How Fields Transform

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1 States

A Lorentz transformation Λ is implemented by a unitary operator $U(\Lambda)$ which replaces the state $|p,\sigma\rangle$ of a massive particle of momentum p and spin σ along the z-axis by the state

$$U(\Lambda)|p,\sigma\rangle = \sqrt{\frac{(\Lambda p)^0}{p^0}} \sum_{s'} D^{(j)}_{s'\sigma}(W(\Lambda,p))\,|\Lambda p, s'\rangle \tag{1}$$

where $W(\Lambda,p)$ is a Wigner rotation

$$W(\Lambda,p) = L^{-1}(\Lambda p)\Lambda L(p) \tag{2}$$

and $L(p)$ is a standard Lorentz transformation that takes $(m,\vec{0})$ to p.

2 Creation operators

The vacuum is invariant under Lorentz transformations and translations

$$U(\Lambda,a)|0\rangle = |0\rangle. \tag{3}$$

A creation operator $a^\dagger(p,\sigma)$ makes the state $|p,\sigma\rangle$ from the vacuum state $|0\rangle$

$$|p\sigma\rangle = a^\dagger(p,\sigma)|0\rangle. \tag{4}$$

The creation and annihilation operators obey either the commutation relation

$$[a(p,s), a^\dagger(p',s')]_- = a(p,s)\,a^\dagger(p',s') - a^\dagger(p',s')\,a(p,s) = \delta_{ss'}\,\delta^{(3)}(\boldsymbol{p}-\boldsymbol{p}') \tag{5}$$

or the anticommutation relation

$$[a(p,s), a^\dagger(p',s')]_+ = a(p,s)\,a^\dagger(p',s') + a^\dagger(p',s')\,a(p,s) = \delta_{ss'}\,\delta^{(3)}(\boldsymbol{p}-\boldsymbol{p}'). \tag{6}$$

The two kinds of relations are written together as

$$[a(p,s), a^\dagger(p',s')]_\mp = a(p,s)\, a^\dagger(p',s') \mp a^\dagger(p',s')\, a(p,s) = \delta_{ss'}\, \delta^{(3)}(\boldsymbol{p}-\boldsymbol{p}'). \tag{7}$$

A bracket $[A,B]$ with no signed subscript is interpreted as a commutator.

Equations (1 & 4) give

$$U(\Lambda)a^\dagger(p,\sigma)|0\rangle = \sqrt{\frac{(\Lambda p)^0}{p^0}} \sum_{s'} D^{(j)}_{s'\sigma}(W(\Lambda,p))\, a^\dagger(\Lambda p, s')|0\rangle. \tag{8}$$

And (3) gives

$$U(\Lambda)a^\dagger(p,\sigma)U^{-1}(\Lambda)|0\rangle = \sqrt{\frac{(\Lambda p)^0}{p^0}} \sum_{s'} D^{(j)}_{s'\sigma}(W(\Lambda,p))\, a^\dagger(\Lambda p, s')|0\rangle. \tag{9}$$

SW in chapter 4 concludes that

$$U(\Lambda)a^\dagger(p,\sigma)U^{-1}(\Lambda) = \sqrt{\frac{(\Lambda p)^0}{p^0}} \sum_{s'} D^{(j)}_{s'\sigma}(W(\Lambda,p))\, a^\dagger(\Lambda p, s'). \tag{10}$$

If $U(\Lambda,b)$ follows Λ by a translation by b, then

$$\begin{aligned} U(\Lambda,b)a^\dagger(p,\sigma)U^{-1}(\Lambda,b) &= e^{-i(\Lambda p)\cdot a}\sqrt{\frac{(\Lambda p)^0}{p^0}} \sum_{s'} D^{(j)}_{s'\sigma}(W(\Lambda,p))\, a^\dagger(\Lambda p, s') \\ &= e^{-i(\Lambda p)\cdot a}\sqrt{\frac{(\Lambda p)^0}{p^0}} \sum_{s'} D^{\dagger(j)}_{s'\sigma}(W^{-1}(\Lambda,p))\, a^\dagger(\Lambda p, s') \\ &= e^{-i(\Lambda p)\cdot a}\sqrt{\frac{(\Lambda p)^0}{p^0}} \sum_{s'} D^{*(j)}_{\sigma s'}(W^{-1}(\Lambda,p))\, a^\dagger(\Lambda p, s') \end{aligned} \tag{11}$$

The adjoint of this equation is

$$\begin{aligned} U(\Lambda,b)a(p,\sigma)U^{-1}(\Lambda,b) &= e^{i(\Lambda p)\cdot a}\sqrt{\frac{(\Lambda p)^0}{p^0}} \sum_{s'} D^{*(j)}_{s'\sigma}(W(\Lambda,p))\, a(\Lambda p, s') \\ &= e^{i(\Lambda p)\cdot a}\sqrt{\frac{(\Lambda p)^0}{p^0}} \sum_{s'} D^{\dagger(j)}_{\sigma s'}(W(\Lambda,p))\, a(\Lambda p, s') \\ &= e^{i(\Lambda p)\cdot a}\sqrt{\frac{(\Lambda p)^0}{p^0}} \sum_{s'} D^{(j)}_{\sigma s'}(W^{-1}(\Lambda,p))\, a(\Lambda p, s'). \end{aligned} \tag{12}$$

These equations (11 & 12) are (5.1.11 & 5.1.12) of SW.

3 How fields transform

The "positive frequency" part of a field is a linear combination of annihilation operators

$$\psi_\ell^+(x) = \sum_\sigma \int d^3p \, u_\ell(x;p,\sigma)\, a(p,\sigma). \tag{13}$$

The "negative frequency" part of a field is a linear combination of creation operators of the antiparticles

$$\psi_\ell^-(x) = \sum_\sigma \int d^3p \, v_\ell(x;p,\sigma)\, b^\dagger(p,\sigma). \tag{14}$$

To have the fields (13 & 14) transform properly under Poincaré transformations

$$\begin{aligned}
U(\Lambda,a)\psi_\ell^+(x)U^{-1}(\Lambda,a) &= \sum_{\bar\ell} D_{\ell\bar\ell}(\Lambda^{-1})\psi_{\bar\ell}^+(\Lambda x + a) \\
&= \sum_{\bar\ell} D_{\ell\bar\ell}(\Lambda^{-1}) \sum_\sigma \int d^3p \, u_{\bar\ell}(\Lambda x + a; p,\sigma)\, a(p,\sigma) \\
U(\Lambda,a)\psi_\ell^-(x)U^{-1}(\Lambda,a) &= \sum_{\bar\ell} D_{\ell\bar\ell}(\Lambda^{-1})\psi_{\bar\ell}^-(\Lambda x + a) \\
&= \sum_{\bar\ell} D_{\ell\bar\ell}(\Lambda^{-1} \sum_\sigma \int d^3p \, v_{\bar\ell}(\Lambda x + a; p,\sigma)\, b^\dagger(p,\sigma)
\end{aligned} \tag{15}$$

the spinors $u_\ell(x;p,\sigma)$ and $v_\ell(x;p,\sigma)$ must obey certain rules which we'll now determine.

First (12 & 13) give

$$\begin{aligned}
U(\Lambda,a)\psi_\ell^+(x)U^{-1}(\Lambda,a) &= U(\Lambda,a)\sum_\sigma \int d^3p \, u_\ell(x;p,\sigma)\, a(p,\sigma)U^{-1}(\Lambda,a) \\
&= \sum_\sigma \int d^3p \, u_\ell(x;p,\sigma)\, U(\Lambda,a)a(p,\sigma)U^{-1}(\Lambda,a) \\
&= \sum_\sigma \int d^3p \, u_\ell(x;p,\sigma)\, e^{i(\Lambda p)\cdot a}\sqrt{\frac{(\Lambda p)^0}{p^0}} \sum_{s'} D^{(j)}_{\sigma s'}(W^{-1}(\Lambda,p))\, a(\Lambda p, s').
\end{aligned} \tag{16}$$

Now we use the identity

$$\frac{d^3p}{p^0} = \frac{d^3(\Lambda p)}{(\Lambda p)^0} \tag{17}$$

to turn (16) into

$$\begin{aligned}
U(\Lambda,a)\psi_\ell^+(x)U^{-1}(\Lambda,a) = \sum_\sigma \int & d^3(\Lambda p)\, u_\ell(x;p,\sigma)\, e^{i(\Lambda p)\cdot a} \\
& \times \sqrt{\frac{p^0}{(\Lambda p)^0}} \sum_{s'} D^{(j)}_{\sigma s'}(W^{-1}(\Lambda,p))\, a(\Lambda p, s').
\end{aligned} \tag{18}$$

Similarly (11, 14, & 17) give

$$
\begin{aligned}
U(\Lambda, a)\psi_\ell^-(x)U^{-1}(\Lambda, a) &= U(\Lambda, a)\sum_\sigma \int d^3p\, v_\ell(x; p, \sigma)\, b^\dagger(p, \sigma)U^{-1}(\Lambda, a) \\
&= \sum_\sigma \int d^3p\, v_\ell(x; p, \sigma)\, U(\Lambda, a)b^\dagger(p, \sigma)U^{-1}(\Lambda, a) \\
&= \sum_\sigma \int d^3p\, v_\ell(x; p, \sigma)\, e^{-i(\Lambda p)\cdot a}\sqrt{\frac{(\Lambda p)^0}{p^0}}\sum_{s'} D^{*(j)}_{\sigma s'}(W^{-1}(\Lambda, p))\, b^\dagger(\Lambda p, s') \\
&= \sum_\sigma \int d^3(\Lambda p)\, v_\ell(x; p, \sigma)\, e^{-i(\Lambda p)\cdot a}\sqrt{\frac{p^0}{(\Lambda p)^0}}\sum_{s'} D^{*(j)}_{\sigma s'}(W^{-1}(\Lambda, p))\, b^\dagger(\Lambda p, s').
\end{aligned}
\tag{19}
$$

So to get the fields to transform as in (15), equations (18 & 19) say that we need

$$
\begin{aligned}
\sum_{\bar\ell} D_{\ell\bar\ell}(\Lambda^{-1})\psi^+_{\bar\ell}(\Lambda x + a) &= \sum_{\bar\ell} D_{\ell\bar\ell}(\Lambda^{-1})\sum_\sigma \int d^3p\, u_{\bar\ell}(\Lambda x + a; p, \sigma)\, a(p, \sigma) \\
&= \sum_{\bar\ell} D_{\ell\bar\ell}(\Lambda^{-1})\sum_\sigma \int d^3(\Lambda p)\, u_{\bar\ell}(\Lambda x + a; \Lambda p, \sigma)\, a(\Lambda p, \sigma) \\
&= \sum_\sigma \int d^3(\Lambda p)\, u_\ell(x; p, \sigma)\, e^{i(\Lambda p)\cdot a} \\
&\quad \times \sqrt{\frac{p^0}{(\Lambda p)^0}}\sum_{s'} D^{(j)}_{\sigma s'}(W^{-1}(\Lambda, p))\, a(\Lambda p, s') \\
&= \sum_{s'} \int d^3(\Lambda p)\, u_\ell(x; p, s')\, e^{i(\Lambda p)\cdot a} \\
&\quad \times \sqrt{\frac{p^0}{(\Lambda p)^0}}\sum_{\sigma} D^{(j)}_{s'\sigma}(W^{-1}(\Lambda, p))\, a(\Lambda p, \sigma)
\end{aligned}
\tag{20}
$$

and

$$
\begin{aligned}
\sum_{\bar\ell} D_{\ell\bar\ell}(\Lambda^{-1})\psi^-_{\bar\ell}(\Lambda x+a) &= \sum_{\bar\ell} D_{\ell\bar\ell}(\Lambda^{-1}) \sum_\sigma \int d^3p\, v_{\bar\ell}(\Lambda x+a;p,\sigma)\, b^\dagger(p,\sigma) \\
&= \sum_{\bar\ell} D_{\ell\bar\ell}(\Lambda^{-1}) \sum_\sigma \int d^3(\Lambda p)\, v_{\bar\ell}(\Lambda x+a;\Lambda p,\sigma)\, b^\dagger(\Lambda p,\sigma) \\
&= \sum_\sigma \int d^3(\Lambda p)\, v_\ell(x;p,\sigma)\, e^{-i(\Lambda p)\cdot a} \qquad (21) \\
&\quad \times \sqrt{\frac{p^0}{(\Lambda p)^0}} \sum_{s'} D^{*(j)}_{\sigma s'}(W^{-1}(\Lambda,p))\, b^\dagger(\Lambda p,s') \\
&= \sum_{s'} \int d^3(\Lambda p)\, v_\ell(x;p,s')\, e^{-i(\Lambda p)\cdot a} \\
&\quad \times \sqrt{\frac{p^0}{(\Lambda p)^0}} \sum_{\sigma} D^{*(j)}_{s'\sigma}(W^{-1}(\Lambda,p))\, b^\dagger(\Lambda p,\sigma).
\end{aligned}
$$

Equating coefficients of the red annihilation and blue creation operators, we find that the fields will transform properly if the spinors u and v satisfy the rules

$$
\sum_{\bar\ell} D_{\ell\bar\ell}(\Lambda^{-1})\, u_{\bar\ell}(\Lambda x+a;\Lambda p,\sigma) = \sqrt{\frac{p^0}{(\Lambda p)^0}} \sum_{s'} D^{(j)}_{s'\sigma}(W^{-1}(\Lambda,p)) u_\ell(x;p,s')\, e^{i(\Lambda p)\cdot a} \qquad (22)
$$

$$
\sum_{\bar\ell} D_{\ell\bar\ell}(\Lambda^{-1}) v_{\bar\ell}(\Lambda x+a;\Lambda p,\sigma) = \sqrt{\frac{p^0}{(\Lambda p)^0}} \sum_{s'} D^{*(j)}_{s'\sigma}(W^{-1}(\Lambda,p)) v_\ell(x;p,s') e^{-i(\Lambda p)\cdot a} \qquad (23)
$$

which differ from SW's by an interchange of the subscripts σ, s' on the rotation matrices $D^{(j)}$. (I think SW has a typo there.) If we multiply both sides of these equations (22 & 23) by the two kinds of D matrices, then we get first

$$
\begin{aligned}
\sum_{\bar\ell,\ell} D_{\ell'\ell}(\Lambda) D_{\ell\bar\ell}(\Lambda^{-1})\, u_{\bar\ell}(\Lambda x+a;\Lambda p,\sigma) &= u_{\ell'}(\Lambda x+a;\Lambda p,\sigma) \\
&= \sqrt{\frac{p^0}{(\Lambda p)^0}} \sum_{s',\ell} D^{(j)}_{s'\sigma}(W^{-1}(\Lambda,p)) D_{\ell'\ell}(\Lambda) u_\ell(x;p,s')\, e^{i(\Lambda p)\cdot a} \qquad (24) \\
\sum_{\bar\ell,\ell} D_{\ell'\ell}(\Lambda) D_{\ell\bar\ell}(\Lambda^{-1}) v_{\bar\ell}(\Lambda x+a;\Lambda p,\sigma) &= v_{\ell'}(\Lambda x+a;\Lambda p,\sigma) \\
&= \sqrt{\frac{p^0}{(\Lambda p)^0}} \sum_{s',\ell} D^{*(j)}_{s'\sigma}(W^{-1}(\Lambda,p)) D_{\ell'\ell}(\Lambda) v_\ell(x;p,s') e^{-i(\Lambda p)\cdot a} \qquad (25)
\end{aligned}
$$

and then with $W \equiv W(\Lambda, p)$

$$
\begin{aligned}
&\sum_\sigma D^{(j)}_{\sigma \bar{s}}(W) u_{\ell'}(\Lambda x + a; \Lambda p, \sigma) \\
&\qquad = \sqrt{\frac{p^0}{(\Lambda p)^0}} \sum_{s', \sigma, \ell} D^{(j)}_{s' \sigma}(W^{-1}) D^{(j)}_{\sigma \bar{s}}(W) D_{\ell' \ell}(\Lambda) u_\ell(x; p, s') \, e^{i(\Lambda p) \cdot a} \\
&\qquad = \sqrt{\frac{p^0}{(\Lambda p)^0}} \sum_{\ell} D_{\ell' \ell}(\Lambda) u_\ell(x; p, \bar{s}) \, e^{i(\Lambda p) \cdot a}
\end{aligned} \tag{26}
$$

$$
\begin{aligned}
&\sum_\sigma D^{*(j)}_{\sigma \bar{s}}(W) v_{\ell'}(\Lambda x + a; \Lambda p, \sigma) \\
&\qquad = \sqrt{\frac{p^0}{(\Lambda p)^0}} \sum_{\sigma, s', \ell} D^{*(j)}_{s' \sigma}(W^{-1}) D^{*(j)}_{\sigma \bar{s}}(W) D_{\ell' \ell}(\Lambda) v_\ell(x; p, s') e^{-i(\Lambda p) \cdot a} \\
&\qquad = \sqrt{\frac{p^0}{(\Lambda p)^0}} \sum_{\ell} D_{\ell' \ell}(\Lambda) v_\ell(x; p, \bar{s}) e^{-i(\Lambda p) \cdot a}
\end{aligned} \tag{27}
$$

which are equations (5.1.13 & 5.1.14) of SW:

$$
\begin{aligned}
\sum_{\bar{s}} u_{\bar{\ell}}(\Lambda x + a; \Lambda p, \bar{s}) D^{(j)}_{\bar{s} \sigma}(W(\Lambda, p)) &= \sqrt{\frac{p^0}{(\Lambda p)^0}} \sum_\ell D_{\bar{\ell} \ell}(\Lambda) u_\ell(x; p, \sigma) \, e^{i(\Lambda p) \cdot a} \\
\sum_{\bar{s}} v_{\bar{\ell}}(\Lambda x + a; \Lambda p, \bar{s}) D^{*(j)}_{\bar{s} \sigma}(W(\Lambda, p)) &= \sqrt{\frac{p^0}{(\Lambda p)^0}} \sum_\ell D_{\bar{\ell} \ell}(\Lambda) v_\ell(x; p, \sigma) e^{-i(\Lambda p) \cdot a}.
\end{aligned} \tag{28}
$$

These are the equations that determine the spinors u and v up to a few arbitrary phases.

4 Translations

When $\Lambda = I$, the D matrices are equal to unity, and these last equations (28) say that for $x = 0$

$$
\begin{aligned}
u_\ell(a; p, \sigma) &= u_\ell(0; p, \sigma) \, e^{ip \cdot a} \\
v_\ell(a; p, \sigma) &= v_\ell(0; p, \sigma) e^{-ip \cdot a}.
\end{aligned} \tag{29}
$$

Thus the spinors u and v depend upon spacetime by the usual phase $e^{\pm ip \cdot x}$

$$
\begin{aligned}
u_\ell(x; p, \sigma) &= (2\pi)^{-3/2} u_\ell(p, \sigma) \, e^{ip \cdot x} \\
v_\ell(x; p, \sigma) &= (2\pi)^{-3/2} v_\ell(p, \sigma) e^{-ip \cdot x}
\end{aligned} \tag{30}
$$

in which the 2π's are conventional. The fields therefore are Fourier transforms:

$$
\begin{aligned}
\psi_\ell^+(x) &= (2\pi)^{-3/2} \sum_\sigma \int d^3p \, e^{ip\cdot x} u_\ell(p,\sigma) \, a(p,\sigma) \\
\psi_\ell^-(x) &= (2\pi)^{-3/2} \sum_\sigma \int d^3p \, e^{-ip\cdot x} v_\ell(p,\sigma) \, b^\dagger(p,\sigma)
\end{aligned}
\tag{31}
$$

and every field of mass m obeys the Klein-Gordon equation

$$
(\nabla^2 - \partial_0^2 - m^2) \, \psi_\ell(x) = (\Box - m^2) \, \psi_\ell(x) = 0. \tag{32}
$$

Since $\exp[i(\Lambda p \cdot (\Lambda x + a))] = \exp(ip \cdot x + i\Lambda p \cdot a)$, the conditions (28) simplify to

$$
\begin{aligned}
\sum_{\bar{s}} u_{\bar{\ell}}(\Lambda p, \bar{s}) D^{(j)}_{\bar{s}\sigma}(W(\Lambda,p)) &= \sqrt{\frac{p^0}{(\Lambda p)^0}} \sum_\ell D_{\bar{\ell}\ell}(\Lambda) u_\ell(p,\sigma) \\
\sum_{\bar{s}} v_{\bar{\ell}}(\Lambda p, \bar{s}) D^{*(j)}_{\bar{s}\sigma}(W(\Lambda,p)) &= \sqrt{\frac{p^0}{(\Lambda p)^0}} \sum_\ell D_{\bar{\ell}\ell}(\Lambda) v_\ell(p,\sigma)
\end{aligned}
\tag{33}
$$

for all Lorentz transformations Λ.

5 Boosts

Set $p = k = (m, \vec{0})$ and $\Lambda = L(q)$ where $L(q)k = q$. So $L(p) = 1$ and

$$
W(\Lambda, p) \equiv L^{-1}(\Lambda p)\Lambda L(p) = L^{-1}(q) L(q) = 1. \tag{34}
$$

Then the equations (33) are

$$
\begin{aligned}
u_{\bar{\ell}}(q,\sigma) &= \sqrt{\frac{m}{q^0}} \sum_\ell D_{\bar{\ell}\ell}(L(q)) u_\ell(\vec{0},\sigma) \\
v_{\bar{\ell}}(q,\sigma) &= \sqrt{\frac{m}{q^0}} \sum_\ell D_{\bar{\ell}\ell}(L(q)) v_\ell(\vec{0},\sigma).
\end{aligned}
\tag{35}
$$

Thus a spinor at finite momentum is given by a representation $D(\Lambda)$ of the Lorentz group (see the online notes of chapter 10 of my book for its finite-dimensional nonunitary representations) acting on the spinor at zero 3-momentum $p = k = (m, \vec{0})$. We need to find what these spinors are.

6 Rotations

Now set $p = k = (m, \vec{0})$ and $\Lambda = R$ a rotation so that $W = R$. For rotations, the spinor conditions (33) are

$$\begin{aligned}\sum_{\bar{s}} u_{\bar{\ell}}(\vec{0}, \bar{s}) D^{(j)}_{\bar{s}\sigma}(R) &= \sum_{\ell} D_{\bar{\ell}\ell}(R) u_{\ell}(\vec{0}, \sigma) \\ \sum_{\bar{s}} v_{\bar{\ell}}(\vec{0}, \bar{s}) D^{*(j)}_{\bar{s}\sigma}(R) &= \sum_{\ell} D_{\bar{\ell}\ell}(R) v_{\ell}(\vec{0}, \sigma).\end{aligned} \tag{36}$$

The representations $D^{(j)}_{\bar{s}\sigma}(R)$ of the rotation group are $(2j+1) \times (2j+1)$-dimensional unitary matrices. For a rotation of angle θ about the $\vec{\theta} = \boldsymbol{\theta}$ axis, they are the ones taught in courses on quantum mechanics (and discussed in the notes of chapter 10)

$$D^{(j)}_{\bar{s}\sigma}(\boldsymbol{\theta}) = \left[e^{-i\boldsymbol{\theta}\cdot\boldsymbol{J}^{(j)}} \right]_{\bar{s}\sigma} \tag{37}$$

where $[J_a, J_b] = i\epsilon_{abc} J_c$. The representations $D_{\bar{\ell}\ell}(R)$ of the rotation group are finite-dimensional unitary matrices. For a rotation of angle θ about the $\vec{\theta} = \boldsymbol{\theta}$ axis, they are

$$D_{\bar{\ell}\ell}(\boldsymbol{\theta}) = \left[e^{-i\boldsymbol{\theta}\cdot\boldsymbol{\mathcal{J}}} \right]_{\bar{\ell}\ell} \tag{38}$$

in which $[\mathcal{J}_a, \mathcal{J}_b] = i\epsilon_{abc}\mathcal{J}$. For tiny rotations, the conditions (36) require (because of the complex conjugation of the antiparticle condition) that the spinors obey the rules

$$\begin{aligned}\sum_{\bar{s}} u_{\bar{\ell}}(\vec{0}, \bar{s}) (J^{(j)}_a)_{\bar{s}\sigma} &= \sum_{\ell} (\mathcal{J}_a)_{\bar{\ell}\ell} u_{\ell}(\vec{0}, \sigma) \\ \sum_{\bar{s}} v_{\bar{\ell}}(\vec{0}, \bar{s}) (-J_a)^{*(j)}_{\bar{s}\sigma}) &= \sum_{\ell} (\mathcal{J}_a)_{\bar{\ell}\ell} v_{\ell}(\vec{0}, \sigma)\end{aligned} \tag{39}$$

for $a = 1, 2, 3$.

7 Spin-zero fields

Spin-zero fields have no spin or Lorentz indexes. So the boost conditions (207) merely require that $u(q) = \sqrt{m/q^0}\, u(0)$ and $v(q) = \sqrt{m/q^0}\, v(0)$. The conventional normalization is $u(0) = 1/\sqrt{2m}$ and $v(0) = 1/\sqrt{2m}$. The spin-zero spinors then are

$$u(p) = (2p^0)^{-1/2} \quad \text{and} \quad v(p) = (2p^0)^{-1/2}. \tag{40}$$

For simpicity, let's first consider a neutral scalar field so that $b(p, s) = a(p, s)$. The definitions (13) and (14) of the positive-frequency and negative-frequency fields and their behavior (30) under translations then give us

$$
\begin{aligned}
\phi^+(x) &= \int \frac{d^3p}{\sqrt{(2\pi)^3 2p^0}}\, a(p)\, e^{ip\cdot x} \\
\phi^-(x) &= \int \frac{d^3p}{\sqrt{(2\pi)^3 2p^0}}\, a^\dagger(p)\, e^{-ip\cdot x}.
\end{aligned}
\tag{41}
$$

Note that

$$
\left[\phi^\pm(x)\right]^\dagger = \phi^\mp(x). \tag{42}
$$

Since $[a(p), a(p')]_\pm = 0$, it follows that

$$
[\phi^+(x), \phi^+(y)]_\mp = 0 \quad \text{and} \quad [\phi^-(x), \phi^-(y)]_\mp = 0 \tag{43}
$$

whatever the values of x and y as long as we use commutators for bosons and anticommutators for fermions.

But the commutation relation

$$
[a(p,s), a^\dagger(q,t)]_\mp = \delta_{st}\, \delta^{(3)}(\boldsymbol{p} - \boldsymbol{q}) \tag{44}
$$

makes the commutator

$$
\begin{aligned}
[\phi^+(x), \phi^-(y)]_\mp &= \int \frac{d^3p d^3p'}{(2\pi)^3 \sqrt{2p^0 2p'^0}}\, e^{ip\cdot x} e^{-ip'\cdot y}\, \delta^3(\boldsymbol{p} - \boldsymbol{p}') \\
&= \int \frac{d^3p}{(2\pi)^3 2p^0}\, e^{ip\cdot(x-y)} = \Delta_+(x-y)
\end{aligned}
\tag{45}
$$

nonzero even for $(x-y)^2 > 0$ as we'll now verify.

For space-like x, the Lorentz-invariant function $\Delta_+(x)$ can only depend upon $x^2 > 0$ since the time x^0 and its sign are not Lorentz invariant. So we choose a Lorentz frame with $x^0 = 0$ and $|\boldsymbol{x}| = \sqrt{x^2}$. In this frame,

$$
\begin{aligned}
\Delta_+(x) &= \int \frac{d^3p}{(2\pi)^3 2\sqrt{\boldsymbol{p}^2 + m^2}}\, e^{i\boldsymbol{p}\cdot\boldsymbol{x}} \\
&= \int \frac{p^2 dp\ d\cos\theta}{(2\pi)^2 2\sqrt{\boldsymbol{p}^2 + m^2}}\, e^{ipx\cos\theta}
\end{aligned}
\tag{46}
$$

where $p = |\boldsymbol{p}|$ and $x = |\boldsymbol{x}|$. Now

$$
\int d\cos\theta\, e^{ipx\cos\theta} = \left(e^{ipx} - e^{-ipx}\right)/(ipx) = 2\sin(px)/(px), \tag{47}
$$

so the integral (46) is

$$
\Delta_+(x) = \frac{1}{4\pi^2 x} \int_0^\infty \frac{\sin(px)\, p dp}{\sqrt{p^2 + m^2}} \tag{48}
$$

with $u \equiv p/m$

$$\Delta_+(x) = \frac{m}{4\pi^2 x} \int_0^\infty \frac{\sin(mxu)\, udu}{\sqrt{\boldsymbol{u}^2+1}} = \frac{m}{4\pi^2 x}\, K_1(mx^2) \tag{49}$$

a Hankel function.

To get a Lorentz-invariant, causal theory, we use the arbitrary parameters κ and λ setting

$$\phi(x) = \kappa\phi^+(x) + \lambda\phi^-(x) \tag{50}$$

Now the adjoint rule (42) and the commutation relations (45 and 45) give

$$\begin{aligned}
[\phi(x), \phi^\dagger(y)]_\mp &= [\kappa\phi^+(x) + \lambda\phi^-(x), \kappa^*\phi^-(y) + \lambda\phi^+(y)]_\mp \\
&= |\kappa|^2[\phi^+(x), \phi^-(y)]_\mp + |\lambda|^2[\phi^-(x), \phi^+(y)]_\mp \\
&= |\kappa|^2\, \Delta_+(x-y) \mp |\lambda|^2\, \Delta_+(y-x) \\
[\phi(x), \phi(y)]_\mp &= [\kappa\phi^+(x) + \lambda\phi^-(x), \kappa\phi^+(y) + \lambda\phi^-(y)]_\mp \\
&= \kappa\lambda\left([\phi^+(x), \phi^-(y)]_\mp + [\phi^-(x), \phi^+(y)]_\mp\right) \\
&= \kappa\lambda\left(\Delta_+(x-y) \mp \Delta_+(y-x)\right).
\end{aligned} \tag{51}$$

But when $(x-y)^2 > 0$, $\Delta_+(x-y) = \Delta_+(y-x)$. Thus these conditions are

$$\begin{aligned}
[\phi(x), \phi^\dagger(y)]_\mp &= \left(|\kappa|^2 \mp |\lambda|^2\right)\, \Delta_+(x-y) \\
[\phi(x), \phi(y)]_\mp &= \kappa\lambda\Delta_+(x-y)(1 \mp 1).
\end{aligned} \tag{52}$$

The first of these equations implies that we choose the minus sign and so that we use commutation relations and not anticommutation relations for spin-zero fields. This is the **spin-statistics theorem** for spin-zero fields. SW proves the theorem for arbitrary massive fields in section 5.7.

We also must set

$$|\kappa| = |\lambda|. \tag{53}$$

The second equation then is automatically satisfied. The common magnitude and the phases of κ and λ are arbitrary, so we choose $\kappa = \lambda = 1$. We then have

$$\phi(x) = \phi^+(x) + \phi^-(x) = \phi^+(x) + \phi^{+\dagger}(x) = \phi^\dagger(x). \tag{54}$$

Now the interaction density $\mathcal{H}(x)$ will commute with $\mathcal{H}(y)$ for $(x-y)^2 > 0$, and we have a chance of having a Lorentz-invariant, causal theory.

8 Conserved charges

If the field ϕ adds and deletes charged particles, an interaction $\mathcal{H}(x)$ that is a polynomial in ϕ will not commute with the charge operator Q because ϕ^+ will lower the charge and

ϕ^- will raise it. The standard way to solve this problem is to start with two hermitian fields ϕ_1 and ϕ_2 of the same mass. One defines a complex scalar field as a complex linear combination of the two fields

$$\begin{aligned}\phi(x) &= \frac{1}{\sqrt{2}}\left(\phi_1(x) + i\phi_2(x)\right) \\ &= \int \frac{d^3p}{\sqrt{(2\pi)^3 2p^0}} \left[\frac{1}{\sqrt{2}}\left(a_1(p) + i a_2(p)\right) e^{ip\cdot x} + \frac{1}{\sqrt{2}}\left(a_1^\dagger(p) + i a_2^\dagger(p)\right) e^{-ip\cdot x}\right].\end{aligned} \tag{55}$$

Setting

$$a(p) = \frac{1}{\sqrt{2}}\left(a_1(p) + i a_2(p)\right) \quad \text{and} \quad b^\dagger(p) = \frac{1}{\sqrt{2}}\left(a_1^\dagger(p) + i a_2^\dagger(p)\right) \tag{56}$$

so that

$$b(p) = \frac{1}{\sqrt{2}}\left(a_1(p) - i a_2(p)\right) \quad \text{and} \quad a^\dagger(p) = \frac{1}{\sqrt{2}}\left(a_1^\dagger(p) - i a_2^\dagger(p)\right) \tag{57}$$

we have

$$\phi(x) = \int \frac{d^3p}{\sqrt{(2\pi)^3 2p^0}} \left[a(p)\, e^{ip\cdot x} + b^\dagger(p)\, e^{-ip\cdot x}\right] \tag{58}$$

and

$$\phi^\dagger(x) = \int \frac{d^3p}{\sqrt{(2\pi)^3 2p^0}} \left[b(p)\, e^{ip\cdot x} + a^\dagger(p)\, e^{-ip\cdot x}\right]. \tag{59}$$

Since the commutation relations of the real creation and annihilation operators are for $i, j = 1, 2$

$$[a_i(p), a_j^\dagger(p')] = \delta_{ij}\, \delta^3(\boldsymbol{p} - \boldsymbol{p}') \quad \text{and} \quad [a_i(p), a_j(p')] = 0 = [a_i^\dagger(p), a_j^\dagger(p')] \tag{60}$$

the commutation relations of the complex creation and annihilation operators are

$$[a(p), a^\dagger(p')] = \delta^3(\boldsymbol{p} - \boldsymbol{p}') \quad \text{and} \quad [b(p), b^\dagger(p')] = \delta^3(\boldsymbol{p} - \boldsymbol{p}') \tag{61}$$

with all other commutators vanishing.

Now $\phi(x)$ lowers the charge of a state by q if $a^\dagger$ adds a particle of charge q and if $b^\dagger$ adds a particle of charge $-q$. Similarly, $\phi^\dagger(x)$ raises the charge of a state by q

$$[Q, \phi(x)] = -q\phi(x) \quad \text{and} \quad [Q, \phi^\dagger(x)] = q\phi^\dagger(x). \tag{62}$$

So an interaction with as many $\phi(x)$'s as $\phi^\dagger(x)$'s conserves charge.

9 Parity, charge conjugation, and time reversal

If the unitary operator P represents parity on the creation operators

$$\mathsf{P}a_1^\dagger(\boldsymbol{p})\mathsf{P}^{-1} = \eta\, a_1^\dagger(-\boldsymbol{p}) \quad \text{and} \quad \mathsf{P}a_2^\dagger(\boldsymbol{p})\mathsf{P}^{-1} = \eta\, a_2^\dagger(-\boldsymbol{p}) \tag{63}$$

with the same phase η. Then

$$\mathsf{P}a_1(\boldsymbol{p})\mathsf{P}^{-1} = \eta^* a_1(-\boldsymbol{p}) \quad \text{and} \quad \mathsf{P}a_2(\boldsymbol{p})\mathsf{P}^{-1} = \eta^* a_2(-\boldsymbol{p}) \tag{64}$$

and so both

$$\mathsf{P}a^\dagger(\boldsymbol{p})\mathsf{P}^{-1} = \eta a^\dagger(-\boldsymbol{p}) \quad \text{and} \quad \mathsf{P}a(\boldsymbol{p})\mathsf{P}^{-1} = \eta^* a(-\boldsymbol{p}) \tag{65}$$

and

$$\mathsf{P}b^\dagger(\boldsymbol{p})\mathsf{P}^{-1} = \eta b^\dagger(-\boldsymbol{p}) \quad \text{and} \quad \mathsf{P}b(\boldsymbol{p})\mathsf{P}^{-1} = \eta^* b(-\boldsymbol{p}). \tag{66}$$

Thus if the field

$$\phi_1(x) = \int \frac{d^3p}{\sqrt{(2\pi)^3 2p^0}} \left[a_1(p)\, e^{ip\cdot x} + a_1^\dagger(p)\, e^{-ip\cdot x} \right] \tag{67}$$

or $\phi_2(x)$, or the complex field (58) is to go into a multiple of itself under parity, then we need $\eta = \eta^*$ so that η is real. Then the fields transform under parity as

$$\begin{aligned} \mathsf{P}\phi_1(x)\mathsf{P}^{-1} &= \eta^* \phi_1(x^0, -\boldsymbol{x}) = \eta\phi_1(x^0, -\boldsymbol{x}) \\ \mathsf{P}\phi_2(x)\mathsf{P}^{-1} &= \eta^* \phi(x^0, -\boldsymbol{x}) = \eta\phi_2(x^0, -\boldsymbol{x}) \\ \mathsf{P}\phi(x)\mathsf{P}^{-1} &= \eta^* \phi(x^0, -\boldsymbol{x}) = \eta\phi(x^0, -\boldsymbol{x}). \end{aligned} \tag{68}$$

Since $\mathsf{P}^2 = I$, we must have $\eta = \pm 1$. SW allows for a more general phase by having parity act with the same phase on a and $b^\dagger$. Both schemes imply that the parity of a hermitian field is ± 1 and that the state

$$|ab\rangle = \int d^3p\, f(\boldsymbol{p}^2)\, a^\dagger(\boldsymbol{p})\, b^\dagger(-\boldsymbol{p})\, |0\rangle \tag{69}$$

has even or positive parity, $\mathsf{P}|ab\rangle = |ab\rangle$.

Charge conjugation works similarly. If the unitary operator C represents charge conjugation on the creation operators

$$\mathsf{C}a_1^\dagger(\boldsymbol{p})\mathsf{C}^{-1} = \xi a_1^\dagger(\boldsymbol{p}) \quad \text{and} \quad \mathsf{C}a_2^\dagger(\boldsymbol{p})\mathsf{C}^{-1} = -\,\xi a_2^\dagger(\boldsymbol{p}) \tag{70}$$

with the same phase ξ. Then

$$\mathsf{C}a_1(\boldsymbol{p})\mathsf{C}^{-1} = \xi^* a_1(\boldsymbol{p}) \quad \text{and} \quad \mathsf{C}a_2(\boldsymbol{p})\mathsf{C}^{-1} = -\,\xi^* a_2(\boldsymbol{p}) \tag{71}$$

and so since $a = (a_1 + ia_2)/\sqrt{2}$ and $b = (a_1 - ia_2)/\sqrt{2}$

$$\mathsf{C}a(\boldsymbol{p})\mathsf{C}^{-1} = \xi^*\, b(\boldsymbol{p}) \quad \text{and} \quad \mathsf{C}b(\boldsymbol{p})\mathsf{C}^{-1} = \xi^* a(\boldsymbol{p}) \tag{72}$$

and since $a^\dagger = (a_1^\dagger - ia_2^\dagger)/\sqrt{2}$ and $b^\dagger = (a_1^\dagger + ia_2^\dagger)/\sqrt{2}$

$$\mathsf{C}a^\dagger(\boldsymbol{p})\mathsf{C}^{-1} = \xi\, b^\dagger(\boldsymbol{p}) \quad \text{and} \quad \mathsf{C}b^\dagger(\boldsymbol{p})\mathsf{C}^{-1} = \xi a^\dagger(\boldsymbol{p}). \tag{73}$$

Thus under charge conjugation, the field (58) becomes

$$\mathsf{C}\phi(x)\mathsf{C}^{-1} = \int \frac{d^3p}{\sqrt{(2\pi)^3 2p^0}} \left[\, \xi^*\, b(p)\, e^{ip\cdot x} + \xi\, a^\dagger(p)\, e^{-ip\cdot x}\right] \tag{74}$$

and so if it is to go into a multiple of itself or of its adjoint under charge conjugation then we need $\xi = \xi^*$ so that ξ is real. We then get

$$\mathsf{C}\phi(x)\mathsf{C}^{-1} = \xi^*\, \phi^\dagger(x) = \xi\, \phi^\dagger(x). \tag{75}$$

Since $\mathsf{C}^2 = I$, we must have $\xi = \pm 1$. SW allows for a more general phase by having charge conjugation act with the same phase on a and $b^\dagger$. Both schemes imply that the charge-conjugation parity of a hermitian field is ± 1 and that the state

$$|ab\rangle = \int d^3p\, f(\boldsymbol{p}^2)\, a^\dagger(\boldsymbol{p})\, b^\dagger(\boldsymbol{p})\, |0\rangle \tag{76}$$

has even or positive charge-conjugation parity, $\mathsf{C}|ab\rangle = |ab\rangle$.

The time-reversal operator T is antilinear and antiunitary. So if

$$\begin{aligned} \mathsf{T}a_1(\boldsymbol{p})\mathsf{T}^{-1} &= \zeta^* a_1(-\boldsymbol{p}) \quad \text{and} \quad \mathsf{T}a_2(\boldsymbol{p})\mathsf{T}^{-1} = -\,\zeta^* a_2(-\boldsymbol{p}) \\ \mathsf{T}a_1^\dagger(\boldsymbol{p})\mathsf{T}^{-1} &= \zeta a_1^\dagger(-\boldsymbol{p}) \quad \text{and} \quad \mathsf{T}a_2^\dagger(\boldsymbol{p})\mathsf{T}^{-1} = -\,\zeta a_2^\dagger(-\boldsymbol{p}) \end{aligned} \tag{77}$$

then

$$\begin{aligned} \mathsf{T}a(\boldsymbol{p})\mathsf{T}^{-1} &= \mathsf{T}\frac{1}{\sqrt{2}}\big(a_1(\boldsymbol{p}) + ia_2(\boldsymbol{p})\big)\mathsf{T}^{-1} = \frac{1}{\sqrt{2}}\big(\mathsf{T}a_1(\boldsymbol{p})\mathsf{T}^{-1} - i\mathsf{T}a_2(\boldsymbol{p})\mathsf{T}^{-1}\big) \\ &= \zeta^* \frac{1}{\sqrt{2}}\big(a_1(-\boldsymbol{p}) + ia_2(-\boldsymbol{p})\big) = \zeta^*\, a(-\boldsymbol{p}) \end{aligned} \tag{78}$$

and

$$\begin{aligned} \mathsf{T}b^\dagger(\boldsymbol{p})\mathsf{T}^{-1} &= \mathsf{T}\frac{1}{\sqrt{2}}\big(a_1^\dagger(\boldsymbol{p}) + ia_2^\dagger(\boldsymbol{p})\big)\mathsf{T}^{-1} = \frac{1}{\sqrt{2}}\big(\mathsf{T}a_1^\dagger(\boldsymbol{p})\mathsf{T}^{-1} - i\mathsf{T}a_2^\dagger(\boldsymbol{p})\mathsf{T}^{-1}\big) \\ &= \zeta \frac{1}{\sqrt{2}}\big(a_1^\dagger(-\boldsymbol{p}) + ia_2^\dagger(-\boldsymbol{p})\big) = \zeta\, b^\dagger(-\boldsymbol{p}) \end{aligned} \tag{79}$$

then one has

$$
\begin{aligned}
\mathsf{T}\phi(x)\mathsf{T}^{-1} &= \mathsf{T}\int \frac{d^3p}{\sqrt{(2\pi)^3 2p^0}} \left[a(p)\, e^{ip\cdot x} + b^\dagger(p)\, e^{-ip\cdot x} \right] \mathsf{T}^{-1} \\
&= \int \frac{d^3p}{\sqrt{(2\pi)^3 2p^0}} \left[\mathsf{T}a(p)\mathsf{T}^{-1}\, e^{-ip\cdot x} + \mathsf{T}b^\dagger(p)\mathsf{T}^{-1}\, e^{ip\cdot x} \right] \\
&= \int \frac{d^3p}{\sqrt{(2\pi)^3 2p^0}} \left[\zeta^*\, a(-\boldsymbol{p})\, e^{-ip\cdot x} + \zeta\, b^\dagger(-\boldsymbol{p})\, e^{ip\cdot x} \right].
\end{aligned} \tag{80}
$$

So if ζ is real, then after replacing $-\boldsymbol{p}$ by $\boldsymbol{p}$, we get

$$
\begin{aligned}
\mathsf{T}\phi(x)\mathsf{T}^{-1} &= \zeta^* \int \frac{d^3p}{\sqrt{(2\pi)^3 2p^0}} \left[a(\boldsymbol{p})\, e^{ip^0x^0 + i\boldsymbol{p}\cdot\boldsymbol{x}} + b^\dagger(\boldsymbol{p})\, e^{-ip^0x^0 + i\boldsymbol{p}\cdot\boldsymbol{x}} \right] \\
&= \zeta^*\phi(-x^0, \boldsymbol{x}) = \zeta\phi(-x^0, \boldsymbol{x}).
\end{aligned} \tag{81}
$$

Since $\mathsf{T}^2 = I$, the phase $\zeta = \pm 1$. SW lets ζ be complex but defined only for complex scalar fields and not for their real and imaginary parts.

10 Vector fields

Vector fields transform like the 4-vector x^i of spacetime. So

$$
D_{\bar{\ell}\ell}(\Lambda) = \Lambda^{\bar{\ell}}{}_{\ell} \tag{82}
$$

for $\bar{\ell}, \ell = 0, 1, 2, 3$. Again we start with a hermitian field labelled by $i = 0, 1, 2, 3$

$$
\begin{aligned}
\phi^{+i}(x) &= (2\pi)^{-3/2} \sum_s \int d^3p\, e^{ip\cdot x} u^i(p,s)\, a(p,s) \\
\phi^{-i}(x) &= (2\pi)^{-3/2} \sum_s \int d^3p\, e^{-ip\cdot x} v^i(p,s)\, a^\dagger(p,s).
\end{aligned} \tag{83}
$$

The boost conditions (207) say that

$$
\begin{aligned}
u^i(p,s) &= \sqrt{\frac{m}{p^0}} \sum_k L(p)^i{}_k u^k(\vec{0}, s) \\
v^i(p,s) &= \sqrt{\frac{m}{p^0}} \sum_k L(p)^i{}_k v^k(\vec{0}, s).
\end{aligned} \tag{84}
$$

The rotation conditions (39) give

$$
\begin{aligned}
\sum_{\bar{s}} u^i(\vec{0}, \bar{s}) (J_a^{(j)})_{\bar{s}s} &= \sum_k (\mathcal{J}_a)^i{}_k u^k(\vec{0}, s) \\
-\sum_{\bar{s}} v^i(\vec{0}, \bar{s}) (J_a^{*(j)})_{\bar{s}s} &= \sum_k (\mathcal{J}_a)^i{}_k v^k(\vec{0}, s).
\end{aligned} \tag{85}
$$

The $(2j+1)\times(2j+1)$ matrices $(J_a^{(j)})_{\bar{s}s}$ are the generators of the $(2j+1)\times(2j+1)$ representation of the rotation group. (See my online notes on group theory.) You learned that

$$\sum_{a=1}^{3}\left[(J_a^{(j)})^2\right]_{\bar{s}s'} = \sum_{a=1}^{3}\sum_{s=-j}^{j}(J_a^{(j)})_{\bar{s}s}\,(J_a^{(j)})_{ss'} = j(j+1)\delta_{\bar{s}s'} \tag{86}$$

and that

$$J_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}, \quad J_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}0 & -i & 0\\ i & 0 & -i\\ 0 & i & 0\end{pmatrix}, \quad J_3 = \begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{pmatrix} \tag{87}$$

in courses on quantum mechanics.

For $k = 1,2,3$, the three 4×4 matrices $(\mathcal{J}_k)^i{}_j$ are the generators of rotations in the vector representation of the Lorentz group. Their nonzero components are

$$(\mathcal{J}_k)^i{}_j = -\,i\epsilon_{ijk} \tag{88}$$

for $i,j,k = 1,2,3$, while $(\mathcal{J}_k)^0{}_0 = 0$, $(\mathcal{J}_k)^0{}_j = 0$, and $(\mathcal{J}_k)^i{}_0 = 0$ for $i,j,k = 1,2,3$. So

$$(\mathcal{J}^2)^i{}_j = 2\delta^i{}_j \tag{89}$$

with $(\mathcal{J}^2)^0{}_0 = 0$, $(\mathcal{J}^2)^0{}_j = 0$, and $(\mathcal{J}^2)^i{}_0 = 0$ for $i,j = 1,2,3$. Apart from a factor of i, the $\mathcal{J}_k$'s are the 4×4 matrices $J_a = iR_a$ of my online notes on the Lorentz group

$$\mathcal{J}_1 = \begin{pmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & -i\\ 0 & 0 & i & 0\end{pmatrix} \quad \mathcal{J}_2 = \begin{pmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & i\\ 0 & 0 & 0 & 0\\ 0 & -i & 0 & 0\end{pmatrix} \quad \mathcal{J}_3 = \begin{pmatrix}0 & 0 & 0 & 0\\ 0 & 0 & -i & 0\\ 0 & i & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}. \tag{90}$$

Since $(\mathcal{J}_a)^0{}_k = 0$ for $a,k = 1,2,3$, the spin conditions (85) give for $i = 0$

$$\sum_{\bar{s}} u^0(\vec{0},\bar{s})(J_a^{(j)})_{\bar{s}s} = 0 \quad \text{and} \quad -\sum_{\bar{s}} v^0(\vec{0},\bar{s})(J_a^{*(j)})_{\bar{s}s} = 0. \tag{91}$$

Multiplying these equations from the right by $(J_a^{(j)})_{ss'}$ while summing over $a = 1,2,3$ and using the formula (86) $[(\boldsymbol{J}^{(j)})^2]_{ss'} = j(j+1)\,\delta_{ss'}$, we find

$$j(j+1)\,u^0(\vec{0},s) = 0 \quad \text{and} \quad j(j+1)\,v^0(\vec{0},s) = 0. \tag{92}$$

Thus $u^0(\vec{0},\sigma)$ and $v^0(\vec{0},\sigma)$ can be anything if the field represents particles of spin $j = 0$, but $u^0(\vec{0},\sigma)$ and $v^0(\vec{0},\sigma)$ must both vanish if the field represents particles of spin $j > 0$.

Now we set $i = 1,2,3$ in the spin conditions (85) and again multiply from the right by $(J_a^{(j)})_{ss'}$ while summing over $a = 1,2,3$ and using the formula (86) $(\boldsymbol{J}^{(j)})^2 = j(j+1)$. The Lorentz rotation matrices generate a $j = 1$ representation of the group of rotations.

$$\sum_{ka=1}^{3} (\mathcal{J}_a)^i{}_k \, (\mathcal{J}_a)^k{}_\ell = j(j+1)\, \delta^i_\ell = 2\delta^i_\ell. \tag{93}$$

So the remaining conditions on the fields are

$$\begin{aligned}
j(j+1)\, u^i(\vec{0},s') &= \sum_{\bar{s}sa} u^i(\vec{0},\bar{s})(J_a^{(j)})_{\bar{s}s}\, (J_a^{(j)})_{ss'} = \sum_{ksa} (\mathcal{J}_a)^i{}_k u^k(\vec{0},s)\, (J_a^{(j)})_{ss'} \\
&= \sum_{k\ell a} (\mathcal{J}_a)^i{}_k \, (\mathcal{J}_a)^k{}_\ell \, u^\ell(\vec{0},s') = \sum_k 2\, \delta^i_\ell \, u^\ell(\vec{0},s') = 2\, u^i(\vec{0},s') \\
j(j+1)\, v^i(\vec{0},s') &= \sum_{\bar{s}sa} v^i(\vec{0},\bar{s})(J_a^{*(j)})_{\bar{s}s}\, (J_a^{*(j)})_{ss'} = \sum_{ksa} (\mathcal{J}_a)^i{}_k v^k(\vec{0},s)\, (J_a^{(j)})_{ss'} \\
&= \sum_{k\ell a} (\mathcal{J}_a)^i{}_k \, (\mathcal{J}_a)^k{}_\ell \, v^\ell(\vec{0},s') = \sum_k 2\, \delta^i_\ell \, v^\ell(\vec{0},s') = 2\, v^i(\vec{0},s').
\end{aligned} \tag{94}$$

Thus if $j = 0$, then for $i = 1,2,3$ both $u^i(\vec{0},s)$ and $v^i(\vec{0},s)$ must vanish, while if $j > 0$, then since $j(j+1) = 2$, the spin j must be unity, $j = 1$.

11 Vector field for spin-zero particles

The only nonvanishing components are constants taken conventionally as

$$u^0(\vec{0}) = i\sqrt{m/2} \quad \text{and} \quad v^0(\vec{0}) = -\,i\sqrt{m/2}. \tag{95}$$

At finite momentum the boost conditions (207) give them as

$$u^\mu(\vec{p}) = ip^\mu/\sqrt{2p^0} \quad \text{and} \quad v^\mu(\vec{p}) = -\,ip^\mu/\sqrt{2p^0}. \tag{96}$$

The vector field $\phi^\mu(x)$ of a spin-zero particle is then the derivative of a scalar field $\phi(x)$

$$\phi^\mu(x) = \partial^\mu \phi(x) = \int \frac{d^3p}{\sqrt{(2\pi)^3 2p^0}} \left[ip^\mu \, a(p)\, e^{ip\cdot x} - ip^\mu \, b^\dagger(p)\, e^{-ip\cdot x} \right] \tag{97}$$

12 Vector field for spin-one particles

We start with the $s = 0$ spinors $u^i(\vec{0},0)$ and $v^i(\vec{0},0)$ and note that since $(J_3^{(j)})_{\bar{s}0} = 0$, the $a = 3$ rotation conditions (85) imply that

$$(\mathcal{J}_3)^i{}_k \, u^k(\vec{0},0) = iR_3\, u^i(\vec{0},0) = 0 \quad \text{and} \quad (\mathcal{J}_3)^i{}_k \, v^k(\vec{0},0) = iR_3\, v^i(\vec{0},0) = 0. \tag{98}$$

Referring back to the explicit formulas for the generators of rotations and setting $u, v = (0, x, y, z)$ we see that

$$\mathcal{J}_3\, u = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ -iy \\ x \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{99}$$

and

$$\mathcal{J}_3\, v = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ -iy \\ x \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{100}$$

Thus only the 3-component z can be nonzero. The conventional choice is

$$u^\mu(\vec{0}, 0) = v^\mu(\vec{0}, 0) = \frac{1}{\sqrt{2m}} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{101}$$

We now form the linear combinations of the rotation conditions (85) that correspond to the raising and lowering matrices $J^{(1)}_\pm = J^{(1)}_1 \pm i J^{(1)}_2$

$$J^{(1)}_+ = \sqrt{2} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad J^{(1)}_- = \sqrt{2} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \tag{102}$$

Their Lorentz counterparts are

$$\mathcal{J}^{(1)}_\pm = \mathcal{J}^{(1)}_1 \pm i \mathcal{J}^{(1)}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mp 1 \\ 0 & 0 & 0 & -i \\ 0 & \pm 1 & i & 0 \end{pmatrix}. \tag{103}$$

In these terms, the rotation conditions (85) for the $j = 1$ spinors $u^i(\vec{0}, s)$ are

$$\sum_{\bar{s}} u^i(\vec{0}, \bar{s}) (J^{(1)}_\pm)_{\bar{s}s} = \sum_k (\mathcal{J}_\pm)^i{}_k u^k(\vec{0}, s). \tag{104}$$

But

$$J^{*(1)}_1 \pm i J^{*(1)}_2 = J^{(1)}_1 \mp i J^{(1)}_2 = J_\mp. \tag{105}$$

So the rotation conditions (85) for the $j = 1$ spinors $v^i(\vec{0}, s)$ are

$$-\sum_{\bar{s}} v^i(\vec{0}, \bar{s}) (J^{(1)}_\mp)_{\bar{s}s} = \sum_k (\mathcal{J}_\pm)^i{}_k v^k(\vec{0}, s). \tag{106}$$

So for the plus sign and the choice $s = 0$, the condition (104) gives $u^i(\vec{0}, 1)$ as

$$\sum_{\bar{s}} u^i(\vec{0}, \bar{s})\, J^{(1)}_{+\bar{s}0} = \sqrt{2}\, u^i(\vec{0}, 1) = (\mathcal{J}_+)^i{}_k u^k(\vec{0}, 0) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -i \\ 0 & 1 & i & 0 \end{pmatrix} \frac{1}{\sqrt{2m}} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{107}$$

or

$$u^i(\vec{0}, 1) = \frac{1}{2\sqrt{m}} \begin{pmatrix} 0 \\ -1 \\ -i \\ 0 \end{pmatrix}. \tag{108}$$

Similarly, the minus sign and the choice $s = 0$ give for $u^i(\vec{0}, -1)$

$$\sum_{\bar{s}} u^i(\vec{0}, \bar{s})\, J^{(1)}_{-\bar{s}0} = \sqrt{2}\, u^i(\vec{0}, -1) = (\mathcal{J}_-)^i{}_k u^k(\vec{0}, 0) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -i \\ 0 & -1 & i & 0 \end{pmatrix} \frac{1}{\sqrt{2m}} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{109}$$

or

$$u^i(\vec{0}, -1) = \frac{1}{2\sqrt{m}} \begin{pmatrix} 0 \\ 1 \\ -i \\ 0 \end{pmatrix}. \tag{110}$$

The rotation condition (106) for the $j = 1$ spinors $v^i(\vec{0}, s)$ with the minus sign and the choice $s = 0$ gives

$$-\sum_{\bar{s}} v^i(\vec{0}, \bar{s}) J^{(1)}_{-\bar{s}0} = -\sqrt{2}\, v^i(\vec{0}, -1) = (\mathcal{J}_+)^i{}_k v^k(\vec{0}, 0) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -i \\ 0 & 1 & i & 0 \end{pmatrix} \frac{1}{\sqrt{2m}} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{111}$$

or

$$v^i(\vec{0}, -1) = \frac{1}{2\sqrt{m}} \begin{pmatrix} 0 \\ 1 \\ i \\ 0 \end{pmatrix}. \tag{112}$$

Similarly, the plus sign and the choice $s = 0$ give

$$-\sum_{\bar{s}} v^i(\vec{0}, \bar{s}) J^{(1)}_{+\bar{s}0} = -\sqrt{2}\, v^i(\vec{0}, 1) = (\mathcal{J}_-)^i{}_k v^k(\vec{0}, 0) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -i \\ 0 & -1 & i & 0 \end{pmatrix} \frac{1}{\sqrt{2m}} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{113}$$

or

$$v^i(\vec{0},1) = \frac{1}{2\sqrt{m}} \begin{pmatrix} 0 \\ -1 \\ i \\ 0 \end{pmatrix}. \tag{114}$$

The boost conditions (207) now give for $i,k = 0,1,2,3$

$$u^i(\vec{p},s) = v^{*i}(\vec{p},s) = \sqrt{m/p^0}\, L^i{}_k(\vec{p})\, u^k(\vec{0},s) = e^i(\vec{p},s)/\sqrt{2p^0} \tag{115}$$

where

$$e^i(\vec{p},s) = L^i{}_k(\vec{p})\, e^k(\vec{0},s) \tag{116}$$

and

$$e(\vec{0},0) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad e(\vec{0},1) = -\frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ i \\ 0 \end{pmatrix}, \quad \text{and} \quad e(\vec{0},-1) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -i \\ 0 \end{pmatrix}. \tag{117}$$

A single massive vector field is then

$$\phi^i(x) = \phi^{+i}(x) + \phi^{-i}(x) = \sum_{s=-1}^{1} \int \frac{d^3p}{\sqrt{(2\pi)^3 2p^0}} e^i(\vec{p},s)\, a(\vec{p},s) e^{ip\cdot x} + e^{*i}(\vec{p},s)\, a^\dagger(\vec{p},s) e^{-ip\cdot x}. \tag{118}$$

The commutator/anticommutator of the positive and negative frequency parts of the field is

$$[\phi^{+i}(x), \phi^{-k}(y)]_\mp = \int \frac{d^3p}{\sqrt{(2\pi)^3 2p^0}} e^{ip\cdot(x-y)}\, \Pi^{ik}(\vec{p}) \tag{119}$$

where Π is a sum of outer products of 4-vectors

$$\Pi^{ik}(\vec{p}) = \sum_{s=-1}^{1} e^i(\vec{p},s) e^{*k}(\vec{p},s). \tag{120}$$

At $\vec{p} = 0$, the matrix Π is the unit matrix on the spatial coordinates

$$\Pi(\vec{0}) = \sum_{s=-1}^{1} e^i(\vec{0},s) e^{*k}(\vec{0},s) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{121}$$

So $\Pi(\vec{p})$ is

$$\Pi(\vec{p}) = L\Pi(0)L^{\mathsf{T}} = L\eta L^{\mathsf{T}} + L \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} L^{\mathsf{T}} \tag{122}$$

or

$$\Pi(\vec{p})^{ik} = \eta^{ik} + p^i p^k/m^2. \tag{123}$$

This equation lets us write the commutator (124) in terms of the Lorentz-invariant function $\Delta_+(x-y)$ (45) as

$$\begin{aligned}
[\phi^{+i}(x), \phi^{-k}(y)]_{\mp} &= (\eta^{ik} - \partial^i \partial^k/m^2) \int \frac{d^3p}{\sqrt{(2\pi)^3 2p^0}} e^{ip\cdot(x-y)} \\
&= (\eta^{ik} - \partial^i \partial^k/m^2)\, \Delta_+(x-y).
\end{aligned} \tag{124}$$

As for a scalar field, we set

$$v^i(x) = \kappa\, \phi^{+i}(x) + \lambda\, \phi^{-i}(x) \tag{125}$$

and find for $(x-y)^2 > 0$ since $\Delta_+(x-y) = \Delta_+(y-x)$ for x, y spacelike

$$\begin{aligned}
[v(x), v^\dagger(y)]_{\mp} &= \left(|\kappa|^2 \mp |\lambda|^2\right) (\eta^{ik} - \partial^i \partial^k/m^2)\, \Delta_+(x-y) \\
[v(x), v(y)]_{\mp} &= (1 \mp 1)\, \kappa\lambda(\eta^{ik} - \partial^i \partial^k/m^2)\, \Delta_+(x-y).
\end{aligned} \tag{126}$$

So we must choose the minus sign and set $|\kappa| = |\lambda|$. So then

$$v^i(x) = v^{+i}(x) + v^{-i}(x) = v^{+i}(x) + v^{+i\dagger}(x) \tag{127}$$

is real. This is a second example of the spin-statistics theorem.

If two such fields have the same mass, then we can combine them as we combined scalar fields

$$v^i(x) = v_1^{+i}(x) + i v_2^{-i}(x). \tag{128}$$

These fields obey the Klein-Gordon equation

$$(\Box - m^2) v^i(x) = 0. \tag{129}$$

And since both

$$p^i = L^i{}_j k^j \quad \text{and} \quad e^k(\vec{p}) = L^k{}_\ell e^\ell(0) \tag{130}$$

it follows that

$$p \cdot e(\vec{p}) = k \cdot e(0) = 0. \tag{131}$$

So the field v^i also obeys the rule

$$\partial_i\, v^i(x) = 0. \tag{132}$$

These equations (131) and 132 are like those of the electromagnetic field in Lorentz gauge. But one can't get quantum electrodynamics as the $m \to 0$ limit of just any such theory. For the interaction $\mathcal{H} = J_i\, v^i$ would lead to a rate for v-boson production like

$$J_i J_k \Pi^{ik}(\vec{p}) \tag{133}$$

which diverges as $m \to 0$ because of the $p^i p^k/m^2$ term in $\Pi^{ik}(\vec{p})$. One can avoid this divergence by requiring that $\partial_i J^i = 0$ which is current conservation.

Under parity, charge conjugation, and time reversal, a vector field transforms as

$$\begin{aligned}
\mathsf{P} v^a(x) \mathsf{P}^{-1} &= -\eta^* \mathcal{P}^a{}_b v^b(\mathcal{P}x) \\
\mathsf{C} v^a(x) \mathsf{C}^{-1} &= \xi^* v^{a\dagger}(x) \\
\mathsf{T} v^a(x) \mathsf{T}^{-1} &= \zeta^* \mathcal{P}^a{}_b v^b(-\mathcal{P}x).
\end{aligned} \tag{134}$$

13 Lorentz group

The Lorentz group $O(3,1)$ is the set of all linear transformations L that leave invariant the Minkowski inner product

$$xy \equiv \boldsymbol{x} \cdot \boldsymbol{y} - x^0 y^0 = x^{\mathsf{T}} \eta y \tag{135}$$

in which η is the diagonal matrix

$$\eta = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{136}$$

So L is in $O(3,1)$ if for all 4-vectors x and y

$$(Lx)^{\mathsf{T}} \eta L\, y = x^{\mathsf{T}} L^{\mathsf{T}} \eta\, L y = x^{\mathsf{T}} \eta\, y. \tag{137}$$

Since x and y are arbitrary, this condition amounts to

$$L^{\mathsf{T}} \eta\, L = \eta. \tag{138}$$

Taking the determinant of both sides and recalling that $\det A^{\mathsf{T}} = \det A$ and that $\det(AB) = \det A \det B$, we have

$$(\det L)^2 = 1. \tag{139}$$

So $\det L = \pm 1$, and every Lorentz transformation L has an inverse. Multiplying (138) by η, we get

$$\eta L^{\mathsf{T}} \eta L = \eta^2 = I \tag{140}$$

which identifies L^{-1} as

$$L^{-1} = \eta L^{\mathsf{T}} \eta. \tag{141}$$

The subgroup of $O(3,1)$ with $\det L = 1$ is the proper Lorentz group $SO(3,1)$. The subgroup of $SO(3,1)$ that leaves invariant the sign of the time component of timelike vectors is the **proper orthochronous Lorentz group** $SO^+(3,1)$.

To find the Lie algebra of $SO^+(3,1)$, we take a Lorentz matrix $L = I + \omega$ that differs from the identity matrix I by a tiny matrix ω and require L to obey the condition (138) for membership in the Lorentz group

$$\left(I + \omega^{\mathsf{T}}\right) \eta \, (I + \omega) = \eta + \omega^{\mathsf{T}} \eta + \eta \, \omega + \omega^{\mathsf{T}} \omega = \eta. \tag{142}$$

Neglecting $\omega^{\mathsf{T}} \omega$, we have $\omega^{\mathsf{T}} \eta = -\eta \, \omega$ or since $\eta^2 = I$

$$\omega^{\mathsf{T}} = \, - \, \eta \, \omega \, \eta. \tag{143}$$

This equation says that under transposition the time-time and space-space elements of ω change sign, while the time-space and spacetime elements do not. That is, the tiny matrix ω is for infinitesimal $\boldsymbol{\theta}$ and $\boldsymbol{\lambda}$ a linear combination

$$\omega = \boldsymbol{\theta} \cdot \boldsymbol{R} + \boldsymbol{\lambda} \cdot \boldsymbol{B} \tag{144}$$

of three antisymmetric space-space matrices

$$R_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad R_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad R_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{145}$$

and of three symmetric time-space matrices

$$B_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad B_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad B_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{146}$$

all of which satisfy condition (143). The three R_ℓ are 4×4 versions of the familiar rotation generators; the three B_ℓ generate Lorentz boosts.

If we write $L = I + \omega$ as

$$L = I - i\theta_\ell \, iR_\ell - i\lambda_\ell \, iB_\ell \equiv I - i\theta_\ell J_\ell - i\lambda_\ell K_\ell \tag{147}$$

then the three matrices $J_\ell = iR_\ell$ are imaginary and antisymmetric, and therefore hermitian. But the three matrices $K_\ell = iB_\ell$ are imaginary and symmetric, and so are antihermitian. The 4×4 matrix $L = \exp(i\theta_\ell J_\ell - i\lambda_\ell K_\ell)$ is **not unitary** because the Lorentz group is **not compact**.

14 Gamma matrices and Clifford algebras

In component notation, $L = I + \omega$ is

$$L^a{}_b = \delta^a{}_b + \omega^a{}_b, \tag{148}$$

the matrix η is $\eta_{cd} = \eta^{cd}$, and $\omega^{\mathsf{T}} = -\eta\,\omega\,\eta$ is

$$\omega^a{}_b = (\omega^{\mathsf{T}})_b{}^a = -(\eta\omega\eta)_b{}^a = -\eta_{bc}\,\omega^c{}_d\,\eta^{da} = -\omega_{bd}\,\eta^{da} = -\omega_b{}^a. \tag{149}$$

Lowering index a we get

$$\omega_{eb} = \eta_{ea}\,\omega^a{}_b = -\omega_{bd}\,\eta^{da}\,\eta_{ea} = -\omega_{bd}\,\delta^d{}_e = -\omega_{be} \tag{150}$$

That is, ω_{ab} is antisymmetric

$$\omega_{ab} = -\omega_{ba}. \tag{151}$$

A representation of the Lorentz group is generated by matrices $D(L)$ that represent matrices L close to the identity matrix by sums over $a, b = 0, 1, 2, 3$

$$D(L) = 1 + \frac{i}{2}\omega_{ab}\,\mathcal{J}^{ab}. \tag{152}$$

The generators $\mathcal{J}^{ab}$ must obey the commutation relations

$$i[\mathcal{J}^{ab}, \mathcal{J}^{cd}] = \eta^{bc}\,\mathcal{J}^{ad} - \eta^{ac}\,\mathcal{J}^{bd} - \eta^{da}\,\mathcal{J}^{cb} + \eta^{db}\,\mathcal{J}^{ca}. \tag{153}$$

A remarkable representation of these commutation relations is provided by matrices γ^a that obey the anticommutation relations

$$\{\gamma^a, \gamma^b\} = 2\,\eta^{ab}. \tag{154}$$

One sets

$$\mathcal{J}^{ab} = -\frac{i}{4}\,[\gamma^a, \gamma^b] \tag{155}$$

where η is the usual flat-space metric (136). Any four 4×4 matrices that satisfy these anticommutation relations form a set of Dirac gamma matrices. They are not unique. Is S is any nonsingular 4×4 matrix, then the matrices

$$\gamma'^a = S\,\gamma^a\,S^{-1} \tag{156}$$

also are a set of Dirac's gamma matrices.

Any set of matrices obeying the anticommutation relations (154) for any $n \times n$ diagonal matrix η with entries that are ± 1 is called a **Clifford algebra**.

As a homework problem, show that

$$[\mathcal{J}^{ab}, \gamma^c] = -i\,\gamma^a\,\eta^{bc} + i\,\gamma^b\,\eta^{ac}. \tag{157}$$

One can use these commutation relations to derive the commutation relations (153) of the Lorentz group.

The gamma matrices are a vectors in the sense that for L near the identity

$$\begin{aligned} D(L)\,\gamma^c\,D^{-1}(L) &\approx (I + \frac{i}{2}\omega_{ab}\,\mathcal{J}^{ab})\,\gamma^c\,(I - \frac{i}{2}\omega_{ab}\,\mathcal{J}^{ab}) \\ &= \gamma^c + \frac{i}{2}\omega_{ab}\,[\mathcal{J}^{ab}, \gamma^c] \\ &= \gamma^c + \frac{i}{2}\omega_{ab}\,(-i\,\gamma^a\,\eta^{bc} + i\,\gamma^b\,\eta^{ac}) \\ &= \gamma^c + \frac{1}{2}\omega_{ab}\,\gamma^a\,\eta^{bc} - \frac{1}{2}\omega_{ab}\,\gamma^b\,\eta^{ac} \\ &= \gamma^c - \frac{1}{2}\eta^{cb}\,\omega_{ba}\,\gamma^a\, - \frac{1}{2}\,\eta^{ca}\,\omega_{ab}\,\gamma^b \\ &= \gamma^c - \frac{1}{2}\,\omega^c{}_a\,\gamma^a\, - \frac{1}{2}\,\omega^c{}_b\,\gamma^b \\ &= \gamma^c - \omega^c{}_a\,\gamma^a \\ &= \gamma^c + \omega_a{}^c\,\gamma^a \\ &= (\delta_a{}^c + \omega_a{}^c)\,\gamma^a \\ &= L_a{}^c\,\gamma^a \end{aligned} \tag{158}$$

in which we used (149) to write $-\,\omega^c{}_a = \omega_a{}^c$. The finite ω form is

$$D(L)\,\gamma^a\,D^{-1}(L) = L_c{}^a\,\gamma^c. \tag{159}$$

The unit matrix is a scalar

$$D(L)\,I\,D^{-1}(L) = I. \tag{160}$$

The generators of the Lorenz group form an antisymmetric tensor

$$D(L)\,\mathcal{J}^{ab}\,D^{-1}(L) = L_c{}^a\,L_d{}^b\,\mathcal{J}^{cd}. \tag{161}$$

Out of four gamma matrices, one can also make totally antisymmetric tensors of rank-3 and rank-4

$$A^{abc} \equiv \gamma^{[a}\,\gamma^b\,\gamma^{c]} \quad \text{and} \quad B^{abcd} \equiv \gamma^{[a}\,\gamma^b\,\gamma^c\,\gamma^{d]} \tag{162}$$

where the brackets mean that one inserts appropriate minus signs so as to achieve total antisymmetry. Since there are only four γ matrices in four spacetime dimensions, any rank-5 totally antisymmetric tensor made from them must vanish, $C^{abcde} = 0$.

Notation: The parity transformation is

$$\beta = i\gamma^0. \tag{163}$$

It flips the spatial gamma matrices but not the temporal one

$$\beta\,\gamma^i\,\beta^{-1} = -\gamma^i \quad \text{and} \quad \beta\,\gamma^0\,\beta^{-1} = \gamma^0. \tag{164}$$

It flips the generators of boosts but not those of rotations

$$\beta\,\mathcal{J}^{i0}\,\beta^{-1} = -\mathcal{J}^{i0} \quad \text{and} \quad \beta\,\mathcal{J}^{ik}\,\beta^{-1} = \mathcal{J}^{ik}. \tag{165}$$

15 Dirac's gamma matrices

Weinberg's chosen set of Dirac matrices is

$$\gamma^0 = -i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -\gamma^{0\dagger} \quad \text{and} \quad \gamma^i = -i\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} = \gamma^{i\dagger} \tag{166}$$

in which the σ's are Pauli's 2×2 hermitian matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \text{and} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{167}$$

which are the gamma matrices of 3-dimensional spacetime. With this choice of γ's, the matrix β is

$$\beta = i\,\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \beta^\dagger. \tag{168}$$

In spacetimes of five dimensions, the fifth gamma matrix γ^4 which traditionally is called $\gamma^5 = \gamma_5$ is

$$\gamma^5 = \gamma_5 = -i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{169}$$

It anticommutes with all four Dirac gammas and its square is unity, as it must if it is to be the fifth gamma in 5-space:

$$\{\gamma^a, \gamma^b\} = 2\eta^{ab} \tag{170}$$

for $a, b = 0, 1, 2, 3, 4$ with $\eta^{44} = 1$ and $\eta^{40} = \eta^{04} = 0$.

With Weinberg's choice of γ's, the Lorentz boosts are

$$\begin{aligned} \mathcal{J}^{i0} = -\frac{i}{4}\,[\gamma^i, \gamma^0] &= -\frac{i}{4}\left[-i\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, -i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right] \\ &= \frac{i}{4}\left[\begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix} - \begin{pmatrix} -\sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}\right] = \frac{i}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix}. \end{aligned} \tag{171}$$

The Lorentz rotation matrices are

$$
\begin{aligned}
\mathcal{J}^{ik} = &-\frac{i}{4}\,[\gamma^i,\gamma^k] = -\frac{i}{4}\left[-i\begin{pmatrix}0 & \sigma^i\\ -\sigma^i & 0\end{pmatrix}, -i\begin{pmatrix}0 & \sigma^k\\ -\sigma^k & 0\end{pmatrix}\right]\\
&= \frac{i}{4}\left[\begin{pmatrix}-\sigma^i\sigma^k & 0\\ 0 & -\sigma^i\sigma^k\end{pmatrix} - \begin{pmatrix}-\sigma^k\sigma^i & 0\\ 0 & -\sigma^k\sigma^i\end{pmatrix}\right]\\
&= \frac{-i}{4}\begin{pmatrix}[\sigma^i,\sigma^k] & 0\\ 0 & [\sigma^i,\sigma^k]\end{pmatrix} = \frac{-i}{4}\begin{pmatrix}2i\epsilon_{ikj}\sigma^j & 0\\ 0 & 2i\epsilon_{ikj}\sigma^j\end{pmatrix}\\
&= \frac{1}{2}\,\epsilon_{ikj}\begin{pmatrix}\sigma^j & 0\\ 0 & \sigma^j\end{pmatrix}.
\end{aligned}
\tag{172}
$$

The Dirac representation of the Lorentz group is reducible, as SW's choice of gamma matrices makes apparent. The Dirac rotation matrices are

$$
\mathcal{J}_i = \frac{1}{2}\begin{pmatrix}\sigma^i & 0\\ 0 & \sigma^i\end{pmatrix}. \tag{173}
$$

Some useful relations are

$$
\beta\,\gamma^{a\dagger}\,\beta = -\gamma^a, \quad \beta\,\mathcal{J}^{ab\dagger}\,\beta = \mathcal{J}^{ab} \quad \text{and} \quad \beta\,D(L)^\dagger\,\beta = D(L)^{-1} \tag{174}
$$

as well as

$$
\beta\,\gamma_5^\dagger\,\beta = -\gamma_5 \quad \text{and} \quad \beta(\gamma_5\gamma^a)^\dagger\,\beta = -\gamma_5\gamma^a. \tag{175}
$$

16 Dirac fields

The positive- and negative-frequency parts of a Dirac field are

$$
\begin{aligned}
\psi_\ell^+(x) &= (2\pi)^{-3/2}\sum_s \int d^3p\;\; u_\ell(\vec{p},s)\,e^{ip\cdot x}\,a(\vec{p},s)\\
\psi_\ell^-(x) &= (2\pi)^{-3/2}\sum_s \int d^3p\;\; v_\ell(\vec{p},s)\,e^{-ip\cdot x}\,b^\dagger(\vec{p},s).
\end{aligned}
\tag{176}
$$

The rotation conditions (39) are

$$
\begin{aligned}
\sum_{\bar{s}} u_{\bar{\ell}}(\vec{0},\bar{s})(J_i^{(j)})_{\bar{s}s} &= \sum_\ell (\mathcal{J}_i)_{\bar{\ell}\ell}u_\ell(\vec{0},s)\\
\sum_{\bar{s}} v_{\bar{\ell}}(\vec{0},\bar{s})(-J_i^{(j)})^*_{\bar{s}s}) &= \sum_\ell (\mathcal{J}_i)_{\bar{\ell}\ell}v_\ell(\vec{0},s).
\end{aligned}
\tag{177}
$$

The Dirac rotation matrices (173) are

$$
\mathcal{J}_i = \frac{1}{2}\begin{pmatrix}\sigma^i & 0\\ 0 & \sigma^i\end{pmatrix} \tag{178}
$$

so we set the four values $\ell, \bar{\ell} = 1, 2, 3, 4$ to $\ell = (m, \pm)$ with $m = \pm\frac{1}{2}$. And we consider $u_\ell(s)$ to be $u^+_m(s)$ stacked upon $u^-_m(s)$ and similarly take $v_\ell(s)$ to be $v^+_{m\pm}(s)$ above $v^-_{m\pm}(s)$ where $u^\pm_m(s)$ and $v^\pm_{m\pm}(s)$ are, a priori, $2 \times (2j + 1)$-dimensional matrices with indexes $m = \pm 1/2$ and $s = -j, \ldots, j$. That is,

$$\begin{pmatrix} u_1(s) \\ u_2(s) \\ u_3(s) \\ u_4 v \end{pmatrix} = \begin{pmatrix} u^+_{+1/2}(s) \\ u^+_{-1/2}(s) \\ u^-_{+1/2}(s) \\ u^-_{-1/2}(s) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} v_1(s) \\ v_2(s) \\ v_3(s) \\ v_4(s) \end{pmatrix} = \begin{pmatrix} v^+_{+1/2}(s) \\ v^+_{-1/2}(s) \\ v^-_{+1/2}(s) \\ v^-_{-1/2}(s) \end{pmatrix}. \tag{179}$$

We then have four equations

$$\begin{aligned} \sum_{\bar{s}} u^+_{\bar{m}}(\vec{0}, \bar{s})(J^{(j)}_i)_{\bar{s}s} &= \sum_m \tfrac{1}{2}\, \sigma^i_{\bar{m}m} u^+_m(\vec{0}, s) \\ \sum_{\bar{s}} u^-_{\bar{m}}(\vec{0}, \bar{s})(J^{(j)}_i)_{\bar{s}s} &= \sum_m \tfrac{1}{2}\, \sigma^i_{\bar{m}m} u^-_m(\vec{0}, s) \\ \sum_{\bar{s}} v^+_{\bar{m}}(\vec{0}, \bar{s})(-J_i)^{*(j)}_{\bar{s}s}) &= \sum_m \tfrac{1}{2}\, \sigma^i_{\bar{m}m} v^+_m(\vec{0}, s) \\ \sum_{\bar{s}} v^-_{\bar{m}}(\vec{0}, \bar{s})(-J_i)^{*(j)}_{\bar{s}s}) &= \sum_m \tfrac{1}{2}\, \sigma^i_{\bar{m}m} v^-_m(\vec{0}, s). \end{aligned} \tag{180}$$

SW defines the four $2 \times (2j + 1)$ matrices

$$\begin{aligned} U^+_{ms} &= u^+_m(\vec{0}, s) \quad \text{and} \quad U^-_{ms} = u^-_m(\vec{0}, s) \\ V^+_{ms} &= v^+_m(\vec{0}, s) \quad \text{and} \quad V^-_{ms} = v^-_m(\vec{0}, s). \end{aligned} \tag{181}$$

in terms of which the four Dirac rotation conditions (180) are

$$\begin{aligned} U^+\, J^{(j)}_i &= \tfrac{1}{2}\, \sigma_i\, U^+ \quad \text{and} \quad U^-\, J^{(j)}_i = \tfrac{1}{2}\, \sigma_i\, U^- \\ V^+\, (-J^{*(j)}_i) &= \tfrac{1}{2}\, \sigma_i\, V^+ \quad \text{and} \quad V^-\, (-J^{*(j)}_i) = \tfrac{1}{2}\, \sigma_i\, V^-. \end{aligned} \tag{182}$$

Taking the complex conjugate of the second of these equations, we get

$$\begin{aligned} -J^{(j)}_i &= V^{+*-1}(\tfrac{1}{2}\sigma^{i*})V^{+*} = V^{+*-1}(-\tfrac{1}{2}\sigma_2\, \sigma^i\, \sigma_2)V^{+*} \\ -J^{(j)}_i &= V^{-*-1}(\tfrac{1}{2}\sigma^{i*})V^{-*} = V^{-*-1}(-\tfrac{1}{2}\sigma_2\, \sigma^i\, \sigma_2)V^{-*} \end{aligned} \tag{183}$$

or more simply

$$\begin{aligned} J^{(j)}_i &= (\sigma_2 V^{+*})^{-1}\, \tfrac{1}{2}\sigma^i\, (\sigma_2 V^{+*}) \\ J^{(j)}_i &= (\sigma_2 V^{-*})^{-1}\, \tfrac{1}{2}\sigma^i\, (\sigma_2 V^{-*}). \end{aligned} \tag{184}$$

The 2×2 Pauli matrices $\vec{\sigma}$ and the $(2j+1)\times(2j+1)$ matrices $\vec{J}^{(j)}$ both generate irreducible representations of the rotation group. So by writing

$$U^+ \, J_i^{(j)} \, J_k^{(j)} = \tfrac{1}{2}\,\sigma_i \, U^+ \, J_k^{(j)} = \; = \tfrac{1}{2}\,\sigma_i \, \tfrac{1}{2}\,\sigma_k \, U^+ \tag{185}$$

and similar equations for U^-, V^+, V^-, we see that

$$\begin{aligned} U^+ \, D^{(j)}(\vec{\theta}) &= U^+ \, e^{-i\theta\cdot\vec{J}^{(j)}} = e^{-i\vec{\theta}\cdot\vec{\sigma}} \, U^+ = D^{(1/2)}(\vec{\theta}) \, U^+ \\ U^- \, D^{(j)}(\vec{\theta}) &= U^- \, e^{-i\theta\cdot\vec{J}^{(j)}} = e^{-i\vec{\theta}\cdot\vec{\sigma}} \, U^- = D^{(1/2)}(\vec{\theta}) \, U^- \end{aligned} \tag{186}$$

and similar equations for $V^\pm$.

$$\begin{aligned} \sigma_2 V^{+*} \, D^{(j)}(\vec{\theta}) &= \sigma_2 V^{+*} \, e^{-i\theta\cdot\vec{J}^{(j)}} = e^{-i\vec{\theta}\cdot\vec{\sigma}} \, \sigma_2 V^{+*} = D^{(1/2)}(\vec{\theta}) \, \sigma_2 V^{+*} \\ \sigma_2 V^{-*} \, D^{(j)}(\vec{\theta}) &= \sigma_2 V^{-*} \, e^{-i\theta\cdot\vec{J}^{(j)}} = e^{-i\vec{\theta}\cdot\vec{\sigma}} \, \sigma_2 V^{-*} = D^{(1/2)}(\vec{\theta}) \, \sigma_2 V^{-*}. \end{aligned} \tag{187}$$

Now recall Schur's lemma (section 10.7 of PM):

> Part 1: If D_1 and D_2 are inequivalent, irreducible representations of a group G, and if $D_1(g)A = AD_2(g)$ for some matrix A and for all $g \in G$, then the matrix A must vanish, $A = 0$.
>
> Part 2: If for a finite-dimensional, irreducible representation $D(g)$ of a group G, we have $D(g)A = AD(g)$ for some matrix A and for all $g \in G$, then $A = cI$. That is, any matrix that commutes with every element of a finite-dimensional, irreducible representation must be a multiple of the identity matrix.

Part 1 tells us that $D^{(j)}(\vec{\theta})$ and $D^{(1/2)}(\vec{\theta})$ must be equivalent. So $j = 1/2$ and $2j + 1 = 2$. A Dirac field must represent particles of spin 1/2.

Part 2 then says that the matrices $U^\pm$ must be multiples of the 2×2 identity matrix

$$U^+ = c_+ \, I \quad \text{and} \quad U^- = c_- \, I \tag{188}$$

and that the matrices $\sigma_2 V^{\pm *}$ must be multiples of the 2×2 identity matrix

$$\sigma_2 V^{+*} = d'_+ \, I \quad \text{and} \quad \sigma_2 V^{-*} = d'_- \, I \tag{189}$$

or more simply

$$V^+ = \, - \, i d_+ \, \sigma_2 \quad \text{and} \quad V^- = \, - \, i d_- \, \sigma_2. \tag{190}$$

That is,

$$v_m^+(\vec{0}, s) = d_+ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad v_m^-(\vec{0}, s) = d_- \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{191}$$

Going back to $\ell = (m, \pm)$ by using the index code (179)

$$\begin{bmatrix} u_1(s) \\ u_2(s) \\ u_3(s) \\ u_4(s) \end{bmatrix} = \begin{bmatrix} u^+_{+1/2}(s) \\ u^+_{-1/2}(s) \\ u^-_{+1/2}(s) \\ u^-_{-1/2}(s) \end{bmatrix} = \begin{bmatrix} c_+\,\delta_{1/2,s} \\ c_+\,\delta_{-1/2,s} \\ c_-\,\delta_{1/2,s} \\ c_-\,\delta_{-1/2,s} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} v_1(s) \\ v_2(s) \\ v_3(s) \\ v_4(s) \end{bmatrix} = \begin{bmatrix} v^+_{+1/2}(s) \\ v^+_{-1/2}(s) \\ v^-_{1/2}(s) \\ v^-_{-1/2}(s) \end{bmatrix} = \begin{bmatrix} d_+\,\delta_{-1/2,s} \\ d_+\,\delta_{1/2,s} \\ d_-\,\delta_{-1/2,s} \\ d_-\,\delta_{+1/2,s} \end{bmatrix}. \tag{192}$$

we have for the u's

$$u^+_{1/2}(1/2) = c_+ \quad \text{and} \quad u^+_{-1/2}(1/2) = 0 \tag{193}$$

$$u^-_{1/2}(1/2) = c_- \quad \text{and} \quad u^-_{-1/2}(1/2) = 0 \tag{194}$$

$$u^+_{1/2}(-1/2) = 0 \quad \text{and} \quad u^+_{-1/2}(-1/2) = c_+ \tag{195}$$

$$u^-_{1/2}(-1/2) = 0 \quad \text{and} \quad u^-_{-1/2}(-1/2) = c_- \tag{196}$$

$$v^+_{1/2}(1/2) = 0 \quad \text{and} \quad v^+_{-1/2}(1/2) = d_+ \tag{197}$$

$$v^-_{1/2}(1/2) = 0 \quad \text{and} \quad v^-_{-1/2}(1/2) = d_- \tag{198}$$

$$v^+_{1/2}(-1/2) = \,-\,d_+ \quad \text{and} \quad v^+_{-1/2}(-1/2) = 0 \tag{199}$$

$$v^-_{1/2}(-1/2) = \,-\,d_- \quad \text{and} \quad v^-_{-1/2}(-1/2) = 0 \tag{200}$$

So

$$u(\vec{0}, m = \tfrac{1}{2}) = \begin{bmatrix} u^+_{1/2}(1/2) \\ u^+_{-1/2}(1/2) \\ u^-_{1/2}(1/2) \\ u^-_{-1/2}(1/2) \end{bmatrix} = \begin{bmatrix} c_+ \\ 0 \\ c_- \\ 0 \end{bmatrix} \quad \text{and} \quad u(\vec{0}, m = -\tfrac{1}{2}) = \begin{bmatrix} u^+_{1/2}(-1/2) \\ u^+_{-1/2}(-1/2) \\ u^-_{1/2}(-1/2) \\ u^-_{-1/2}(-1/2) \end{bmatrix} = \begin{bmatrix} 0 \\ c_+ \\ 0 \\ c_- \end{bmatrix},$$
$$v(\vec{0}, m = \tfrac{1}{2}) = \begin{bmatrix} v^+_{1/2}(1/2) \\ v^+_{-1/2}(1/2) \\ v^-_{1/2}(1/2) \\ v^-_{-1/2}(1/2) \end{bmatrix} = \begin{bmatrix} 0 \\ d_+ \\ 0 \\ d_- \end{bmatrix} \quad \text{and} \quad v(\vec{0}, m = -\tfrac{1}{2}) = \begin{bmatrix} v^+_{1/2}(-1/2) \\ v^+_{-1/2}(-1/2) \\ v^-_{1/2}(-1/2) \\ v^-_{-1/2}(-1/2) \end{bmatrix} = -\begin{bmatrix} d_+ \\ 0 \\ d_- \\ 0 \end{bmatrix}. \tag{201}$$

To put more constraints on $c_\pm$ and $d_\pm$, we recall that under parity

$$\mathsf{P}a(\vec{p}, s)\mathsf{P}^{-1} = \eta_a^*\, a(-\vec{p}, s) \quad \text{and} \quad \mathsf{P}b^\dagger(\vec{p}, s)\mathsf{P}^{-1} = \eta_b\, \beta^\dagger(-\vec{p}, s) \tag{202}$$

and so

$$
\begin{aligned}
\mathsf{P}\psi_\ell^+(x)\mathsf{P}^{-1} &= (2\pi)^{-3/2}\sum_s \int d^3p \;\; u_\ell(\vec{p},s)\, e^{ip\cdot x}\, \eta_a^*\, a(-\vec{p},s) \\
&= (2\pi)^{-3/2}\sum_s \int d^3p \;\; u_\ell(-\vec{p},s)\, e^{ip\cdot \mathcal{P}x}\, \eta_a^*\, a(\vec{p},s) \\
\mathsf{P}\psi_\ell^-(x)\mathsf{P}^{-1} &= (2\pi)^{-3/2}\sum_s \int d^3p \;\; v_\ell(\vec{p},s)\, e^{-ip\cdot x}\, \eta_b\, b^\dagger(-\vec{p},s) \\
&= (2\pi)^{-3/2}\sum_s \int d^3p \;\; v_\ell(-\vec{p},s)\, e^{-ip\cdot \mathcal{P}x}\, \eta_b\, b^\dagger(\vec{p},s).
\end{aligned}
\tag{203}
$$

We recall the relations (174)

$$
\beta\,\gamma^{a\dagger}\,\beta = -\,\gamma^a, \quad \beta\,\mathcal{J}^{ab\dagger}\,\beta = \mathcal{J}^{ab}, \quad \text{and} \quad \beta\,D(L)^\dagger\,\beta = D(L)^{-1} \tag{204}
$$

and in particular, since $\mathcal{J}^{0i\dagger} = -\,\mathcal{J}^{0i}$, the rule

$$
\beta\mathcal{J}^{0i}\beta = \mathcal{J}^{0i\dagger} = -\mathcal{J}^{0i}. \tag{205}
$$

We also have the pseudounitarity relation

$$
\beta\,D^\dagger(L)\,\beta = D^{-1}(L). \tag{206}
$$

In general spinors at finite momentum are related to those at zero momentum by

$$
\begin{aligned}
u_{\bar\ell}(q,s) &= \sqrt{\frac{m}{q^0}}\sum_\ell D_{\bar\ell\ell}(L(q))u_\ell(\vec{0},s) \\
v_{\bar\ell}(q,s) &= \sqrt{\frac{m}{q^0}}\sum_\ell D_{\bar\ell\ell}(L(q))v_\ell(\vec{0},s)
\end{aligned}
\tag{207}
$$

which for Dirac spinors is

$$
\begin{aligned}
u(p,s) &= \sqrt{\frac{m}{p^0}}\,D(L(p))\,u(\vec{0},s) \\
v(p,s) &= \sqrt{\frac{m}{p^0}}\,D(L(p))\,v(\vec{0},s).
\end{aligned}
\tag{208}
$$

So now by using the boost rule (205) we have

$$
u_\ell(-\vec{p},s) = \sqrt{m/p^0}D(L(-\vec{p}))u(0,s) = \sqrt{m/p^0}D(L(\vec{p}))^{-1}u(0,s) \tag{209}
$$

$$
= \sqrt{m/p^0}\,\beta\,D(L(\vec{p}))\,\beta\,u(0,s) \tag{210}
$$

and

$$v_\ell(-\vec{p}, s) = \sqrt{m/p^0} D(L(-\vec{p})) v(0, s) = \sqrt{m/p^0} D(L(\vec{p}))^{-1} v(0, s) \tag{211}$$
$$= \sqrt{m/p^0}\, \beta\, D(L(\vec{p}))\, \beta\, v(0, s). \tag{212}$$

So under parity

$$\begin{aligned} \mathsf{P}\psi^+(x)\mathsf{P}^{-1} &= (2\pi)^{-3/2} \sum_s \int d^3p\ \ \sqrt{m/p^0}\, \beta\, D(L(\vec{p}))\, \beta\, u(0, s)\, e^{ip\cdot\mathcal{P}x}\, \eta_a^*\, a(\vec{p}, s) \\ \mathsf{P}\psi^-(x)\mathsf{P}^{-1} &= (2\pi)^{-3/2} \sum_s \int d^3p\ \ \sqrt{m/p^0}\, \beta\, D(L(\vec{p}))\, \beta\, v(0, s)\, e^{-ip\cdot\mathcal{P}x}\, \eta_b\, b^\dagger(\vec{p}, s). \end{aligned} \tag{213}$$

So to have $\mathsf{P}\psi_\ell^\pm(x)\mathsf{P}^{-1} \propto \psi_\ell^\pm(x)$, we need

$$\beta\, u(0, s) = b_u\, u(0, s) \quad \text{and} \quad \beta\, v(0, s) = b_v\, u(0, s). \tag{214}$$

We then get

$$\mathsf{P}\psi_\ell^+(t, \vec{x})\mathsf{P}^{-1} = b_u\, \beta\, \eta_a^*\, \psi_\ell^+(t, -\vec{x}) \quad \text{and} \quad \mathsf{P}\psi_\ell^-(t, \vec{x})\mathsf{P}^{-1} = b_v\, \beta\, \eta_b\, \psi_\ell^-(t, -\vec{x}). \tag{215}$$

Here since $\mathsf{P}^2 = 1$, these factors are just signs, $b_u^2 = b_v^2 = 1$. The eigenvalue equations (214) tell us that $c_- = b_u\, c_+$ and that $d_- = b_v\, d_+$. So rescaling the fields we get

$$\begin{aligned} u(\vec{0}, m = \tfrac{1}{2}) &= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ b_u \\ 0 \end{bmatrix} \quad \text{and} \quad u(\vec{0}, m = -\tfrac{1}{2}) = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 0 \\ b_u \end{bmatrix}, \\ v(\vec{0}, m = \tfrac{1}{2}) &= \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 0 \\ b_v \end{bmatrix} \quad \text{and} \quad v(\vec{0}, m = -\tfrac{1}{2}) = \frac{-1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ b_v \\ 0 \end{bmatrix}. \end{aligned} \tag{216}$$

If the annihilation and creation operators $a(p, s)$ and $a^\dagger(p, s)$ obey the rule

$$[a(p, s), a^\dagger(p', s')]_\mp = \delta_{ss'} \delta^3(\vec{p} - \vec{p}') \tag{217}$$

and if the field is the sum of the positive- and negative-frequency parts (176)

$$\begin{aligned} \psi_\ell^+(x) &= (2\pi)^{-3/2} \sum_s \int d^3p\ \ u_\ell(\vec{p}, s)\, e^{ip\cdot x}\, a(\vec{p}, s) \\ \psi_\ell^-(x) &= (2\pi)^{-3/2} \sum_s \int d^3p\ \ v_\ell(\vec{p}, s)\, e^{-ip\cdot x}\, b^\dagger(\vec{p}, s) \end{aligned} \tag{218}$$

with arbitrary coefficients κ and λ

$$\psi_\ell(x) = \kappa \psi^+_\ell(x) + \lambda \psi^-_\ell(x) \tag{219}$$

then

$$\begin{aligned}
[\psi_\ell(x), \psi^\dagger_{\ell'}(y)]_\mp &= [\kappa\psi^+_\ell(x) + \lambda\psi^-_\ell(x), \kappa^*\psi^{+\dagger}_{\ell'}(y) + \lambda^*\psi^{-\dagger}_{\ell'}(y)] \\
&= \int \frac{d^3p}{(2\pi)^3} \sum_s \left[|\kappa|^2\, u_\ell(\vec{p},s)\, u^*_{\ell'}(\vec{p},s) e^{ip\cdot(x-y)} \mp |\lambda|^2\, v_\ell(\vec{p},s)\, v^*_{\ell'}(\vec{p},s) e^{-ip\cdot(x-y)} \right] \\
&= \int \frac{d^3p}{(2\pi)^3} \sum_s \left[|\kappa|^2\, N_{\ell\ell'}(p)\, e^{ip\cdot(x-y)} \mp |\lambda|^2\, N_{\ell\ell'}(p)\, e^{-ip\cdot(x-y)} \right]
\end{aligned} \tag{220}$$

where

$$\begin{aligned}
N_{\ell\ell'}(p) &= \sum_s u_\ell(\vec{p},s)\, u^*_{\ell'}(\vec{p},s) \\
M_{\ell\ell'}(p) &= \sum_s v_\ell(\vec{p},s)\, v^*_{\ell'}(\vec{p},s).
\end{aligned} \tag{221}$$

When $\vec{p} = 0$, these matrices are

$$\begin{aligned}
N_{\ell\ell'}(0) &= \sum_s u_\ell(\vec{0},s)\, u^*_{\ell'}(\vec{0},s) \\
N(0) &= \frac{1}{2}\begin{bmatrix} 1 \\ 0 \\ b_u \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & b_u & 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 \\ 1 \\ 0 \\ b_u \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & b_u \end{bmatrix} \\
&= \frac{1}{2}\begin{bmatrix} 1 & 0 & b_u & 0 \\ 0 & 0 & 0 & 0 \\ b_u & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & b_u \\ 0 & 0 & 0 & 0 \\ 0 & b_u & 0 & 1 \end{bmatrix} = \frac{1 + b_u\,\beta}{2}
\end{aligned} \tag{222}$$

and

$$\begin{aligned}
M_{\ell\ell'}(0) &= \sum_s v_\ell(\vec{0},s)\, v^*_{\ell'}(\vec{0},s) \\
M(0) &= \frac{1}{2}\begin{bmatrix} 0 \\ 1 \\ 0 \\ b_v \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & b_v \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 \\ 0 \\ b_v \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & b_v & 0 \end{bmatrix} \\
&= \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & b_v \\ 0 & 0 & 0 & 0 \\ 0 & b_v & 0 & 1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 & 0 & b_v & 0 \\ 0 & 0 & 0 & 0 \\ b_v & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \frac{1 + b_v\,\beta}{2}.
\end{aligned} \tag{223}$$

So then using the boost relations (208) we find

$$
\begin{aligned}
N(\vec{p}) &= \sum_s u_\ell(\vec{p},s)\, u^*_{\ell'}(\vec{p},s) = \frac{m}{p^0} D(L(p)) \sum_s u_\ell(\vec{0},s)\, u^*_{\ell'}(\vec{0},s) D^\dagger(L(p)) \\
&= \frac{m}{2p^0}\, D(L(p))\, (1 + b_u\, \beta)\, D^\dagger(L(p)) \\
M(\vec{p}) &= \sum_s v_\ell(\vec{p},s)\, v^*_{\ell'}(\vec{p},s) = \frac{m}{p^0} D(L(p)) \sum_s v_\ell(\vec{0},s)\, v^*_{\ell'}(\vec{0},s) D^\dagger(L(p)) \\
&= \frac{m}{2p^0}\, D(L(p))\, (1 + b_v\, \beta)\, D^\dagger(L(p)).
\end{aligned} \tag{224}
$$

The pseudounitarity relation (206)

$$\beta\, D^\dagger(L)\, \beta = D^{-1}(L). \tag{225}$$

gives

$$\beta\, D^\dagger(L) = D^{-1}(L)\, \beta \tag{226}$$

which implies that

$$D(L)\, \beta\, D^\dagger(L) = \beta. \tag{227}$$

The pseudounitarity relation also says that

$$D^\dagger(L) = \beta\, D^{-1}(L)\, \beta \tag{228}$$

so that

$$D(L)\, D^\dagger(L) = D(L)\, \beta\, D^{-1}(L)\, \beta. \tag{229}$$

Also since the gammas form a 4-vector (159)

$$D(L)\, \gamma^a\, D^{-1}(L) = L_c{}^a\, \gamma^c \tag{230}$$

and since $\beta = i\gamma^0$, we have

$$D(L(p))\, \beta\, D^{-1}(L(p)) = D(L(p))\, i\gamma^0\, D^{-1}(L(p)) = iL_c{}^0(p)\, \gamma^c = -iL^c{}_0 \gamma_c. \tag{231}$$

Now

$$p^a = L^a{}_b(p) k^b = L^a{}_0(p) m \tag{232}$$

so

$$D(L(p))\, \beta\, D^{-1}(L(p)) = -i\, p^c \gamma_c / m \tag{233}$$

which implies that

$$D(L)\, D^\dagger(L) = \, - i\, (p^c \gamma_c / m)\, \beta. \tag{234}$$

Thus

$$
\begin{aligned}
N(\vec{p}) &= \frac{m}{2p^0}\, D(L(p))\,(1+b_u\,\beta)\,D^\dagger(L(p)) = \frac{m}{2p^0}\,[(-i\,p^c\gamma_c/m)\,\beta + b_u\,\beta] \\
&= \frac{1}{2p^0}\,[-i\,p^c\gamma_c + b_u\,m]\;\beta
\end{aligned}
\tag{235}
$$

and

$$
\begin{aligned}
M(\vec{p}) &= \frac{m}{2p^0}\, D(L(p))\,(1+b_v\,\beta)\,D^\dagger(L(p)) = \frac{m}{2p^0}\,[(-i\,p^c\gamma_c/m)\,\beta + b_v\,\beta] \\
&= \frac{1}{2p^0}\,[-i\,p^c\gamma_c + b_v\,m]\;\beta.
\end{aligned}
\tag{236}
$$

We now put the spin sums (235) and (236) in the (anti)commutator (220) and get

$$
\begin{aligned}
[\psi_\ell(x), \psi^\dagger_{\ell'}(y)]_\mp = & \int \frac{d^3p}{(2\pi)^3 2p^0}\Big[|\kappa|^2\,[(-\,i\,p^c\gamma_c + b_u\,m)\beta]_{\ell\ell'}\;e^{ip\cdot(x-y)} \\
& \mp|\lambda|^2\,[(-\,i\,p^c\gamma_c + b_v\,m)\beta]_{\ell\ell'}\;e^{-ip\cdot(x-y)}\Big] \\
= & \,|\kappa|^2\,[(-\,\partial_c\gamma^c + b_u\,m)\beta]_{\ell\ell'}\int \frac{d^3p}{(2\pi)^3 2p^0} e^{ip\cdot(x-y)} \\
& \mp|\lambda|^2\,[(-\,\partial_c\gamma^c + b_v\,m)\beta]_{\ell\ell'}\;\int \frac{d^3p}{(2\pi)^3 2p^0} e^{-ip\cdot(x-y)} \\
= & \,|\kappa|^2\,[(-\,\partial_c\gamma^c + b_u\,m)\beta]_{\ell\ell'}\,\Delta_+(x-y) \\
& \mp|\lambda|^2\,[(-\,\partial_c\gamma^c + b_v\,m)\beta]_{\ell\ell'}\;\Delta_+(y-x).
\end{aligned}
\tag{237}
$$

Recall that for $(x-y)^{>}0$, i.e. spacelike, $\Delta_+(x-y) = \Delta_+(y-x)$. So its first derivatives are odd. So for $x-y$ spacelike

$$
\begin{aligned}
[\psi_\ell(x), \psi^\dagger_{\ell'}(y)]_\mp = & \,|\kappa|^2\,[(-\,\partial_c\gamma^c + b_u\,m)\beta]_{\ell\ell'}\,\Delta_+(x-y) \\
& \mp|\lambda|^2\,[(\partial_c\gamma^c + b_v\,m)\beta]_{\ell\ell'}\;\Delta_+(x-y) \\
= & \,\big(|\kappa|^2 \pm |\lambda|^2\big)[(-\,\partial_c\gamma^c)\beta]_{\ell\ell'}\Delta_+(x-y) \\
& + \big(|\kappa|^2 b_u \mp |\lambda|^2\,b_v\big)m\,\beta_{\ell\ell'}\,\Delta_+(x-y).
\end{aligned}
\tag{238}
$$

To get the first term to vanish, we need to choose the lower sign (that is, use anticommutators) and set $|\kappa| = |\lambda|$. To get the second term to be zero, we must set $b_u = -\,b_v$. We may adjust κ and and b_u so that

$$
\kappa = \lambda \quad \text{and} \quad b_u = -\,b_v = 1.
\tag{239}
$$

In particular, a spin-one-half field must obey anticommutation relations

$$
[\psi_\ell(x), \psi^\dagger_{\ell'}(y)]_+ \equiv \{\psi_\ell(x), \psi^\dagger_{\ell'}(y)\} = 0 \quad \text{for} \quad (x-y)^2 > 0.
\tag{240}
$$

Finally then, the Dirac field is

$$\psi_\ell(x) = \sum_s \int \frac{d^3p}{(2\pi)^{3/2}} \left[u_\ell(\vec{p},s)\, e^{ip\cdot x}\, a(\vec{p},s) + v_\ell(\vec{p},s)\, e^{-ip\cdot x}\, b^\dagger(\vec{p},s) \right]. \tag{241}$$

The zero-momentum spinors are

$$\begin{aligned} u(\vec{0}, m=\tfrac{1}{2}) &= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad u(\vec{0}, m=-\tfrac{1}{2}) = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \\ v(\vec{0}, m=\tfrac{1}{2}) &= \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} \quad \text{and} \quad v(\vec{0}, m=-\tfrac{1}{2}) = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}. \end{aligned} \tag{242}$$

The spin sums are

$$\begin{aligned} \left[N(\vec{p})\right]_{\ell m} &= \sum_s u_\ell(\vec{p},s) u_m^*(\vec{p},s) = \left[\frac{1}{2p^0} \left(-i\, p^c \gamma_c + m\right)\, \beta \right]_{\ell m} \\ \left[M(\vec{p})\right]_{\ell m} &= \sum_s v_\ell(\vec{p},s) v_m^*(\vec{p},s) = \left[\frac{1}{2p^0} \left(-i\, p^c \gamma_c - m\right)\, \beta \right]_{\ell m}. \end{aligned} \tag{243}$$

The Dirac anticommutator is

$$[\psi_\ell(x), \psi^\dagger_{\ell'}(y)]_+ \equiv \{\psi_\ell(x), \psi^\dagger_{\ell'}(y)\} = [(- \partial_c \gamma^c + m)\beta]_{\ell\ell'}\, \Delta_+(x-y). \tag{244}$$

Two standard abbreviations are

$$\beta \equiv i\gamma^0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad \overline{\psi} \equiv \psi^\dagger \beta = i\psi^\dagger \gamma^0 = \begin{bmatrix} \psi_\ell^* & \psi_r^* \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \psi_r^* & \psi_\ell^* \end{bmatrix}. \tag{245}$$

A Majorana fermion is represented by a field like

$$\psi_\ell(x) = \sum_s \int \frac{d^3p}{(2\pi)^{3/2}} \left[u_\ell(\vec{p},s)\, e^{ip\cdot x}\, a(\vec{p},s) + v_\ell(\vec{p},s)\, e^{-ip\cdot x}\, a^\dagger(\vec{p},s) \right]. \tag{246}$$

Since $\mathcal{C} = \gamma_2 \beta$ it follows that $\mathcal{C}^{-1} = \beta\gamma_2$ and so that $\mathcal{J}^{*ab} = -\beta\mathcal{C}\mathcal{J}^{ab}\mathcal{C}^{-1}\beta = -\beta\gamma_2\beta\mathcal{J}^{ab}\beta\gamma_2\beta$. But $\beta\gamma_2\beta = -\beta\beta\gamma_2 = -i^2\gamma_0^2\gamma_2 = -\gamma_2$. So $\mathcal{J}^{*ab} = -\gamma_2\mathcal{J}^{ab}\gamma_2$. Thus

$$D^*(L) = e^{-i\omega_{ab}\mathcal{J}^{*ab}} = e^{-i\omega_{ab}(-\gamma_2\mathcal{J}^{ab}\gamma_2)} = \gamma_2 e^{i\omega_{ab}\mathcal{J}^{ab}}\gamma_2 = \gamma_2\, D(L)\, \gamma_2. \tag{247}$$

Now with SW's γ's,

$$\gamma_2\, u(\vec{0}, \pm\tfrac{1}{2}) = v(\vec{0}, \pm\tfrac{1}{2}) \quad \text{and} \quad \gamma_2\, v(\vec{0}, \pm\tfrac{1}{2}) = u(\vec{0}, \pm\tfrac{1}{2}). \tag{248}$$

Thus the hermitian conjugate of a Majorana field is

$$
\begin{aligned}
\psi^*(x) &= \sum_s \int \frac{d^3p}{(2\pi)^{3/2}} \left[u^*(\vec{p},s)\, e^{-ip\cdot x}\, a^\dagger(\vec{p},s) + v^*(\vec{p},s)\, e^{ip\cdot x}\, a(\vec{p},s)\right] \\
&= \sum_s \int \frac{d^3p}{(2\pi)^{3/2}} \left[v^*(\vec{p},s)\, e^{ip\cdot x}\, a(\vec{p},s) + u^*(\vec{p},s)\, e^{-ip\cdot x}\, a^\dagger(\vec{p},s)\right] \\
&= \sum_s \int \frac{d^3p}{(2\pi)^{3/2}} \left[D^*(L(p))\, v^*(\vec{0},s)\, e^{ip\cdot x}\, a(\vec{p},s) + D^*(L(p))\, u^*(\vec{0},s)\, e^{-ip\cdot x}\, a^\dagger(\vec{p},s)\right] \\
&= \gamma_2 \sum_s \int \frac{d^3p}{(2\pi)^{3/2}} \left[D(L(p))\gamma_2 v(\vec{0},s)\, e^{ip\cdot x}\, a(\vec{p},s) + D(L(p))\gamma_2 u(\vec{0},s)\, e^{-ip\cdot x}\, a^\dagger(\vec{p},s)\right] \\
&= \gamma_2 \sum_s \int \frac{d^3p}{(2\pi)^{3/2}} \left[D(L(p)) u(\vec{0},s)\, e^{ip\cdot x}\, a(\vec{p},s) + D(L(p)) v(\vec{0},s)\, e^{-ip\cdot x}\, a^\dagger(\vec{p},s)\right] \\
&= \gamma_2 \sum_s \int \frac{d^3p}{(2\pi)^{3/2}} \left[u(\vec{p},s)\, e^{ip\cdot x}\, a(\vec{p},s) + v(\vec{p},s)\, e^{-ip\cdot x}\, a^\dagger(\vec{p},s)\right] = \gamma_2\, \psi(x).
\end{aligned}
\tag{249}
$$

The parity rules (250) now are

$$
\mathsf{P}\psi^+_\ell(t,\vec{x})\mathsf{P}^{-1} = \beta\, \eta_a^*\, \psi^+_\ell(t,-\vec{x}) \quad \text{and} \quad \mathsf{P}\psi^-_\ell(t,\vec{x})\mathsf{P}^{-1} = -\,\beta\, \eta_b\, \psi^-_\ell(t,-\vec{x}). \tag{250}
$$

So to have a Dirac field survive a parity transformation, we need the phase of the particle to be minus the complex conjugate of the phase of the antiparticle

$$
\eta_a^* = -\,\eta_b \quad \text{or} \quad \eta_b = -\,\eta_a^*. \tag{251}
$$

So the intrinsic parity of a particle-antiparticle state is odd. So negative-parity bospns like $\pi^0, \rho_0, J/\psi$ can be interpreted as s-wave bound states of quark-antiquark pairs. Under parity a Dirac field goes as

$$
\mathsf{P}\psi(t,\vec{x})\mathsf{P}^{-1} = \eta^*\, \beta\, \psi(t,-\vec{x}). \tag{252}
$$

If a Dirac particle is the same as its antiparticle, then its intrinsic parity must be odd under complex conjugation, $\eta = -\eta^*$. So the intrinsic parity of a Majorana fermion must be imaginary

$$
\eta = \pm\, i. \tag{253}
$$

But this means that if we express a Dirac field ψ as a complex linear combination

$$
\psi = \frac{1}{\sqrt{2}}(\chi_1 + i\chi_2) \tag{254}
$$

of two Majorana fields with intrinsic parities $\eta_1^* = \pm i$ and $\eta_2^* = \pm i$, then under parity

$$\mathsf{P}\psi(t,\vec{x})\mathsf{P}^{-1} = \frac{1}{\sqrt{2}}\Big(\eta_1^*\,\beta\,\phi_1(t,-\vec{x}) + i\eta_2^*\,\beta\,\phi_2(t,-\vec{x})\Big) \tag{255}$$

so we need $\eta_1^* = \eta_2^*$ to have

$$\mathsf{P}\psi(t,\vec{x})\mathsf{P}^{-1} = \frac{1}{\sqrt{2}}\Big(\eta_1^*\,\beta\,\phi_1(t,-\vec{x}) + i\eta_2^*\,\beta\,\phi_2(t,-\vec{x})\Big) = \eta^*\,\beta\,\psi(t,-\vec{x}). \tag{256}$$

But in that case the Dirac field has intrinsic parity $\eta = \pm i$.

The equation (233) that shows how beta goes under $D(L(p))$

$$D(L(p))\,\beta\,D^{-1}(L(p)) = \,-\,i\,p^c\gamma_c/m \tag{257}$$

tells us that the spinors (208)

$$u(p,s) = \sqrt{\frac{m}{p^0}}\,D(L(p))\,u(\vec{0},s) \quad \text{and} \quad v(p,s) = \sqrt{\frac{m}{p^0}}\,D(L(p))\,v(\vec{0},s) \tag{258}$$

are eigenstates of $-i\,p^c\gamma_c/m$ with eigenvalues ± 1

$$\begin{aligned}
(\,-\,i\,p^c\gamma_c/m)\,u(p,s) &= D(L(p))\,\beta\,D^{-1}(L(p))\,\sqrt{\frac{m}{p^0}}\,D(L(p))\,u(\vec{0},s)\\
&= \sqrt{\frac{m}{p^0}}\,D(L(p))\,\beta\;u(\vec{0},s) = \sqrt{\frac{m}{p^0}}\,D(L(p))\;u(\vec{0},s) = u(p,s)\\
(\,-\,i\,p^c\gamma_c/m)\,v(p,s) &= D(L(p))\,\beta\,D^{-1}(L(p))\,\sqrt{\frac{m}{p^0}}\,D(L(p))\,v(\vec{0},s)\\
&= \sqrt{\frac{m}{p^0}}\,D(L(p))\,\beta\;v(\vec{0},s) = \,-\,\sqrt{\frac{m}{p^0}}\,D(L(p))\;v(\vec{0},s) = \,-\,v(p,s).
\end{aligned} \tag{259}$$

So

$$(i\,p^c\gamma_c + m)u(p,s) = 0 \quad \text{and} \quad (-i\,p^c\gamma_c + m)v(p,s) = 0 \tag{260}$$

which implies that a Dirac field obeys Dirac's equation

$$\begin{aligned}
(\gamma^a\,\partial_a + m)\psi(x) &= (\gamma^a\,\partial_a + m)\sum_s\int\frac{d^3p}{(2\pi)^{3/2}}\;\Big[u(\vec{p},s)\,e^{ip\cdot x}\,a(\vec{p},s) + v(\vec{p},s)\,e^{-ip\cdot x}\,b^\dagger(\vec{p},s)\Big]\\
&= \sum_s\int\frac{d^3p}{(2\pi)^{3/2}}\;\Big[(\gamma^a\,\partial_a + m)u(\vec{p},s)\,e^{ip\cdot x}\,a(\vec{p},s)\\
&\quad + (\gamma^a\,\partial_a + m)v(\vec{p},s)\,e^{-ip\cdot x}\,b^\dagger(\vec{p},s)\Big]\\
&= \sum_s\int\frac{d^3p}{(2\pi)^{3/2}}\;\Big[(i\gamma^a\,p_a + m)u(\vec{p},s)\,e^{ip\cdot x}\,a(\vec{p},s)\\
&\quad + (-i\gamma^a\,p_a + m)v(\vec{p},s)\,e^{-ip\cdot x}\,b^\dagger(\vec{p},s)\Big] = 0.
\end{aligned} \tag{261}$$

SW shows that under complex conjugation

$$u^*(p,s) = -\,\beta\mathcal{C}v(p,s) \quad \text{and} \quad u^*(p,s) = -\,\beta\mathcal{C}u(p,s). \tag{262}$$

So for a Dirac field to survive charge conjugation, the particle-antiparticle phases must be related

$$\xi_b = \xi_a^*. \tag{263}$$

Then under charge conjugation a Dirac field goes as

$$\mathsf{C}\,\psi(x)\,\mathsf{C}^{-1} = -\,\xi^*\,\beta\mathcal{C}\,\psi^*(x). \tag{264}$$

If a Dirac particle is the same as its antiparticle, then ξ must be real (and η imaginary), $\xi = \pm 1$, and must satisfy the reality condition

$$\psi(x) = -\,\beta\,\mathcal{C}\,\psi^*(x). \tag{265}$$

Suppose a particle and its antiparticle form a bound state

$$|\Phi\rangle = \sum_{ss'} \int d^3p\, d^3p' \chi(p,s;p',s')\, a^\dagger(p,s)\, b^\dagger(p's')\, |0\rangle. \tag{266}$$

Under charge conjugation

$$\begin{aligned}
\mathsf{C}\,|\Phi\rangle &= \xi_a\xi_b \sum_{ss'} \int d^3p\, d^3p' \chi(p,s;p',s')\, b^\dagger(p,s)\, a^\dagger(p's')\, |0\rangle \\
&= -\,\xi_a\xi_b \sum_{ss'} \int d^3p\, d^3p' \chi(p,s;p',s')\, a^\dagger(p's')\; b^\dagger(p,s)\, |0\rangle \\
&= -\,\xi_a\xi_b \sum_{ss'} \int d^3p\, d^3p' \chi(p',s';p,s)\, a^\dagger(p,s)\; b^\dagger(p',s')\, |0\rangle \\
&= -\,\xi_a\xi_b\, |\Phi\rangle = -\xi_a\xi_a^*\, |\Phi\rangle = -|\Phi\rangle.
\end{aligned} \tag{267}$$

The intrinsic charge-conjugation parity of a bound state of a particle and its antiparticle is odd.