Action

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1 Lagrangians and hamiltonians

A transformation is a symmetry of a theory if the **action** is invariant or changes by a surface term. So we choose to work with actions that are symmetrical. The action is normally an integral over spacetime of an **action density** often called a **lagrangian**. Often the action density itself is invariant under the transformation of the symmetry.

There are procedures, sometimes clumsy procedures, for computing the hamiltonian from the lagrangian. The hamiltonian often is not invariant under the transformation of the symmetry. So it's very hard to find a suitably symmetrical theory by starting with a hamiltonian. But once one has a hamiltonian, one can compute scattering amplitudes energies, and states with these energies.

2 Canonical variables

In quantum mechanics, we use the equal-time commutation relations

$$[q_i, p_k] = i\delta_{ik}, \quad [q_i, q_k] = 0, \quad \text{and} \quad [p_i, p_k] = 0. \tag{1}$$

In general, the operators q_i and $q_i(t) = e^{iHt} q_i e^{-iHt}$ do not commute. Quantum field theory promotes these equal-time commutation relations to ones in which the indexes i and k denote different points of space

$$\begin{aligned} &[q^n(\vec{x}, t), p_m(\vec{y}, t)]_\mp = i\delta(\vec{x} - \vec{y})\delta^n_m, \\ &[q^n(\vec{x}, t), q^m(\vec{y}, t)]_\mp = 0, \quad \text{and} \quad [p_n(\vec{x}, t), p_m(\vec{y}, t)]_\mp = 0. \end{aligned} \tag{2}$$

The commutator of a real scalar field

$$\phi(x) = \int \frac{d^3p}{\sqrt{(2\pi)^3 2p^0}} \left[a(\vec{p})\, e^{ip\cdot x} + a^\dagger(\vec{p})\, e^{-ip\cdot x}\right] \tag{3}$$

is

$$[\phi(x), \phi(y)] = \Delta(x-y) = \int \frac{d^3p}{(2\pi)^3 2p^0} \left(e^{ip\cdot(x-y)} - e^{-ip\cdot(x-y)}\right). \tag{4}$$

At equal times, one has

$$\begin{aligned} \Delta(\vec{x}-\vec{y}, 0) = 0, \quad \frac{\partial}{\partial x^0} \Delta(x-y)|_{x^0=y^0} = -i\,\delta^3(\vec{x}-\vec{y}), \quad \text{and} \\ \frac{\partial^2}{\partial x^0 \partial y^0} \Delta(x-y)|_{x^0=y^0} = 0. \end{aligned} \tag{5}$$

So a real field ϕ and its time derivative $\dot{\phi}$ satisfy the equal-time commutation relations (2)

$$\begin{aligned} &[\phi(\vec{x}, t), \dot{\phi}(\vec{y}, t)]_- = i\,\delta^3(\vec{x}-\vec{y}), \\ &[\phi(\vec{x}, t), \phi(\vec{y}, t)]_- = 0, \quad \text{and} \quad [\dot{\phi}(\vec{x}, t), \dot{\phi}(\vec{y}, t)]_- = 0. \end{aligned} \tag{6}$$

A complex scalar field

$$\phi(x) = \frac{1}{\sqrt{2}}\left[\phi_1(x) + i\phi_2(x)\right] \tag{7}$$

obeys the commutation relations

$$\begin{aligned} [\phi(x), \phi^\dagger(y)]_- &= \frac{1}{2}\,[\phi_1(x) + i\phi_2(x), \phi_1(x) - i\phi_2(x)] \\ &= \frac{1}{2}\left([\phi_1(x), \phi_1(x)] + [\phi_2(x), \phi_2(x)]\right) = \Delta(x-y) \end{aligned} \tag{8}$$

and

$$\begin{aligned} [\phi(x), \phi(y)]_- &= \frac{1}{2}\,[\phi_1(x) + i\phi_2(x), \phi_1(x) + i\phi_2(x)] \\ &= \frac{1}{2}\left([\phi_1(x), \phi_1(x)] - [\phi_2(x), \phi_2(x)]\right) = 0. \end{aligned} \tag{9}$$

So the complex scalar fields $\phi(x)$ and $\dot{\phi}^\dagger(x)$ obey the equal-time commutation relations (6)

$$[\phi(\vec{x}, t), \dot{\phi}^\dagger(\vec{y}, t)]_- = i\,\delta^3(\vec{x}-\vec{y}) \quad \text{and} \quad [\phi(\vec{x}, t), \phi^\dagger(\vec{y}, t)]_- = 0 \tag{10}$$

$$[\dot{\phi}(\vec{x}, t), \dot{\phi}(\vec{y}, t)] = 0 \quad \text{and} \quad [\dot{\phi}^\dagger(\vec{x}, t), \dot{\phi}^\dagger(\vec{y}, t)] = 0 \tag{11}$$

$$[\phi(\vec{x}, t), \phi(\vec{y}, t)] = 0 \quad \text{and} \quad [\phi^\dagger(\vec{x}, t), \phi^\dagger(\vec{y}, t)] = 0. \tag{12}$$