{ij} x_j - b_\ell x_\ell - c}$$

$$= \left(\prod_k \sqrt{\frac{2\pi}{a_k}} \right) e^{\frac{1}{2} b^T A^{-1} b - c}$$

$$= \left(\det \frac{A}{2\pi} \right)^{-\frac{1}{2}} e^{\frac{1}{2} b^T A^{-1} b - c}$$

what is stationary point of arg of exp?

$$0 = -A_{ij} x_j - b_i \qquad b = -Ax$$

$x_0 = -A^{-1} b$ at that point

$$-\frac{1}{2} x_0 A x_0 - b x_0 - c = -\frac{1}{2} b^T A^{-1T} A A^{-1} x_0 + b A^{-1} b - c$$

$$= -\frac{1}{2} b^T A^{-1T} b + b A^{-1} b - c$$

$$= +\frac{1}{2} b^T A^{-1} b - c \qquad \text{so}$$

$$\mathcal{I} = \int \prod dx_i \; e^{-\frac{1}{2} x^T A x - bx - c}$$

$$= \left(\det \frac{A}{2\pi} \right)^{-\frac{1}{2}} e^{\frac{1}{2} b^T A^{-1} b - c} = \left(\det \frac{A}{2\pi} \right)^{-\frac{1}{2}} e^{-\frac{1}{2} x_0^T A x_0 - b x_0 - c}$$

$$I_{kl} = \int \prod_{i=1}^{n} dx_i \quad x_k x_l \, e^{-\frac{1}{2} x_r A_{rs} x_s}$$

$$\mathcal{J} = \int \pi dx_i \; e^{-\frac{1}{2} x^T A x - b^T x - c}$$

$$= \left(\det \frac{A}{2\pi}\right)^{-\frac{1}{2}} e^{\frac{1}{2} b^T A^{-1} b - c}$$

So

$$\int \pi dx_i \; e^{-\frac{1}{2} x^T A x - b^T x} = \left(\det \frac{A}{2\pi}\right)^{-\frac{1}{2}} e^{\frac{1}{2} b^T A^{-1} b}$$

So

$$\int \pi dx_i \; x_k x_l e^{-\frac{1}{2} x^T A x} = \frac{\partial^2}{\partial b_k \partial b_l} \int \pi dx_i \; e^{-\frac{1}{2} x^T A x - b^T x} \Big|_{b=0}$$

$$= \left(\det \frac{A}{2\pi}\right)^{-\frac{1}{2}} A^{-1}_{kl}$$

$$\int \pi dx_i \; x_k x_l x_r x_s e^{-\frac{1}{2} x^T A x} = \frac{\partial^4}{\partial b_k \partial b_l \partial b_r \partial b_s} \int \pi dx_i \, e^{-\frac{1}{2} x^T A x - b^T x} \Big|_{b=0}$$

$$= \frac{\partial^4}{\partial b_k \partial b_l \partial b_r \partial b_s} \left(\det \frac{A}{2\pi}\right)^{-\frac{1}{2}} e^{\frac{1}{2} b^T A^{-1} b}$$

$$\text{let } I_0 = \left(\det \frac{A}{2\pi}\right)^{-\frac{1}{2}}$$

$$= \frac{\partial^3}{\partial b_k \partial b_\ell \partial b_r} \left(I_0 \, A^{-1}_{si} b_i \, e^{\frac{1}{2} b^T A^{-1} b} \right) \Big|_{b=0}$$

$$= I_0 \frac{\partial^2}{\partial b_k \partial b_\ell} \left[A^{-1}_{sr} + A^{-1}_{si} b_i \, A^{-1}_{rj} b_j \right] e^{\frac{1}{2} b^T A^{-1} b} \Big|_{b=0}$$

$$= I_0 \frac{\partial}{\partial b_k} \Big[A^{-1}_{sr} A^{-1}_{\ell i} b_i + A^{-1}_{s\ell} A^{-1}_{rj} b_j + A^{-1}_{si} b_i A^{-1}_{r\ell} + A^{-1}_{si} b_i \, A^{-1}_{rj} b_j \, A^{-1}_{\ell k} b_k \Big] e^{\frac{1}{2} b^T A b} \Big|_{b=0}$$

$$= I_0 \left(A^{-1}_{sr} A^{-1}_{\ell k} + A^{-1}_{s\ell} A^{-1}_{rk} + A^{-1}_{sk} A^{-1}_{r\ell} \right)$$

$$= \left[\det\left(\frac{A}{2\pi}\right) \right]^{-\frac{1}{2}} \left(A^{-1}_{sr} A^{-1}_{\ell k} + A^{-1}_{s\ell} A^{-1}_{rk} + A^{-1}_{sk} A^{-1}_{r\ell} \right)$$

In general

$$\int \prod dx_i \; x_{k_1} \cdots x_{k_{2n}} \, e^{-\frac{1}{2} x^T A x} = \left[\det\left(\frac{A}{2\pi}\right) \right]^{-\frac{1}{2}} \sum_{\text{pairings}} \prod_{\text{pairs}} \left(A^{-1} \right)_{\text{paired indices}}$$