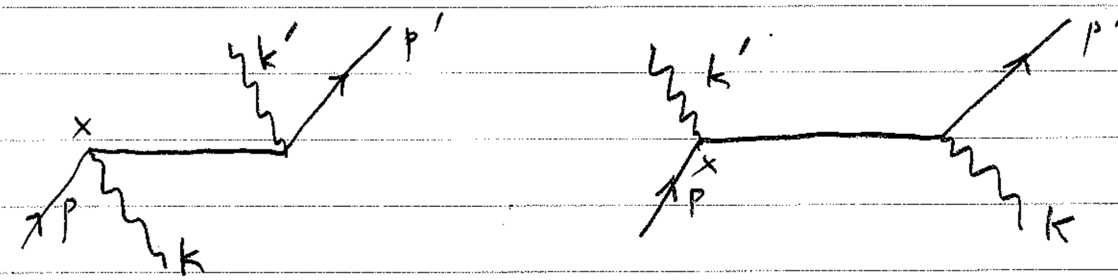
Compton Scattering with extra ± for fermions / bosons

$$-i\Delta_{\ell m}(x,y) = \langle 0|T\{\psi_\ell(x)\,\psi^\dagger_m(y)\}|0\rangle$$

$$= -i\int \frac{d^4q}{(2\pi)^4}\,\frac{P^{(L)}_{\ell m}(q)\,e^{iq(x-y)}}{q^2+m^2-i\epsilon}$$

$$P^{(L)}_{\ell m}(q) = P_{\ell m}(q) = \left[(-i\not{q}+m)\beta\right]_{\ell m}$$

$$\langle p'k'|S|pk\rangle = \frac{(-i)^2}{2}\langle p'k'|\,T\!\iint d^4x\,d^4y\,(ie)^2(\bar\psi\not{A}\psi)_x(\bar\psi\not{A}\psi)_y$$
$$|pk\rangle$$




take $p$ absorbed at $x$

$$= e^2\langle p'k'|\,T\int d^4x\,d^4y\,(\bar\psi\not{A}\psi)_x\,(\bar\psi\not{A}\psi)_y\,|pk\rangle$$

$$\psi_\alpha(x) = \sum_s\int\frac{d^3p}{(2\pi)^{3/2}}\left[u_\alpha(ps)\,a(p,s)\,e^{ipx} + v_\alpha(ps)\,a_c^\dagger(ps)\,e^{-ipx}\right]$$

$$A_\mu(x) = \int\frac{d^3p}{\sqrt{(2\pi)^3 2p^0}}\sum_s\left[e^{ipx}\,e^\mu(p,s)\,a(ps) + e^{-ipx}\,e^{*\mu}(p,s)\,a^\dagger(p,s)\right]$$

$$S = e^2 \langle 0| a(p's') a(k',\lambda') \int d^4x\, d^4y$$

$$\theta(x-y) (\bar\Psi \not{A} \Psi)_x (\bar\Psi \not{A} \Psi)_y$$

$$+ \theta(y-x) (\bar\Psi \not{A} \Psi)_y (\bar\Psi \not{A} \Psi)_x \quad a^+(p,s) a^+(k,\lambda) |0\rangle$$

$$= \frac{e^2 \langle 0| a a'}{(2\pi)^{3/2}} \int d^4x\, d^4y\; e^{ipx} \Big[ (\bar\Psi \not{A})_x u(p,s)\, \theta(x-y) (\bar\Psi \not{A} \Psi)_y$$

$$+ \theta(y-x) (\bar\Psi \not{A} \Psi)_y (\bar\Psi \not{A})_x u(p,s) \Big] a^+(k\lambda)|0\rangle$$

$$= \frac{e^2}{(2\pi)^3} \langle 0| a(k',\lambda') \int d^4x\, d^4y\, e^{ipx - ip'y} \Big[ -\theta(x-y) (\bar\Psi \not{A})_x u\, \bar u' (\not{A} \Psi)_y$$

$$+ \theta(y-x) \bar u' (\not{A}\Psi)_y (\bar\Psi \not{A})_x u \Big] a^+(k,\lambda)|0\rangle$$

$$= \frac{e^2}{(2\pi)^6\, 2\sqrt{k^0 k^{0\prime}}} \langle 0| \int d^4x\, d^4y\, e^{ipx - ip'y} \Big\{ \Big[ -\theta(x-y)\, \bar\Psi_x \not{\epsilon}\, u\, \bar u' \not{\epsilon}'^* \Psi_y$$

$$+ \theta(y-x)\, \bar u' \not{\epsilon}'^* \Psi_y\, \bar\Psi_x \not{\epsilon}\, u \Big] e^{ikx - ik'y}$$

$$+ \Big[ -\theta(x-y)\, \bar\Psi_x \not{\epsilon}'^* u\, \bar u' \not{\epsilon}\, \Psi_y$$

$$+ \theta(y-x)\, \bar u' \not{\epsilon}\, \Psi_y\, \bar\Psi_x \not{\epsilon}'^* u \Big] e^{iky - ik'x} \quad |0\rangle$$

$$S = \frac{e^2}{(2\pi)^6\, 2\sqrt{k^0 k^{0\prime}}} \int d^4x\, d^4y \Big[ e^{ipx - ip'y + ikx - ik'y}$$

$$-i \Delta_{\alpha\beta}(y-x)\, (\bar{u}'\not{\epsilon}'^{*})_\alpha (\not{\epsilon} u)_\beta$$

$$+ e^{ipx - ip'y - ik'x + iky}\, \bar{u}'\not{\epsilon}(-i)\Delta(y-x)\not{\epsilon}'^{*} u \Big]$$

$$= \frac{-i e^2}{(2\pi)^6\, 2\sqrt{k^0 k^{0\prime}}} \int d^4x\, d^4y\, e^{ipx - ip'y}$$

$$\Big[ e^{ikx - ik'y}\, \bar{u}'\not{\epsilon}'^{*} \Delta(y-x) \not{\epsilon} u$$

$$+ e^{-ik'x + iky}\, \bar{u}'\not{\epsilon}\, \Delta(y-x)\, \not{\epsilon}'^{*} u \Big]$$

$$= \frac{-i e^2}{(2\pi)^{10}\, 2\sqrt{k^0 k^{0\prime}}} \int d^4x\, d^4y\, d^4q\, \frac{e^{ipx - ip'y + iq(y-x)}}{q^2 + m^2 - i\epsilon}$$

$$\Big[ e^{ikx - ik'y}\, \bar{u}'\not{\epsilon}'^{*}(-i\not{q} + m)\not{\epsilon} u$$

$$+ e^{-ik'x + iky}\, \bar{u}'\not{\epsilon}(-i\not{q} + m)\not{\epsilon}'^{*} u \Big]$$

$$S = \frac{-ie^2}{(2\pi)^2}\frac{1}{2\sqrt{k^0 k^{0\prime}}}\int d^4q \Big[ \delta^4(p-q+k)\,\delta^4(-p'+q-k')\,\frac{\bar{u}'\,\not\epsilon'^*(-i\not q+m)\not\epsilon\,u}{q^2+m^2-i\epsilon}$$

$$+\,\delta^4(p-q-k')\,\delta^4(-p'+q+k)\,\frac{\bar{u}'\not\epsilon(-i\not q+m)\not\epsilon'^*u}{q^2+m^2-i\epsilon}\Big]$$

$$= \frac{-ie^2\,\delta^4(p+k-p'-k')}{(2\pi)^2\,2\sqrt{k^0k^{0\prime}}}\,\bar{u}'\Big[\frac{\not\epsilon'^*(-i(\not p+\not k)+m)\not\epsilon}{(p+k)^2+m^2-i\epsilon}$$

$$+\,\frac{\not\epsilon\,(-i(\not p-\not k')+m)\not\epsilon'^*}{(p-k')^2+m^2-i\epsilon}\Big]u$$

The denominators are, since $p^2=-m^2$ and $k^2=0$,

$$(p+k)^2+m^2 = -m^2+2p\cdot k+m^2 = 2p\cdot k$$

$$(p-k')^2+m^2 = -m^2-2p\cdot k'+m^2 = -2p\cdot k'$$

In the lab. frame $p=(\vec{0},m)$, and so these are

$$2p\cdot k = -2m\omega \quad \text{and} \quad -2p\cdot k' = 2m\omega'.$$

Neither vanishes, and we drop the $-i\epsilon$'s.

By (3.3.2)

$$S = -2\pi i\, \delta^4(p+k-p'-k')\, M \qquad \text{so}$$

$$M = \frac{e^2}{4(2\pi)^3\sqrt{k^0 k^{0\prime}}}\, \bar{u}(p's')\left\{ \not{e}^{*\prime} \frac{[-i(\not{p}+\not{k})+m]}{p\cdot k}\not{e} \right.$$

$$\left. - \not{e}\frac{[-i(\not{p}-\not{k}')+m]\not{e}^{*\prime}}{p\cdot k'} \right\} u(p,s).$$

By (3.4.15)

$$d\sigma = (2\pi)^4 u^{-1} |M|^2 \delta^4(p'+k'-p-k)\, d^3p'\, d^3k'$$

where $\quad u = \dfrac{|p\cdot k|}{p^0 k^0}$

which in the lab. frame is

$$u = \frac{m\omega}{m\omega} = 1.$$

So the $\delta^3$ cancels the $d^3p'$ leaving

$$d\sigma = (2\pi)^4 |M|^2 \delta\left(\sqrt{(k-k')^2+m^2} + \omega' - m - \omega\right) d^3k'$$

The energy condition is

$$\sqrt{\omega^2+\omega'^2-2\omega\omega'\cos\theta+m^2} = m+\omega-\omega'$$

one finds

$$\omega' \equiv \omega_c(\theta) = \frac{m}{m + \omega(1-\cos\theta)} .$$

Now

$$\int \delta(f(E) - x)\, df(E) = \int \delta(E-E_x)\, dE = 1$$

$$= \int \delta(f(E)-x)\, \frac{df}{dE}\, dE \qquad \text{so}$$

$$\delta(f(E)-x) = \frac{\delta(E-E_x)}{|df/dE|}$$

Here $f(E) = p'^0(\omega') + \omega'$

$$\frac{d(p^{0\prime}+\omega')}{d\omega'} = \frac{d}{d\omega'}\left(\sqrt{\omega^2+\omega'^2-2\omega\omega'\cos\theta+m^2} + \omega'\right)$$

$$= 1 + \frac{\omega' - \omega\cos\theta}{\sqrt{\omega^2+\omega'^2-2\omega\omega'\cos\theta+m^2}} = \left|\frac{\omega'-\omega\cos\theta}{p^0} + 1\right|$$

So

$$\delta(p^{0\prime}+\omega'-m-\omega) = \frac{\delta(\omega'-\omega_c(\theta))}{\left|\frac{\omega'-\omega\cos\theta}{p^{0\prime}} + 1\right|} = \frac{p^{0\prime}\,\delta(\omega'-\omega_c(\theta))}{|\omega'-\omega\cos\theta+p^{0\prime}|}$$

$$= \frac{p^{0\prime}\,\delta(\omega'-\omega_c(\theta))}{|m+\omega(1-\cos\theta)|} = \frac{p^{0\prime}\,\delta(\omega'-\omega_c(\theta))}{m\omega/\omega'}$$

or

$$\delta\left(p^{0\prime}+\omega'-p^0\cdot\omega\right)=\frac{p^{0\prime}\omega'}{m\omega}\,\delta\left(\omega'-\omega_c(\theta)\right)$$

$$d^3k' = \omega'^2\, d\omega\, d\Omega \qquad \text{so}$$

$$d\sigma = (2\pi)^4 |M|^2 \frac{p^{0\prime}\omega'^3}{m\,\omega}\, d\Omega$$

with

$$p^{0\prime} = m+\omega-\omega' \quad \text{and} \quad \omega' = \omega_c(\theta).$$

Usually, we average over initial spins & polarizations and sum over final ones.

$$M = \bar{u}'_\alpha A_{\alpha\beta} u_\beta$$

$$|M|^2 = MM^* = \bar{u}'_\alpha A_{\alpha\beta} u_\beta\, u^\dagger_\gamma (A^\dagger\beta)_{\gamma\delta} u'_\delta$$

$$= \bar{u}'_\alpha A_{\alpha\beta} u_\beta\, \bar{u}_\gamma (\beta A^\dagger\beta)_{\gamma\delta}\, u'_\delta$$

$$= A_{\alpha\beta}\, u_\beta\, \bar{u}_\gamma\, (\beta A^\dagger \beta)_{\gamma\delta}\, u'_\delta\, \bar{u}'_\alpha$$

$$= \mathrm{Tr}\; A\, u\bar{u}\, \beta A^\dagger \beta\, u\bar{u}$$

$$\sum_s u_\beta(ps)\,\bar{u}_\gamma(ps) = \frac{(-i\not{p}+m)_{\beta\gamma}}{2p^0}$$

$$\sum_{s'} u_\delta(p's')\,\bar{u}_\alpha(p's') = \frac{(-i\not{p}'+m)_{\delta\alpha}}{2p^0}$$

Our $\gamma$'s are

$$\gamma^0 = -i\begin{pmatrix}0&1\\1&0\end{pmatrix} \qquad \vec{\gamma} = -i\begin{pmatrix}0 & \vec{\sigma}\\ -\vec{\sigma} & 0\end{pmatrix}$$

(5.4.17)

So

$$\gamma^{0\dagger} = -\gamma^0 \qquad \vec{\gamma}^{\,\dagger} = \vec{\gamma}$$

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$$

so

$$\beta\gamma^{\mu\dagger}\beta = -\gamma^\mu$$

Now

$$A = \not{e}^{*\prime}\,[-i(\not{p}+\not{k})+m]\,\not{e}\,/\,p\cdot k$$

$$-\not{e}\,[-i(\not{p}-\not{k}')+m]\,\not{e}^{*\prime}/\,p\cdot k'$$

and

$$\beta\not{p}^\dagger\beta = \beta p_\mu\gamma^{\mu\dagger}\beta = -\not{p}$$

$$\beta\not{e}^\dagger\beta = \beta e^*_\mu\gamma^{\mu\dagger}\beta = -\not{e}^*$$

$$\beta(\not{e}^*)^\dagger\beta = \beta e_\mu\gamma^{\mu\dagger}\beta = -\not{e}$$

So

$$A^\dagger = \not{\epsilon}^\dagger \left[+i\left(\not{p}^\dagger + \not{k}^\dagger\right) + m\right] \left(\not{\epsilon}^{*\prime}\right)^\dagger / p\cdot k$$

$$- \left(\not{\epsilon}^{*\prime}\right)^\dagger \left[ i \left(\not{p}^\dagger - \not{k}'\right) + m\right] \left(\not{\epsilon}\right)^\dagger / p\cdot k'$$

So

$$\beta A^\dagger \beta = \beta \not{\epsilon}^\dagger \beta\, \beta \left[ i\left(\not{p}^\dagger + \not{k}^\dagger\right) + m\right] \beta\, \beta \left(\not{\epsilon}^{*\prime}\right)^\dagger \beta / p\cdot k$$

$$- \beta \left(\not{\epsilon}^{*\prime}\right)^\dagger \beta\beta \left[ i\left(\not{p}^\dagger - \not{k}'^\dagger\right) + m\right] \beta\beta\, \not{\epsilon}^\dagger / p\cdot k'$$

$$= - \not{\epsilon}^* \left[ -i\left(\not{p} + \not{k}\right) + m\right] \left(-\not{\epsilon}'\right) / p\cdot k$$

$$+ \not{\epsilon}' \left[ -i\left(\not{p} - \not{k}'\right) + m\right] \left(-\not{\epsilon}^*\right) / p\cdot k'$$

So

$$\sum_{ss'} |M|^2 = \frac{e^4}{64(2\pi)^6 \omega\omega' p^0 p^{0\prime}}$$

$$\times \mathrm{Tr}\left[ \left\{ \not{\epsilon}^{*\prime} \frac{\left[-i\left(\not{p}+\not{k}\right)+m\right]}{p\cdot k} \not{\epsilon} - \not{\epsilon} \frac{\left[-i\left(\not{p}-\not{k}'\right)+m\right]}{p\cdot k'} \not{\epsilon}^{*\prime} \right\} \right.$$

$$\left(-i\not{p}+m\right) \left\{ \not{\epsilon}^* \frac{\left[-i\left(\not{p}+\not{k}\right)+m\right]}{p\cdot k} \not{\epsilon}' - \not{\epsilon}' \frac{\left[-i\left(\not{p}-\not{k}'\right)+m\right]}{p\cdot k'} \not{\epsilon}^{*} \right\}$$

$$\left. \left(-i\not{p}'+m\right) \right]$$

In Coulomb's gauge in lab. frame

$$e \cdot p = e^* \cdot p = e' \cdot p = e'^* \cdot p = 0$$

since $e^0 = e'^0 = 0$ & $\vec{p} = 0$.

$$\not{f}\not{g} = f_\mu g_\nu \gamma^\mu \gamma^\nu$$

$$= f_\mu g_\nu (-\gamma^\nu \gamma^\mu + 2\eta^{\mu\nu})$$

$$= -\not{g}\not{f} + 2 fg \quad (\not{f}^2 = f \cdot f) \quad \text{so}$$

$$(-i\not{p} + m)\not{e}(-i\not{p} + m) = \not{e}(i\not{p} + m)(-i\not{p} + m)$$

$$= \not{e}(\not{p}^2 + m^2) = \not{e}(-m^2 + m^2) = 0.$$

This works also for $e'$, $e^*$ & $e'^*$.

One gets

$$\sum_{ss'} |M|^2 = \frac{e^4}{64(2\pi)^6 \omega\omega' p^0 p^{0\prime}}$$

$$\times \mathrm{Tr}\left[\left(\frac{\not{e}^{*\prime}\not{k}\not{e}}{p \cdot k} + \frac{\not{e}\not{k}'\not{e}'^*}{p \cdot k'}\right)(-i\not{p} + m)\right.$$

$$\left.\times \left(\frac{\not{e}^*\not{k}\not{e}'}{p \cdot k} + \frac{\not{e}'\not{k}'\not{e}^*}{p \cdot k'}\right)(-i\not{p}' + m)\right] \qquad (8.7.23)$$

Now
$$\mathrm{Tr}\left(\gamma_{\mu_1}\gamma_{\mu_2}\cdots\gamma_{\mu_{2n+1}}\right) = 0$$

because

$$\gamma_5\gamma_\mu\gamma_5^{-1} = -\gamma_\mu$$

where $\gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \gamma_5^{-1}$

Thus

$$\mathrm{Tr}\,\gamma_{a_1}\gamma_{a_2}\cdots\gamma_{a_{2n+1}}$$
$$= \mathrm{Tr}\,\gamma_5\gamma_{a_1}\gamma_5\gamma_5\gamma_{a_2}\gamma_5^2\gamma_{a_3}\cdots\gamma_5\gamma_{a_{2n+1}}\gamma_5$$
$$= (-1)^{2n+1}\,\mathrm{Tr}\,\gamma_{a_1}\gamma_{a_2}\cdots\gamma_{a_{2n+1}}.$$

The appendix to ch 8 shows that

$$2\,\mathrm{Tr}\,\gamma_\mu\gamma_\nu = 2\eta_{\mu\nu}\mathrm{Tr}\,1 = 8\eta_{\mu\nu}$$
$$\mathrm{Tr}\,\gamma_\mu\gamma_\nu = 4\eta_{\mu\nu}$$
$$\mathrm{Tr}\,\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma = 4\left(\eta_{\mu\nu}\eta_{\rho\sigma} - \eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho}\right)$$

etc.

Also $e\cdot p = 0 \quad k\cdot k = k'\cdot k' = 0 \quad e\cdot e^* = e'\cdot e'^* = 1$

We will use real $e$'s to further simplify the results. We then drop all $*$'s.

$$\not{e}\not{p}\not{e} = -\not{p}\not{e}\not{e} = -\not{p}$$

$$\not{k}\not{p}\not{k} = -\not{k}\not{k}\not{p} + \not{k}\, 2p\cdot k = 2\not{k}\, p\cdot k$$

So

$$T_1 \equiv \mathrm{Tr}\, \not{e}'\not{k}\not{e}\not{p}\not{e}\not{k}\not{e}'\not{p}'$$

$$= -\mathrm{Tr}\, \not{e}'\not{k}\not{p}\not{k}\not{e}'\not{p}'$$

$$= -2p\cdot k\ \mathrm{Tr}\, \not{e}'\not{k}\not{e}'\not{p}'$$

$$= -8p\cdot k\left(2e'\cdot k\, e'\cdot p' - k\cdot p'\right)$$

$$e'\cdot p' = e'\cdot(p+k-k') = e'\cdot k$$

$$k\cdot p' = -\tfrac{1}{2}(p'-k)^2 - \tfrac{1}{2}m^2 = -\tfrac{1}{2}(p-k')^2 - \tfrac{1}{2}m^2 = p\cdot k'$$

So

$$T_1 = -16\, p\cdot k\,(e'\cdot k)^2 + 8\, p\cdot k\ p\cdot k'$$

Eventually, one has

$$\sum_{ss'} |M|^2 = \frac{e^4}{64(2\pi)^6 \omega\omega' p^0 p'^0} \left[ \frac{8(k \cdot k')^2}{k \cdot p \;\; k' \cdot p} + 32(e \cdot e')^2 \right]$$

in lab. frame

$$k \cdot k' = \omega\omega'(\cos\theta - 1) = m\omega\omega'\left(\frac{1}{\omega} - \frac{1}{\omega'}\right)$$

$$p \cdot k = -m\omega \qquad p \cdot k' = -m\omega' \qquad \text{so}$$

$$\frac{1}{2}\sum_{ss'} d\sigma = \frac{e^4 \omega'^2 d\Omega}{64\pi^2 m^2 \omega^2}\left[\frac{\omega}{\omega'} + \frac{\omega'}{\omega} - 2 + 4(e \cdot e')^2\right]$$

Klein & Nishina 1929

Average over initial polarization

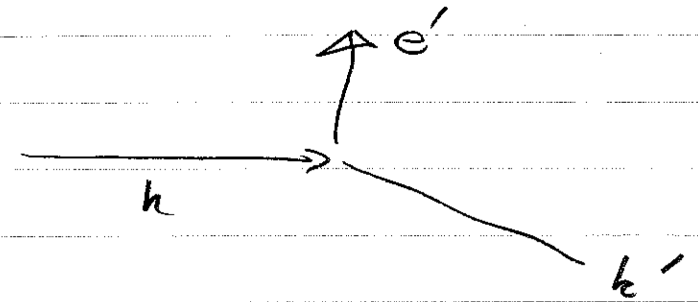
$$\frac{1}{2}\sum_{\lambda} e_i e_j = \frac{1}{2}\left(\delta_{ij} - \hat{k}_i \hat{k}_j\right)$$

$$\frac{1}{4}\sum_{ss'\lambda} d\sigma = \frac{e^4 \omega'^2 d\Omega}{64\pi^2 m^2 \omega^2}\left(\frac{\omega}{\omega'} + \frac{\omega'}{\omega} - 2(\hat{k} \cdot e')^2\right)$$

which is highest when $\hat{k} \cdot e' = 0$,

that is, $e' \cdot k = e' \cdot k' = 0$.

So $e'$ tends to be $\perp$ to plane of scattering




Usually one sums over the $e'$'s:

$$\frac{1}{4}\sum_{ss'\lambda\lambda'} d\sigma = \frac{e^4 \omega'^2 d\Omega}{32\pi^2 m^2 \omega^2}\left(\frac{\omega}{\omega'}+\frac{\omega'}{\omega}-1+\cos^2\theta\right)$$

where $\hat{k}\cdot\hat{k}' = \cos\theta$.

In non-rel case, $\omega \ll m$, then $\omega \approx \omega'$ and

$$\frac{1}{4}\sum d\sigma = \frac{e^4 d\Omega}{32\pi^2 m^2}\left(1+\cos^2\theta\right)$$

$$\frac{1}{4}\sum\int d\Omega = \sigma = \frac{16\pi}{3}\,\frac{e^4}{32\pi^2 m^2}$$

$$= \frac{e^4}{6\pi m^2} = \sigma_T$$

Thomson