Chapter 7

A classical Lagrange density with a given symmetry when canonically quantized leads to a quantum theory that exhibits the symmetry, if that is possible.

Canonical Variables

The free fields of chapter 5 are systems of $q^n(\vec{x},t)$ and canonical conjugates $p_n(\vec{x},t)$ that satisfy the canonical (anti) commutation relations

$$[q^n(\vec{x},t),\ p_{\bar{n}}(\vec{y},t)]_{\mp} = i\,\delta^3(\vec{x}-\vec{y})\,\delta^n_{\bar{n}}$$

$$[q^n(\vec{x},t),\ q^{\bar{n}}(\vec{y},t)]_{\mp} = 0$$

$$[p_n(\vec{x},t),\ p_{\bar{n}}(\vec{y},t)]_{\mp} = 0$$

Note the pairs are at <u>equal</u> <u>times</u>, $t_x = t_y = t$.

For example the real scalar field $\phi(x)$ for the $a^c = a$, spin-zero case

$$\phi(x) = \phi^{(+)}(x) + \phi^{(-)}(x) \qquad \text{satisfies (5.2.6)}$$

$$[\phi^{(+)}(x),\ \phi^{(-)}(y)]_- = \Delta_+(x-y) \qquad \text{so}$$

$$[\phi(x),\phi(y)]_- = [\phi^{(+)}(x),\phi^{(-)}(y)]_- + [\phi^{(-)}(x),\phi^{(+)}(y)]_-$$

$$= \Delta_+(x-y) - \Delta_+(y-x) = \Delta(x-y)$$

by the definition (5.2.13) of Δ,

$$\Delta(x) = \Delta_+(x) - \Delta_+(-x) = \int \frac{d^3p}{2p^0(2\pi)^3}\left(e^{ipx} - e^{-ipx}\right).$$

Here $p^0 = \sqrt{\vec{p}^2 + m^2}$.

Now $\Delta(\vec{x}, 0) = 0$ and

$$\dot{\Delta}(\vec{x}, 0) = \frac{\partial}{\partial t} \Delta(\vec{x}, t)\Big|_{t=0}$$

$$= \int \frac{d^3p}{2p^0(2\pi)^3} \left(-ip^0 e^{ipx} + ip^0 e^{-ipx} \right)$$

$$= -i \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\cdot\vec{x}} = -i\, \delta^{(3)}(\vec{x}).$$

So $\Delta(\vec{x}-\vec{y}, 0) = 0$ and thus

$$[\phi(\vec{x}, t), \phi(\vec{y}, t)]_- = \Delta(\vec{x}-\vec{y}, 0) = 0.$$

And

$$[\phi(x, t), \frac{\partial}{\partial t} \phi(y, t)]_- = \frac{\partial}{\partial y^0} \Delta(x-y)$$

$$= -\frac{\partial}{\partial x^0 - y^0} \Delta(x-y)\Big|_{x^0 - y^0 = 0} = i\, \delta^{(3)}(\vec{x}-\vec{y}).$$

So $[\phi(x, t), \dot{\phi}(y, t)]_- = i\, \delta^{(3)}(\vec{x}-\vec{y})$.

Finally

$$[\dot{\phi}(x, t), \dot{\phi}(y, t)] = \frac{\partial^2}{\partial x^0 \partial y^0} \Delta(x-y)\Big|_{x^0 = y^0}$$

$$= -\int \frac{d^3p}{2p^0(2\pi)^3} (-ip^0)^2 e^{ipx} - (ip^0)^2 e^{-ipx} = 0.$$

So $[\dot{\phi}(x, t), \dot{\phi}(y, t)]_- = 0$.

So the canonical variables are

$$q(\vec{x},t) = \phi(\vec{x},t) \quad \text{and} \quad p(\vec{x},t) = \dot{\phi}(\vec{x},t).$$

The complex spin-zero field by (5.2.12) satisfies

$$[\phi(x), \phi^\dagger(y)]_- = \Delta(x-y) \text{ and } [\phi(x,t), \phi(y,t)]_- = 0$$

So

$$[\phi(x,t), \phi^\dagger(y,t)]_- = 0$$

and

$$[\phi(x,t), \frac{\partial}{\partial t}\phi^\dagger(y,t)]_- = i\,\delta^{(3)}(\vec{x}-\vec{y}) = -[\dot{\phi}, \phi^\dagger] = [\phi^\dagger, \dot{\phi}].$$

because

$$\dot{\Delta}(\vec{x},0) = -i\,\delta^3(\vec{x}),$$

Also $[\dot{\phi}(x,t), \dot{\phi}^\dagger(y,t)]_- = 0.$

So we may take $q(x,t) = \phi(x,t)$ and $p(x,t) = \dot{\phi}^\dagger(x,t)$.

Or we may use real variables

$$\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) \qquad \phi_1 = \frac{1}{\sqrt{2}}(\phi + \phi^\dagger) \qquad \phi_2 = \frac{1}{i\sqrt{2}}(\phi - \phi^\dagger)$$

Then

$$[\phi_i(x,t), \phi_j(y,t)]_- = 0$$

$$[\phi_1(x,t), \dot{\phi}_1(y,t)]_- = \frac{1}{2}[\phi(x,t) + \phi^\dagger(x,t), \dot{\phi}(y,t) + \dot{\phi}^\dagger(y,t)]$$

$$= i\,\delta^{(3)}(\vec{x}-\vec{y}) \qquad \text{while}$$

$$[\phi_1(x,t), \dot{\phi}_2(y,t)] = \frac{1}{2i}[\phi(x,t) + \phi^\dagger(x,t), \dot{\phi}(y,t) - \dot{\phi}^\dagger(y,t)]$$

$$= \frac{1}{2i}\left[i\,\delta(x-y) - i\,\delta(x-y)\right] = 0$$

$$[\phi_2(x,t), \dot{\phi}_2(y,t)] = -\frac{1}{2}[\phi(x,t) - \phi^\dagger(x,t), \dot{\phi}(y,t) - \dot{\phi}^\dagger(y,t)] = i\,\delta(\vec{x}-\vec{y}).$$

We may make a complex field ϕ out of two real fields

$$\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) \quad \text{with} \quad q^i = \phi_i \qquad p_j = \dot{\phi}_j.$$

and $\quad \phi^\dagger = \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2)$

Then $[\phi(x,t), \phi(y,t)]_- = \frac{1}{2}[\phi_1(x,t) + i\phi_2(x,t), \phi_1(y,t) + i\phi_2(y,t)]_- = 0$

and

$$[\phi(x,t), \phi^\dagger(y,t)] = \frac{1}{2}[\phi_1 + i\phi_2, \phi_1 - i\phi_2]_- = 0$$

while

$$[\phi(x,t), \dot{\phi}^\dagger(y,t)] = \frac{1}{2}[\phi_1(x,t) + i\phi_2(x,t), \dot{\phi}_1(y,t) - i\dot{\phi}_2(y,t)]_-$$

$$= \frac{1}{2} i \overset{3}{\delta}(x-y) + \frac{1}{2} i \delta(x-y) = i\,\delta(x-y).$$

But

$$[\phi(x,t), \dot{\phi}(y,t)] = \frac{1}{2}[\phi_1, \dot{\phi}_1] - \frac{1}{2}[\phi_2, \dot{\phi}_2] = 0.$$

So the real canonical variables are

$$q^k(x,t) = \phi_k(x,t) \quad \text{and} \quad p_k(x,t) = \dot{\phi}_k(\vec{x},t)$$

which obey
$$[q^k(x,t), q^{k'}(y,t)]_- = 0$$
$$[p_k(x,t), p_{k'}(y,t)]_- = 0$$
and
$$[q^k(x,t), p_{k'}(y,t)]_- = i\,\delta(x-y).$$

The real vector field of spin one, $v^\mu(x)$,
$$v^\mu(x) = v^{\mu(+)}(x) + v^{\mu(-)}(x)$$
satisfies by (5.3.30) $\quad [v^{\mu +}(x), v^{\nu(-)}(y)]_- = (\eta^{\mu\nu} - \partial^\mu\partial^\nu/m^2)\Delta_+(x-y)$

$$[v^\mu(x,t), v^\nu(y,t)] = [v^{\mu(+)}(x,t) + v^{\mu(-)}(x,t), v^{\nu(+)}(y,t) + v^{\nu(-)}(y,t)]_-$$
$$= [v^{\mu(+)}(x,t), v^{\nu(-)}(y,t)]_- + [v^{\mu(-)}(x,t), v^{\nu(+)}(y,t)]_-$$
$$= \left(\eta^{\mu\nu} - \frac{\partial^\mu\partial^\nu}{m^2}\right)\Delta_+(x-y) - \left(\eta^{\mu\nu} - \frac{\partial^\mu\partial^\nu}{m^2}\right)\Delta_+(y-x)$$

But by (5.2.13)
$$\Delta_+(x) - \Delta_+(-x) = \Delta(x).$$ So we have

$$[v^\mu(x,t), v^\nu(y,t)]_- = \left(\eta^{\mu\nu} - \frac{\partial^\mu\partial^\nu}{m^2}\right)\Delta(x-y).$$

So

$$[v^i(x,t), v^j(y,t)]_- = \left(\delta^{ij} - \frac{\partial_i\partial_j}{m^2}\right)\Delta(\vec{x}-\vec{y},0)$$
$$= \left(\delta^{ij} - \frac{\partial_i\partial_j}{m^2}\right)0 = 0$$

Let $\quad p_i(\vec{x},t) = \dfrac{\partial v^i(x,t)}{\partial t} + \dfrac{\partial v^0(x,t)}{\partial x^i} = \partial_0 v^i + \partial_i v^0$
$$= \partial^i v^0 - \partial^0 v^i.$$

So if $q^i(\vec{x},t) = v^i(\vec{x},t)$, then

$[q^i(x,t), q^j(y,t)]_- = 0$ and

$$[q^i(x,t), p_j(y,t)]_- = \left[v^i(x,t), \frac{\partial}{\partial y^0} v^j(y,t) + \partial_j v^0(y,t)\right]_-$$

$$= \frac{\partial}{\partial y^0}\left(\delta^{ij} - \frac{\partial^i\partial^j}{m^2}\right)\Delta(x-y) + \frac{\partial}{\partial y^j}\left(-\frac{\partial^i\partial^0}{m^2}\right)\Delta(x-y)$$

$$= i\delta^{ij}\delta^{(3)}(x-y) + \frac{\partial_y^0\partial^i\partial^j}{m^2}\Delta(x-y) + \frac{\partial^i\partial^j\partial^0}{m^2}\Delta(x-y)$$

$$= i\delta^{ij}\delta^{(3)}(\vec{x}-\vec{y}) - \frac{\partial^0\partial^i\partial^j}{m^2}\Delta(x-y) + \frac{\partial^i\partial^j\partial^0}{m^2}\Delta(x-y)$$

$$= i\delta^{ij}\delta^{(3)}(\vec{x}-\vec{y}).$$

And

$$[p_i(x,t), p_j(y,t)]_- = \left[\partial_0 v^i(x,t) + \partial_i v^0(x,t), \partial_0 v^j(y,t) + \partial_j v^0(y,t)\right]_-$$

$$= -\partial_0^2\left(\delta^{ij} - \frac{\partial^i\partial^j}{m^2}\right)\Delta(x-y) - \partial_i\partial_0\left(-\frac{\partial^0\partial^j}{m^2}\right)\Delta(x-y)$$

$$-\partial_i\partial_j\left(\eta^{00} - \frac{\partial_0^2}{m^2}\right)\Delta(x-y) - \partial_0\partial_j\left(-\frac{\partial^i\partial^0}{m^2}\right)\Delta(x-y)$$

$$= -\delta^{ij}\partial_0^2\Delta + \frac{\partial_0^2\partial_i\partial_j}{m^2}\Delta - \frac{\partial_0^2\partial_i\partial_j}{m^2}\Delta + \partial_i\partial_j\Delta + \frac{\partial_0^2\partial_i\partial_j}{m^2}\Delta - \frac{\partial_0^2\partial_i\partial_j}{m^2}\Delta$$

$$= -\delta^{ij}\partial_0^2\Delta(x-y) + \partial_i\partial_j\Delta(x-y) = 0 - 0 = 0 \text{ at } x^0 = y^0 = t.$$

By (5.3.38) $\partial_\mu v^\mu = 0 \Rightarrow \dot{v}^0 = -\nabla \cdot v$

So $\vec{\nabla} \cdot \vec{p} = \nabla \cdot \dot{v} + \Delta v^0 = -\partial_0^2 v^0 + \Delta v^0$

But by (5.3.36) $\Delta v^0 - \partial_0^2 v^0 = \Box v^0 = m^2 v^0$

So

$$\frac{\vec{\nabla} \cdot \vec{p}}{m^2} = v^0$$

$$(\Box - m^2) v^\mu(x) = 0$$

This means that v^0 is a <u>dependent</u> variable.

The complex spin-one vector field $v^\mu(x)$ obeys

(5.3.35)
$$[v^\mu(x,t), v^{\nu\dagger}(y,t)] = \left(\eta^{\mu\nu} - \frac{\partial^\mu \partial^\nu}{m^2}\right) \Delta(x-y)$$

So $q^i(x,t) = v^i(x,t)$ and

$$p_i(x,t) = \dot{v}^{i\dagger}(x,t) + \frac{\partial v^{0\dagger}}{\partial x^i}(x,t).$$

Or use real fields $v_1^\mu = \frac{1}{\sqrt{2}}(v^\mu + v^{\mu\dagger})$

and $v_2^\mu = \frac{1}{2i}(v^\mu - v^{\mu\dagger})$.

$$v^\mu = \frac{1}{\sqrt{2}}(v_1^\mu + i v_2^\mu).$$

Dirac fields.

$(5.5.39) \Rightarrow \quad [\psi_\ell(x,t), \psi^\dagger_{\bar\ell}(y)]_+ = \{[-\gamma^\mu\partial_\mu + m]\beta\}_{\ell\bar\ell}\,\Delta(x-y),$

and $[\psi_\ell(x,t), \psi_{\bar\ell}(y,t)]_+ = 0 \qquad (\psi \text{ complex})$

Since $[\psi_\ell, \psi^\dagger_{\ell'}]_+ \neq 0$ at equal times, we cannot have $q_1 = \psi$ and $q_2 = \psi^*$. We use instead

$$q^n(x) = \psi_n(x) \quad \text{and} \quad p_n(x) = i\psi^\dagger_n(x).$$

Then

$$[q^n(\vec{x},t), q^{\bar n}(y,t)]_+ = 0 \qquad [p_n, p_{n'}]_+ = 0$$

and $[q^n(x,t), p_{\bar n}(y,t)]_+ = i\,[\psi_n(x,t), \psi^\dagger_{\bar n}(y,t)]_+$

$$= i\{[-\gamma^\mu\partial_\mu + m]\beta\}_{n\bar n}\,\Delta(x-y)\Big|_{x^0=y^0}$$

$$= -i(\gamma^0\partial_0\beta)_{n\bar n}\,\Delta(x-y) \qquad\qquad \beta = i\gamma^0, \quad \gamma^{0\,2} = -1$$

$$= (\gamma^0\partial_0\gamma^0)_{n\bar n}\,\Delta(x-y)$$

$$= -\partial_0\Delta(x-y)\,\delta_{n\bar n} = i\,\delta^3(\vec{x}-\vec{y})\,\delta_{n\bar n}.$$

QM functional derivative of a bosonic functional F

$$\frac{\delta F[q(t), p(t)]}{\delta q^n(x,t)} \equiv i\,[p_n(\vec{x},t),\ F[q(t),p(t)]]$$

and

$$\frac{\delta F[q(t), p(t)]}{\delta p_n(x,t)} \equiv i\,[F[q(t),p(t)],\ q^n(\vec{x},t)]$$

For example, $F = \int d^3x\, q^2(x,t)\, p(x,t)$ then

$$\frac{\delta F}{\delta q^n(x,t)} = i\,[p_n(x,t),\ \int d^3y\, q^2(y,t)\, p(y,t)]$$

$$= i\int d^3y\, [p_n(x,t), q^2(y,t)]\ p(y,t)$$

$$[p, q^2] = (pq - qp)q + q(pq - qp) = -iq - iq$$

So

$$\frac{\delta F}{\delta q^n} = i\int d^3y\, -2i\,\delta(x-y)\, q(y,t)\, p(y,t) = 2\,q(x,t)\, p(x,t). \quad = \frac{\partial F}{\partial q}$$

But for fermions L & R get interchanged.

For instance $F = \int d^3y\, \psi_\ell(y,t)\, \psi_\ell^\dagger(y,t)$

$$\frac{\delta F}{\delta p_n} = \frac{1}{i}\frac{\delta F}{\delta \psi_n^\dagger} \equiv i\left[\int d^3y\, \psi_\ell(y,t)\, \psi_\ell^\dagger(y,t),\ \psi_n(x,t)\right]$$

$$= i\int d^3y\ \psi_\ell(y,t)\, \psi_\ell^\dagger(y,t)\, \psi_n(x,t) - \psi_n(x,t)\, \psi_\ell(y,t)\, \psi_\ell^\dagger(y,t)$$

$$\frac{\delta F}{\delta_i \psi_n^\dagger(x,t)} = +i\int dy\, \psi_\ell(y,t)\left[\psi_\ell^\dagger(y,t), \psi_n(x,t)\right]_+$$

$$= +i\int d^3y\, \delta_{\ell n}\, \delta(x-y)\, \psi_\ell(y,t)$$

$$= +i\, \psi_n(\vec{x},t). \qquad \text{Like } \frac{\partial F}{\partial i\psi^\dagger} \text{ from the left}$$

$$\frac{\delta F}{\delta q} = \frac{\delta F}{\delta \psi} = i\left[i\psi^\dagger(x), \int \psi\psi^\dagger dy\right] = -\int dy\, [\psi^\dagger(x),\psi(y)]_+\psi^\dagger(y) = -\psi^\dagger(x) \text{ like } \frac{\partial F}{\partial \psi} \text{ from right}$$

$$\delta F[q(t),p(t)] = \int d^3x \sum_n \delta q^n(x,t)\, \frac{\delta F[q(t),p(t)]}{\delta q^n(x,t)}$$

$$+ \frac{\delta F[q(t),p(t)]}{\delta p_n(x,t)}\, \delta p_n(\vec{x},t)$$

So to get the signs right $F = q^m p^m$ for bosons but $F = p^n q^m$ for fermions (with $L \rightleftarrows R$).

For fermions the choices are:

$$F = \psi_\ell^\dagger \psi_\ell \qquad \text{OR} \qquad F = \psi_\ell \psi_\ell^\dagger$$

$$\frac{\delta F}{\delta p_n} = -i\psi_n = \frac{\partial^L F}{\partial i\psi^\dagger} \qquad\qquad \frac{\delta F}{\delta p_n} = i\psi_n = -\frac{\partial^R F}{\partial i\psi_n^\dagger} = \frac{\partial^L F}{\partial i\psi_n^\dagger}$$

$$\frac{\delta F}{\delta q^n} = \psi_n^\dagger = \frac{\partial^R F}{\partial \psi_n} \qquad\qquad \frac{\delta F}{\delta q^n} = -\psi_n^\dagger = -\frac{\partial^L F}{\partial \psi_n} = \frac{\partial^R F}{\partial \psi_n}$$

Both seem awkward.

H_0 is the generator of time translations

$$q^n(\vec{x},t) = e^{iH_0 t}\, q^n(\vec{x},0)\, e^{-iH_0 t}$$

$$p_n(\vec{x},t) = e^{iH_0 t}\, p_n(\vec{x},0)\, e^{-iH_0 t}$$

So the free-particle operators obey

$$\dot{q}^n(\vec{x},t) = i\,[H_0, q^n(\vec{x},t)] = \frac{\delta H_0}{\delta p_n(x,t)}$$

$$\dot{p}_n(\vec{x},t) = -i\,[p_n(x,t), H_0] = -\frac{\delta H_0}{\delta q^n(\vec{x},t)}$$

as in Hamilton's mechanics.

The free-particle Hamiltonian is

$$H_0 = \sum_{n\sigma} \int d^3k\; a^\dagger(k,\sigma,n)\, a(k,\sigma,n)\sqrt{\vec{k}^2 + m_n^2}$$

Up to a constant, for the real scalar field

$$H_0 = \int d^3x\; \tfrac{1}{2}p^2 + \tfrac{1}{2}(\nabla q)^2 + \tfrac{1}{2}m^2 q^2$$

$$= \tfrac{1}{2}\int d^3x \int \frac{d^3p\, d^3p'}{(2\pi)^3\sqrt{2p'^0 2p^0}} \Big\{ \left[-ip^0 a(p)e^{ipx} + ip^0 a^\dagger(p)e^{-ipx}\right]$$

$$\left[-ip'^0 a(p')e^{ip'x} + ip'^0 a^\dagger(p')e^{-ip'x}\right]$$

$$+\left[i\vec{p}\,a(p)e^{ipx} - i\vec{p}\,a^\dagger(p)e^{-ipx}\right]\left[i\vec{p}'a(p')e^{ip'x} - i\vec{p}'a^\dagger(p')e^{-ip'x}\right]$$

$$+m^2\left[a(p)e^{ipx} + a^\dagger(p)e^{-ipx}\right]\left[a(p')e^{ip'x} + a^\dagger(p')e^{-ip'x}\right]\Big\}$$

$$H_0 = \frac{1}{2}\int \frac{d^3p}{2p^0}\Big\{ a(p)\,a(-\vec{p})\left[-p^{0\,2} + \vec{p}^{\,2} + m^2\right]$$

$$+ a(p)\,a^\dagger(p)\left[p^{0\,2} + \vec{p}^{\,2} + m^2\right]$$

$$+ a^\dagger(p)\,a(p)\left[p^{0\,2} + \vec{p}^{\,2} + m^2\right]$$

$$+ a^\dagger(p)\,a^\dagger(-p)\left[-p^{0\,2} + \vec{p}^{\,2} + m^2\right]\Big\}$$

$$= \frac{1}{2}\int \frac{d^3p}{2p^0}\, 2p^{0\,2}\left(a(p)\,a^\dagger(p) + a^\dagger(p)\,a(p)\right)$$

$$= \frac{1}{2}\int d^3p\; p^0\left(a(p)\,a^\dagger(p) + a^\dagger(p)\,a(p)\right) \qquad \delta^3(0) = \frac{V}{(2\pi)^3}$$

$$= \int d^3p\; p^0\left[a^\dagger(p)\,a(p) + \tfrac{1}{2}\,\delta^3(0)\right] \qquad d^3p = \frac{(2\pi)^3}{V}\sum_n$$

$$= \sum_n p_n^0\left[a^\dagger(p_n)\,a(p_n) + \tfrac{1}{2}\right] \qquad \frac{(2\pi)^3}{V}a^\dagger(p)a(p) = a_n^\dagger a_n$$

↑ zero-point energies.

From this H_0, we derive L_0

$$L_0\left[q(t), \dot{q}(t)\right] = \sum_n \int d^3x\; p_n(x,t)\,\dot{q}^n(x,t) - H_0$$

For the real scalar field, $p = \dot{q} = \dot{\phi}$

$$L_0 = \int d^3x\; p\dot{q} - \tfrac{1}{2}p^2 - \tfrac{1}{2}(\nabla q)^2 - \tfrac{1}{2}m^2 q^2$$

$$= \int d^3x\; -\tfrac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \tfrac{1}{2}m^2\phi^2 .$$

This must be L_0 because it gives the right H_0.

The 'Heisenberg picture' canonical variables are

$$Q^n(x,t) = e^{iHt}\, q^n(x,0)\, e^{-iHt}$$

$$P_n(x,t) = e^{iHt}\, p_n(x,0)\, e^{-iHt}$$

where H is the full hamiltonian.

$$H[Q,P] = e^{iHt}\, H[q(0),p(0)]\, e^{-iHt} = H[q,p].$$

$e^{iHt} \cdots e^{-iHt}$ is a similarity transformation so

$$[Q^n(x,t), P_{\bar{m}}(y,t)]_{\mp} = e^{iHt}\, [q^n(x,0), p_{\bar{m}}(y,0)]_{\mp}\, e^{-iHt}$$

$$= i\,\delta^3(x-y)\,\delta^n_{\bar{m}}.$$

$$[Q^n(x,t), Q^{\bar{m}}(y,t)]_{\mp} = 0 = [P_n(x,t), P_{\bar{m}}(y,t)]_{\mp}$$

But they obey

$$\dot{Q}^n(x,t) = i\,[H, Q^n(x,t)] = \frac{\delta H}{\delta P_n(x,t)} \qquad (7.1.33)$$

$$\dot{P}_n(x,t) = -i\,[P_n(x,t), H] = -\frac{\delta H}{\delta Q^n(x,t)}. \qquad (7.1.34)$$

Ex. $H = \int d^3x \; \frac{1}{2}P^2 + \frac{1}{2}(\nabla Q)^2 + \frac{1}{2}m^2 Q^2 + \mathcal{H}(Q)$

here $\qquad P = \dot{Q}$ as before for the free case

But in general P is not always $\dot{Q}$. One must use H's equations of motion (7.1.33–34) to infer $P(Q, \dot{Q})$.

The Lagrangian Formalism

Pick L with right symmetries and then find H. In fact given H, we could find L. But it is $\mathcal{L}(x)$ that is a scalar.

In general the Lagrangian

$$L = L[\psi(t), \dot{\psi}(t)]$$

is a functional of the general fields $\psi^\ell(x,t)$, $\dot{\psi}^\ell(x,t)$.

$$\Pi_\ell(x,t) \equiv \frac{\delta L[\psi(t), \dot{\psi}(t)]}{\delta \dot{\psi}^\ell(x,t)} \qquad (7.2.1)$$

where we do what we must to make this sensible, which may or may not be (7.1.17–18).

The equations of motion are

$$\dot{\Pi}_\ell(x,t) = \frac{\delta L[\psi(t), \dot{\psi}(t)]}{\delta \psi^\ell(x,t)} \qquad (7.2.2)$$

The *action* is

$$I[\psi] = \int_{-\infty}^{\infty} dt \; L[\psi(t), \dot\psi(t)]$$

$$\delta I[\psi] = \int d^4x \; \frac{\delta L}{\delta \psi^\ell(x)} \delta\psi^\ell(x) + \frac{\delta L}{\delta \dot\psi^\ell(x)} \delta\dot\psi^\ell(x)$$

Let $\delta\psi^\ell = 0$ as $t \to \pm\infty$. Then by parts

$$\delta I[\psi] = \int d^4x \left[\frac{\delta L}{\delta\psi^\ell(x)} - \frac{d}{dt} \frac{\delta L}{\delta\dot\psi^\ell(x)} \right] \delta\psi^\ell(x).$$

So $\delta I[\psi] = 0$ to lowest order if $\delta\psi^\ell = 0$ at $t = \pm\infty$ and if

$$\frac{d}{dt} \frac{\delta L}{\delta\dot\psi^\ell(x)} = \dot\Pi_\ell(x) = \frac{\delta L}{\delta\psi^\ell(x)}$$ which is (7.2.2).

$I[\psi]$ is a scalar usually (always).

We usually take $L[\psi(t), \dot\psi(t)] = \int d^3x \; \mathcal{L}(\psi(x,t), \nabla\psi(x,t), \dot\psi(x,t))$

so that the action is

$$I[\psi] = \int d^4x \; \mathcal{L}(\psi(x), \partial\psi(x)/\partial x^a),$$

which is the case for all commercial field theories.

Now we vary $\psi^\ell(x)$ by $\delta\psi^\ell(x)$ which vanishes on the boundary of d^3x:

$$\delta L = \int d^3x \left[\frac{\partial\mathcal{L}}{\partial\psi^\ell} \delta\psi^\ell + \frac{\partial\mathcal{L}}{\partial\nabla\psi^\ell} \nabla\delta\psi^\ell + \frac{\partial\mathcal{L}}{\partial\dot\psi^\ell} \delta\dot\psi^\ell \right]$$

$$= \int d^3x \left[\left(\frac{\partial\mathcal{L}}{\partial\psi^\ell} - \nabla \cdot \frac{\partial\mathcal{L}}{\partial\nabla\psi^\ell} \right) \delta\psi^\ell + \frac{\partial\mathcal{L}}{\partial\dot\psi^\ell} \delta\dot\psi^\ell \right].$$

So $$\frac{\delta L}{\delta \psi^\ell} = \frac{\partial \mathcal{L}}{\partial \psi^\ell} - \nabla \cdot \frac{\partial \mathcal{L}}{\partial \nabla \psi^\ell}$$

$$\frac{\delta L}{\delta \dot{\psi}^\ell} = \frac{\partial \mathcal{L}}{\partial \dot{\psi}^\ell}$$

So (7.2.2) now is

$$\frac{d}{dt}\frac{\delta L}{\delta \dot{\psi}^\ell} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\psi}^\ell} = \frac{\delta L}{\delta \psi^\ell} = \frac{\partial \mathcal{L}}{\partial \psi^\ell} - \nabla \cdot \frac{\partial \mathcal{L}}{\partial \nabla \psi^\ell}$$

or

$$\frac{\partial}{\partial x^\mu}\frac{\partial \mathcal{L}}{\partial \frac{\partial \psi^\ell}{\partial x^\mu}} = \frac{\partial \mathcal{L}}{\partial \psi^\ell}$$

which are the Euler-Lagrange equations. If $\mathcal{L}$ is a scalar, then these E-L eqs. are Lorentz invariant.

We also want $I = I^*$. Think of the fields as real ($\psi = \psi_1 + i\psi_2$ if $\psi^* \neq \psi$). Say there are N real fields. If I were complex, then there would be $2N$ real equations of motion, the E-L eqns. But we want only N such E-L equations.

The Legendre transformation is used to find H from L or $\mathcal{L}$.

$$H = \sum_{\ell} \int d^3x \; \Pi_{\ell}(x,t)\, \dot{\psi}^{\ell}(x,t) \;-\; L[\psi(t), \dot{\psi}(t)].$$

Now (7.2.1),

$$\Pi_{\ell}(x,t) = \left.\frac{\delta L[\psi(t), \dot{\psi}(t)]}{\delta \dot{\psi}^{\ell}(x,t)}\right|_{\psi},$$

does not always allow $\dot{\psi}^{\ell}(x,t)$ to be expressed uniquely in terms of ψ^{ℓ} and Π^{ℓ}. But H is constructed so that

$$\left.\frac{\delta H}{\delta \dot{\psi}}\right|_{\Pi} = 0.$$

H is taken to be a functional of $\Pi(x,t)$ and $\psi(x,t)$.

$$\left.\frac{\delta H}{\delta \psi^{\ell}(x,t)}\right|_{\Pi} = \int d^3y \sum_{\ell'} \Pi_{\ell'}(y,t) \left.\frac{\delta \dot{\psi}^{\ell'}(y,t)}{\delta \psi^{\ell}(x,t)}\right|_{\Pi} - \left.\frac{\delta L}{\delta \psi^{\ell}(x,t)}\right|_{\dot{\psi}}$$

$$- \int d^3y \sum_{\ell'} \left.\frac{\delta L}{\delta \dot{\psi}^{\ell'}(y,t)}\right|_{\psi} \left.\frac{\delta \dot{\psi}^{\ell'}(y,t)}{\delta \psi^{\ell}(x,t)}\right|_{\Pi}$$

and

$$\left.\frac{\delta H}{\delta \Pi_{\ell}(x,t)}\right|_{\psi} = \dot{\psi}^{\ell}(x,t) + \int d^3y \sum_{\ell'} \Pi_{\ell'}(y,t) \left.\frac{\delta \dot{\psi}^{\ell'}(y,t)}{\delta \Pi^{\ell}(x,t)}\right|_{\psi}$$

$$- \int d^3y \sum_{\ell'} \left.\frac{\delta L}{\delta \dot{\psi}^{\ell'}(y,t)}\right|_{\psi} \left.\frac{\delta \dot{\psi}^{\ell'}(y,t)}{\delta \Pi^{\ell}(x,t)}\right|_{\psi}$$

Since $\left.\frac{\delta L}{\delta \dot{\psi}^{\ell'}(y,t)}\right|_{\psi} = \Pi_{\ell'}(y,t)$, we have

$$\left.\frac{\delta H}{\delta \psi^{\ell}(x,t)}\right|_{\pi} = -\left.\frac{\delta L}{\delta \psi^{\ell}(x,t)}\right|_{\dot{\psi}}$$

and

$$\left.\frac{\delta H}{\delta \Pi_{\ell}(x,t)}\right|_{\psi} = \dot{\psi}^{\ell}(x,t),$$

So (7.2.2), $\dot{\pi}_{\ell}(x,t) = \frac{\delta L}{\delta \psi^{\ell}(x,t)}$,

now implies Hamilton's equation

$$\left.\frac{\delta H}{\delta \psi^{\ell}(x,t)}\right|_{\pi} = -\dot{\Pi}_{\ell}(x,t) \quad (7.2.12) \quad \text{and} \quad \left.\frac{\delta H}{\delta \Pi_{\ell}(x,t)}\right|_{\psi} = \dot{\psi}^{\ell}(x,t). \quad (7.2.13)$$

In the simpler cases, one may identify ψ^{ℓ}, Π_{ℓ} with Q^{n}, P_{n}, and one may impose the canonical commutation relations (7.1.30-32) $[\psi^{\ell}(x,t), \Pi_{\ell'}(y,t)]_{\mp} = i\delta^{\ell}_{\ell'}\,\delta^{3}(x-y)$ so that (7.2.12-13) become like (7.1.33-34)

$$-\dot{\Pi}_{\ell}(x,t) = \frac{\delta H}{\delta \psi^{\ell}(x,t)} = i[\Pi^{\ell}(x,t), H]$$

and

$$\dot{\psi}^{\ell}(x,t) = \frac{\delta H}{\delta \Pi_{\ell}(x,t)} = i[H, \psi^{\ell}(x,t)].$$

For example, if $\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{m^2}{2}\phi^2 - \mathcal{H}(\phi)$,
then the E-L equations

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \frac{\partial \mathcal{L}}{\partial \phi} \quad \text{are} \quad -\partial_\mu \partial^\mu \phi = -m^2\phi - \mathcal{H}'(\phi)$$

or

$$(\Box - m^2)\phi = \mathcal{H}'(\phi) . \qquad \ddot{\phi} = \Delta\phi - m^2\phi - \mathcal{H}'(\phi)$$

$$\Pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi} \qquad (\text{cf } 7.1.36)$$

$$H = \int d^3x \left(\pi\dot{\phi} - \mathcal{L}\right)$$

$$= \int d^3x \ \tfrac{1}{2}\pi^2 + \tfrac{1}{2}(\nabla\phi)^2 + \tfrac{1}{2}m^2\phi^2 + \mathcal{H}(\phi) .$$

But $\dot{V}_0$ is often absent from $\mathcal{L}$
and $\bar{\psi}\gamma^\mu\partial_\mu\psi$ has no $\dot{\psi}^\dagger$. Even if we
write

$$\bar{\psi}\gamma^\mu\partial_\mu\psi = \tfrac{1}{2}\bar{\psi}\gamma^\mu\partial_\mu\psi - \tfrac{1}{2}\partial_\mu\bar{\psi}\gamma^\mu\psi ,$$

by integrating by parts, things are no simpler
because now $\Pi_\psi \sim \psi^\dagger$ but $\Pi_{\psi^\dagger} \sim \psi$.
Let Q^n be those canonical variables that have
$\dot{Q}^n$ in $\mathcal{L}$. Let C^r be those fields that
do not have $\dot{C}^r$ in $\mathcal{L}$.

Then the conjugates of the Q^n are

$$P_n(x,t) = \frac{\delta L[Q(t), \dot{Q}(t), C(t)]}{\delta \dot{Q}^n(x,t)} .$$

The C^r's have no conjugates because $\frac{\delta L}{\delta \dot{C}^r} = 0$.

So H is now

$$H = \sum_n \int d^3x \left(P_n \dot{Q}^n \right) - L[Q(t), \dot{Q}(t), C(t)]$$

in which C^n and $\dot{Q}^l$ must be expressed in terms of Q^l and P_l.

In some cases (7.6) one may avoid actually solving for C^n & $\dot{Q}^l$. In gauge theories one must either pick a gauge (8) or use the F-P tricks of II.

To do perturbation theory, we use the interaction picture.

$$H = H(Q^n, P_n)\Big|_{t=0} = H(q^n, p_n)\Big|_{t=0}$$

$$= H_0(q\ p) + V(q\ p).$$

For example $\quad H = H_0 + V$

$$H_0 = \int d^3x \ \tfrac{1}{2}\Pi^2 + \tfrac{1}{2}(\nabla\Phi)^2 + \tfrac{1}{2}m^2\Phi^2$$

$$V = \int d^3x \ \mathcal{H}(\Phi).$$

Φ, Π Heisenberg fields at t

Pass to interaction rep.

$$V(t) = e^{iH_0 t}\, V\, e^{-iH_0 t}$$

$\phi(x,0) = \Phi(x,0)$
$\pi(x,0) = \Pi(x,0)$

where

$$H_0 = \int d^3x \ \tfrac{1}{2}\pi^2 + \tfrac{1}{2}(\nabla\phi)^2 + \tfrac{1}{2}m^2\phi^2 = e^{iH_0 t} H_0 e^{-iH_0 t}$$

in which

(7.1.21) $$\dot\phi(x,t) = \frac{\delta H_0}{\delta\pi(y,t)} = \pi(x,t) = i[H_0, \phi(x,t)]$$

$$\dot\phi(x,t) = i\int d^3y\, [\tfrac{1}{2}\pi(y,t)^2, \phi(x,t)] = i\int d^3y\, \pi(y,t)(-i\delta(x-y)) = \pi(x,t).$$

(7.1.22) $$\dot\pi(x,t) = -\frac{\delta H_0}{\delta\phi(x,t)} = i[H_0, \pi(x,t)]$$

$$= i\int d^3y\, [\tfrac{1}{2}(\nabla\phi(y,t))^2 + \tfrac{1}{2}m^2\phi^2(y,t), \pi(x,t)]$$

$$= +\Delta\phi - m^2\phi = \ddot\phi$$

So $\quad (\Box + m^2)\phi = 0.$

The general real solution is

$$\phi(x) = \int \frac{d^3p}{(2\pi)^{3/2}} \frac{1}{\sqrt{2p^0}} \left[e^{ipx} a(\vec p) + e^{-ipx} a^\dagger(p) \right]$$

with $p^0 = \sqrt{\vec p^2 + m^2}$ and $a(\vec p)$ to be determined.

$$\pi(x) = \dot\phi = -i\int \frac{d^3p}{(2\pi)^{3/2}} \sqrt{\frac{p^0}{2}} \left[e^{ipx} a(\vec p) - e^{-ipx} a^\dagger(p) \right].$$

Now we want

$$[\phi(x,t),\ \pi(y,t)]_- = i\,\delta(x-y)$$

$$[\phi(x,t),\ \phi(y,t)]_- = 0 = [\pi(x,t),\ \pi(y,t)]_-,$$

so we choose the a's so that

$$[a(\vec{p}),\ a^\dagger(\vec{p}\,')]_- = \delta^{(3)}(p-p')$$

and

$$[a(\vec{p}),\ a(\vec{p}\,')]_- = 0.$$

These formulae also give

$$H_0 = \int d^3k\,\sqrt{m^2+k^2}\,\left(a^\dagger(k)a(k) + \tfrac{1}{2}\right).$$

So in fact the $\mathcal{L}$ was okay.

Say $\delta\mathcal{L} = \partial_\mu F^\mu$. Then

$$\Delta I = \int d^4x\ \partial_\mu F^\mu \equiv \int d\sigma_\mu F^\mu$$

So the action changes only if the fields at the boundary make F^μ nonzero.

In any case the equations of motion remain unchanged because they are derived under the assumption that $\delta\phi = 0$ on the boundary.

Similarly $\delta\mathcal{L} = \nabla\cdot F$ means that

$$\Delta L = \int d^3x\ \nabla\cdot F = \int d\sigma_i F^i$$

which vanishes if $F^i = 0$ on the boundary of space.

Similarly $\delta \mathcal{L} = \nabla \cdot F$ will not affect the field equations.

If $\delta \mathcal{L} = \partial_0 F^0$, then clearly the equations of motion are unaffected.

How about the more general term

$$\Delta L(t) = \int d^3x \, D_{n x}[Q(t)] \, \dot{Q}^n(x, t)$$

ie

$$\Delta \mathcal{L} = D_{n x}(Q(x)) \, \dot{Q}^n(x).$$

$$\Delta P_n(x) = \frac{\delta \Delta L(t)}{\delta \dot{Q}^n(x, t)} = \frac{\partial \Delta \mathcal{L}(x)}{\partial \dot{Q}^n(x)}$$

$$= D_n(Q(x), x).$$

$$\Delta H = \sum_n \int d^3x \, \Delta P_n(x) \dot{Q}^n(x) \quad - \quad \Delta L$$

$$= \int d^3x \sum_n D_{n x}(Q) \dot{Q}^n \quad - \int d^3x \sum_n D_{n x} \dot{Q}^n = 0.$$

So $\Delta H = 0$. But

$$[P_n + \Delta P_n, P_m + \Delta P_m] = [\Delta P_n, P_m] + [P_n, \Delta P_m]$$

$$= [D_n, P_m] + [P_n, D_m] \qquad ([\Delta P_n, \Delta P_m] = 0)$$

$$= i \frac{\delta D_n}{\delta Q_m} - i \frac{\delta D_m}{\delta Q_n}$$ might not vanish.

although $[Q^n, Q^m] = 0$ and

$$[Q^n, P_m + \Delta P_m] = [Q^n, P_m] = i\,\delta^n_m$$

still work.

Now when

$$\Delta L = \frac{dG}{dt} = \int d^3x\, \frac{\delta G}{\delta Q^n}\,\dot{Q}^n$$

or

$$\delta \mathcal{L} = \frac{\delta G}{\delta Q^n}\,\dot{Q}^n = \frac{dG}{dt},$$

then

$$D_n = \frac{\delta G}{\delta Q^n} \quad \text{and so}$$

$$i\,\frac{\delta D_n}{\delta Q_m} - i\,\frac{\delta D_m}{\delta Q_n} = i\left(\frac{\delta^2 G}{\delta Q_m \delta Q_n} - \frac{\delta^2 G}{\delta Q_n \delta Q_m}\right) = 0.$$

So the quantum structure as well as the classical structure of a theory is unaffected by partial integration.

Global Symmetries

go to 352.5 Because the dynamics follows from a variational principle, the L. formalism facilitates the implementation of symmetries in the quantum theory.

Suppose under

$$\psi^\ell(x) \to \psi^{\ell\prime}(x) = \psi^\ell(x) + i\epsilon F^\ell(x)$$

the action

$$I[\psi] = \int dt\, L[\psi(t)] \quad \text{is invariant,}$$

$$0 = \delta I = i\epsilon \int d^4x \frac{\delta I}{\delta \psi^\ell(x)} F^\ell(x)$$

in which ϵ is a constant (hence global).

Now in fact if $\psi^\ell(x)$ obeys the field equations, then $\delta I = 0$ to order ϵ. Here we assume $\delta I = 0$ even if the ψ^ℓ's do not satisfy the field equations. This is a global symmetry.

Consider now the local transformation

$$\psi^\ell(x) \to \psi^{\prime\ell}(x) = \psi^\ell(x) + i\epsilon(x) F^\ell(x)$$

in which $\epsilon(x) \to 0$ as $x \to \infty$. Then since due to the symmetry $\delta I = 0$ for constant ϵ, here $\delta I \neq 0$ but δI has the simple form

$$\delta I = -\int d^4x \; J^\mu(x) \, \frac{\partial \epsilon(x)}{\partial x^\mu}$$

whether or not the fields $\psi^\ell(x)$ satisfy their field equations. This result also follows from the assumption that L involves only the first derivatives of the fields and is invariant for constant ϵ.

But if the fields do obey their field equations, then

$$0 = \delta I = \int d^4x \; \epsilon(x) \, \partial_\mu J^\mu(x).$$

Since $\epsilon(x)$ is arbitrary, at finite x, we have the conservation law

$$0 = \frac{\partial^\mu J(x)}{\partial x^\mu}.$$

Its integral form is

$$0 = \frac{d}{dt} F = \frac{d}{dt} \int d^3x \; J^0(x)$$

$$= \int d^3x \; - \vec{\nabla} \cdot \vec{J} = -\int d\sigma_i \, J^i$$

which vanishes if the fields vanish as $\vec{x} \to \infty$.

This is Noether's theorem: symmetries imply conservation laws.

If $L(t)$ and not just I is invariant under the symmetry transformation, then suppose

$$\psi^\ell(x) \to \psi'^\ell(x) = \psi^\ell(x) + i\epsilon(t)\, F^\ell(x),$$

Then

$$\delta I = i\int dt \int d^3x \left[\frac{\delta L[\psi(t), \dot\psi(t)]}{\delta \psi^\ell(x,t)} \epsilon(t)\, F^\ell(x,t) + \frac{\delta L[\psi,\dot\psi]}{\delta \dot\psi^\ell(x,t)} \frac{d}{dt}\left(\epsilon(t)\, F^\ell(x,t)\right)\right]$$

Now by the symmetry of $L(t)$

$$0 = \int d^3x \quad \frac{\delta L}{\delta \psi^\ell(x,t)} F^\ell(x) + \frac{\delta L}{\delta \dot\psi} \dot F^\ell(x) \quad (\text{at each } t)$$

so in general

$$\delta I = i\int dt \int d^3x \ \frac{\delta L}{\delta \dot\psi^\ell} \dot\epsilon\, F^\ell = -\int d^4x\, J^\mu \partial_\mu \epsilon$$

So $\quad 0 = \delta I = -i\int dt\, d^3x\, \epsilon \dfrac{d}{dt}\left(\dfrac{\delta L}{\delta \dot\psi^\ell} F^\ell\right) = F(t_2) - F(t_1)$

$$F = \int d^3x\, J^0 = -i \int d^3x\, \frac{\delta L}{\delta \dot\psi^\ell} F^\ell .$$

This is invariant for any fields that obey their field equations and vanish at $\vec{x} \to \infty$.

Examples of such symmetries are the translations and rotations.

Under some symmetry transformations, such as isospin or color or other internal symmetries, the Lagrange density $\mathcal{L}$ is itself invariant. Then

$$\delta I = i \int d^4x \; \frac{\partial \mathcal{L}}{\partial \psi^l} F^l \epsilon + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi^l)} \partial_\mu (F^l \epsilon)$$

But the symmetry of $\mathcal{L}$ means that

$$\frac{\partial \mathcal{L}}{\partial \psi^l} F^l \epsilon + \frac{\partial \mathcal{L}}{\partial \psi^l} \epsilon \partial_\mu F^l = 0 \qquad \text{(at each } x\text{)}$$

So for arbitrary fields

$$\delta I = i \int d^4x \; \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi^l)} F^l \partial_\mu \epsilon$$

$$= -i \int d^4x \; \epsilon(x) \; \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi^l)} F^l \right)$$

which vanishes if the fields obey the dynamical equations. So

$$J^\mu(x) = -i \; \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi^l(x))} F^l(x)$$

is conserved: $\partial_\mu J^\mu = 0$.

When $\mathcal{L}(x)$ is invariant under

$$\psi'^{\ell}(x) = \psi^{\ell}(x) + i\,\epsilon(x)^{\ell}\,F^{\ell}(x)$$

then by using Lagrange's equations one has

$$0 = \frac{\partial \mathcal{L}}{\partial \psi^{\ell}}\, i\epsilon F^{\ell} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi^{\ell})}\,\partial_\mu(i\epsilon F^{\ell})$$

$$= \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi^{\ell})}\right) i\epsilon F^{\ell} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi^{\ell})}\,\partial_\mu(i\epsilon F^{\ell})$$

$$= \partial_\mu\left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi^{\ell})}\, i\epsilon F^{\ell}\right]$$

So

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi^{\ell})}\, i\epsilon F^{\ell}$$

is conserved:

$$0 = \partial_\mu j^\mu.$$

The conserved quantity is

$$Q = \int d^3x\, j^0(x) = \int d^3x\, \frac{\partial \mathcal{L}}{\partial \partial_0 \psi^{\ell}}\, i\epsilon F^{\ell}(x)$$

$$= \int d^3x\, \Pi_\ell(x)\, i\epsilon\, F^{\ell}(x)$$

W. uses $-Q$, which is also conserved.

The quantum aspects are clearest when the canonical fields transform into $\vec{x}$-dependent functionals of themselves at the same time.

$$\varphi'^{\ell}(x) = \varphi^{\ell}(x) + i\epsilon(x)\, F^{\ell}[Q(t);\vec{x}]$$

Spatial rotations and translations are examples, as are all infinitesimal internal symmetries. Here F is the conserved generator of the symmetry.

$$F = \int d^3x\, J^0 = -i\int d^3x\, \frac{\partial \mathcal{L}}{\partial \dot{\varphi}^{\ell}}\, F^{\ell}(x)$$

$$= -i\int d^3x\, P_{\ell}(x,t)\, F^{\ell}[Q(t);\vec{x}]$$

F is time independent. So we may choose the t in F to be the same as that of $Q^{n}(t)$:

$$[F, Q^{n}(x,t)]_- = -i\int d^3x'\, [P_{\ell}(x',t), Q^{n}(x,t)]\, F^{\ell}[Q,x]$$

$$= -\int d^3x'\, \delta(x-x')\, \delta^{n}_{\ell}\, F^{\ell}[Q,x]$$

$$= -F^{n}[Q(t);\vec{x}],$$

So F is the generator, that is

$$e^{-i\epsilon F}\, Q^{n}(x,t)\, e^{+i\epsilon F} \simeq Q^{n}(x,t) + i\epsilon\, F^{n}[Q,x].$$

On P_n F gives

$$[F, P_n(\vec{x},t)]_- = -i\int d^3x\, P_{\ell}(x,t)\, [F^{\ell}[Q(t),x], P_n(\vec{x},t)]_-$$

or by

$$[F, P_n(\vec{x}',t)] = \int d^3x' \; P_\ell(x',t) \frac{\delta F^\ell[Q,x']}{\delta Q^n(x,t)}$$

If F^ℓ is linear in Q^m, as is normally the case, then $F^\ell = a^\ell{}_m Q^m$ and

$$[F, P_n(x,t)] = \int d^3x' \; P_\ell(x',t)\, \delta(x-x')\, a^\ell{}_n$$

$$= a^\ell{}_n P_\ell(x,t)$$

while

$$[F, Q^n(x,t)] = -F^n = -a^n{}_m Q^m.$$

One says that P_n transforms contragrediently to Q^n.

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For example, translations leave action invariant,

$$\psi^\ell(x) \to \psi'^\ell(x) = \psi^\ell(x+\epsilon) = \psi^\ell(x) + \epsilon^\mu \partial_\mu \psi^\ell(x),$$

Now we have 4 ϵ^μ's and $F^\ell_\mu = -i\,\partial_\mu \psi^\ell$

So there are 4 J^μ's $\quad J^\mu{}_\nu$

$$\partial_\mu J^\mu{}_\nu = 0$$

Usually we write $\quad J^\mu{}_\nu = T^\mu{}_\nu$

$$\partial_\mu T^\mu{}_\nu = 0$$

The conserved quantities are

$$P_\nu = \int d^3x \; J^0{}_\nu = \int d^3x \; T^0{}_\nu$$

Under $\psi^\ell(x) \to \psi'^\ell(x) = \psi^\ell(x+\epsilon) = \psi^\ell(x) + \epsilon^\mu \partial_\mu \psi^\ell(x)$,
we expect $\mathcal{L}(x)$ to turn into

$$\mathcal{L}'(x) = \mathcal{L}(x+\epsilon) = \mathcal{L}(x) + \epsilon^\mu \partial_\mu \mathcal{L}(x).$$ So

$$\epsilon^\mu \partial_\mu \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \psi^\ell} \epsilon^\mu \partial_\mu \psi^\ell + \frac{\partial \mathcal{L}}{\partial(\partial_\nu \psi^\ell)} \partial_\nu (\epsilon^\mu \partial_\mu \mathcal{L})$$

$$= \partial_\nu \left(\frac{\partial \mathcal{L}}{\partial \partial_\nu \psi^\ell} \right) \epsilon^\mu \partial_\mu \psi^\ell + \frac{\partial \mathcal{L}}{\partial(\partial_\nu \psi^\ell)} \partial_\nu (\epsilon^\mu \partial_\mu \mathcal{L})$$

So

$$0 = \partial_\nu \left[-\epsilon^\mu \delta^\nu_\mu \mathcal{L} + \frac{\partial \mathcal{L}}{\partial(\partial_\nu \psi^\ell)} \epsilon^\mu \partial_\mu \psi^\ell \right]$$ So

setting $$T^\nu{}_\mu = \delta^\nu_\mu \mathcal{L} - \frac{\partial \mathcal{L}}{\partial \partial_\nu \psi^\ell} \partial_\mu \psi^\ell,$$

we see that

$$0 = \partial_\nu T^\nu{}_\mu$$ whence

$$P_\mu = \int d^3x \; T^0{}_\mu$$ is conserved.

$$\vec{P} = \int d^3x \; - \frac{\partial \mathcal{L}}{\partial \dot{\psi}^\ell} \vec{\nabla} \psi^\ell$$

momentum

$$= - \int d^3x \; \Pi_\ell \nabla \psi^\ell = - \int d^3x \; P_\ell \nabla Q^\ell$$

and energy

$$-H = P_0 = \int d^3x\, T^0{}_0 = \int d^3x\, \mathcal{L} - \frac{\partial \mathcal{L}}{\partial \dot\psi^\ell}\dot\psi^\ell$$

or

$$H = \int d^3x\, \Pi_\ell \dot\psi^\ell - \mathcal{L} = \int d^3x\, P_\ell \dot Q^\ell - \mathcal{L}$$

So

$$[\vec P, Q^m(x,t)]_- = -[\int d^3y\, P_\ell(y,t) \nabla Q^\ell(y,t), Q^m(x,t)]_-$$

$$= i \int d^3y\, \delta_\ell^m\, \delta(x-y)\, \nabla_y Q^\ell(y,t)$$

$$= i\, \nabla Q^\ell(x,t)$$

$$[\vec P, P_n(x,t)] = -\int d^3y\, [P_\ell \nabla Q^\ell, P_n(x,t)]$$

$$= -\int d^3y\, [\nabla P_\ell\, Q^\ell, P_n(x,t)]$$

$$= i \int d^3y\, \delta^\ell_n\, \delta(x-y)\, \nabla \Pi_\ell(y,t)$$

$$= i\, \nabla P_n(x,t)$$

Thus

$$\Downarrow \quad [\vec P, G(Q,P)] = i\, \nabla\, G(Q(x), P(x)).$$

$\vec P$ generates translations in space as H does in time

$$[H, G(Q,P)] = -i\, \dot G(Q,P).$$

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for which $\dot{P}_\nu = 0$

as long as the fields vanish as $x \to \infty$.

L is invariant under spatial translations. So 3 P_i's are conserved (7.3.11)

$$P_i = -i \int d^3x \, \frac{\delta L}{\delta \dot{\psi}^\ell} F^\ell{}_i$$

$$= -i \int d^3x \, \frac{\delta L}{\delta \dot{\psi}^\ell} (-i)\, \partial_i \psi^\ell$$

$$= -\int d^3x \, P_\ell \, \partial_i Q^\ell \qquad \text{or}$$

$$\vec{P} = -\int d^3x \, P_\ell \nabla Q^\ell$$

Thus
$$[\vec{P}, Q^n(x,t)]_- = i \vec{\nabla} Q^n(x,t)$$

$$[\vec{P}, P_n(x,t)]_- = -\int d^3x' P_\ell \, [\nabla' Q^\ell(x',t), P_n(x,t)]_-$$

$$= -i \int d^3x' \, P_\ell(x',t) \nabla' \delta^\ell_n \, \delta(x'-x)$$

$$= i \int d^3x' \, \delta(x'-x) \, \nabla' P_n(x',t)$$

$$= i \nabla P_n(x,t).$$

So any $G[Q,P]$ that does not depend upon $\vec{x}$ explicitly

$$[\vec{P}, G(x)] = i \nabla G(x).$$

So $\vec{P}$ generates spatial translations.

The action is invariant under time translations as long as the Lagrangian $L(t)$ depends on time t only through the fields $\psi^\ell(x)$ and not explicitly.

$$\psi'^\ell(x) = \psi^\ell(x) + \epsilon(t)\dot{\psi}^\ell(x)$$

$$\delta I = \int dt \int \left[\frac{\delta L(t)}{\delta \psi^\ell(x)} \epsilon \dot{\psi}^\ell + \frac{\delta L(t)}{\delta \dot{\psi}^\ell(x)} \frac{d}{dt} (\epsilon \dot{\psi}^\ell) \right] d^3x$$

$$= \int dt \int d^3x \left[\frac{\delta L(t)}{\delta \psi^\ell(x)} \dot{\psi}^\ell(x) + \frac{\delta L(t)}{\delta \dot{\psi}^\ell(x)} \ddot{\psi}^\ell(x) \right] \epsilon(t) + \frac{\delta L(t)}{\delta \dot{\psi}^\ell(x)} \dot{\psi}^\ell(x) \dot{\epsilon}(t)$$

$$= \int dt \left[\frac{\partial L}{\partial t} \epsilon(t) + \int d^3x \; P_\ell(x) \dot{\psi}^\ell(x) \dot{\epsilon} \right]$$

$$= \int dt \; \dot{\epsilon} \left[-L + \int d^3x \; P_\ell \dot{\psi}^\ell \right]$$

This vanishes when the fields follow their dynamical equations,

$$0 = \int dt \; \dot{\epsilon} \left[-L + \int d^3x \; P_\ell \dot{\psi}^\ell \right]$$

$$= - \int dt \; \epsilon(t) \frac{d}{dt} \left[-L + \int d^3x \; P_\ell(x) \dot{\psi}^\ell(x) \right]$$

Thus

$$H = -L + \int d^3x \; P_\ell(x) \dot{\psi}^\ell(x)$$

is conserved, $\dot{H} = 0$.

Time translations don't leave $L(t)$ fixed.

H is their generator;

$$[H, G(x)] = -i\dot{G}(x)$$

if $\quad G(x) = G[Q(x), P(x)]$.

Time translations leave I fixed. Suppose

$$I[\psi] = \int d^4x\, \mathcal{L}(x)$$

is fixed under

$$\psi^\ell(x) \to \psi'^\ell(x) = \psi^\ell(x + \epsilon(x)) = \psi^\ell(x) + \epsilon^\mu(x)\partial_\mu \psi^\ell(x)$$

when $\epsilon(x) = \epsilon$. Then

$$\delta I[\psi] = \int d^4x\, \frac{\partial \mathcal{L}}{\partial \psi^\ell}\epsilon^\mu \partial_\mu \psi^\ell + \frac{\partial \mathcal{L}}{\partial(\partial_\nu \psi^\ell)}\partial_\nu(\epsilon^\mu \partial_\mu \psi^\ell)$$

$$= \int d^4x\, \frac{\partial \mathcal{L}}{\partial \psi^\ell}\epsilon^\mu \partial_\mu \psi^\ell + \frac{\partial \mathcal{L}}{\partial(\partial_\nu \psi^\ell)}\epsilon^\mu \partial_\nu \partial_\mu \psi^\ell + \frac{\partial \mathcal{L}}{\partial(\partial_\nu \psi^\ell)}\partial_\mu \psi^\ell \partial_\nu \epsilon^\mu$$

$$= \int d^4x\, \frac{\partial \mathcal{L}}{\partial x^\mu}\epsilon^\mu(x) + \frac{\partial \mathcal{L}}{\partial(\partial_\nu \psi^\ell)}\partial_\mu \psi^\ell \partial_\nu \epsilon^\mu$$

$$= \int d^4x \left(-\delta^\nu_\mu \mathcal{L} + \frac{\partial \mathcal{L}}{\partial(\partial_\nu \psi^\ell)}\partial_\mu \psi^\ell\right)\partial_\nu \epsilon^\mu(x)$$

$$= -\int d^4x\, T^\nu{}_\mu \partial_\nu \epsilon^\mu \quad \text{with}$$

$$T^\nu{}_\mu = \delta^\nu_\mu \mathcal{L} - \frac{\partial \mathcal{L}}{\partial(\partial_\nu \psi^\ell)}\partial_\mu \psi^\ell = J^\nu{}_\mu$$

For $\nu = 0$ and $\mu = i$ we get as before

$$\int d^3x\, J^0{}_i = \int d^3x\, T^0{}_i = -\int d^3x\, P_\ell\, \partial_i Q^\ell = P_i$$

while for $\mu = 0 = \nu$ we get

$$\int d^3x\, J^0{}_0 = \int d^3x\, T^0{}_0 = \int d^3x\, \mathcal{L} - P_\ell \dot{Q}^\ell = -H$$

$$= -P_0 .$$

(NB $T^{\nu\mu}$ is not symmetric; In GR use $\Theta^{\mu\nu}$ of 7.4.)

Suppose $\mathcal{L}(x)$ is invariant under — hermitian

$$Q'^n(x) = Q^n(x) + i\epsilon^a t_a{}^n{}_m Q^m(x)$$ canonical

$$C'^r(x) = C^r(x) + i\epsilon^a T_a{}^r{}_s C^s(x)$$ auxilliary

Then

$$0 = \frac{\partial \mathcal{L}}{\partial Q^n}\, i\epsilon^a t_a{}^n{}_m Q^m + \frac{\partial \mathcal{L}}{\partial(\partial_\mu Q^n)}\, \partial_\mu\left(i\epsilon^a t_a{}^n{}_m Q_m\right)$$

$$+ \frac{\partial \mathcal{L}}{\partial C^r}\, i\epsilon^a T_a{}^r{}_s C^s + \frac{\partial \mathcal{L}}{\partial(\partial_\mu C^r)}\, \partial_\mu\left(i\epsilon^a T_a{}^r{}_s C^s\right)$$

and by Lagrange's equations

$$0 = \partial_\mu\left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu Q^n)}\, i\epsilon^a t_a{}^n{}_m Q_m + \frac{\partial \mathcal{L}}{\partial(\partial_\mu C^r)}\, i\epsilon^a T_a{}^r{}_s C^s\right]$$

If the ϵ's are constants, then the conserved currents are

$$J_a^\mu = -i\frac{\partial \mathcal{L}}{\partial(\partial_\mu Q^n)}\, t_a{}^n{}_m Q^m - i\frac{\partial \mathcal{L}}{\partial(\partial_\mu C^r)}\, T_a{}^r{}_s C^s$$

and

$$0 = \partial_\mu J_a^\mu .$$

The conserved charges are

$$T^a = \int d^3x \, J_a^0$$

$$= -i\int d^3x \, \frac{\partial \mathcal{L}}{\partial(\dot{Q}^n)} \, t_a{}^n{}_m Q^m + \frac{\partial \mathcal{L}}{\partial(\dot{C}^n)} \, T_a{}^n{}_s C^s$$

these awkwardly vanish

$$= -i\int d^3x \, \frac{\partial \mathcal{L}}{\partial \dot{Q}^n} \, t_a{}^n{}_m Q^m$$

$$= -i\int d^3x \, P^n \, t_a{}^n{}_m Q^m.$$

The ETCR's now give

Suppose the action is invariant under

$$Q'^{n}(x) = Q^{n}(x) + i\epsilon^{a}\, t_{a}{}^{n}{}_{m}\, Q^{m}(x)$$

$$C'^{r}(x) = C^{r}(x) + i\epsilon^{a}\, \tau_{a}{}^{r}{}_{s}\, C^{s}(x)$$

on the canonical fields Q^{n} and the auxiliary fields C^{r}. If $L(t)$ is invariant, then (7.3.11) implies that the operator

$$T^{a} = -i\int d^{3}x\; P_{\ell}(x,t)\, t_{a}{}^{\ell}{}_{m}\, Q^{m}(x,t)$$

is conserved. Now the ETCR's give

$$[T^{a}, Q^{n}(x)]_{-} = -i\int d^{3}y\, [P_{\ell}(y,t), Q^{n}(x,t)]\, t_{a}{}^{\ell}{}_{m}\, Q^{m}(y,t)$$

$$= -\int d^{3}y\; \delta^{n}_{\ell}\, \delta(x-y)\; t_{a}{}^{\ell}{}_{m}\, Q^{m}(y,t)$$

$$= -\, t_{a}{}^{n}{}_{m}\, Q^{m}(x,t)$$

And

$$[T_{a}, P_{n}(x)]_{-} = P_{\ell}\, t_{a}{}^{\ell}{}_{n},$$

If t_{a} is diagonal, then Q^{n} and P_{n} respectively lower and raise the value of T^{a} by $t_{a}{}^{n}{}_{n}$.

9/9 Now

$$[T_{a}, T_{b}] = -\int d^{3}x\, d^{3}y\, \left[P_{\ell}\, t_{a}{}^{\ell}{}_{m} Q^{m},\; P_{n}\, t_{b}{}^{n}{}_{o} Q^{o}\right]$$

$$= -\int d^{3}x\, d^{3}y \Big(t_{a}{}^{\ell}{}_{m}\, t_{b}{}^{n}{}_{o} \big(i\,\delta^{m}_{n}\, \delta(x-y)\, P_{\ell} Q^{o}$$

$$- i\, \delta^{o}_{\ell}\, P_{n} Q^{m} \big)\Big)$$

If the t_a's form a Lie algebra with structure constants $f_{ab}{}^c$, that is, if

$$[t_a, t_b] = i f_{ab}{}^c t_c,$$

then

$$[T_a, T_b] = i\int d^3x \; P_n Q^m t_b{}^n{}_\ell t_a{}^\ell{}_m - P_\ell Q^o t_a^\ell{}_n t_b{}^n{}_o$$

$$= -i\int d^3x \; P^T (t_a t_b - t_b t_a) Q$$

$$= -i\int d^3x \; P^T (i f_{ab}{}^c t_c) Q$$

$$= \int d^3x \; f_{ab}{}^c \; P_\ell \; t_c{}^\ell{}_m Q^m$$

$$= i f_{ab}{}^c (-i) \int d^3x \; P_\ell \, t^\ell_c{}_n Q^n$$

$$= i f_{ab}{}^c \, T_c$$

The T_a's generate the group.

If $\mathcal{L}(x)$ is invariant under these transformations then

$$0 = \frac{\partial \mathcal{L}}{\partial Q^n} \, i\epsilon^a t_a{}^n{}_m Q^m + \frac{\partial \mathcal{L}}{\partial(\partial_\mu Q^n)} \partial_\mu (i\epsilon^a t_a{}^n{}_m Q_m)$$

$$+ \frac{\partial \mathcal{L}}{\partial (C^r)} \, i\epsilon^a T_a{}^r{}_s C^s + \frac{\partial \mathcal{L}}{\partial(\partial_\mu C^r)} \partial_\mu (i\epsilon^a T_a{}^r{}_s C^s)$$

and using the dynamics we have

$$0 = \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu Q^n)} i\epsilon^a t^n{}_{am} Q^m + \frac{\partial \mathcal{L}}{\partial(\partial_\mu C^r)} i\epsilon^a T^n_{as} C^s \right]$$

$$= \partial_\mu J^\mu_a \, \epsilon^a$$

Thus the current

$$J_a^\mu = -i \frac{\partial \mathcal{L}}{\partial \partial_\mu Q^n} t_a{}^n{}_m Q^m - i \frac{\partial \mathcal{L}}{\partial(\partial_\mu C^r)} T_a{}^r{}_s C^s$$

is conserved

$$0 = \partial_\mu J_a^\mu$$

for each a.

Suppose $\mathcal{L}(x)$ is invariant under

$$\phi'(x) = \exp\left[i \frac{\sigma_a}{2} \theta^a\right] \phi(x)$$

where $\phi(x) = \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \end{pmatrix}$ both ϕ_i's being complex.

For instance

$$\mathcal{L} = -\partial_\mu \phi^\dagger \partial^\mu \phi - m^2 \phi^\dagger \phi - V(\phi^\dagger \phi).$$

The conserved currents are

$$J_a^\mu = -i \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi^i} \frac{\sigma_a}{2}{}^i{}_j \phi^j = i \partial^\mu \phi^\dagger \frac{\sigma_a}{2} \phi.$$

In general the time components of the currents are

$$J_a^0 = -i P_n t_a{}^n{}_m Q^m = -i P^T t_a Q.$$

In this example $\partial^0 \phi^\dagger = -\dot{\phi}^\dagger = -P_\phi$

$$J_a^0 = -i P_i \frac{\sigma_a}{2}{}^i{}_j \phi^j = -i \dot{\phi}_i^\dagger \frac{\sigma_a}{2}{}^i{}_j \phi^j$$

In general

$$[J_a^0(x,t), Q^n(y,t)] = [-i P_\ell t_a{}^\ell{}_m Q^m, Q^n]$$

$$= - t_a{}^n{}_m Q^m(x,t)\, \delta^3(\vec{x}-\vec{y})$$

And

$$[J_a^0(x,t), P_m(y,t)] = [-i P_\ell t_a{}^\ell{}_n Q^n, P_m]$$

$$= \delta(x-y)\, t_a{}^\ell{}_m P_\ell(\vec{x},t)$$

If the auxiliary fields $C^r(x,t) = C^r(Q(x,t), P(x,t))$ and transform correctly, then t.o

$$[J_a^0(x,t), C^r(y,t)] = -\delta(x-y)\, t_a{}^r{}_s\, C^s(x,t).$$

One may write these rules as

$$[J_a^0(x,t), \psi^\ell(y,t)] = -\delta(x-y)\, t_a{}^\ell{}_{\ell'}\, \psi^{\ell'}(x,t).$$

Such relations are used in 10 for Ward identities.

Lorentz invariance

The action is invariant under

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu = (\delta^\mu{}_\nu + \omega^\mu{}_\nu) x^\nu$$

with $\omega_{\mu\nu} = -\omega_{\nu\mu}$. So we expect 6 conserved currents $M^{\rho\mu\nu}$

$$\partial_\rho M^{\rho\mu\nu} = 0$$

$$M^{\rho\mu\nu} = -M^{\rho\nu\mu}$$

and conserved operators

$$J^{\mu\nu} = \int d^3x \, M^{0\mu\nu}$$

$$\frac{d}{dt} J^{\mu\nu} = 0.$$

How does $\partial_\lambda \phi(x)$ transform?

$$\partial_\lambda \phi(x') = \frac{\partial \phi(x')}{\partial x'^\nu} \frac{\partial x'^\nu}{\partial x^\lambda} = \Lambda^\nu{}_\lambda \, \partial'_\nu \phi(x')$$
$$= (\delta^\nu{}_\lambda + \omega^\nu{}_\lambda) \partial'_\nu \phi(x')$$

So

$$\delta \, \partial_\lambda \phi(x') = \omega^\kappa{}_\lambda \, \partial_\kappa \phi(x')$$

$$U(\Lambda) \psi^\ell(x) U^{-1}(\Lambda) = D_{\ell\ell'}(\Lambda^{-1}) \psi^{\ell'}(\Lambda x)$$

or

$$U(\Lambda^{-1}) \psi^\ell(x) U^{-1}(\Lambda^{-1}) = D_{\ell\ell'}(\Lambda) \psi^{\ell'}(\Lambda^{-1} x)$$

In this case $D_{\ell\ell'}(\Lambda) \psi_{\ell'}(\Lambda^{-1}x) = \left(\delta_{\ell\ell'} + \frac{i}{2} \omega^{\mu\nu} \mathcal{J}^\ell_{\mu\nu\ell'}\right) \psi^{\ell'}(\Lambda^{-1}x)$

So on x we switch to Λ^{-1} (2.3.10)

$$\Lambda^{-1\,\nu}{}_{\nu} = \Lambda_{\nu}{}^{\mu} = \delta_{\nu}{}^{\mu} + \omega_{\nu}{}^{\mu}$$

So derivatives transform as

$$\delta\, \partial_\lambda \phi(x') = \omega_\lambda{}^{\kappa}\, \partial'_\kappa \phi(x')$$

where $x' = \Lambda^{-1} x$.

Anyways

$$\delta\left(\partial_\kappa \psi^\ell\right) = \tfrac{1}{2} i\, \omega^{\mu\nu} \mathcal{J}_{\mu\nu}{}^{\ell}{}_{m}\, \partial_\kappa \psi^m + \omega_\kappa{}^{\lambda}\, \partial_\lambda \psi^\ell \qquad (*)$$

$$U(\Lambda^{-1})\, \mathcal{L}(x)\, U^{-1}(\Lambda^{-1}) = \mathcal{L}(\Lambda^{-1} x)$$

So apart from $x \to x'$, $\mathcal{L}$ is invariant.

That is under $(*)$ with no change in x, $\mathcal{L}(x)$ is invariant, whence

$$0 = \frac{\partial \mathcal{L}}{\partial \psi^\ell} \frac{i}{2} \omega^{\mu\nu} \mathcal{J}_{\mu\nu}{}^{\ell}{}_{m} \psi^m$$
$$+ \frac{\partial \mathcal{L}}{\partial(\partial_\kappa \psi^\ell)} \partial_\kappa \left[\tfrac{1}{2} i\, \omega^{\mu\nu} \mathcal{J}_{\mu\nu}{}^{\ell}{}_{m}\, \partial_\kappa \psi^m + \omega_\kappa{}^{\lambda}\, \partial_\lambda \psi^\ell\right]$$

$$= \partial_\kappa \frac{\partial \mathcal{L}}{\partial(\partial_\kappa \psi^\ell)} \frac{i}{2} \omega^{\mu\nu} \mathcal{J}_{\mu\nu}{}^{\ell}{}_{m} \psi^m$$
$$+ \frac{\partial \mathcal{L}}{\partial(\partial_\kappa \psi^\ell)} \partial_\kappa \left[\tfrac{1}{2} i\, \omega^{\mu\nu} \mathcal{J}_{\mu\nu}{}^{\ell}{}_{m}\, \partial_\kappa \psi^m + \omega_\kappa{}^{\lambda}\, \partial_\lambda \psi^\ell\right].$$

Now $\omega_\kappa{}^\lambda \partial_\lambda \psi^l = \eta_{\kappa\sigma}\, \omega^{\sigma\lambda}\, \partial_\lambda \psi^l$

$$= \frac{1}{2}\eta_{\kappa\sigma}\left(\omega^{\sigma\lambda} - \omega^{\lambda\sigma}\right)\partial_\lambda \psi^l$$

$$= \frac{1}{2}\left(\eta_{\kappa\mu}\,\omega^{\mu\nu}\partial_\nu - \eta_{\kappa\mu}\,\omega^{\nu\mu}\partial_\nu\right)\psi^l$$

$$= \frac{1}{2}\left(\eta_{\kappa\mu}\,\omega^{\mu\nu}\partial_\nu - \eta_{\kappa\nu}\,\omega^{\mu\nu}\partial_\mu\right)\psi^l$$

So cancelling $\omega^{\mu\nu}$, we have

$$0 = \partial_\kappa\left(\frac{\partial \mathcal{L}}{\partial(\partial_\kappa \psi^l)}\right)\frac{1}{2}\,\mathcal{J}_{\mu\nu}{}^l{}_m\,\psi^m$$

$$+ \frac{\partial \mathcal{L}}{\partial(\partial_\kappa \psi^l)}\left[\frac{1}{2}\, i\,\mathcal{J}_{\mu\nu}{}^l{}_m\,\partial_\kappa \psi^m + \frac{1}{2}\left(\eta_{\kappa\mu}\partial_\nu - \eta_{\kappa\nu}\partial_\mu\right)\psi^l\right]$$

$$= \partial_\kappa\left[\frac{1}{2}\frac{\partial \mathcal{L}}{\partial(\partial_\kappa \psi^l)}\,\mathcal{J}_{\mu\nu}{}^l{}_m\,\psi^m\right]$$

$$+ \frac{1}{2}\frac{\partial \mathcal{L}}{\partial(\partial_\kappa \psi^l)}\left(\eta_{\kappa\mu}\partial_\nu - \eta_{\kappa\nu}\partial_\mu\right)\psi^l$$

Recall (7.3.34) $\quad T^\nu{}_\mu = \delta^\nu{}_\mu \mathcal{L} - \dfrac{\partial \mathcal{L}}{\partial(\partial_\nu \psi^l)}\,\partial_\mu \psi^l$

Then

$$T_{\nu\mu} - T_{\mu\nu} = -\frac{\partial \mathcal{L}}{\partial(\partial^\nu \psi^l)}\,\partial_\mu \psi^l + \frac{\partial \mathcal{L}}{\partial(\partial^\mu \psi^l)}\,\partial_\nu \psi^l$$

So

$$0 = \partial_\kappa \left[\frac{i}{2} \frac{\partial \mathcal{L}}{\partial(\partial_\kappa \psi^\ell)} \mathcal{J}_{\mu\nu}{}^\ell{}_m \psi^m \right]$$

$$+ \frac{1}{2} \left(\frac{\partial \mathcal{L}}{\partial(\partial^\mu \psi^\ell)} \partial_\nu \psi^\ell - \frac{\partial \mathcal{L}}{\partial(\partial^\nu \psi^\ell)} \partial_\mu \psi^\ell \right)$$

$$0 = \partial_\kappa [\quad] + \frac{1}{2} (T_{\nu\mu} - T_{\mu\nu}) \qquad (7.4.10)$$

So the A.S. part of $T_{\mu\nu}$ is a total divergence.

The Belinfante tensor Θ is

$$\Theta^{\mu\nu} = T^{\mu\nu} - \frac{i}{2} \partial_\kappa \left[\frac{\partial \mathcal{L}}{\partial(\partial_\kappa \psi^\ell)} \mathcal{J}^{\mu\nu\ell}{}_m \psi^m \right.$$

$$\left. - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi^\ell)} \mathcal{J}^{\kappa\nu\ell}{}_m \psi^m - \frac{\partial \mathcal{L}}{\partial(\partial_\nu \psi^\ell)} \mathcal{J}^{\kappa\mu\ell}{}_m \psi^m \right]$$

in which $[\;]$ is A.S. in μ, κ.

Hence $\partial_\mu \Theta^{\mu\nu} = \partial_\mu T^{\mu\nu} - \frac{1}{2} \partial_\mu \partial_\kappa [\quad]$

$$= \partial_\mu T^{\mu\nu} = 0.$$

And $\qquad \partial_0 [AS]^{00\nu} = 0$

$$\int \Theta^{0\nu} d^3x = \int d^3x \left(T^{0\nu} - \frac{i}{2} \partial_\kappa [A.S.]^{\kappa 0\nu} \right)$$

$$= \int d^3x \left(T^{0\nu} - \frac{i}{2} \partial_i [A.S.]^{i0\nu} \right) = \int d^3x \, T^{0\nu} = P^\nu.$$

Here $P^0 = H$.

So $\Theta^{\mu\nu}$ is as good as $T^{\mu\nu}$. But $\Theta^{\mu\nu}$ is symmetric

$$\Theta^{\mu\nu} - \Theta^{\nu\mu} = T^{\mu\nu} - T^{\nu\mu} - i\,\partial_\kappa \left[\frac{\partial \mathcal{L}}{\partial(\partial_\kappa \psi^\ell)} \mathcal{J}^{\mu\nu\ell}{}_m \psi^m \right]$$

because the second two terms in $\Theta = T - [\;\;]$ are μ, ν symmetric. But by (7.4.10) then

$$\Theta^{\mu\nu} = \Theta^{\nu\mu}.$$

This is the source of the gravitational field.

Since Θ is symmetric,

$$M^{\lambda\mu\nu} \equiv x^\mu \Theta^{\lambda\nu} - x^\nu \Theta^{\lambda\mu}$$

is conserved:

$$\partial_\lambda M^{\lambda\mu\nu} = x^\mu \partial_\lambda \Theta^{\lambda\nu} - x^\nu \partial_\lambda \Theta^{\lambda\mu} + \delta^\mu_\lambda \Theta^{\lambda\nu} - \delta^\nu_\lambda \Theta^{\lambda\mu} = \Theta^{\mu\nu} - \Theta^{\nu\mu} = 0.$$

So the tensor

$$J^{\mu\nu} = \int M^{0\mu\nu} d^3x = \int d^3x \, (x^\mu \Theta^{0\nu} - x^\nu \Theta^{0\mu})$$

is constant.

$J_k = \frac{1}{2} \epsilon_{ijk} J^{ij}$ is both constant and free of any explicit time dependence. So it obeys

$$0 = [H, \vec{J}].$$

By 7.3.28 $[\vec{P}, G(x)] = i\vec{\nabla} G(x)$

So

$$[P_j, J_i] = \tfrac{1}{2}\epsilon_{i\ell k}[P_j, J^{\ell k}]$$

$$= \tfrac{1}{2}\epsilon_{i\ell k}\int d^3x \; x^\ell [P_j, \Theta^{0k}] - x^k [P_j, \Theta^{0\ell}]$$

$$= \tfrac{i}{2}\epsilon_{i\ell k}\int d^3x \; x^\ell \partial_j \Theta^{0k} - x^k \partial_j \Theta^{0\ell}$$

$$= -\tfrac{i}{2}\epsilon_{i\ell k}\int d^3x \; \delta^\ell_j \Theta^{0k} - \delta^k_j \Theta^{0\ell}$$

$$= -\tfrac{i}{2}\int d^3x \; \epsilon_{ijk}\Theta^{0k} - \epsilon_{i\ell j}\Theta^{0\ell}$$

$$= -i\epsilon_{ijk}\int d^3x \; \Theta^{0k} = -i\epsilon_{ijk} P_k$$

which we usually write as

$$[P_i, J_j] = i\epsilon_{ijk} P_k.$$

The boost $K_k = J^{k0}$ does involve t explicitly but is time independent.

$$K_k = \int d^3x \, (x^k \Theta^{00} - x^0 \Theta^{0k})$$

$$= -t P_k + \int d^3x \; x^k \Theta^{00}(x,t)$$

$$0 = \dot{\vec{K}} = -\vec{P} + i[H, \vec{K}]$$

So $\quad [H, \vec{K}] = -i\vec{P}. \qquad (2.4.24)$

(7.3.28) is $[\vec{P}, G] = i\nabla G.$ so

$$[P_j, K_k] = \int d^3y\, y_k [P_j, \Theta^{00}] = i\int d^3y\, y_k \partial_j \Theta^{00}$$

$$= i\int d^3y\, y_k \frac{\partial \Theta^{00}}{\partial y_j} = -i\int d^3y\, \delta^k_j\, \Theta^{00}$$

$$= -i\,\delta_{kj} H$$

That is

$$[P_j, K_k] = -i\,\delta_{jk} H. \qquad (2.4.22)$$

In general $\vec{K}$ is smooth, ie,

$$\langle \varphi | e^{iH_0 t} \int d^3x\, \vec{x}\, \Theta^{00}_V\, e^{-iH_0 t} | \varphi' \rangle \to 0 \text{ as } t \to \pm\infty$$

where Θ^{00}_V is the non-H_0 part of Θ^{00}.

This smoothness and $[H, K] = -i\vec{P}$ leads to the Lorentz invariance of the S-matrix.

The J^{ij}'s are

$$J^{ij} = \int d^3x\, (x^i \Theta^{0j} - x^j \Theta^{0i})$$

$$= \int d^3x \Big(x^i \Big\{ T^{0j} - \tfrac{i}{2}\partial_k \Big[\frac{\partial \mathcal{L}}{\partial(\partial_k \psi^\ell)} (\mathcal{J}^{0j})^\ell{}_m \psi^m$$

$$- \frac{\partial \mathcal{L}}{\partial(\partial_0 \psi^\ell)} (\mathcal{J}^{kj})^\ell{}_m \psi^m - \frac{\partial \mathcal{L}}{\partial(\partial_j \psi^\ell)} (\mathcal{J}^{k0})^\ell{}_m \psi^m \Big] - \binom{i\to j}{j\to i} \Big)$$

$$J^{ij} = \int d^3x \;\; x^i T^{0j} + \frac{i}{2}\left[\frac{\partial \mathcal{L}}{\partial(\partial_i \psi^\ell)} \mathcal{J}^{0j\ell}{}_m \psi^m \right.$$

$$\left. - \frac{\partial \mathcal{L}}{\partial \dot{\psi}^\ell} \mathcal{J}^{ij\ell}{}_m \psi^m + \frac{\partial \mathcal{L}}{\partial \partial_j \psi^\ell} (\mathcal{J}^{0i})^\ell{}_m \psi^m \right] - \left(\begin{smallmatrix} i \to j \\ j \to i \end{smallmatrix} \right)$$

$$= \int d^3x \;\; x^i T^{0j} - x^j T^{0i} - i \frac{\partial \mathcal{L}}{\partial \dot{\psi}^\ell} \mathcal{J}^{ij\ell}{}_m \psi^m$$

$$= \int d^3x \;\; x^i\left(-\frac{\partial \mathcal{L}}{\partial \dot{\psi}^\ell} \partial_j \psi^\ell\right) + x^j \frac{\partial \mathcal{L}}{\partial \dot{\psi}^\ell} \partial_i \psi^\ell - i \frac{\partial \mathcal{L}}{\partial \dot{\psi}^\ell} \mathcal{J}^{ij\ell}{}_m \psi^m$$

$$= \int d^3x \;\; \frac{\partial \mathcal{L}}{\partial \dot{\psi}^\ell}\left(-x^i \partial_j \psi^\ell + x^j \partial_i \psi^\ell - i (\mathcal{J}^{ij})^\ell{}_m \psi^m\right)$$

$$= \int d^3x \;\; P_n\left(-x^i \partial_j Q^n + x^j \partial_i Q^n - i (\mathcal{J}^{ij})^n{}_m Q^m\right)$$

No auxilliary fields here. So

$$[J^{ij}, Q^n(x)] = i x^i \partial_j Q^n - i x^j \partial_i Q^n - \mathcal{J}^{ij\,n}{}_m Q^m$$

$$= -i\left(-x_i \partial_j + x_j \partial_i\right) Q^n - (\mathcal{J}^{ij})^n{}_m Q^m$$

And

$$[J^{ij}, P_n(x)] = \int d^3y \;\; P_{n'}(y) \left[-i y_i \partial_j Q^{n'}(y) + y_j \partial_i Q^{n'}(y) \right.$$

$$\left. - i (\mathcal{J}^{ij})^{n'}{}_m Q^m(y),\; P_n(x) \right]$$

$$= \int d^3y \left[(y_i \partial_j - y_j \partial_i) P_n(y) - i P_{n'}(y) \mathcal{J}^{ij\,n'}{}_n \right] i\delta(x-y)$$

$$= i (x_i \partial_j - x_j \partial_i) P_n(x) + P_{n'}(x) \mathcal{J}^{ij\,n'}{}_n$$

Note there were no C^{r}'s here. But J^{i0} mixes Q^{i} with Q^{0} and so Q^{i}'s with C^{r}'s. So the full derivation of $[J^{\mu\nu}, J^{\rho\sigma}]$ requires a case-by-case derivation.

The Interaction Picture

Scalar field with derivative coupling

$$\mathcal{L} = -\tfrac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \tfrac{1}{2}m^2\phi^2 - J^\mu\partial_\mu\phi - \mathcal{H}(\phi)$$

where J^μ may be external or composed of other fields

$$\Pi = \frac{\partial\mathcal{L}}{\partial\dot\phi} = \dot\phi - J^0 \qquad \text{(H's fields)}$$

Now

$$H = \int d^3x\,\left(\Pi\dot\phi - \mathcal{L}\right)$$

$$= \int d^3x\;\; \Pi(\Pi + J^0) + \tfrac{1}{2}(\nabla\phi)^2 - \tfrac{1}{2}(\Pi + J^0)^2 + \tfrac{1}{2}m^2\phi^2$$
$$+ \vec{J}\cdot\nabla\phi + J^0(\Pi + J^0) + \mathcal{H}(\phi))$$

$$= \int d^3x\;\; \tfrac{1}{2}\Pi^2 + \Pi J^0 + \tfrac{1}{2}(\nabla\phi)^2 + \tfrac{1}{2}m^2\phi^2 + \mathcal{H}(\phi))$$
$$+ \tfrac{1}{2}J^{0\,2} + \vec{J}\cdot\vec{\nabla}\phi$$

We choose to do pert. theory. We know H_0.

$$H = H_0 + V \qquad H_0 = \int d^3x\;\tfrac{1}{2}\Pi^2 + \tfrac{1}{2}(\nabla\phi)^2 + \tfrac{1}{2}m^2\phi^2$$

So $\quad V = \int d^3x\;\; \Pi J^0 + \vec{\nabla}\phi\cdot\vec{J} + \tfrac{1}{2}J^{0\,2} + \mathcal{H}(\phi)$.

All fields on this page are Heisenberg fields.

The interaction fields $\phi(x,t)$ and $\pi(x,t)$ are the Heisenberg fields at $t=0$ and evolve with t via H_0. H_0 is fixed in the int. picture

Surely OK → as in (7.2.25-35)

$$H_0 = H_0(t) = \int d^3x \ \tfrac{1}{2}\pi^2 + \tfrac{1}{2}(\nabla\phi)^2 + \tfrac{1}{2}m^2\phi^2$$

$$V(t) = \int d^3x \ \pi J^0 + \nabla\phi\cdot J + \tfrac{1}{2}J^{0\,2} + \mathcal{H}(\phi).$$

ϕ, π are int. picture

Now

$$\Pi = \dot{\Phi} - J^0 = i[H, \Phi] - J^0 = i[H_0, \Phi]$$

$$\pi = \Pi = i[H_0, \Phi] = i[H_0, \phi] = \dot{\phi} \quad \text{at } t=0.$$

So in $V(t)$ we may set

$$\pi(\vec{x},0) = i[H_0, \phi(x,0)] = \dot{\phi}$$

and then evolve π with H_0. Then $\pi(x,t) = \dot{\phi}(x,t)$ in which both follow H_0's time evolution.

$$V(t) = \int d^3x \ \partial_\mu\phi J^\mu + \tfrac{1}{2}J^{0\,2} + \mathcal{H}(\phi).$$

in which the need for the non-covariant term $\frac{1}{2}J^{0\,2}$ cancels a non-covariant term in the $\partial\phi$ propagator as explained in 6.2, Eq. (6.2.27).

Vector field of spin one (H's fields)

$$\mathcal{L} = -\tfrac{1}{2}\alpha\, \partial_\mu V_\nu\, \partial^\mu V^\nu - \tfrac{1}{2}\beta\, \partial_\mu V_\nu\, \partial^\nu V^\mu - \tfrac{1}{2} m^2 V_\mu V^\mu + J_\mu V^\mu$$

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial V^\nu_{,\mu}} = \partial_\mu\left(-\alpha\, \partial^\mu V_\nu - \beta\, \partial_\nu V^\mu\right) = \frac{\partial \mathcal{L}}{\partial V^\nu} = -m^2 V_\nu + J_\nu$$

$$-\alpha\, \Box V_\nu - \beta\, \partial_\nu (\partial_\mu V^\mu) + m^2 V_\nu = J_\nu$$

$$-(\alpha+\beta)\, \Box\, \partial_\lambda V^\lambda + m^2 \partial_\lambda V^\lambda = \partial_\lambda J^\lambda$$

This describes a scalar field $\partial_\lambda V^\lambda$ of mass^2 $m^2/(\alpha+\beta)$ and source $\partial_\lambda J^\lambda/(\alpha+\beta)$.

To prevent this field from being dynamical, we set

$$\alpha = -\beta = 1 \quad \text{so that}$$

$$\mathcal{L} = -\tfrac{1}{2}\, \partial_\mu V_\nu\, \partial^\mu V^\nu + \tfrac{1}{2}\, \partial_\mu V_\nu\, \partial^\nu V^\mu - \tfrac{1}{2} m^2 V_\mu V^\mu + J_\mu V^\mu.$$

We set $F_{\mu\nu} \equiv \partial_\mu V_\nu - \partial_\nu V_\mu$, whence

$$\begin{aligned} F_{\mu\nu} F^{\mu\nu} &= \partial_\mu V_\nu\, \partial^\mu V^\nu + \partial_\nu V_\mu\, \partial^\nu V^\mu \\ &\quad - \partial_\mu V_\nu\, \partial^\nu V^\mu - \partial_\nu V_\mu\, \partial^\mu V^\nu \\ &= 2\partial_\mu V_\nu\, \partial^\mu V^\nu - 2\, \partial_\mu V_\nu\, \partial^\nu V^\mu. \end{aligned}$$

So

$$\mathcal{L} = -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} - \tfrac{1}{2} m^2 V_\mu V^\mu + J_\mu V^\mu.$$

Now $\dfrac{\partial \mathcal{L}}{\partial \dot{V}_\mu} = -F^{0\mu} = 0$ iff $\mu = 0$.

So the V^i's for $i=1,2,3$ are canonical fields with

$$\Pi^i = \frac{\partial \mathcal{L}}{\partial \dot{V}_i} = -F^{0i} = F^{i0} = \partial^i V^0 - \partial^0 V^i \qquad (0)$$

and $F^{00} = 0$ — no $\dot{V}^0$ in $\mathcal{L}$. So V^0 is an auxiliary field. It's field equation is

$$0 = \frac{\partial \mathcal{L}}{\partial V_0} = -m^2 V^0 + J^0 = \partial_i \frac{\partial \mathcal{L}}{\partial \partial_i V_0} = -\partial_i F^{i0}$$

or

$$\partial_i F^{i0} = m^2 V^0 - J^0 \qquad \text{like Gauss's law}$$

which has no $\ddot{V}^\mu$ terms at all. It's a constraint, which we use to solve for V^0

$$V^0 = \frac{1}{m^2}\left(J^0 + \partial_i F^{i0} \right)$$

$$= \frac{1}{m^2}\left(\nabla \cdot \Pi + J^0 \right).$$

Now

$$H = \int d^3x \; \Pi \cdot \dot{V} - \mathcal{L}$$

By (0), $\Pi = \nabla V^0 + \dot{V}^i$. So $\dot{V}^i = \vec{\Pi} - \vec{\nabla} V^0$ or $\qquad (*)$

$$\dot{\vec{V}} = \vec{\Pi} - \frac{1}{m^2} \vec{\nabla}\left(\vec{\nabla} \cdot \vec{\Pi} + J^0 \right).$$

So

$$H = \int d^3x\ \Pi \cdot \dot{V} - \mathcal{L}$$

$$= \int d^3x\ \vec{\Pi} \cdot \left(\vec{\Pi} - \frac{1}{m^2} \vec{\nabla}(\nabla \cdot \Pi + J^0) \right) - \frac{\Pi^2}{2} + \frac{1}{2}(\nabla \times V)^2 - \frac{1}{2} m^{-2} (\nabla \cdot \Pi + J^0)^2$$

$$+ \frac{1}{2} m^2 V^2 - J \cdot V + \frac{J^0}{m^2} (\nabla \cdot \Pi + J^0)$$

$$= \int d^3x\ \frac{1}{2} \vec{\Pi}^2 + \frac{1}{2}(\vec{\nabla} \times \vec{V})^2 + \frac{1}{2} m^2 \vec{V}^2 + \frac{1}{2m^2} (\nabla \cdot \Pi)^2$$

$$+ \int d^3x\ \frac{J^0 \nabla \cdot \Pi}{m^2} - J \cdot V + \frac{J^{0\,2}}{2m^2}$$

$$= H_0 + V$$ So far all fields are Heisenberg fields.

In terms of I.P. fields at $t = 0$

$$H_0 = \int d^3x\ \frac{1}{2} \pi^2 + \frac{1}{2m^2} (\nabla \cdot \pi)^2 + \frac{1}{2} (\nabla \times v)^2 + \frac{m^2}{2} v^2$$

$$V = \int d^3x\ -J \cdot v + m^{-2} J^0 \nabla \cdot \pi + \frac{1}{2m^2} J^{0\,2}.$$

So

$$\dot{v} = i[H_0, v] = \vec{\pi} - \frac{1}{m^2} \nabla(\nabla \cdot \pi) \text{ in the I. P.} \qquad (7.5.20)$$

To evaluate

$$\dot{\pi} = i[H_0, \pi], \quad -m^2 v$$

we first write $\quad - \nabla \times \nabla$

$$\int d^3x\ (\nabla \times v)^2 = \int d^3x\ \epsilon_{ijk} \partial_j v_k \epsilon_{ilm} \partial_l v_m$$

$$= \int d^3x\ (\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}) \partial_j v_k \partial_l v_m = \int d^3x\ \partial_j v_k \partial_j v_k - \partial_j v_k \partial_k v_j$$

$$\int d^3x \, (\nabla \times v)^2 = \int d^3x \, -v_k \, \partial_j^2 v_k - (\partial_k v_k)\, \partial_j v_j$$

$$= \int d^3x \, -v_k \, \partial_j^2 v_k + v_k \, \partial_k \partial_j v_j$$

$$= \int d^3x \, -\vec{v} \cdot \Delta \vec{v} + v \cdot \nabla (\nabla \cdot v)$$

Then

$$\dot{\pi} = i\,[H_0, \pi] = + \Delta \vec{v} - \vec{\nabla}(\vec{\nabla} \cdot \vec{v}) - m^2 v \qquad (**)$$

in the I. P. At $t=0$ $\pi = \Pi = i[H, \vec{V}] + \vec{\nabla} V^0$

$$= i\,[H_0, \vec{V}] + i\,[V, \vec{V}] + \vec{\nabla} V^0$$

So $\pi = \dot{v} - \frac{\nabla J^0}{m^2} + \nabla V^0$ by $(*)$

We define $v^0 \equiv V^0 - \frac{J^0}{m^2}$. Then $\pi = \dot{v} + \vec{\nabla} v^0$ and

$$v^0 = V^0 - \frac{J^0}{m^2} = \frac{\nabla \cdot \Pi}{m^2} + \frac{J^0}{m^2} - \frac{J^0}{m^2} = \frac{\nabla \cdot \Pi}{m^2} = \frac{\nabla \cdot \pi}{m^2} \text{ at } t=0.$$

$$\dot{v} = \pi - m^{-2} \nabla (\nabla \cdot \pi) = \pi - \nabla v^0 \quad \text{and}$$

$$\pi = \dot{v} + \nabla v^0 \qquad \text{as in (7.5.14)}$$

So

$$v^0 = m^{-2} \nabla \cdot \pi = m^{-2} \left(\nabla \cdot \dot{v} + \Delta v^0 \right) \qquad t=0$$

$$\Delta v^0 + \nabla \cdot \dot{v} - m^2 v^0 = 0 \quad \text{, a constraint, and}$$

$$\dot{\pi} = \ddot{v} + \nabla \dot{v}^0 = \Delta v - \nabla(\nabla \cdot v) - m^2 v \quad \text{by } (**)$$

gives

$$\Delta v - \nabla(\nabla \cdot v) - \ddot{v} - \nabla \dot{v}^0 - m^2 v = 0 \qquad \text{(eqm)}$$

(eom) $\Rightarrow$

$$\Box v^i - \partial_i \partial_j v_j - \partial_i \partial_0 v^0 - m^2 v^i = 0$$

$$\Box v^i - \partial^i (\partial_\nu v^\nu) - m^2 v^i = 0$$

(const) $\Rightarrow$ $\Delta v^0 - \partial_0^2 v^0 + \partial_0^2 v^0 + \partial_j \partial_0 v^j - m^2 v^0 = 0$

$$\Box v^0 - \partial^0 \partial_0 v^0 - \partial^0 \partial_j v^j - m^2 v^0 = \Box v^0 - \partial^0 \partial_\nu v^\nu - m^2 v^0 = 0$$

$$\int d^3x\ (\nabla\times v)^2 = \int d^3x\ -v_k\,\partial_j^2 v_k - \partial_k v_k\,\partial_j v_j$$

$$= \int d^3x\ -v_k\,\partial_j^2 v_k + v_k\,\partial_k\partial_j v_j = \int d^3x\left(-v\cdot\nabla^2 v + v\cdot\nabla(\nabla\cdot v)\right)$$

Then $\dot{\pi} = i[H_0, \pi]$

$$= i\int d^3x\left[-\tfrac{1}{2}v\cdot\nabla^2 v + \tfrac{v}{2}\cdot\nabla(\nabla\cdot v) + \tfrac{m^2 v^2}{2},\ \pi\right]$$

$$= \nabla^2 v - \nabla(\nabla\cdot v) - m^2 v \qquad (7.5.21)$$

Now V^0 does not appear in H_0 or in V. We may define

$$v^0 = m^{-2}\,\nabla\cdot\pi. \qquad (7.5.22)$$

Then (7.5.20) gives $\qquad \dot{v} = \pi - \nabla v^0 \qquad (7.5.23)$

or by (22)

$$\nabla\cdot\dot{v} = \nabla\cdot\pi - \nabla^2 v^0 = m^2 v^0 - \nabla^2 v^0 \quad \text{ie}$$

$$\nabla^2 v^0 + \nabla\cdot\dot{v} - m^2 v^0 = 0 \qquad (\text{con})$$

and (21) & (23) give

$$\dot{\pi} = \ddot{v} + \nabla\dot{v}^0 = \nabla^2 v - \nabla(\nabla\cdot v) - m^2 v \quad \text{or}$$

$$\nabla^2 v - \nabla(\nabla\cdot v) - \ddot{v} - \nabla\dot{v}^0 - m^2 v = 0 \quad \text{or}$$

$$\Box v^i - \partial_i\partial_j v_j - \partial_i\partial_0 v^0 - m^2 v^i = 0 \quad \text{or}$$

$$\Box v^i - \partial_i\partial_\nu v^\nu - m^2 v^i = 0 \qquad (\text{eom})$$

(con)$_0^0$: $\nabla^2 v^0 + \nabla \cdot \dot{v} - m^2 v^0 = 0$ or

$$\Box v^0 - \partial^0 \partial_0 v^0 - \partial^0 \partial_j v^j - m^2 v^0 = 0 \quad \text{or}$$

$$\Box v^0 - \partial^0(\partial_\nu v^\nu) - m^2 v^0 = 0 \qquad (\text{con}')$$

So (com) & (con') now give

$$\Box v^\mu - \partial^\mu \partial_\nu v^\nu - m^2 v^\mu = 0 \qquad (24)$$

$$\Box \partial_\mu v^\mu - \Box \partial_\nu v^\nu - m^2 \partial_\mu v^\mu = -m^2 \partial_\mu v^\mu = 0 \qquad (25)$$

whence

$$(\Box - m^2) v^\mu = 0 \qquad (26)$$

and

$$\partial_\mu v^\mu = 0 \qquad (25)$$

So $\quad \Box v^\mu - \partial^\mu \partial_\nu v^\nu - m^2 v^\mu = 0$

$$\Box \partial_\mu v^\mu - \Box \partial_\nu v^\nu - m^2 \partial_\mu v^\mu = 0$$

or $\quad \partial_\mu v^\mu = 0$ whence $\quad (25)$

$$(\Box - m^2) v^\mu = 0 . \quad (26)$$

A real vector field obeying (25-26) may be written as

$$v^\mu(x) = \sum_\sigma \int \frac{d^3 p}{\sqrt{2\pi^3 2p^0}} \left\{ e^\mu(p,\sigma)\, a(p\sigma)\, e^{ipx} + e^{\mu *}(p\sigma)\, a^\dagger(p\sigma)\, e^{-ipx} \right\}$$

with $p^0 = \sqrt{\vec{p}^{\,2} + m^2}$ and for $\sigma = 1, 0, -1$,

$$p_\mu\, e^\mu(p,\sigma) = 0$$

and

$$\sum_\sigma e^\mu(p\sigma)\, e^{\nu *}(p\sigma) = \eta^{\mu\nu} + \frac{p^\mu p^\nu}{m^2} ,$$

which is (5.3.28-29).

In fact v and π satisfy at equal times

$$[v^i(x,t), \pi^j(y,t)]_{et} = i\, \delta_{ij}\, \delta(x-y)$$

$$[v^i, v^j]_{e.t.} = [\pi^i, \pi^j]_{e.t.} = 0$$

as long as

$$[a(p,\sigma), a^\dagger(p',\sigma')] = \delta_{\sigma\sigma'}\, \delta^3(p-p')$$

$$[a(p\sigma), a(p'\sigma')] = 0 .$$

Since the expansion (7.5.27) for $v^\mu(x)$ was derived in 5, these results show that H_0 is correct. One may show that

$$H_0 = \sum_\sigma \int d^3p \; p^0 \left(a^\dagger(p,\sigma) a(p,\sigma) + \frac{1}{2} \right).$$

Finally by using $v^0 = \nabla \cdot \pi / m^2$, we may write V as

$$V = \int d^3x \; - J_\mu v^\mu + \frac{1}{2m^2} \left(J^0 \right)^2$$

in which the term $J^{0\,2}$ cancels a funny term in the propagator of v^μ as shown in 6.

Dirac

$$\mathcal{L} = -\bar{\psi}(\gamma^\mu \partial_\mu + m)\psi - \mathcal{H}(\bar{\psi}, \psi).$$

$\mathcal{L}$ is real apart from a total divergence.

$$\mathcal{L} - \mathcal{L}^\dagger = -\bar{\psi}\gamma^\mu\partial_\mu\psi + \partial_\mu\psi^\dagger\gamma^{\mu\dagger}(i\gamma^0)^\dagger\psi$$

$$= -\bar{\psi}\gamma^\mu\partial_\mu\psi + \partial_\mu\psi^\dagger i\gamma^0 i\gamma^0\gamma^{\mu\dagger}(i\gamma^0)^\dagger\psi$$

$(i\gamma^0)^\dagger = \beta^\dagger = \beta \qquad \beta\gamma^{\mu\dagger}\beta = -\gamma^\mu$

$$\mathcal{L} - \mathcal{L}^\dagger = -\bar{\psi}\gamma^\mu\partial_\mu\psi - \partial_\mu\psi^\dagger\beta\gamma^\mu\psi$$

$$= -\partial_\mu(\bar{\psi}\gamma^\mu\psi).$$

So $\mathcal{L}$ and $\mathcal{L}^\dagger$ generate the same field equations.

$$\Pi = \frac{\partial^R \mathcal{L}}{\partial\dot{\psi}} = -\bar{\psi}\gamma^0 = -\psi^\dagger i\gamma^0\gamma^0 = i\psi^\dagger$$

$$H = \int d^3x\ \Pi\dot{\psi} - \mathcal{L} = \int d^3x\ \Pi\dot{\psi} + \bar{\psi}\gamma^0\dot{\psi} + \bar{\psi}(\vec{\gamma}\cdot\vec{\nabla} + m)\psi + \mathcal{H}$$

$$= \int d^3x\ \Pi\dot{\psi} - \Pi\dot{\psi} + \bar{\psi}(\gamma\cdot\nabla + m)\psi + \mathcal{H}$$

$$= \int d^3x\ \bar{\psi}(\gamma\cdot\nabla + m)\psi + \mathcal{H}$$

$$= H_0 + V \qquad \Pi\gamma^0 = \bar{\psi}$$

$$H_0 = \int d^3x\ \Pi\gamma^0(\gamma\cdot\nabla + m)\psi$$

$$V = \int d^3x\ \mathcal{H}(\bar{\psi}, \psi).$$

$\underset{HP}{\Pi = -\bar{\Psi}\gamma^0}$ means $\underset{I.P.}{\pi = -\bar{\psi}\gamma^0}$

In the I. P.

$$\dot\psi_\alpha = i[H_0,\psi_\alpha] = i\int d^3x'[\pi\,\gamma^0(\gamma\cdot\nabla+m)\psi,\psi_\alpha(x)]$$

$$= i\int d^3x' \; -\pi_\beta(x')\psi_\alpha(x)\left(\gamma^0(\gamma\cdot\nabla+m)\psi(x')\right)_\beta$$

$$-\psi_\alpha(x)\,\pi_\beta(x')\left(\gamma^0(\gamma\cdot\nabla+m)\psi(x')\right)_\beta$$

$$= \int d^3x' \; \delta_{\alpha\beta}\,\delta(x-x')\,\left(\gamma^0(\gamma\cdot\nabla+m)\psi(x')\right)_\beta$$

$$= [\gamma^0(\gamma\cdot\nabla+m)]_{\alpha\gamma}\,\psi_\gamma(x).$$

$$\dot\psi = \gamma^0(\gamma\cdot\nabla+m)\psi$$

$$\gamma^0\partial_0\psi = -(\gamma\cdot\nabla+m)\psi$$

$$(\gamma^\mu\partial_\mu+m)\psi = 0$$

At equal times $[\psi_\alpha(x'),\pi_\beta(x)]_+ = i[\psi_\alpha(x'),\psi^\dagger_\beta(x)] = i\delta_{\alpha\beta}\delta(\vec x-\vec x')$

$$\dot\pi = i[H_0,\pi] = i\int d^3x'\,[\pi\,\gamma^0(\gamma\cdot\nabla+m)\psi(x')\,\pi(x)$$

$$+\pi(x')\gamma^0(\gamma\cdot\nabla+m)\,\pi(x)\,\psi(x')$$

$$= -\pi\,\gamma^0(-\gamma\cdot\overleftarrow\nabla+m)$$ which, since $\pi = -\bar\psi\gamma^0$, is

$$-\dot{\bar\psi}\gamma^0 = \bar\psi\,\gamma^{0\,2}(-\gamma\cdot\overleftarrow\nabla+m)$$

$$i\dot\psi^\dagger = -\bar\psi\;(-\gamma\cdot\overleftarrow\nabla+m)$$

$$= -\psi^\dagger\beta(\beta\vec\gamma^\dagger\cdot\overleftarrow\nabla\beta+m) = -\psi^\dagger\beta\,\beta(\gamma^\dagger\cdot\overleftarrow\nabla+m)\beta$$

$$i\dot\psi^\dagger\beta = -\psi^\dagger(\gamma^\dagger\cdot\overleftarrow\nabla+m) \qquad \beta = i\gamma^0$$

$$-\dot\psi^\dagger\gamma^0 = -\psi^\dagger(\gamma^\dagger\cdot\overleftarrow\nabla+m)$$

$$\dot\psi^\dagger\gamma^{0\dagger} = -\psi^\dagger(\gamma^\dagger\cdot\overleftarrow\nabla+m) \qquad 0 = \partial_\mu\psi^\dagger\gamma^{\mu\dagger}+\psi^\dagger m$$

which is $[0 = (\gamma^\mu\partial_\mu+m)\psi]^\dagger$.

The general solution of $0 = (\not\partial + m)\psi$ is

$$\psi(x) = \int \frac{d^3p}{(2\pi)^{3/2}} \sum_\sigma \left[u(p\sigma) e^{ipx} a(p\sigma) + v(p\sigma) e^{-ipx} a_c^+(p\sigma) \right]$$

in which $p^0 = \sqrt{p^2+m^2}$; a and a_c^+ are operator coefficients; and $u(p,\sigma)$ are two independent solutions of

$$(i p_\mu \gamma^\mu + m) u(p\sigma) = 0$$

and $v(p\sigma)$ are two of

$$(-i \gamma^\mu p_\mu + m) v(p\sigma) = 0$$

normalized so that

$$\sum_\sigma u(p\sigma)\bar{u}(p\sigma) = \frac{-i\gamma^\mu p_\mu + m}{2p^0}$$

$$\sum_\sigma v(p\sigma)\bar{v}(p\sigma) = -\frac{(i\gamma^\mu p_\mu + m)}{2p^0}$$

Here $i\gamma^\mu p_\mu$ has e.v.'s $\pm m$.

$i p_\mu \gamma^\mu u = -m u$ $\qquad\qquad -i p_\mu \gamma^\mu v = m v$

$(-i\not p + m)u = 2m u$ $\qquad\qquad (i\not p + m)v = 2m v$

$2m \sum u\bar{u} = (-i\not p + m)\sum u\bar{u}$ $\qquad\qquad 2m\sum v\bar{v} = (i\not p + m)\sum v\bar{v}$

$\sum u\bar{u} = \frac{-i\not p + m}{2m}\sum u\bar{u} = \frac{-i\not p + m}{2m}\frac{2m}{2p^0}$ $\qquad\qquad \sum v\bar{v} = \frac{i\not p + m}{2m}\left(\frac{-2m}{2p^0}\right)$

$= \frac{-i\not p + m}{2p^0}$ $\qquad\qquad = -\frac{(i\not p + m)}{2p^0}$

$$\mathrm{tr} \sum u \bar{u} \beta = \sum u^\dagger u = 2 = \mathrm{tr} \frac{(-ip_0\gamma^0 + m) i\gamma^0}{2p^0}$$

$$= \frac{\mathrm{tr} -p_0 I}{2p^0} = \frac{1}{2} \mathrm{tr}\, I = 2$$

$$\mathrm{tr} \sum v \bar{v} \beta = \sum v^\dagger v = 2 = \mathrm{tr}\left[-\frac{(i\not{p} + m)}{2p^0} i\gamma^0 \right]$$

$$= \frac{1}{2p^0} \mathrm{tr} -ip_0 \gamma^0 i \gamma^0 = \frac{1}{2p^0} \mathrm{tr}\, p_0 \gamma^{0^2} = \frac{4p^0}{2p^0} = 2$$

which verifys the work done in 5.

To have the ETAC's $(\pi = -\bar{\psi}\gamma^0$ or $-\pi\gamma^0 = \bar{\psi})$

$$[\psi_\alpha(x,t), \bar{\psi}_\beta(y,t)]_+ = [\psi_\alpha(x,t), \pi_\gamma(y,t)]_+ \gamma^0_{\gamma\beta}$$

$$= i\delta(x-y)\delta_{\alpha\gamma}\gamma^0_{\gamma\beta} = i\gamma^0_{\alpha\beta}\delta(x-y)$$

and $[\psi_\alpha(x,t), \psi^\dagger_\beta(y,t)]_+ = \delta_{\alpha\beta}\delta(\vec{x}-\vec{y})$

$$[\psi_\alpha(x,t), \psi_\beta(y,t)]_+ = 0$$

we choose $[a(p,\sigma), a^\dagger(p'\sigma')]_+ = \delta_{\sigma\sigma'}\delta^3(p-p')$

$[a_c(p,\sigma), a_c^\dagger(p'\sigma')]_+ = \delta_{\sigma\sigma'}\delta^3(p-p')$

$[a,a']_+ = [a_c, a_c']_+ = 0 = [a, a_c'] = [a, a_c'^\dagger]$

as in 5. So (7.5.37) is right H_0.

$$H_0 = \sum_\sigma \int d^3p\, p^0 \left(a^\dagger(p\sigma) a(p\sigma) - a_c(p\sigma) a_c^\dagger(p\sigma) \right)$$

$$= \sum_\sigma \int d^3p\, p^0 \left(a^\dagger(p,\sigma) a(p\sigma) + a_c^\dagger(p\sigma) a_c(p\sigma) - \delta^3(p-p) \right)$$

only gravity sees

Dirac Brackets

Primary constraints are imposed (eg as gauge conditions) or arise from $\mathcal{L}$. Thus if $\dot{\psi}^l$ does not appear in L, then

$$\Pi_l = \frac{\partial L}{\partial \dot{\psi}^l} = 0.$$

In general primary constraints arise when the equations

$$\Pi_l = \frac{\partial L}{\partial \dot{\psi}^l}$$

cannot be solved for the $\dot{\psi}^l$'s, ie., when the matrix

$$M_{lm} = \frac{\partial^2 \mathcal{L}}{\partial \dot{\psi}^l \partial \dot{\psi}^m} = \frac{\partial \Pi_l}{\partial \dot{\psi}_m} = \frac{\partial \Pi_m}{\partial \dot{\psi}_l}$$

is singular, $\det M = 0$. Such $\mathcal{L}$ are irregular.

Secondary constraints arise from the field equations and the primary constraints. For the massive vector field, since $\Pi_0 = 0$, the 0th field equation is

$$\partial_i \Pi_i = m^2 V^0 - J^0,$$

In E&M we have Gauss's law $\nabla \cdot E = \rho = J^0$, which is a secondary constraint.