Physics 523: Quantum Field Theory I

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Quantum fields and special relativity

1. States

A Lorentz transformation $\Lambda$ is implemented by a unitary operator $U(\Lambda)$ which replaces the state $|p, \sigma\rangle$ of a massive particle of momentum $p$ and spin $\sigma$ along the $z$-axis by the state

$$U(\Lambda)|p, \sigma\rangle = \sqrt{\frac{(\Lambda p)^0}{p^0}} \sum_{s'} D^{(j)}_{s'\sigma}(W(\Lambda, p)) |\Lambda p, s'\rangle \quad (1.1)$$

where $W(\Lambda, p)$ is a Wigner rotation

$$W(\Lambda, p) = L^{-1}(\Lambda p)\Lambda L(p) \quad (1.2)$$

and $L(p)$ is a standard Lorentz transformation that takes $(m, \vec{0})$ to $p$.

1.2 Creation operators

The vacuum is invariant under Lorentz transformations and translations

$$U(\Lambda, a)|0\rangle = |0\rangle. \quad (1.3)$$

A creation operator $a^\dagger(p, \sigma)$ makes the state $|p, \sigma\rangle$ from the vacuum state $|0\rangle$

$$|p\sigma\rangle = a^\dagger(p, \sigma)|0\rangle. \quad (1.4)$$

The creation and annihilation operators obey either the commutation relation

$$[a(p, s), a^\dagger(p', s')]_+ = a(p, s) a^\dagger(p', s') - a^\dagger(p', s') a(p, s) = \delta_{ss'} \delta^{(3)}(p - p') \quad (1.5)$$
or the anticommutation relation

\[ [a(p, s), a^\dagger(p', s')]_\mp = a(p, s) a^\dagger(p', s') + a^\dagger(p', s') a(p, s) = \delta_{ss'} \delta^{(3)}(p - p'). \]

(1.6)

The two kinds of relations are written together as

\[ [a(p, s), a^\dagger(p', s')]_\pm = a(p, s) a^\dagger(p', s') \mp a^\dagger(p', s') a(p, s) = \delta_{ss'} \delta^{(3)}(p - p'). \]

(1.7)

A bracket \([A, B]\) with no signed subscript is interpreted as a commutator.

Equations (1.1 & 1.4) give

\[ U(\Lambda) a^\dagger(p, \sigma) |0\rangle = \sqrt{(\Lambda p)^0 / p^0} \sum_{s'} D^{(j)}_{s'\sigma}(W(\Lambda, p)) a^\dagger(\Lambda p, s') |0\rangle. \]

(1.8)

And (1.3) gives

\[ U(\Lambda) a^\dagger(p, \sigma) U^{-1}(\Lambda) |0\rangle = \sqrt{(\Lambda p)^0 / p^0} \sum_{s'} D^{(j)}_{s'\sigma}(W(\Lambda, p)) a^\dagger(\Lambda p, s') |0\rangle. \]

(1.9)

SW in chapter 4 concludes that

\[ U(\Lambda) a^\dagger(p, \sigma) U^{-1}(\Lambda) = \sqrt{(\Lambda p)^0 / p^0} \sum_{s'} D^{(j)}_{s'\sigma}(W(\Lambda, p)) a^\dagger(\Lambda p, s'). \]

(1.10)

If \(U(\Lambda, b)\) follows \(\Lambda\) by a translation by \(b\), then

\[ U(\Lambda, b) a^\dagger(p, \sigma) U^{-1}(\Lambda, b) = e^{-i(\Lambda p)\cdot a} \sqrt{(\Lambda p)^0 / p^0} \sum_{s'} D^{(j)}_{s'\sigma}(W(\Lambda, p)) a^\dagger(\Lambda p, s') \]

\[ = e^{-i(\Lambda p)\cdot a} \sqrt{(\Lambda p)^0 / p^0} \sum_{s'} D^{(j)}_{s'\sigma}(W^{-1}(\Lambda, p)) a^\dagger(\Lambda p, s') \]

\[ = e^{-i(\Lambda p)\cdot a} \sqrt{(\Lambda p)^0 / p^0} \sum_{s'} D^{(j)}_{s's'}(W^{-1}(\Lambda, p)) a^\dagger(\Lambda p, s'). \]

(1.11)
1.3 How fields transform

The adjoint of this equation is

\[ U(\Lambda, b)a(p, \sigma)U^{-1}(\Lambda, b) = e^{i(\Lambda p) \cdot a} \sqrt{p^0} \sum_{s'} D^{(j)}_{s'}(W(\Lambda, p)) a(\Lambda p, s') \]

\[ = e^{i(\Lambda p) \cdot a} \sqrt{p^0} \sum_{s'} D^{(j)}_{s'}(W(\Lambda, p)) a(\Lambda p, s') \]

\[ = e^{i(\Lambda p) \cdot a} \sqrt{p^0} \sum_{s'} D^{(j)}_{s'}(W^{-1}(\Lambda, p)) a(\Lambda p, s'). \]

These equations (1.11 & 1.12) are (5.1.11 & 5.1.12) of SW.

1.3 How fields transform

The “positive frequency” part of a field is a linear combination of annihilation operators

\[ \psi^+_\ell(x) = \sum_{\sigma} \int d^3p \ u_\ell(x; p, \sigma) a(p, \sigma). \]  

(1.13)

The “negative frequency” part of a field is a linear combination of creation operators of the antiparticles

\[ \psi^-\ell(x) = \sum_{\sigma} \int d^3p \ v_\ell(x; p, \sigma) b^\dagger(p, \sigma). \]  

(1.14)

To have the fields (1.13 & 1.14) transform properly under Poincaré transformations

\[ U(\Lambda, a)\psi^+_\ell(x)U^{-1}(\Lambda, a) = \sum_\ell D_{\ell\ell}(\Lambda^{-1})\psi^+_\ell(\Lambda x + a) \]

\[ = \sum_\ell D_{\ell\ell}(\Lambda^{-1}) \sum_{\sigma} \int d^3p \ u_\ell(\Lambda x + a; p, \sigma) a(p, \sigma) \]

\[ U(\Lambda, a)\psi^-\ell(x)U^{-1}(\Lambda, a) = \sum_\ell D_{\ell\ell}(\Lambda^{-1})\psi^-\ell(\Lambda x + a) \]

\[ = \sum_\ell D_{\ell\ell}(\Lambda^{-1}) \sum_{\sigma} \int d^3p \ v_\ell(\Lambda x + a; p, \sigma) b^\dagger(p, \sigma) \]  

(1.15)

the spinors \( u_\ell(x; p, \sigma) \) and \( v_\ell(x; p, \sigma) \) must obey certain rules which we’ll now determine.
Quantum fields and special relativity

First (1.12 & 1.13) give

\[ U(\Lambda, a)\psi^+(x)U^{-1}(\Lambda, a) = U(\Lambda, a) \sum_\sigma \int d^3p \ u_\ell(x; p, \sigma) a(p, \sigma) U^{-1}(\Lambda, a) \]

\[ = \sum_\sigma \int d^3p \ u_\ell(x; p, \sigma) U(\Lambda, a) a(p, \sigma) U^{-1}(\Lambda, a) \]

\[ = \sum_\sigma \int d^3p \ u_\ell(x; p, \sigma) e^{i(\Lambda p) \cdot a} \sqrt{\frac{(\Lambda p)^0}{p^0}} \sum_{s'} D_{\sigma s'}^{(j)}(W^{-1}(\Lambda, p)) a(\Lambda p, s'). \]

(1.16)

Now we use the identity

\[ \frac{d^3p}{p^0} = \frac{d^3(\Lambda p)}{(\Lambda p)^0} \]

(1.17)

to turn (1.16) into

\[ U(\Lambda, a)\psi^+ (x)U^{-1}(\Lambda, a) = \sum_\sigma \int d^3(\Lambda p) \ u_\ell(x; p, \sigma) e^{i(\Lambda p) \cdot a} \]

\[ \times \sqrt{\frac{p^0}{(\Lambda p)^0}} \sum_{s'} D_{\sigma s'}^{(j)}(W^{-1}(\Lambda, p)) a(\Lambda p, s'). \]  

(1.18)

Similarly (1.11, 1.14, & 1.17) give

\[ U(\Lambda, a)\psi^- (x)U^{-1}(\Lambda, a) = U(\Lambda, a) \sum_\sigma \int d^3p \ v_\ell(x; p, \sigma) b^\dagger(p, \sigma) U^{-1}(\Lambda, a) \]

\[ = \sum_\sigma \int d^3p \ v_\ell(x; p, \sigma) U(\Lambda, a) b^\dagger(p, \sigma) U^{-1}(\Lambda, a) \]

\[ = \sum_\sigma \int d^3p \ v_\ell(x; p, \sigma) e^{-i(\Lambda p) \cdot a} \sqrt{\frac{(\Lambda p)^0}{p^0}} \sum_{s'} D_{\sigma s'}^{*(j)}(W^{-1}(\Lambda, p)) b^\dagger(\Lambda p, s') \]

\[ = \sum_\sigma \int d^3(\Lambda p) \ v_\ell(x; p, \sigma) e^{-i(\Lambda p) \cdot a} \sqrt{\frac{p^0}{(\Lambda p)^0}} \sum_{s'} D_{\sigma s'}^{*(j)}(W^{-1}(\Lambda, p)) b^\dagger(\Lambda p, s'). \]

(1.19)

So to get the fields to transform as in (1.15), equations (1.18 & 1.19) say
1.3 How fields transform

that we need

\[ \sum_{\ell} D_{\ell \ell}(\Lambda^{-1}) \psi^{\dagger}_{\ell}(\Lambda x + a) = \sum_{\ell} D_{\ell \ell}(\Lambda^{-1}) \sum_{\sigma} \int d^3 p \, u_{\ell}(\Lambda x + a; p, \sigma) a(p, \sigma) \]

\[ = \sum_{\ell} D_{\ell \ell}(\Lambda^{-1}) \sum_{\sigma} \int d^3 (\Lambda p) \, u_{\ell}(\Lambda x + a; \Lambda p, \sigma) a(\Lambda p, \sigma) \]

\[ = \sum_{\sigma} \int d^3 (\Lambda p) \, u_{\ell}(x; p, \sigma) e^{i(\Lambda p) \cdot a} \quad (1.20) \]

\[ \times \sqrt{\frac{p^0}{(\Lambda p)^0}} \sum_{s'} D^{(j)}_{s' \sigma}(W^{-1}(\Lambda, p)) a(\Lambda p, s') \]

\[ = \sum_{s'} \int d^3 (\Lambda p) \, u_{\ell}(x; p, s') e^{i(\Lambda p) \cdot a} \]

\[ \times \sqrt{\frac{p^0}{(\Lambda p)^0}} \sum_{\sigma} D^{(j)}_{s' \sigma}(W^{-1}(\Lambda, p)) a(\Lambda p, \sigma) \]

and

\[ \sum_{\ell} D_{\ell \ell}(\Lambda^{-1}) \psi^{\dagger}_{\ell}(\Lambda x + a) = \sum_{\ell} D_{\ell \ell}(\Lambda^{-1}) \sum_{\sigma} \int d^3 p \, v_{\ell}(\Lambda x + a; p, \sigma) b^{\dagger}(p, \sigma) \]

\[ = \sum_{\ell} D_{\ell \ell}(\Lambda^{-1}) \sum_{\sigma} \int d^3 (\Lambda p) \, v_{\ell}(\Lambda x + a; \Lambda p, \sigma) b^{\dagger}(\Lambda p, \sigma) \]

\[ = \sum_{\sigma} \int d^3 (\Lambda p) \, v_{\ell}(x; p, \sigma) e^{-i(\Lambda p) \cdot a} \quad (1.21) \]

\[ \times \sqrt{\frac{p^0}{(\Lambda p)^0}} \sum_{s'} D^{(j)}_{s' \sigma}(W^{-1}(\Lambda, p)) b^{\dagger}(\Lambda p, s') \]

\[ = \sum_{s'} \int d^3 (\Lambda p) \, v_{\ell}(x; p, s') e^{-i(\Lambda p) \cdot a} \]

\[ \times \sqrt{\frac{p^0}{(\Lambda p)^0}} \sum_{\sigma} D^{(j)}_{s' \sigma}(W^{-1}(\Lambda, p)) b^{\dagger}(\Lambda p, \sigma) . \]

Equating coefficients of the red annihilation and blue creation operators, we find that the fields will transform properly if the spinors \( u \) and \( v \) satisfy the
Quantum fields and special relativity

\[ \sum_{\ell} D_{\ell\ell}(\Lambda^{-1}) u_{\ell}(\Lambda x + a; \Lambda p, \sigma) = \sqrt{\frac{p^0}{(\Lambda p)^0}} \sum_{s'} D^{(j)}_{s',\sigma}(W^{-1}(\Lambda, p)) u_{\ell}(x; p, s') e^{i(Ap) \cdot a} \] (1.22)

\[ \sum_{\ell} D_{\ell\ell}(\Lambda^{-1}) v_{\ell}(\Lambda x + a; \Lambda p, \sigma) = \sqrt{\frac{p^0}{(\Lambda p)^0}} \sum_{s'} D^{(j)}_{s',\sigma}(W^{-1}(\Lambda, p)) v_{\ell}(x; p, s') e^{-i(Ap) \cdot a} \] (1.23)

which differ from SW’s by an interchange of the subscripts \( \sigma, s' \) on the rotation matrices \( D^{(j)} \). (I think SW has a typo there.) If we multiply both sides of these equations (1.22 & 1.23) by the two kinds of \( D \) matrices, then we get first

\[ \sum_{\ell, \ell'} D_{\ell\ell}(\Lambda) D_{\ell'\ell'}(\Lambda^{-1}) u_{\ell}(\Lambda x + a; \Lambda p, \sigma) = u_{\ell'}(\Lambda x + a; \Lambda p, \sigma) \]

\[ = \sqrt{\frac{p^0}{(\Lambda p)^0}} \sum_{s', \ell} D^{(j)}_{s',\sigma}(W^{-1}(\Lambda, p)) D_{\ell\ell}(\Lambda) u_{\ell}(x; p, s') e^{i(Ap) \cdot a} \] (1.24)

\[ \sum_{\ell, \ell'} D_{\ell\ell}(\Lambda) D_{\ell'\ell'}(\Lambda^{-1}) v_{\ell}(\Lambda x + a; \Lambda p, \sigma) = v_{\ell'}(\Lambda x + a; \Lambda p, \sigma) \]

\[ = \sqrt{\frac{p^0}{(\Lambda p)^0}} \sum_{s', \ell} D^{(j)}_{s',\sigma}(W^{-1}(\Lambda, p)) D_{\ell\ell}(\Lambda) v_{\ell}(x; p, s') e^{-i(Ap) \cdot a} \] (1.25)
and then with $W \equiv W(\Lambda, p)$

$$
\sum_{\sigma} D^{(j)}_{\sigma \bar{s}}(W) u_{\ell'}(\Lambda x + a; \Lambda p, \sigma) = \sqrt{p^0_0} \sum_{s'} D^{(j)}_{s' \sigma}(W^{-1}) D^{(j)}_{\sigma \bar{s}}(W) D_{\ell' \ell}(\Lambda) u_{\ell}(x; p, s') e^{i(\Lambda p) \cdot a}
\]

$$
= \sqrt{p^0_0} \sum_{s', \sigma} D_{\ell' \ell}(\Lambda) u_{\ell}(x; p, \bar{s}) e^{i(\Lambda p) \cdot a}
\]

$$
(1.26)
\]

$$
\sum_{\sigma} D^{*(j)}_{\sigma \bar{s}}(W) v_{\ell'}(\Lambda x + a; \Lambda p, \sigma) = \sqrt{p^0_0} \sum_{s', \sigma, \ell} D^{*(j)}_{s' \sigma}(W^{-1}) D^{*(j)}_{\sigma \bar{s}}(W) D_{\ell' \ell}(\Lambda) v_{\ell}(x; p, s') e^{-i(\Lambda p) \cdot a}
\]

$$
= \sqrt{p^0_0} \sum_{\ell} D_{\ell' \ell}(\Lambda) v_{\ell}(x; p, \bar{s}) e^{-i(\Lambda p) \cdot a}
\]

$$
(1.27)
\]

which are equations (5.1.13 & 5.1.14) of SW:

$$
\sum_{s} u_{\ell}(\Lambda x + a; \Lambda p, \bar{s}) D^{(j)}_{s \bar{s}}(W(\Lambda, p)) = \sqrt{p^0_0} \sum_{\ell} D_{\ell' \ell}(\Lambda) u_{\ell}(x; p, \sigma) e^{i(\Lambda p) \cdot a}
\]

$$
\sum_{s} v_{\ell}(\Lambda x + a; \Lambda p, \bar{s}) D^{*(j)}_{s \bar{s}}(W(\Lambda, p)) = \sqrt{p^0_0} \sum_{\ell} D_{\ell' \ell}(\Lambda) v_{\ell}(x; p, \sigma) e^{-i(\Lambda p) \cdot a}.
\]

$$
(1.28)
\]

These are the equations that determine the spinors $u$ and $v$ up to a few arbitrary phases.

1.4 Translations

When $\Lambda = I$, the $D$ matrices are equal to unity, and these last equations (1.28) say that for $x = 0$

$$
\begin{align*}
    u_{\ell}(a; p, \sigma) &= u_{\ell}(0; p, \sigma) e^{ip \cdot a} \\
    v_{\ell}(a; p, \sigma) &= v_{\ell}(0; p, \sigma) e^{-ip \cdot a}.
\end{align*}
\]

$$
(1.29)
\]

Thus the spinors $u$ and $v$ depend upon spacetime by the usual phase $e^{\pm ip \cdot x}$

$$
\begin{align*}
    u_{\ell}(x; p, \sigma) &= (2\pi)^{-3/2} u_{\ell}(p, \sigma) e^{ip \cdot x} \\
    v_{\ell}(x; p, \sigma) &= (2\pi)^{-3/2} v_{\ell}(p, \sigma) e^{-ip \cdot x}.
\end{align*}
\]

$$
(1.30)
\]
Quantum fields and special relativity

In which the $2\pi$’s are conventional. The fields therefore are Fourier transforms:

\[
\psi_+^\ell(x) = (2\pi)^{-3/2} \sum_\sigma \int d^3 p \, e^{ip\cdot x} u_\ell(p, \sigma) a(p, \sigma)
\]

\[
\psi_-^\ell(x) = (2\pi)^{-3/2} \sum_\sigma \int d^3 p \, e^{-ip\cdot x} v_\ell(p, \sigma) b^\dagger(p, \sigma)
\]

and every field of mass $m$ obeys the Klein-Gordon equation

\[
(\nabla^2 - \partial_0^2 - m^2) \psi_\ell(x) = 0.
\]

Since $\exp[i(\Lambda p \cdot (\Lambda x + a))] = \exp(ip \cdot x + i\Lambda p \cdot a)$, the conditions simplify to

\[
\sum_s u_\ell(\Lambda p, \bar{s}) D_s^{(j)}(W(\Lambda, p)) = \sqrt{\frac{p^0}{(\Lambda p)^0}} \sum_\ell D_{\bar{\ell}\ell}(\Lambda) u_\ell(p, \sigma)
\]

\[
\sum_s v_\ell(\Lambda p, \bar{s}) D_s^{* (j)}(W(\Lambda, p)) = \sqrt{\frac{p^0}{(\Lambda p)^0}} \sum_\ell D_{\bar{\ell}\ell}(\Lambda) v_\ell(p, \sigma)
\]

for all Lorentz transformations $\Lambda$.

1.5 Boosts

Set $p = k = (m, \vec{0})$ and $\Lambda = L(q)$ where $L(q)k = q$. So $L(p) = 1$ and

\[
W(\Lambda, p) = L^{-1}(\Lambda p) A L(p) = L^{-1}(q) L(q) = 1.
\]

Then the equations are

\[
u_\ell(q, \sigma) = \sqrt{\frac{m}{q^0}} \sum_\ell D_{\bar{\ell}\ell}(L(q)) u_\ell(\bar{0}, \sigma)
\]

\[
v_\ell(q, \sigma) = \sqrt{\frac{m}{q^0}} \sum_\ell D_{\bar{\ell}\ell}(L(q)) v_\ell(\bar{0}, \sigma).
\]

Thus a spinor at finite momentum is given by a representation $D(\Lambda)$ of the Lorentz group (see the online notes of chapter 10 of my book for its finite-dimensional nonunitary representations) acting on the spinor at zero 3-momentum $p = k = (m, \vec{0})$. We need to find what these spinors are.
1.6 Rotations

Now set $p = k = (m, \vec{0})$ and $\Lambda = R$ a rotation so that $W = R$. For rotations, the spinor conditions (1.33) are

$$
\sum_{\bar{s}} u_{\ell}(\vec{0}, \bar{s}) D_{s\bar{s}}^{(j)}(R) = \sum_{\ell} D_{\ell\ell}(R) u_{\ell}(\vec{0}, \sigma) \quad (1.36)
$$

$$
\sum_{\bar{s}} v_{\ell}(\vec{0}, \bar{s}) D_{s\bar{s}}^{*(j)}(R) = \sum_{\ell} D_{\ell\ell}(R) v_{\ell}(\vec{0}, \sigma).
$$

The representations $D_{s\bar{s}}^{(j)}(R)$ of the rotation group are $(2j + 1) \times (2j + 1)$-dimensional unitary matrices. For a rotation of angle $\theta$ about the $\vec{\theta} = \theta$ axis, they are the ones taught in courses on quantum mechanics (and discussed in the notes of chapter 10)

$$
D_{s\bar{s}}^{(j)}(\theta) = \left[ e^{-i\theta \cdot J^{(j)}} \right]_{s\bar{s}} \quad (1.37)
$$

where $[J_a, J_b] = i\epsilon_{abc} J_c$. The representations $D_{\ell\ell}(R)$ of the rotation group are finite-dimensional unitary matrices. For a rotation of angle $\theta$ about the $\vec{\theta} = \theta$ axis, they are

$$
D_{\ell\ell}(\theta) = \left[ e^{-i\theta \cdot J} \right]_{\ell\ell} \quad (1.38)
$$

in which $[J_a, J_b] = i\epsilon_{abc} J$. For tiny rotations, the conditions (1.36) require (because of the complex conjugation of the antiparticle condition) that the spinors obey the rules

$$
\sum_{\bar{s}} u_{\ell}(\vec{0}, \bar{s})(J_a^{(j)})_{s\bar{s}} = \sum_{\ell} (J_a)_{\ell\ell} u_{\ell}(\vec{0}, \sigma) \quad (1.39)
$$

$$
\sum_{\bar{s}} v_{\ell}(\vec{0}, \bar{s})(-J_a)_{s\bar{s}} = \sum_{\ell} (J_a)_{\ell\ell} v_{\ell}(\vec{0}, \sigma)
$$

for $a = 1, 2, 3$.

1.7 Spin-zero fields

Spin-zero fields have no spin or Lorentz indexes. So the boost conditions (1.210) merely require that $u(q) = \sqrt{m/q^0} u(0)$ and $v(q) = \sqrt{m/q^0} v(0)$. The conventional normalization is $u(0) = 1/\sqrt{2m}$ and $v(0) = 1/\sqrt{2m}$. The spin-zero spinors then are

$$
u(p) = (2p^0)^{-1/2} \quad \text{and} \quad v(p) = (2p^0)^{-1/2} \quad (1.40)
$$

For simplicity, let’s first consider a neutral scalar field so that $b(p, s) = 0$. ...
Quantum fields and special relativity

The definitions (1.13) and (1.14) of the positive-frequency and negative-frequency fields and their behavior (1.30) under translations then give us

$$\phi^+ (x) = \int \frac{d^3 p}{\sqrt{(2\pi)^3 2 p^0}} a(p) e^{ip \cdot x}$$

$$\phi^- (x) = \int \frac{d^3 p}{\sqrt{(2\pi)^3 2 p^0}} a^\dagger (p) e^{-ip \cdot x}. \quad (1.41)$$

Note that

$$[\phi^\pm (x)]^\dagger = \phi^{\mp (x)}.$$ \hspace{1cm} (1.42)

Since $[a(p), a(p')]_\pm = 0$, it follows that

$$[\phi^+ (x), \phi^+ (y)]_\mp = 0 \quad \text{and} \quad [\phi^- (x), \phi^- (y)]_\mp = 0 \quad (1.43)$$

whatever the values of $x$ and $y$ as long as we use commutators for bosons and anticommutators for fermions.

But the commutation relation

$$[a(p, s), a^\dagger (q, t)]_\mp = \delta_{st} \delta^{(3)}(p - q) \quad (1.44)$$

makes the commutator

$$[\phi^+ (x), \phi^-(y)]_\mp = \int \frac{d^3 p d^3 p'}{(2\pi)^3 \sqrt{2 p^0 2 p'^0}} e^{ip \cdot x} e^{-ip' \cdot y} \delta^{(3)}(p - p')$$

$$= \int \frac{d^3 p}{(2\pi)^3 2 p^0} e^{ip \cdot (x - y)} = \Delta_+ (x - y) \quad (1.45)$$

nonzero even for $(x - y)^2 > 0$ as we’ll now verify.

For space-like $x$, the Lorentz-invariant function $\Delta_+ (x)$ can only depend upon $x^2 > 0$ since the time $x^0$ and its sign are not Lorentz invariant. So we choose a Lorentz frame with $x^0 = 0$ and $|x| = \sqrt{x^2}$. In this frame,

$$\Delta_+ (x) = \int \frac{d^3 p}{(2\pi)^3 2 \sqrt{p^2 + m^2}} e^{ipx}$$

$$= \int \frac{p^2 dp \, d \cos \theta}{(2\pi)^2 2 \sqrt{p^2 + m^2}} e^{ipx \cos \theta} \quad (1.46)$$

where $p = |p|$ and $x = |x|$. Now

$$\int d \cos \theta e^{ipx \cos \theta} = (e^{ipx} - e^{-ipx}) / (ipx) = 2 \sin (px) / (px), \quad (1.47)$$

so the integral (1.46) is

$$\Delta_+ (x) = \frac{1}{4\pi^2 x} \int_0^\infty \frac{\sin (px) \, p \, dp}{\sqrt{p^2 + m^2}} \quad (1.48)$$
1.7 Spin-zero fields

With \( u \equiv p/m \)

\[
\Delta_+(x) = \frac{m}{4\pi^2x} \int_0^\infty \frac{\sin(mxu)\,udu}{\sqrt{u^2 + 1}} = \frac{m}{4\pi^2x} K_1(mx^2) \tag{1.49}
\]

a Hankel function.

To get a Lorentz-invariant, causal theory, we use the arbitrary parameters \( \kappa \) and \( \lambda \) setting

\[
\phi(x) = \kappa \phi^+(x) + \lambda \phi^-(x) \tag{1.50}
\]

Now the adjoint rule (1.42) and the commutation relations (1.45 and 1.45) give

\[
\begin{align*}
[\phi(x), \phi^\dagger(y)] &= [\kappa \phi^+(x) + \lambda \phi^-(x), \kappa^* \phi^-(y) + \lambda^* \phi^+(y)] \\
&= |\kappa|^2 [\phi^+(x), \phi^-(y)] + |\lambda|^2 [\phi^-(x), \phi^+(y)] \\
&= |\kappa|^2 \Delta_+(x-y) + |\lambda|^2 \Delta_+(y-x) \\
[\phi(x), \phi(y)] &= [\kappa \phi^+(x) + \lambda \phi^-(x), \kappa \phi^+(y) + \lambda \phi^-(y)] \\
&= \kappa \lambda \left([\phi^+(x), \phi^-(y)] + [\phi^-(x), \phi^+(y)]\right) \\
&= \kappa \lambda \left(\Delta_+(x-y) - \Delta_+(y-x)\right). \tag{1.51}
\end{align*}
\]

But when \( (x-y)^2 > 0, \Delta_+(x-y) = \Delta_+(y-x) \). Thus these conditions are

\[
\begin{align*}
[\phi(x), \phi^\dagger(y)] &= (|\kappa|^2 + |\lambda|^2) \Delta_+(x-y) \\
[\phi(x), \phi(y)] &= \kappa \lambda \Delta_+(x-y)(1 + 1). \tag{1.52}
\end{align*}
\]

The first of these equations implies that we choose the minus sign and so that we use commutation relations and not anticommutation relations for spin-zero fields. This is the spin-statistics theorem for spin-zero fields. SW proves the theorem for arbitrary massive fields in section 5.7.

We also must set

\[
|\kappa| = |\lambda|. \tag{1.53}
\]

The second equation then is automatically satisfied. The common magnitude and the phases of \( \kappa \) and \( \lambda \) are arbitrary, so we choose \( \kappa = \lambda = 1 \). We then have

\[
\phi(x) = \phi^+(x) + \phi^-(x) = \phi^+(x) + \phi^+(x) = \phi^+(x). \tag{1.54}
\]

Now the interaction density \( \mathcal{H}(x) \) will commute with \( \mathcal{H}(y) \) for \( (x-y)^2 > 0 \), and we have a chance of having a Lorentz-invariant, causal theory.

The field (1.54)

\[
\phi(x) = \int \frac{d^3p}{\sqrt{(2\pi)^32p^0}} \left[a(p) e^{ip\cdot x} + a^\dagger(p) e^{-ip\cdot x}\right] \tag{1.55}
\]
obey the **Klein-Gordon equation**

\[
(\nabla^2 - \partial_0^2 - m^2) \phi(x) \equiv (\Box - m^2) \phi(x) = 0. \quad (1.56)
\]

### 1.8 Conserved charges

If the field \( \phi \) adds and deletes charged particles, an interaction \( \mathcal{H}(x) \) that is a polynomial in \( \phi \) will not commute with the charge operator \( Q \) because \( \phi^+ \) will lower the charge and \( \phi^- \) will raise it. The standard way to solve this problem is to start with two hermitian fields \( \phi_1 \) and \( \phi_2 \) of the same mass. One defines a complex scalar field as a complex linear combination of the two fields

\[
\phi(x) = \frac{1}{\sqrt{2}} (\phi_1(x) + i\phi_2(x))
\]

\[
= \int \frac{d^3p}{\sqrt{(2\pi)^32p^0}} \left[ \frac{1}{\sqrt{2}} (a_1(p) + ia_2(p)) e^{ip\cdot x} + \frac{1}{\sqrt{2}} (a_1^+(p) + ia_2^+(p)) e^{-ip\cdot x} \right].
\]

Setting

\[
a(p) = \frac{1}{\sqrt{2}} (a_1(p) + ia_2(p)) \quad \text{and} \quad b^\dagger(p) = \frac{1}{\sqrt{2}} (a_1^+(p) + ia_2^+(p)) \quad (1.58)
\]

so that

\[
b(p) = \frac{1}{\sqrt{2}} (a_1(p) - ia_2(p)) \quad \text{and} \quad a^\dagger(p) = \frac{1}{\sqrt{2}} (a_1^+(p) - ia_2^+(p)) \quad (1.59)
\]

we have

\[
\phi(x) = \int \frac{d^3p}{\sqrt{(2\pi)^32p^0}} \left[ a(p) e^{ip\cdot x} + b^\dagger(p) e^{-ip\cdot x} \right] \quad (1.60)
\]

and

\[
\phi^\dagger(x) = \int \frac{d^3p}{\sqrt{(2\pi)^32p^0}} \left[ b(p) e^{ip\cdot x} + a^\dagger(p) e^{-ip\cdot x} \right]. \quad (1.61)
\]

Since the commutation relations of the real creation and annihilation operators are for \( i, j = 1, 2 \)

\[
[a_i(p), a_j^\dagger(p')] = \delta_{ij} \delta^3(p - p') \quad \text{and} \quad [a_i(p), a_j(p')] = 0 = [a_i^\dagger(p), a_j^\dagger(p')]
\]

(1.62)

the commutation relations of the complex creation and annihilation operators are

\[
[a(p), a^\dagger(p')] = \delta^3(p - p') \quad \text{and} \quad [b(p), b^\dagger(p')] = \delta^3(p - p')
\]

(1.63)
1.9 Parity, charge conjugation, and time reversal

with all other commutators vanishing.

Now \( \phi(x) \) lowers the charge of a state by \( q \) if \( a^\dagger \) adds a particle of charge \( q \) and if \( b^\dagger \) adds a particle of charge \(-q\). Similarly, \( \phi^\dagger(x) \) raises the charge of a state by \( q \)

\[
[Q, \phi(x)] = -q \phi(x) \quad \text{and} \quad [Q, \phi^\dagger(x)] = q \phi^\dagger(x). \tag{1.64}
\]

So an interaction with as many \( \phi(x) \)'s as \( \phi^\dagger(x) \)'s conserves charge.

1.9 Parity, charge conjugation, and time reversal

If the unitary operator \( P \) represents parity on the creation operators

\[
P a_1^\dagger(p) P^{-1} = \eta a_1^\dagger(-p) \quad \text{and} \quad P a_2^\dagger(p) P^{-1} = \eta a_2^\dagger(-p) \tag{1.65}
\]

with the same phase \( \eta \). Then

\[
P a_1(p) P^{-1} = \eta^* a_1(-p) \quad \text{and} \quad P a_2(p) P^{-1} = \eta^* a_2(-p) \tag{1.66}
\]

and so both

\[
P a_1^\dagger(p) P^{-1} = \eta a_1(-p) \quad \text{and} \quad P a(p) P^{-1} = \eta^* a(-p) \tag{1.67}
\]

and

\[
P b_1^\dagger(p) P^{-1} = \eta b_1(-p) \quad \text{and} \quad P b(p) P^{-1} = \eta^* b(-p). \tag{1.68}
\]

Thus if the field

\[
\phi_1(x) = \int \frac{d^3p}{\sqrt{(2\pi)^3 2p^0}} \left[ a_1(p) e^{ip \cdot x} + a_1^\dagger(p) e^{-ip \cdot x} \right] \tag{1.69}
\]

or \( \phi_2(x) \), or the complex field \( \phi(x) \) is to go into a multiple of itself under parity, then we need \( \eta = \eta^* \) so that \( \eta \) is real. Then the fields transform under parity as

\[
P \phi_1(x) P^{-1} = \eta^* \phi_1(x^0, -x) = \eta \phi_1(x^0, -x) \tag{1.70}
\]

\[
P \phi_2(x) P^{-1} = \eta^* \phi_2(x^0, -x) = \eta \phi_2(x^0, -x)
\]

\[
P \phi(x) P^{-1} = \eta^* \phi(x^0, -x) = \eta \phi(x^0, -x).
\]

Since \( P^2 = I \), we must have \( \eta = \pm 1 \). SW allows for a more general phase by having parity act with the same phase on \( a \) and \( b^\dagger \). Both schemes imply that the parity of a hermitian field is \( \pm 1 \) and that the state

\[
|ab\rangle = \int d^3p f(p^2) a_1^\dagger(p) b_1^\dagger(-p) |0\rangle \tag{1.71}
\]

has even or positive parity, \( P|ab\rangle = |ab\rangle \).
Charge conjugation works similarly. If the unitary operator $C$ represents charge conjugation on the creation operators

$$Ca_1^\dagger(p)C^{-1} = \xi a_1^\dagger(p) \quad \text{and} \quad Ca_2^\dagger(p)C^{-1} = -\xi a_2^\dagger(p) \quad (1.72)$$

with the same phase $\xi$. Then

$$Ca_1(p)C^{-1} = \xi^* a_1(p) \quad \text{and} \quad Ca_2(p)C^{-1} = -\xi^* a_2(p) \quad (1.73)$$

and so since $a = (a_1 + ia_2)/\sqrt{2}$ and $b = (a_1 - ia_2)/\sqrt{2}$

$$Ca(p)C^{-1} = \xi^* b(p) \quad \text{and} \quadCb(p)C^{-1} = \xi^* a(p) \quad (1.74)$$

and since $a^\dagger = (a_1^\dagger - ia_2^\dagger)/\sqrt{2}$ and $b^\dagger = (a_1^\dagger + ia_2^\dagger)/\sqrt{2}$

$$Ca^\dagger(p)C^{-1} = \xi b^\dagger(p) \quad \text{and} \quadCb^\dagger(p)C^{-1} = \xi a^\dagger(p) \quad (1.75)$$

Thus under charge conjugation, the field $(1.60)$ becomes

$$C\phi(x)C^{-1} = \int \frac{d^3p}{\sqrt{(2\pi)^32^3}} \left[ \xi^* b(p) e^{ipx} + \xi a^\dagger(p) e^{-ipx} \right] \quad (1.76)$$

and so if it is to go into a multiple of itself or of its adjoint under charge conjugation then we need $\xi = \xi^*$ so that $\xi$ is real. We then get

$$C\phi(x)C^{-1} = \xi^* \phi^\dagger(x) = \xi \phi^\dagger(x) \quad (1.77)$$

Since $C^2 = I$, we must have $\xi = \pm 1$. SW allows for a more general phase by having charge conjugation act with the same phase on $a$ and $b^\dagger$. Both schemes imply that the charge-conjugation parity of a hermitian field is $\pm 1$ and that the state

$$|ab\rangle = \int d^3p f(p^2) a^\dagger(p) b^\dagger(p)|0\rangle \quad (1.78)$$

has even or positive charge-conjugation parity, $\zeta|ab\rangle = |ab\rangle$.

The time-reversal operator $T$ is antilinear and antunitary. So if

$$Ta_1(p)T^{-1} = \zeta^* a_1(-p) \quad \text{and} \quadTa_2(p)T^{-1} = -\zeta^* a_2(-p) \quad (1.79)$$

$$Ta_1^\dagger(p)T^{-1} = \zeta a_1^\dagger(-p) \quad \text{and} \quadTa_2^\dagger(p)T^{-1} = -\zeta a_2^\dagger(-p)$$

then

$$Ta(p)T^{-1} = T \frac{1}{\sqrt{2}} (a_1(p) + ia_2(p))T^{-1} = \frac{1}{\sqrt{2}} (Ta_1(p)T^{-1} - iTa_2(p)T^{-1})$$

$$= \zeta^* \frac{1}{\sqrt{2}} (a_1(-p) + ia_2(-p)) = \zeta^* a(-p) \quad (1.80)$$
and
\[ T b^i(p) T^{-1} = \frac{1}{\sqrt{2}} \left( a_1^i(p) + i a_2^i(p) \right) T^{-1} = \frac{1}{\sqrt{2}} (T a_1^i(p) T^{-1} - i T a_2^i(p) T^{-1}) \]
\[ = \zeta \frac{1}{\sqrt{2}} (a_1^i(-p) + i a_2^i(-p)) = \zeta b^i(-p) \]
(1.81)

then one has
\[ T \phi(x) T^{-1} = T \int \frac{d^3p}{\sqrt{(2\pi)^3} 2p^0} \left[ a(p) e^{ip \cdot x} + b^i(p) e^{-ip \cdot x} \right] T^{-1} \]
\[ = \int \frac{d^3p}{\sqrt{(2\pi)^3} 2p^0} \left[ T a(p) T^{-1} e^{-ip \cdot x} + T b^i(p) T^{-1} e^{ip \cdot x} \right] \]
(1.82)
\[ = \int \frac{d^3p}{\sqrt{(2\pi)^3} 2p^0} \left[ \zeta^* a(-p) e^{ip \cdot x} + \zeta b^i(-p) e^{ip \cdot x} \right]. \]

So if \( \zeta \) is real, then after replacing \( -p \) by \( p \), we get
\[ T \phi(x) T^{-1} = \zeta^* \int \frac{d^3p}{\sqrt{(2\pi)^3} 2p^0} \left[ a(p) e^{ip \cdot x} + b^i(p) e^{-ip \cdot x} \right] \]
\[ = \zeta^* \phi(-x^0, x) = \zeta \phi(-x^0, x). \]
(1.83)

Since \( T^2 = I \), the phase \( \zeta = \pm 1 \). SW lets \( \zeta \) be complex but defined only for complex scalar fields and not for their real and imaginary parts.

### 1.10 Vector fields

Vector fields transform like the 4-vector \( x^i \) of spacetime. So
\[ D_{\ell \ell}(\Lambda) = \Lambda^{\ell}_{\ell} \]
(1.84)

for \( \ell, \ell = 0, 1, 2, 3 \). Again we start with a hermitian field labelled by \( i = 0, 1, 2, 3 \)
\[ \phi^{+i}(x) = (2\pi)^{-3/2} \sum_s \int d^3p \ e^{ip \cdot x} u^i(p, s) a(p, s) \]
\[ \phi^{-i}(x) = (2\pi)^{-3/2} \sum_s \int d^3p \ e^{-ip \cdot x} u^i(p, s) a^i(p, s). \]
(1.85)
The boost conditions (1.210) say that
\[ u^i(p, s) = \sqrt{\frac{m}{p^0}} \sum_k L(p)^i_k u^k(\vec{0}, s) \]
\[ v^i(p, s) = \sqrt{\frac{m}{p^0}} \sum_k L(p)^i_k v^k(\vec{0}, s). \]  
(1.86)

The rotation conditions (1.39) give
\[ \sum \bar{s} u^i(\vec{0}, \bar{s}) (J^{(j)}_a)_{\bar{s}s} = \sum_k (J^{(j)}_a)^i_k u^k(\vec{0}, s) \]
\[ - \sum \bar{s} v^i(\vec{0}, \bar{s}) (J^{*(j)}_a)_{\bar{s}s} = \sum_k (J^{(j)}_a)^i_k v^k(\vec{0}, s). \]  
(1.87)

The \((2j+1) \times (2j+1)\) matrices \((J^{(j)}_a)_{\bar{s}s}\) are the generators of the \((2j+1) \times (2j+1)\) representation of the rotation group. (See my online notes on group theory.) You learned that
\[ \sum_{a=1}^3 \left[ (J^{(j)}_a)^2 \right]_{\bar{s}s'} = \sum_{a=1}^3 \sum_{s=-j}^j (J^{(j)}_a)^i_s (J^{(j)}_a)^{i's'} = j(j+1)\delta_{ss'} \]
(1.88)
and that
\[ J_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad J_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \]  
(1.89)
in courses on quantum mechanics.

For \(k = 1, 2, 3\), the three \(4 \times 4\) matrices \((J^{(j)}_k)^i_j\) are the generators of rotations in the vector representation of the Lorentz group. Their nonzero components are
\[ (J^{(j)}_k)^i_j = -i\epsilon_{ijk} \]  
(1.90)
for \(i,j,k = 1,2,3\), while \((J^{(j)}_k)^i_0 = 0\), \((J^{(j)}_k)^0_j = 0\), and \((J^{(j)}_k)^i_0 = 0\) for \(i,j,k = 1,2,3\). So
\[ (J^{(j)}_k)^i_j = 2\delta^i_j \]  
(1.91)
with \((J^{(j)}_k)^0_0 = 0\), \((J^{(j)}_k)^0_j = 0\), and \((J^{(j)}_k)^i_0 = 0\) for \(i,j = 1,2,3\). Apart from a factor of \(i\), the \(J_k\)'s are the \(4 \times 4\) matrices \(J_a = iR_a\) of my online notes on
the Lorentz group

\[ J_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \]  

(1.92)

Since \((J_a)^0_k = 0\) for \(a, k = 1, 2, 3\), the spin conditions \([1.87]\) give for \(i = 0\)

\[ \sum_s u^0(\vec{0}, s)(J_{a(j)}^{ss}) = 0 \quad \text{and} \quad -\sum_s v^0(\vec{0}, s)(J_{a(j)}^{ss}) = 0. \]  

(1.93)

Multiplying these equations from the right by \((J_{a(j)}^{ss'})\) while summing over \(a = 1, 2, 3\) and using the formula \([1.88]\) \(\left((J_{a(j)}^{j})^2\right)_{ss'} = j(j + 1) \delta_{ss'}\), we find

\[ j(j + 1) u^0(\vec{0}, s) = 0 \quad \text{and} \quad j(j + 1) v^0(\vec{0}, s) = 0. \]  

(1.94)

Thus \(u^0(\vec{0}, \sigma)\) and \(v^0(\vec{0}, \sigma)\) can be anything if the field represents particles of spin \(j = 0\), but \(u^0(\vec{0}, \sigma)\) and \(v^0(\vec{0}, \sigma)\) must both vanish if the field represents particles of spin \(j > 0\).

Now we set \(i = 1, 2, 3\) in the spin conditions \([1.87]\) and again multiply from the right by \((J_{a(j)}^{ss'})\) while summing over \(a = 1, 2, 3\) and using the formula \([1.88]\) \((J_{a(j)}^{j})^2 = j(j + 1)\). The Lorentz rotation matrices generate a \(j = 1\) representation of the group of rotations.

\[ \sum_{k=1}^{3} (J_a)^i_k (J_a)^{j_k} = j(j + 1) \delta_{j_k} = 2\delta_{j_k}. \]  

(1.95)

So the remaining conditions on the fields are

\[ j(j + 1) u^i(\vec{0}, s') = \sum_{ssa} u^i(\vec{0}, s)(J_{a(j)}^{ss}) (J_{a(j)}^{ss'}) = \sum_{k=s} (J_a)^i_k u^k(\vec{0}, s) (J_{a(j)}^{ss'}) = \sum_{k=s} (J_a)^i_k u^k(\vec{0}, s') = 2 \delta_{i} u^i(\vec{0}, s') \]

\[ j(j + 1) v^i(\vec{0}, s') = \sum_{ssa} v^i(\vec{0}, s)(J_{a(j)}^{ss}) (J_{a(j)}^{ss'}) = \sum_{k=s} (J_a)^i_k v^k(\vec{0}, s) (J_{a(j)}^{ss'}) = \sum_{k=s} (J_a)^i_k v^k(\vec{0}, s') = 2 \delta_{i} v^i(\vec{0}, s'). \]  

(1.96)

Thus if \(j = 0\), then for \(i = 1, 2, 3\) both \(u^i(\vec{0}, s)\) and \(v^i(\vec{0}, s)\) must vanish, while if \(j > 0\), then since \(j(j + 1) = 2\), the spin \(j\) must be unity, \(j = 1\).
Quantum fields and special relativity

1.11 Vector field for spin-zero particles

The only nonvanishing components are constants taken conventionally as

\[ u^0(\vec{0}) = i\sqrt{m/2} \quad \text{and} \quad v^0(\vec{0}) = -i\sqrt{m/2}. \]  \hfill (1.97)

At finite momentum the boost conditions (1.210) give them as

\[ u^\mu(\vec{p}) = ip^\mu/\sqrt{2p^0} \quad \text{and} \quad v^\mu(\vec{p}) = -ip^\mu/\sqrt{2p^0}. \]  \hfill (1.98)

The vector field \( \phi^\mu(x) \) of a spin-zero particle is then the derivative of a scalar field \( \phi(x) \)

\[ \phi^\mu(x) = \partial^\mu \phi(x) = \int \frac{d^3p}{\sqrt{(2\pi)^32p^0}} \left[ ip^\mu a(p) e^{ip\cdot x} - ip^\mu b^\dagger(p) e^{-ip\cdot x} \right] \]  \hfill (1.99)

1.12 Vector field for spin-one particles

We start with the \( s = 0 \) spinors \( u^i(\vec{0},0) \) and \( v^i(\vec{0},0) \) and note that since \((J^3)_{i0} = 0\), the \( a = 3 \) rotation conditions (1.87) imply that

\[ (J^3)^i_k u^k(\vec{0},0) = iR^i_3 u^i(\vec{0},0) = 0 \quad \text{and} \quad (J^3)^i_k v^k(\vec{0},0) = iR^i_3 v^i(\vec{0},0) = 0. \]  \hfill (1.100)

Referring back to the explicit formulas for the generators of rotations and setting \( u, v = (0, x, y, z) \) we see that

\[ J^3 u = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ -iy \\ x \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \]  \hfill (1.101)

and

\[ J^3 v = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ -iy \\ x \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \]  \hfill (1.102)

Thus only the 3-component \( z \) can be nonzero. The conventional choice is

\[ u^\mu(\vec{0},0) = v^\mu(\vec{0},0) = \frac{1}{\sqrt{2m}} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \]  \hfill (1.103)

We now form the linear combinations of the rotation conditions (1.87)
that correspond to the raising and lowering matrices $J^{(1)}_{\pm} = J^{(1)}_1 \pm iJ^{(1)}_2$

\[
J^{(1)}_+ = \sqrt{2} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad J^{(1)}_- = \sqrt{2} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{1.104}
\]

Their Lorentz counterparts are

\[
J^{(1)}_{\pm} = J^{(1)}_1 \pm iJ^{(1)}_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \mp 1 \\ 0 & \pm 1 & 0 \end{pmatrix}. \tag{1.105}
\]

In these terms, the rotation conditions (1.87) for the $j = 1$ spinors $u^i(\vec{0}, s)$ are

\[
\sum_s u^i(\vec{0}, \bar{s})(J^{(1)}_{\pm})_{ss} = \sum_k (J^{(1)}_{\pm})^i_j u^k(\vec{0}, s). \tag{1.106}
\]

But

\[
J^{(1)}_1 \pm iJ^{(1)}_2 = J^{(1)}_1 \mp iJ^{(1)}_2 = J^\mp.
\]

So the rotation conditions (1.87) for the $j = 1$ spinors $v^i(\vec{0}, s)$ are

\[
-\sum_s v^i(\vec{0}, \bar{s})(J^{(1)}_\mp)_{ss} = \sum_k (J^{(1)}_{\mp})^i_j v^k(\vec{0}, s). \tag{1.108}
\]

So for the plus sign and the choice $s = 0$, the condition (1.106) gives $u^i(\vec{0}, 1)$ as

\[
\sum_s u^i(\vec{0}, s) J^{(1)}_{+s0} = \sqrt{2} u^i(\vec{0}, 1) = (J^+_*)_k u^k(\vec{0}, 0) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -i \\ 0 & 1 & i & 0 \end{pmatrix} \frac{1}{\sqrt{2m}} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}
\]

or

\[
u^i(\vec{0}, 1) = \frac{1}{2\sqrt{m}} \begin{pmatrix} 0 \\ -1 \\ -i \\ 0 \end{pmatrix} \tag{1.109}
\]
Similarly, the minus sign and the choice $s = 0$ give for $u^i(\vec{0}, -1)$

$$
\sum_s u^i(\vec{0}, \bar{s}) J^1_{|s,0} = \sqrt{2} u^i(\vec{0}, -1) = (J_-)^i_k u^k(\vec{0}, 0) = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & -i \\
0 & -1 & i & 0
\end{pmatrix} \frac{1}{\sqrt{2m}} \begin{pmatrix}
0 \\
0 \\
-\frac{1}{\sqrt{2m}} \\
1
\end{pmatrix}
$$

(1.111)

or

$$
u^i(\vec{0}, -1) = \frac{1}{2\sqrt{m}} \begin{pmatrix}
0 \\
1 \\
-1 \\
0
\end{pmatrix}.
$$

(1.112)

The rotation condition (1.108) for the $j = 1$ spinors $v^i(\vec{0}, s)$ with the minus sign and the choice $s = 0$ gives

$$
-\sum_s v^i(\vec{0}, \bar{s}) J^1_{|s,0} = -\sqrt{2} v^i(\vec{0}, -1) = (J_+)^i_k v^k(\vec{0}, 0) = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 0 & -i \\
0 & 1 & i & 0
\end{pmatrix} \frac{1}{\sqrt{2m}} \begin{pmatrix}
0 \\
0 \\
-\frac{1}{\sqrt{2m}} \\
1
\end{pmatrix}
$$

(1.113)

or

$$
v^i(\vec{0}, -1) = \frac{1}{2\sqrt{m}} \begin{pmatrix}
0 \\
1 \\
i \\
0
\end{pmatrix}.
$$

(1.114)

Similarly, the plus sign and the choice $s = 0$ give

$$
-\sum_s v^i(\vec{0}, \bar{s}) J^1_{|s,0} = -\sqrt{2} v^i(\vec{0}, 1) = (J_-)^i_k v^k(\vec{0}, 0) = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & -i \\
0 & -1 & i & 0
\end{pmatrix} \frac{1}{\sqrt{2m}} \begin{pmatrix}
0 \\
0 \\
-\frac{1}{\sqrt{2m}} \\
1
\end{pmatrix}
$$

(1.115)

or

$$
v^i(\vec{0}, 1) = \frac{1}{2\sqrt{m}} \begin{pmatrix}
0 \\
1 \\
-1 \\
0
\end{pmatrix}.
$$

(1.116)

The boost conditions (1.210) now give for $i, k = 0, 1, 2, 3$

$$u^i(p, s) = v^s(i, p, s) = \sqrt{m/p^0} L^i_k(p) u^k(\vec{0}, s) = e^i(p, s)/\sqrt{2p^0}
$$

(1.117)
1.12 Vector field for spin-one particles

where

\[ e^i(\vec{p}, s) = L^i_k(\vec{p}) e^k(\vec{0}, s) \]  \hspace{1cm} (1.118)

and

\[ e(\vec{0},0) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad e(\vec{0},1) = -\frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ i \end{pmatrix}, \quad \text{and} \quad e(\vec{0},-1) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -i \end{pmatrix}. \]  \hspace{1cm} (1.119)

A single massive vector field is then

\[ \phi^i(x) = \phi^{+i}(x) + \phi^{-i}(x) = \sum_{s=-1}^{1} \int \frac{d^3p}{\sqrt{(2\pi)^32p^0}} e^i(\vec{p}, s) a(\vec{p}, s) e^{ip\cdot x} + e^{*i}(\vec{p}, s) a^{\dagger}(\vec{p}, s) e^{-ip\cdot x}. \]  \hspace{1cm} (1.120)

The commutator/anticommutator of the positive and negative frequency parts of the field is

\[ [\phi^{+i}(x), \phi^{-k}(y)]_\mp = \int \frac{d^3p}{\sqrt{(2\pi)^32p^0}} e^{ip\cdot (x-y)} \Pi^{ik}(\vec{p}) \]  \hspace{1cm} (1.121)

where \( \Pi \) is a sum of outer products of 4-vectors

\[ \Pi^{ik}(\vec{p}) = \sum_{s=-1}^{1} e^i(\vec{p}, s) e^{*k}(\vec{p}, s). \]  \hspace{1cm} (1.122)

At \( \vec{p} = 0 \), the matrix \( \Pi \) is the unit matrix on the spatial coordinates

\[ \Pi(\vec{0}) = \sum_{s=-1}^{1} e^i(\vec{0}, s) e^{*k}(\vec{0}, s) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \]  \hspace{1cm} (1.123)

So \( \Pi(\vec{p}) \) is

\[ \Pi(\vec{p}) = L\Pi(0)L^T = L\eta L^T + L \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} L^T \]  \hspace{1cm} (1.124)

or

\[ \Pi(\vec{p})^{ik} = \eta^{ik} + p^i p^k/m^2. \]  \hspace{1cm} (1.125)
This equation lets us write the commutator (1.126) in terms of the Lorentz-invariant function $\Delta_+(x-y)$ as

$$[[\phi^+(x), \phi^-(y)]_\mp = (\eta^{ik} - \partial^i \partial^k / m^2) \int \frac{d^3 p}{(2\pi)^3 2 \mu} e^{ip(x-y)} \Delta_+(x-y). \tag{1.126}$$

As for a scalar field, we set

$$v^i(x) = \kappa \phi^+(x) + \lambda \phi^-(x) \tag{1.127}$$

and find for $(x - y)^2 > 0$ since $\Delta_+(x-y) = \Delta_+(y-x)$ for $x, y$ spacelike

$$[v(x), v^i(y)]_\mp = (-|\kappa|^2 \mp |\lambda|^2) (\eta^{ik} - \partial^i \partial^k / m^2) \Delta_+(x-y) \tag{1.128}$$

So we must choose the minus sign and set $|\kappa| = |\lambda|$. So then

$$v^i(x) = v^{+i}(x) + v^{-i}(x) = v^{+i}(x) + v^{+i}(x) \tag{1.129}$$

is real. This is a second example of the spin-statistics theorem.

If two such fields have the same mass, then we can combine them as we combined scalar fields

$$v^i(x) = v^{+i}_1(x) + iv^{+i}_2(x). \tag{1.130}$$

These fields obey the Klein-Gordon equation

$$((\Box - m^2)v^i(x) = 0. \tag{1.131}$$

And since both

$$p^j = L^j_k k^i \quad \text{and} \quad e^k(\vec{p}) = L^k_l e^l(0) \tag{1.132}$$

it follows that

$$p \cdot e(\vec{p}) = k \cdot e(0) = 0. \tag{1.133}$$

So the field $v^i$ also obeys the rule

$$\partial_\mu v^i(x) = 0. \tag{1.134}$$

These equations (1.133) and (1.134) are like those of the electromagnetic field in Lorentz gauge. But one can’t get quantum electrodynamics as the $m \to 0$ limit of just any such theory. For the interaction $H = J_i v^i$ would lead to a rate for $v$-boson production like

$$J_i J_k \Pi^{ik}(\vec{p}) \tag{1.135}$$

which diverges as $m \to 0$ because of the $p^i p^k / m^2$ term in $\Pi^{ik}(\vec{p})$. One can
avoid this divergence by requiring that $\partial_i J^i = 0$ which is current conservation.

Under parity, charge conjugation, and time reversal, a vector field transforms as

$$P v^a(x) P^{-1} = - \eta^* a^b v^b (P x)$$
$$C v^a(x) C^{-1} = \xi^* v^a(x)$$
$$T v^a(x) T^{-1} = \zeta^* \bar{v}^a (P x).$$

\[1.136\]

### 1.13 Lorentz group

The Lorentz group $O(3,1)$ is the set of all linear transformations $L$ that leave invariant the Minkowski inner product

$$xy \equiv x \cdot y - x^0 y^0 = x^T \eta y$$

in which $\eta$ is the diagonal matrix

$$\eta = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.$$  \[1.138\]

So $L$ is in $O(3,1)$ if for all 4-vectors $x$ and $y$

$$(Lx)^T \eta L y = x^T L^T \eta Ly = x^T \eta y.$$  \[1.139\]

Since $x$ and $y$ are arbitrary, this condition amounts to

$$L^T \eta L = \eta.$$  \[1.140\]

Taking the determinant of both sides and recalling that $\det A^T = \det A$ and that $\det(AB) = \det A \det B$, we have

$$(\det L)^2 = 1.$$  \[1.141\]

So $\det L = \pm 1$, and every Lorentz transformation $L$ has an inverse. Multiplying \[1.140\] by $\eta$, we get

$$\eta L^T \eta L = \eta^2 = I$$

which identifies $L^{-1}$ as

$$L^{-1} = \eta L^T \eta.$$  \[1.143\]
The subgroup of $O(3,1)$ with $\det L = 1$ is the proper Lorentz group $SO(3,1)$. The subgroup of $SO(3,1)$ that leaves invariant the sign of the time component of timelike vectors is the **proper orthochronous Lorentz group** $SO^+(3,1)$.

To find the Lie algebra of $SO^+(3,1)$, we take a Lorentz matrix $L = I + \omega$ that differs from the identity matrix $I$ by a tiny matrix $\omega$ and require $L$ to obey the condition (1.140) for membership in the Lorentz group

\[
(I + \omega^T) \eta (I + \omega) = \eta + \omega^T \eta + \eta \omega + \omega^T \omega = \eta. \tag{1.144}
\]

Neglecting $\omega^T \omega$, we have $\omega^T \eta = -\eta \omega$ or since $\eta^2 = I$

\[
\omega^T = - \eta \omega \eta. \tag{1.145}
\]

This equation implies that the matrix $\omega_{ab}$ is antisymmetric when both indexes are down

\[
\omega_{ab} = -\omega_{ba}. \tag{1.146}
\]

To see why, we write it (1.145) as $\omega^e = -\eta_{eb} \omega^{bc} \eta_{ce}$ and the multiply both sides by $\eta_{de}$ so as to get $\omega_{da} = \eta_{de} \omega^e = -\eta_{eb} \omega^{bc} \eta_{ce} \eta_{de} = -\omega_{ac} \delta^c_d = -\omega_{ad}$.

The key equation (1.145) also tells us that under transposition the time-time and space-space elements of $\omega$ change sign, while the time-space and spacetime elements do not. That is, the tiny matrix $\omega$ is for infinitesimal $\theta$ and $\lambda$ a linear combination

\[
\omega = \theta \cdot R + \lambda \cdot B \tag{1.147}
\]

of three antisymmetric space-space matrices

\[
R_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad R_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{1.148}
\]

and of three symmetric time-space matrices

\[
B_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad B_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{1.149}
\]

all of which satisfy condition (1.145). The three $R_\ell$ are $4 \times 4$ versions of the familiar rotation generators; the three $B_\ell$ generate Lorentz boosts.
If we write $L = I + \omega$ as
\[ L = I - i\theta \ell R^{\ell} - i\lambda^{\ell} B^{\ell} \equiv I - i\theta \ell J^{\ell} - i\lambda^{\ell} K^{\ell} \tag{1.150} \]
then the three matrices $J^{\ell} = iR^{\ell}$ are imaginary and antisymmetric, and therefore hermitian. But the three matrices $K^{\ell} = iB^{\ell}$ are imaginary and symmetric, and so are antihermitian. The $4 \times 4$ matrix $L = \exp(i\theta^{\ell} J^{\ell} - i\lambda^{\ell} K^{\ell})$ is not unitary because the Lorentz group is not compact.

### 1.14 Gamma matrices and Clifford algebras

In component notation, $L = I + \omega$ is
\[ L^a_b = \delta^a_b + \omega^a_b, \tag{1.151} \]
the matrix $\eta$ is $\eta_{cd} = \eta^{cd}$, and $\omega^T = -\eta \omega \eta$ is
\[ \omega^a_b = (\omega^T)^a_b = - (\eta \omega)^a_b = - \eta_{be} \omega^c_d \eta^{da} = - \omega_{bd} \eta^{da} = - \omega^a_b. \tag{1.152} \]
Lowering index $a$ we get
\[ \omega_{eb} = \eta_{ea} \omega^a_b = - \omega_{bd} \eta^{da} \eta_{ea} = - \omega_{bd} \delta^d_e = - \omega_{be} \tag{1.153} \]
That is, $\omega_{ab}$ is antisymmetric
\[ \omega_{ab} = - \omega_{ba}. \tag{1.154} \]

A representation of the Lorentz group is generated by matrices $D(L)$ that represent matrices $L$ close to the identity matrix by sums over $a, b = 0, 1, 2, 3$
\[ D(L) = 1 + \frac{i}{2} \omega_{ab} \mathcal{J}^{ab}. \tag{1.155} \]
The generators $\mathcal{J}^{ab}$ must obey the commutation relations
\[ i[\mathcal{J}^{ab}, \mathcal{J}^{cd}] = \eta^{bc} \mathcal{J}^{ad} - \eta^{ac} \mathcal{J}^{bd} - \eta^{da} \mathcal{J}^{cb} + \eta^{db} \mathcal{J}^{ca}. \tag{1.156} \]
A remarkable representation of these commutation relations is provided by matrices $\gamma^a$ that obey the anticommutation relations
\[ \{\gamma^a, \gamma^b\} = 2 \eta^{ab}. \tag{1.157} \]
One sets
\[ \mathcal{J}^{ab} = - \frac{i}{4} [\gamma^a, \gamma^b] \tag{1.158} \]
where $\eta$ is the usual flat-space metric (1.138). Any four $4 \times 4$ matrices that
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satisfy these anticommutation relations form a set of Dirac gamma matrices. They are not unique. Is \( S \) is any nonsingular \( 4 \times 4 \) matrix, then the matrices

\[
\gamma^a = S \gamma^a S^{-1}
\]

(1.159)

also are a set of Dirac’s gamma matrices.

Any set of matrices obeying the anticommutation relations (1.157) for any \( n \times n \) diagonal matrix \( \eta \) with entries that are \( \pm 1 \) is called a Clifford algebra.

As a homework problem, show that

\[
[J_{ab}, \gamma^c] = -i \gamma^a \eta^{bc} + i \gamma^b \eta^{ac}.
\]

(1.160)

One can use these commutation relations to derive the commutation relations (1.156) of the Lorentz group.

The gamma matrices are a vectors in the sense that for \( L \) near the identity

\[
D(L) \gamma^c D^{-1}(L) \approx (I + \frac{i}{2} \omega_{ab} J^{ab}) \gamma^c (I - \frac{i}{2} \omega_{ab} J^{ab})
\]

\[
= \gamma^c + \frac{i}{2} \omega_{ab} [J^{ab}, \gamma^c]
\]

\[
= \gamma^c + \frac{i}{2} \omega_{ab} (i \gamma^a \eta^{bc} + i \gamma^b \eta^{ac})
\]

\[
= \gamma^c + \frac{1}{2} \omega_{ab} \gamma^a \eta^{bc} - \frac{1}{2} \omega_{ab} \gamma^b \eta^{ac}
\]

\[
= \gamma^c - \frac{1}{2} \eta^{cb} \omega_{ba} \gamma^a - \frac{1}{2} \eta^{ca} \omega_{ab} \gamma^b
\]

\[
= \gamma^c - \frac{1}{2} \omega^c a \gamma^a - \frac{1}{2} \omega^c b \gamma^b
\]

\[
= \gamma^c - \omega^c a \gamma^a
\]

\[
= \gamma^c + \omega^c a \gamma^a
\]

\[
= (\delta^c_a + \omega^c_a) \gamma^a
\]

\[
= L^c_a \gamma^a
\]

in which we used (1.152) to write \( -\omega^c_a = \omega^c_a \). The finite \( \omega \) form is

\[
D(L) \gamma^a D^{-1}(L) = L^c_a \gamma^c.
\]

(1.162)

The unit matrix is a scalar

\[
D(L) I D^{-1}(L) = I.
\]

(1.163)

The generators of the Lorenz group form an antisymmetric tensor

\[
D(L) J^{ab} D^{-1}(L) = L^c_a L^b_d J^{cd}.
\]

(1.164)
1.15 Dirac’s gamma matrices

Out of four gamma matrices, one can also make totally antisymmetric tensors of rank-3 and rank-4

\[ A^{abc} \equiv \gamma^a \gamma^b \gamma^c \quad \text{and} \quad B^{abcd} \equiv \gamma^a \gamma^b \gamma^c \gamma^d \]  \hspace{1cm} (1.165)

where the brackets mean that one inserts appropriate minus signs so as to achieve total antisymmetry. Since there are only four \( \gamma \) matrices in four spacetime dimensions, any rank-5 totally antisymmetric tensor made from them must vanish, \( C^{abcde} = 0 \).

Notation: The parity transformation is

\[ \beta = i\gamma^0. \] \hspace{1cm} (1.166)

It flips the spatial gamma matrices but not the temporal one

\[ \beta \gamma^i \beta^{-1} = -\gamma^i \quad \text{and} \quad \beta \gamma^0 \beta^{-1} = \gamma^0. \] \hspace{1cm} (1.167)

It flips the generators of boosts but not those of rotations

\[ \beta \mathcal{J}^{i0} \beta^{-1} = -\mathcal{J}^{i0} \quad \text{and} \quad \beta \mathcal{J}^{ik} \beta^{-1} = \mathcal{J}^{ik}. \] \hspace{1cm} (1.168)

1.15 Dirac’s gamma matrices

Weinberg’s chosen set of Dirac matrices is

\[ \gamma^0 = -i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -\gamma^{0\dagger} \quad \text{and} \quad \gamma^i = -i \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} = \gamma^{i\dagger} \] \hspace{1cm} (1.169)

in which the \( \sigma \)'s are Pauli’s 2 \( \times \) 2 hermitian matrices

\[ \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \text{and} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \] \hspace{1cm} (1.170)

which are the gamma matrices of 3-dimensional spacetime. With this choice of \( \gamma \)'s, the matrix \( \beta \) is

\[ \beta = i\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \beta^{\dagger}. \] \hspace{1cm} (1.171)

In spacetimes of five dimensions, the fifth gamma matrix \( \gamma^4 \) which traditionally is called \( \gamma^5 = \gamma_5 \) is

\[ \gamma^5 = \gamma_5 = -i\gamma^0 \gamma^1 \gamma^2 \gamma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \] \hspace{1cm} (1.172)

It anticommutes with all four Dirac gammas and its square is unity, as it must if it is to be the fifth gamma in 5-space:

\[ \{\gamma^a, \gamma^b\} = 2\eta^{ab}. \] \hspace{1cm} (1.173)
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for $a, b = 0, 1, 2, 3, 4$ with $\eta^{44} = 1$ and $\eta^{40} = \eta^{04} = 0$.

With Weinberg’s choice of $\gamma$’s, the Lorentz boosts are

$$J^{i0} = -\frac{i}{4} [\gamma^i, \gamma^0] = -\frac{i}{4} \left[ -i \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, -i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right] = -\frac{i}{4} \left( \begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix} - \begin{pmatrix} -\sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix} \right) = \frac{i}{2} \begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix}. \quad (1.174) $$

The Lorentz rotation matrices are

$$J^{ik} = -\frac{i}{4} [\gamma^i, \gamma^k] = -\frac{i}{4} \left[ -i \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix}, -i \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \right] = -\frac{i}{4} \left( \begin{pmatrix} \sigma^i & \sigma^k \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} -\sigma^k & \sigma^i \\ 0 & 0 \end{pmatrix} \right) = -\frac{i}{4} \sum_{ij} \epsilon_{ijk} \begin{pmatrix} 2i\epsilon_{ikj} \sigma^j \\ 0 \end{pmatrix}. \quad (1.175) $$

The Dirac representation of the Lorentz group is reducible, as SW’s choice of gamma matrices makes apparent. The Dirac rotation matrices are

$$J_i = \frac{1}{2} \begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}. \quad (1.176) $$

Some useful relations are

$$\beta \gamma^a \beta^\dagger = -\gamma^a, \quad \beta J^{ab\dagger} \beta = J^{ab} \quad \text{and} \quad \beta D(L)^\dagger \beta = D(L)^{-1} \quad (1.177) $$

as well as

$$\beta \gamma_5^\dagger \beta = -\gamma_5 \quad \text{and} \quad \beta (\gamma_5 \gamma^a)^\dagger \beta = -\gamma_5 \gamma^a. \quad (1.178) $$

1.16 Dirac fields

The positive- and negative-frequency parts of a Dirac field are

$$\psi_+^+(x) = (2\pi)^{-3/2} \sum_s \int d^3p \ u_+(\vec{p}, s) e^{ip\cdot x} a(\vec{p}, s)$$

$$\psi_-^-(x) = (2\pi)^{-3/2} \sum_s \int d^3p \ u_-(\vec{p}, s) e^{-ip\cdot x} b(\vec{p}, s). \quad (1.179)$$
The rotation conditions (1.39) are
\[
\sum_s u_\ell (\bar{\ell}, s) (J^{(\ell)}_i)^{ss} = \sum_\ell (\mathcal{J}_i)_{\bar{\ell}\ell} u_{\ell}(\bar{\ell}, s)
\]
\[
\sum_s v_\ell (\bar{\ell}, s) (-J^{(\ell)}_i)^{ss} = \sum_\ell (\mathcal{J}_i)_{\bar{\ell}\ell} v_{\ell}(\bar{\ell}, s).
\]
(1.180)

The Dirac rotation matrices (1.176) are
\[
\mathcal{J}_i = \frac{1}{2} \left( \sigma^i \begin{array}{cc} \sigma^i & 0 \\ 0 & \sigma^i \end{array} \right)
\]
so we set the four values \( \ell, \bar{\ell} = 1, 2, 3, 4 \) to \( \ell = (m, \pm) \) with \( m = \pm \frac{1}{2} \).

And we consider \( u_\ell(s) \) to be \( u^m_m(s) \) stacked upon \( u^-_m(s) \) and similarly take \( v_\ell(s) \) to be \( v^m_m(s) \) above \( v^-_m(s) \) where \( u^m_m(s) \) and \( v^m_m(s) \) are, a priori, \( 2 \times (2j + 1) \)-dimensional matrices with indexes \( m = \pm 1/2 \) and \( s = -j, \ldots, j \). That is,
\[
\begin{pmatrix}
  u_1(s) \\
  u_2(s) \\
  u_3(s) \\
  u_4(s)
\end{pmatrix} = \begin{pmatrix}
  u^+_1/2(s) \\
  u^-_1/2(s) \\
  u^+_1/2(s) \\
  u^-_1/2(s)
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
  v_1(s) \\
  v_2(s) \\
  v_3(s) \\
  v_4(s)
\end{pmatrix} = \begin{pmatrix}
  v^+_1/2(s) \\
  v^-_1/2(s) \\
  v^+_1/2(s) \\
  v^-_1/2(s)
\end{pmatrix}.
\]
(1.182)

We then have four equations
\[
\sum_s u^+_m(\bar{\ell}, s) (J^{(\ell)}_i)^{ss} = \sum_m \frac{1}{2} \sigma^j_{mm} u^+_m(\bar{\ell}, s)
\]
\[
\sum_s u^-_m(\bar{\ell}, s) (J^{(\ell)}_i)^{ss} = \sum_m \frac{1}{2} \sigma^j_{mm} u^-_m(\bar{\ell}, s)
\]
\[
\sum_s v^+_m(\bar{\ell}, s) (-J^{(\ell)}_i)^{ss} = \sum_m \frac{1}{2} \sigma^j_{mm} v^+_m(\bar{\ell}, s)
\]
\[
\sum_s v^-_m(\bar{\ell}, s) (-J^{(\ell)}_i)^{ss} = \sum_m \frac{1}{2} \sigma^j_{mm} v^-_m(\bar{\ell}, s).
\]
(1.183)

SW defines the four \( 2 \times (2j + 1) \) matrices
\[
U^+_m = u^+_m(\bar{\ell}, s) \quad \text{and} \quad U^-_m = u^-_m(\bar{\ell}, s)
\]
\[
V^+_m = v^+_m(\bar{\ell}, s) \quad \text{and} \quad V^-_m = v^-_m(\bar{\ell}, s).
\]
(1.184)

in terms of which the four Dirac rotation conditions (1.183) are
\[
U^+ J^{(\ell)}_i = \frac{1}{2} \sigma^i U^+ \quad \text{and} \quad U^- J^{(\ell)}_i = \frac{1}{2} \sigma^i U^-
\]
\[
V^+ (-J^{(\ell)}_i) = \frac{1}{2} \sigma^i V^+ \quad \text{and} \quad V^- (-J^{(\ell)}_i) = \frac{1}{2} \sigma^i V^-.
\]
(1.185)
Taking the complex conjugate of the second of these equations, we get

\[-J_i^{(j)} = V^{+s-1}(\frac{1}{2}\sigma^{is})V^{+s} = V^{+s-1}(-\frac{1}{2}\sigma_2 \sigma^i \sigma_2)V^{+s}\]

\[-J_i^{(j)} = V^{-s-1}(\frac{1}{2}\sigma^{is})V^{-s} = V^{-s-1}(-\frac{1}{2}\sigma_2 \sigma^i \sigma_2)V^{-s}\]

(1.186)

or more simply

\[J_i^{(j)} = (\sigma_2 V^{+s})^{-1} \frac{1}{2}\sigma^i (\sigma_2 V^{+s})\]

\[J_i^{(j)} = (\sigma_2 V^{-s})^{-1} \frac{1}{2}\sigma^i (\sigma_2 V^{-s})\]

(1.187)

The 2 × 2 Pauli matrices \(\vec{\sigma}\) and the \((2j+1) \times (2j+1)\) matrices \(\mathbf{J}^{(j)}\) both generate irreducible representations of the rotation group. So by writing

\[U^+ J_i^{(j)} J_k^{(j)} = \frac{1}{2} \sigma_i U^+ J_k^{(j)} = \frac{1}{2} \sigma_i \frac{1}{2} \sigma_k U^+\]

(1.188)

and similar equations for \(U^-, V^+, V^-\), we see that

\[U^+ D_j^{(j)}(\vec{\theta}) = U^+ e^{-i\vec{\theta} \cdot \mathbf{J}^{(j)}} = U^+ D^{(1/2)}(\vec{\theta}) U^+\]

\[U^- D_j^{(j)}(\vec{\theta}) = U^- e^{-i\vec{\theta} \cdot \mathbf{J}^{(j)}} = U^- D^{(1/2)}(\vec{\theta}) U^-\]

(1.189)

and similar equations for \(V^\pm\).

\[\sigma_2 V^{+s} D_j^{(j)}(\vec{\theta}) = \sigma_2 V^{+s} e^{-i\vec{\theta} \cdot \mathbf{J}^{(j)}} = e^{-i\vec{\theta} \cdot \mathbf{J}^{(j)}} \sigma_2 V^{+s} = D^{(1/2)}(\vec{\theta}) \sigma_2 V^{+s}\]

\[\sigma_2 V^{-s} D_j^{(j)}(\vec{\theta}) = \sigma_2 V^{-s} e^{-i\vec{\theta} \cdot \mathbf{J}^{(j)}} = e^{-i\vec{\theta} \cdot \mathbf{J}^{(j)}} \sigma_2 V^{-s} = D^{(1/2)}(\vec{\theta}) \sigma_2 V^{-s}\]

(1.190)

Now recall Schur’s lemma (section 10.7 of PM):

Part 1: If \(D_1\) and \(D_2\) are inequivalent, irreducible representations of a group \(G\), and if \(D_1(g)A = AD_2(g)\) for some matrix \(A\) and for all \(g \in G\), then the matrix \(A\) must vanish, \(A = 0\).

Part 2: If for a finite-dimensional, irreducible representation \(D(g)\) of a group \(G\), we have \(D(g)A = AD(g)\) for some matrix \(A\) and for all \(g \in G\), then \(A = cI\). That is, any matrix that commutes with every element of a finite-dimensional, irreducible representation must be a multiple of the identity matrix.

Part 1 tells us that \(D_j^{(j)}(\vec{\theta})\) and \(D^{(1/2)}(\vec{\theta})\) must be equivalent. So \(j = 1/2\) and \(2j + 1 = 2\). A Dirac field must represent particles of spin 1/2.

Part 2 then says that the matrices \(U^\pm\) must be multiples of the 2 × 2 identity matrix

\[U^+ = c_+ I\quad \text{and} \quad U^- = c_- I\]

(1.191)

and that the matrices \(\sigma_2 V^{\pm s}\) must be multiples of the 2 × 2 identity matrix

\[\sigma_2 V^{+s} = d_+ I\quad \text{and} \quad \sigma_2 V^{-s} = d_- I\]

(1.192)
or more simply
\[ V^+ = -id_+ \sigma_2 \quad \text{and} \quad V^- = -id_- \sigma_2. \] (1.193)

That is,
\[ v_m^+(\bar{0},s) = d_+ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad v_m^-(\bar{0},s) = d_- \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \] (1.194)

Going back to \( \ell = (m, \pm) \) by using the index code \((1.182)
\[
\begin{bmatrix}
u_1(s) \\ v_2(s) \\ u_3(s) \\ v_4(s)
\end{bmatrix} = \begin{bmatrix}
u_{+1/2}(s) \\ u_{-1/2}(s) \\ u_{+1/2}(s) \\ u_{-1/2}(s)
\end{bmatrix} = \begin{bmatrix}
c_+ \delta_{1/2,s} \\ c_+ \delta_{-1/2,s} \\ c_- \delta_{1/2,s} \\ c_- \delta_{-1/2,s}
\end{bmatrix}
\text{and}
\begin{bmatrix}
u_1(s) \\ v_2(s) \\ v_3(s) \\ v_4(s)
\end{bmatrix} = \begin{bmatrix}
u_{+1/2}(s) \\ v_{-1/2}(s) \\ v_{+1/2}(s) \\ v_{-1/2}(s)
\end{bmatrix} = \begin{bmatrix}
d_+ \delta_{-1/2,s} \\ d_+ \delta_{1/2,s} \\ d_- \delta_{1/2,s} \\ d_- \delta_{-1/2,s}
\end{bmatrix}. \]

we have for the \( u \)'s
\[ u_{+1/2}(1/2) = c_+ \quad \text{and} \quad u_{-1/2}(1/2) = 0 \] (1.196)
\[ u_{1/2}(-1/2) = 0 \quad \text{and} \quad u_{-1/2}(-1/2) = c_- \] (1.197)
\[ u_{-1/2}(1/2) = 0 \quad \text{and} \quad u_{+1/2}(1/2) = c_+ \] (1.198)
\[ u_{-1/2}(-1/2) = 0 \quad \text{and} \quad u_{1/2}(-1/2) = c_- \] (1.199)
\[ v_{+1/2}(1/2) = 0 \quad \text{and} \quad v_{1/2}(1/2) = d_+ \] (1.200)
\[ v_{1/2}(1/2) = 0 \quad \text{and} \quad v_{+1/2}(1/2) = d_- \] (1.201)
\[ v_{1/2}(-1/2) = -d_+ \quad \text{and} \quad v_{+1/2}(-1/2) = 0 \] (1.202)
\[ v_{1/2}(-1/2) = -d_- \quad \text{and} \quad v_{-1/2}(-1/2) = 0 \] (1.203)

So
\[
\begin{align*}
u(\bar{0}, m = \frac{1}{2}) &= \begin{bmatrix}
u_{+1/2}(1/2) \\ u_{-1/2}(1/2) \\ u_{+1/2}(1/2) \\ u_{-1/2}(1/2)
\end{bmatrix} = \begin{bmatrix}
c_+ \\ 0 \\ c_- \\ 0
\end{bmatrix} \quad \text{and} \quad \nu(\bar{0}, m = -\frac{1}{2}) &= \begin{bmatrix}
u_{+1/2}(-1/2) \\ u_{-1/2}(-1/2) \\ u_{+1/2}(-1/2) \\ u_{-1/2}(-1/2)
\end{bmatrix} = \begin{bmatrix}
c_+ \\ 0 \\ c_- \\ 0
\end{bmatrix},

\text{and}
\begin{align*}
v(\bar{0}, m = \frac{1}{2}) &= \begin{bmatrix}
v_{+1/2}(1/2) \\ v_{-1/2}(1/2) \\ v_{+1/2}(1/2) \\ v_{-1/2}(1/2)
\end{bmatrix} = \begin{bmatrix}
0 \\ d_+ \\ 0 \\ d_-
\end{bmatrix} \quad \text{and} \quad \nu(\bar{0}, m = -\frac{1}{2}) &= \begin{bmatrix}
v_{+1/2}(-1/2) \\ v_{-1/2}(-1/2) \\ v_{+1/2}(-1/2) \\ v_{-1/2}(-1/2)
\end{bmatrix} = \begin{bmatrix}
d_+ \\ 0 \\ d_- \\ 0
\end{bmatrix}. \]

\] (1.204)
To put more constraints on $c_\pm$ and $d_\pm$, we recall that under parity
\[
\mathcal{P} a(\vec{p}, s) \mathcal{P}^{-1} = \eta^*_a a(-\vec{p}, s) \quad \text{and} \quad \mathcal{P} b(\vec{p}, s) \mathcal{P}^{-1} = \eta^*_b b(\vec{p}, s)
\]
and so
\[
\mathcal{P} \psi^\dagger(x) \mathcal{P}^{-1} = (2\pi)^{-3/2} \sum_s \int d^3 p \ u(\vec{p}, s) e^{ip \cdot x} \eta^*_a a(-\vec{p}, s)
\]
\[
= (2\pi)^{-3/2} \sum_s \int d^3 p \ u(-\vec{p}, s) e^{ip \cdot P x} \eta^*_a a(\vec{p}, s)
\]
\[
\mathcal{P} \psi(x) \mathcal{P}^{-1} = (2\pi)^{-3/2} \sum_s \int d^3 p \ v(\vec{p}, s) e^{-ip \cdot x} \eta^*_b b(\vec{p}, s)
\]
\[
= (2\pi)^{-3/2} \sum_s \int d^3 p \ v(-\vec{p}, s) e^{-ip \cdot P x} \eta^*_b b(\vec{p}, s).
\]
We recall the relations (1.177)
\[
\beta \gamma^{a1} \beta = -\gamma^a, \quad \beta \mathcal{J}^{ab} \beta = \mathcal{J}^{ab}, \quad \text{and} \quad \beta \mathcal{D}(L) \beta = \mathcal{D}(L)^{-1}
\]
and in particular, since $\mathcal{J}^{0i} = -\mathcal{J}^{0i}$, the rule
\[
\beta \mathcal{J}^{0i} \beta = \mathcal{J}^{0i} = -\mathcal{J}^{0i}.
\]
We also have the pseudounitarity relation
\[
\beta \mathcal{D}^\dagger(L) \beta = D^{-1}(L).
\]
In general spinors at finite momentum are related to those at zero momentum by
\[
u_\ell(q, s) = \sqrt{m/q^0} \sum_\ell D_{\ell\ell}(L(q)) u_\ell(0, s)
\]
\[
v_\ell(q, s) = \sqrt{m/q^0} \sum_\ell D_{\ell\ell}(L(q)) v_\ell(0, s)
\]
which for Dirac spinors is
\[
u(p, s) = \sqrt{m/p^0} D(L(p)) u(0, s)
\]
\[
v(p, s) = \sqrt{m/p^0} D(L(p)) v(0, s).
\]
So now by using the boost rule (1.208) we have
\[
u_\ell(-\vec{p}, s) = \sqrt{m/p^0} D(L(-\vec{p})) u(0, s) = \sqrt{m/p^0} D(L(p))^{-1} u(0, s)
\]
\[
= \sqrt{m/p^0} \beta D(L(p)) \beta u(0, s)
\]
and

\[ v_\ell(-\vec{p}, s) = \sqrt{m/\vec{p}^2} D(L(-\vec{p})) v(0, s) = \sqrt{m/\vec{p}^2} D(L(\vec{p}))^{-1} v(0, s) \] (1.124)
\[ = \sqrt{m/\vec{p}^2} \beta D(L(\vec{p})) \beta v(0, s). \] (1.125)

So under parity

\[ P \psi^+ (x) P^{-1} = (2\pi)^{-3/2} \sum_s \int d^3 \vec{p} \sqrt{m/\vec{p}^2} \beta D(L(\vec{p})) \beta u(0, s) e^{ip \cdot x} \eta_a a(\vec{p}, s) \]
\[ P \psi^- (x) P^{-1} = (2\pi)^{-3/2} \sum_s \int d^3 \vec{p} \sqrt{m/\vec{p}^2} \beta D(L(\vec{p})) \beta v(0, s) e^{-ip \cdot x} \eta_b b^\dagger(\vec{p}, s). \] (1.216)

So to have \( P \psi^\pm (x) P^{-1} \propto \psi^\pm (x) \), we need

\[ \beta u(0, s) = b_u u(0, s) \quad \text{and} \quad \beta v(0, s) = b_v u(0, s). \] (1.217)

We then get

\[ P \psi^+_\ell (t, \vec{x}) P^{-1} = b_u \beta \eta_a \psi^+_\ell (t, -\vec{x}) \quad \text{and} \quad P \psi^-_\ell (t, \vec{x}) P^{-1} = b_v \beta \eta_b \psi^-_\ell (t, -\vec{x}). \] (1.218)

Here since \( P^2 = 1 \), these factors are just signs, \( b_u^2 = b_v^2 = 1 \). The eigenvalue equations (1.217) tell us that \( c_- = b_u c_+ \) and that \( d_- = b_v d_+ \). So rescaling the fields we get

\[ u(\vec{0}, m = \frac{1}{2}) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ b_u \\ 0 \end{bmatrix} \quad \text{and} \quad u(\vec{0}, m = -\frac{1}{2}) = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 0 \\ b_u \end{bmatrix}, \]
\[ v(\vec{0}, m = \frac{1}{2}) = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ b_v \end{bmatrix} \quad \text{and} \quad v(\vec{0}, m = -\frac{1}{2}) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ b_v \end{bmatrix}. \] (1.219)

If the annihilation and creation operators \( a(p, s) \) and \( a^\dagger(p, s) \) obey the rule

\[ [a(p, s), a^\dagger(p', s')]_{\mp} = \delta_{ss'} \delta^3(\vec{p} - \vec{p}') \] (1.220)

and if the field is the sum of the positive- and negative-frequency parts (2.55)

\[ \psi^+_\ell (x) = (2\pi)^{-3/2} \sum_s \int d^3 \vec{p} \ u_\ell(\vec{p}, s) e^{ip \cdot x} a(\vec{p}, s) \]
\[ \psi^-_\ell (x) = (2\pi)^{-3/2} \sum_s \int d^3 \vec{p} \ v_\ell(\vec{p}, s) e^{-ip \cdot x} b^\dagger(\vec{p}, s) \] (1.221)
with arbitrary coefficients $\kappa$ and $\lambda$

$$\psi_\ell(x) = \kappa \psi^\dagger_\ell(x) + \lambda \psi^-(x) \quad (1.222)$$

then

$$[\psi_\ell(x), \psi^\dagger_\ell(y)] = [\kappa \psi^\dagger_\ell(x) + \lambda \psi^-(x), \kappa^* \psi^\dagger_\ell(y) + \lambda^* \psi^-(y)]$$

$$= \int \frac{d^3p}{(2\pi)^3} \sum_s \left[ |\kappa|^2 u_\ell(p, s) u^*_\ell(p, s) e^{ip\cdot(x-y)} \right.$$

$$\left. \mp |\lambda|^2 v_\ell(p, s) v^*_\ell(p, s) e^{-ip\cdot(x-y)} \right]$$

$$= \int \frac{d^3p}{(2\pi)^3} \sum_s \left[ |\kappa|^2 N_{\ell\ell'}(p) e^{ip\cdot(x-y)} \mp |\lambda|^2 N_{\ell\ell'}(p) e^{-ip\cdot(x-y)} \right] \quad (1.223)$$

where

$$N_{\ell\ell'}(p) = \sum_s u_\ell(p, s) u^*_\ell(p, s)$$

$$M_{\ell\ell'}(p) = \sum_s v_\ell(p, s) v^*_\ell(p, s). \quad (1.224)$$

When $\vec{p} = 0$, these matrices are

$$N_{\ell\ell'}(0) = \sum_s u_\ell(\vec{0}, s) u^*_\ell(\vec{0}, s)$$

$$N(0) = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ b_u & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & b_u & 0 \\ 0 & b_u & 0 & 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 1 & 0 \\ b_u & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & b_u \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (1.225)$$

$$= \frac{1}{2} \begin{bmatrix} 1 & 0 & b_u & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & b_u \end{bmatrix} = \frac{1 + b_u \beta}{2}$$
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and

\[ M_{\ell'\ell}(0) = \sum_s v_\ell(\vec{0}, s) v_{\ell'}^*(\vec{0}, s) \]

\[ M(0) = \frac{1}{2} \begin{bmatrix} 0 & 1 & 0 & b_v \\ 0 & 1 & b_v & 0 \\ b_v & 0 & 0 & 0 \\ 0 & b_v & 0 & 1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 1 & 0 & b_v & 0 \\ 0 & 0 & 0 & 0 \\ b_v & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \frac{1 + b_v \beta}{2}. \tag{1.226} \]

So then using the boost relations \((1.211)\) we find

\[ N(\vec{p}) = \sum_s u_\ell(\vec{p}, s) u_{\ell'}^*(\vec{p}, s) = \frac{m}{p^0} D(L(p)) \sum_s u_\ell(\vec{0}, s) u_{\ell'}^*(\vec{0}, s) D^\dagger(L(p)) \]

\[ = \frac{m}{2p^0} D(L(p)) (1 + b_u \beta) D^\dagger(L(p)) \]

\[ M(\vec{p}) = \sum_s v_\ell(\vec{p}, s) v_{\ell'}^*(\vec{p}, s) = \frac{m}{p^0} D(L(p)) \sum_s v_\ell(\vec{0}, s) v_{\ell'}^*(\vec{0}, s) D^\dagger(L(p)) \]

\[ = \frac{m}{2p^0} D(L(p)) (1 + b_v \beta) D^\dagger(L(p)). \tag{1.227} \]

The pseudounitarity relation \((1.209)\)

\[ \beta D^\dagger(L) \beta = D^{-1}(L). \tag{1.228} \]

gives

\[ \beta D^\dagger(L) = D^{-1}(L) \beta \tag{1.229} \]

which implies that

\[ D(L) \beta D^\dagger(L) = \beta. \tag{1.230} \]

The pseudounitarity relation also says that

\[ D^\dagger(L) = \beta D^{-1}(L) \beta \tag{1.231} \]

so that

\[ D(L) D^\dagger(L) = D(L) \beta D^{-1}(L) \beta. \tag{1.232} \]

Also since the gammas form a 4-vector \((1.162)\)

\[ D(L) \gamma^a D^{-1}(L) = L^a_c \gamma^c \tag{1.233} \]
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and since $\beta = i\gamma^0$, we have

$$D(L(p)) \beta D^{-1}(L(p)) = D(L(p)) i\gamma^0 D^{-1}(L(p)) = iL_c^0(p) \gamma^c = -iL_c^0\gamma_c.$$  

(1.234)

Now

$$p^a = L^a_{\ b}(p)k^b = L^a_{\ 0}(p)m$$  

(1.235)

so

$$D(L(p)) \beta D^{-1}(L(p)) = -i p^c \gamma_c / m$$  

(1.236)

which implies that

$$D(L) D^\dagger(L) = -i (p^c \gamma_c / m) \beta.$$  

(1.237)

Thus

$$N(\vec{p}) = \frac{m}{2p^0} D(L(p)) (1 + b_u \beta) D^\dagger(L(p)) = \frac{m}{2p^0} [(-i p^c \gamma_c / m) \beta + b_u \beta]$$

$$= \frac{1}{2p^0} [-i p^c \gamma_c + b_u m] \beta$$  

(1.238)

and

$$M(\vec{p}) = \frac{m}{2p^0} D(L(p)) (1 + b_v \beta) D^\dagger(L(p)) = \frac{m}{2p^0} [(-i p^c \gamma_c / m) \beta + b_v \beta]$$  

$$= \frac{1}{2p^0} [-i p^c \gamma_c + b_v m] \beta.$$  

(1.239)

We now put the spin sums (1.238) and (1.239) in the (anti)commutator (1.223) and get

$$[\psi_\ell(x), \psi_{\ell'}(y)]_+ = \int \frac{d^3p}{(2\pi)^3 2p^0} [\kappa]^2 [(-i p^c \gamma_c + b_u m)\beta]_{\ell\ell'} e^{ip(x-y)}$$

$$+ |\lambda|^2 [(-i p^c \gamma_c + b_v m)\beta]_{\ell\ell'} e^{-ip(x-y)}$$

$$= [\kappa]^2 [(-\partial_c \gamma^c + b_u m)\beta]_{\ell\ell'} \int \frac{d^3p}{(2\pi)^3 2p^0} e^{ip(x-y)}$$

$$+ |\lambda|^2 [(-\partial_c \gamma^c + b_v m)\beta]_{\ell\ell'} \int \frac{d^3p}{(2\pi)^3 2p^0} e^{-ip(x-y)}$$

$$= [\kappa]^2 [(\partial_c \gamma^c + b_u m)\beta]_{\ell\ell'} \Delta_+(x-y)$$

$$+ |\lambda|^2 [(\partial_c \gamma^c + b_v m)\beta]_{\ell\ell'} \Delta_+(y-x).$$  

(1.240)

Recall that for $(x-y)^0 > 0$, i.e. spacelike, $\Delta_+(x-y) = \Delta_+(y-x)$. So its first
derivatives are odd. So for \( x - y \) spacelike

\[
[\psi_\ell(x), \psi^\dagger_{\ell'}(y)]_\mp = |\kappa|^2 \left[ (- \partial_c \gamma^c + b_u m) \beta \right]_{\ell\ell'} \Delta_+(x - y) \\
\mp |\lambda|^2 \left[ (\partial_c \gamma^c + b_v m) \beta \right]_{\ell\ell'} \Delta_+(x - y) \\
= (|\kappa|^2 \pm |\lambda|^2) \left[ (- \partial_c \gamma^c) \beta \right]_{\ell\ell'} \Delta_+(x - y) \\
+ \left( |\kappa|^2 b_u \mp |\lambda|^2 b_v \right) m \beta \Delta_+(x - y). \tag{1.241}
\]

To get the first term to vanish, we need to choose the lower sign (that is, use anticommutators) and set \( |\kappa| = |\lambda| \). To get the second term to be zero, we must set \( b_u = - b_v \). We may adjust \( \kappa \) and \( b_u \) so that \( \kappa = \lambda \) and \( b_u = - b_v = 1 \).

Finally then, the Dirac field is

\[
\psi_\ell(x) = \sum_s \int \frac{d^3p}{(2\pi)^{3/2}} \left[ u_\ell(\vec{p}, s) e^{ip \cdot x} a(\vec{p}, s) + v_\ell(\vec{p}, s) e^{-ip \cdot x} b^\dagger(\vec{p}, s) \right]. \tag{1.244}
\]

The zero-momentum spinors are

\[
u(\vec{0}, m = \frac{1}{2}) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad u(\vec{0}, m = -\frac{1}{2}) = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},
\]

\[
v(\vec{0}, m = \frac{1}{2}) = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \quad \text{and} \quad v(\vec{0}, m = -\frac{1}{2}) = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}. \tag{1.245}
\]

The spin sums are

\[
[N(\vec{p})]_{\ell m} = \sum_s u_\ell(\vec{p}, s) u^*_{m}(\vec{p}, s) = \left[ \frac{1}{2\vec{p}^0} (-i \vec{p}^c \gamma_c + m) \beta \right]_{\ell m}
\]

\[
[M(\vec{p})]_{\ell m} = \sum_s v_\ell(\vec{p}, s) v^*_{m}(\vec{p}, s) = \left[ \frac{1}{2\vec{p}^0} (-i \vec{p}^c \gamma_c - m) \beta \right]_{\ell m}. \tag{1.246}
\]

The Dirac anticommutator is

\[
[\psi_\ell(x), \psi^\dagger_{\ell'}(y)]_+ \equiv \{\psi_\ell(x), \psi^\dagger_{\ell'}(y)\} = \left[ (- \partial_c \gamma^c + m) \beta \right]_{\ell\ell'} \Delta_+(x - y). \tag{1.247}
\]
Two standard abbreviations are

\[ \beta \equiv \imath \gamma^0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad \overline{\psi} \equiv \psi^\dagger \beta = \begin{bmatrix} \psi_r^* & \psi_i^* \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \psi_r^* & \psi_i^* \end{bmatrix}. \] (1.248)

A Majorana fermion is represented by a field like

\[ \psi(x) = \sum_s \int \frac{d^3p}{(2\pi)^{3/2}} \left[ u_\ell(p, s) e^{ip \cdot x} a(p, s) + v_\ell(p, s) e^{-ip \cdot x} a^\dagger(p, s) \right]. \] (1.249)

Since \( \mathcal{C} = \gamma_2 \beta \) it follows that \( \mathcal{C}^{-1} = \beta \gamma_2 \) and so that \( \mathcal{J}^{ab} = -\beta \mathcal{C} \mathcal{J}^{ab} \mathcal{C}^{-1} \beta = -\beta \gamma_2 \beta \mathcal{J}^{ab} \beta \gamma_2 \beta \). But 

\[ \beta \gamma_2 \beta = -\beta \beta \gamma_2 = -i^2 \gamma_0 \gamma_2 = -\gamma_2. \] So \( \mathcal{J}^{ab} = -\gamma_2 \mathcal{J}^{ab} \gamma_2 \). Thus

\[ D^*(L) = e^{-i\omega_{ab} \mathcal{J}^{ab}} = e^{-i\omega_{ab}(-\gamma_2 \mathcal{J}^{ab})} = \gamma_2 e^{i\omega_{ab} \mathcal{J}^{ab}} \gamma_2 = \gamma_2 D(L) \gamma_2. \] (1.250)

Now with SW’s \( \gamma \)'s,

\[ \gamma_2 u(\overline{0}, \pm \frac{1}{2}) = v(\overline{0}, \pm \frac{1}{2}) \quad \text{and} \quad \gamma_2 v(\overline{0}, \pm \frac{1}{2}) = u(\overline{0}, \pm \frac{1}{2}). \] (1.251)

Thus the hermitian conjugate of a Majorana field is

\[ \psi^\dagger(x) = \sum_s \int \frac{d^3p}{(2\pi)^{3/2}} \left[ u^\dagger(p, s) e^{-ip \cdot x} a^\dagger(p, s) + v^\dagger(p, s) e^{ip \cdot x} a(p, s) \right] \]

\[ = \sum_s \int \frac{d^3p}{(2\pi)^{3/2}} \left[ v^\dagger(p, s) e^{ip \cdot x} a(p, s) + u^\dagger(p, s) e^{-ip \cdot x} a^\dagger(p, s) \right] \]

\[ = \sum_s \int \frac{d^3p}{(2\pi)^{3/2}} \left[ D^*(L(p)) v^\dagger(\overline{0}, s) e^{ip \cdot x} a(p, s) + D^*(L(p)) u^\dagger(\overline{0}, s) e^{-ip \cdot x} a^\dagger(p, s) \right] \]

\[ = \gamma_2 \sum_s \int \frac{d^3p}{(2\pi)^{3/2}} \left[ D(L(p)) \gamma_2 u(\overline{0}, s) e^{ip \cdot x} a(p, s) + D(L(p)) \gamma_2 u(\overline{0}, s) e^{-ip \cdot x} a^\dagger(p, s) \right] \]

\[ = \gamma_2 \sum_s \int \frac{d^3p}{(2\pi)^{3/2}} \left[ D(L(p)) u(\overline{0}, s) e^{ip \cdot x} a(p, s) + D(L(p)) v(\overline{0}, s) e^{-ip \cdot x} a^\dagger(p, s) \right] \]

\[ = \gamma_2 \sum_s \int \frac{d^3p}{(2\pi)^{3/2}} \left[ u(\overline{0}, s) e^{ip \cdot x} a(p, s) + v(\overline{0}, s) e^{-ip \cdot x} a^\dagger(p, s) \right] = \gamma_2 \psi(x). \] (1.252)

The parity rules [1.253] now are

\[ \mathbf{P} \psi_+^\dagger(t, \mathbf{x}) \mathbf{P}^{-1} = \beta \eta_+ \psi_+^\dagger(t, -\mathbf{x}) \quad \text{and} \quad \mathbf{P} \psi_-^\dagger(t, \mathbf{x}) \mathbf{P}^{-1} = -\beta \eta_- \psi_-^\dagger(t, -\mathbf{x}). \] (1.253)

So to have a Dirac field survive a parity transformation, we need the phase
of the particle to be minus the complex conjugate of the phase of the antiparticle

$$\eta^*_a = - \eta_b \quad \text{or} \quad \eta_b = - \eta^*_a. \quad (1.254)$$

So the intrinsic parity of a particle-antiparticle state is odd. So negative-parity bosons like $\pi^0, \rho_0, J/\psi$ can be interpreted as s-wave bound states of quark-antiquark pairs. Under parity a Dirac field goes as

$$P\psi(t, \vec{x})P^{-1} = \eta^* \beta \psi(t, -\vec{x}). \quad (1.255)$$

If a Dirac particle is the same as its antiparticle, then its intrinsic parity must be odd under complex conjugation, $\eta = -\eta^*$. So the intrinsic parity of a Majorana fermion must be imaginary

$$\eta = \pm i. \quad (1.256)$$

But this means that if we express a Dirac field $\psi$ as a complex linear combination

$$\psi = \frac{1}{\sqrt{2}}(\chi_1 + i\chi_2) \quad (1.257)$$

of two Majorana fields with intrinsic parities $\eta^*_1 = \pm i$ and $\eta^*_2 = \pm i$, then under parity

$$P\psi(t, \vec{x})P^{-1} = \frac{1}{\sqrt{2}} \left( \eta^*_1 \beta \phi_1(t, -\vec{x}) + i\eta^*_2 \beta \phi_2(t, -\vec{x}) \right) \quad (1.258)$$

so we need $\eta^*_1 = \eta^*_2$ to have

$$P\psi(t, \vec{x})P^{-1} = \frac{1}{\sqrt{2}} \left( \eta^*_1 \beta \phi_1(t, -\vec{x}) + i\eta^*_2 \beta \phi_2(t, -\vec{x}) \right) = \eta^* \beta \psi(t, -\vec{x}). \quad (1.259)$$

But in that case the Dirac field has intrinsic parity $\eta = \pm i$.

The equation (1.236) that shows how beta goes under $D(L(p))$

$$D(L(p)) \beta D^{-1}(L(p)) = -i p^r \gamma_c / m \quad (1.260)$$

tells us that the spinors (1.211)

$$u(p, s) = \sqrt{\frac{m}{p^0}} D(L(p)) u(\tilde{0}, s) \quad \text{and} \quad v(p, s) = \sqrt{\frac{m}{p^0}} D(L(p)) v(\tilde{0}, s) \quad (1.261)$$
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are eigenstates of \( -i p^c \gamma_c / m \) with eigenvalues \( \pm 1 \)

\[
\begin{align*}
\begin{array}{c}
\left(-i p^c \gamma_c / m \right) u(p, s) &= D(L(p)) \beta D^{-1}(L(p)) \sqrt{\frac{m}{p^0}} D(L(p)) u(\vec{0}, s) \\
&= \sqrt{\frac{m}{p^0}} D(L(p)) \beta u(\vec{0}, s) = \sqrt{\frac{m}{p^0}} D(L(p)) u(\vec{0}, s) = u(p, s)
\end{array}
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
\left(-i p^c \gamma_c / m \right) v(p, s) &= D(L(p)) \beta D^{-1}(L(p)) \sqrt{\frac{m}{p^0}} D(L(p)) v(\vec{0}, s) \\
&= \sqrt{\frac{m}{p^0}} D(L(p)) \beta v(\vec{0}, s) = -\sqrt{\frac{m}{p^0}} D(L(p)) v(\vec{0}, s) = -v(p, s).
\end{array}
\end{align*}
\]

So

\[
(i p^c \gamma_c + m) u(p, s) = 0 \quad \text{and} \quad (-i p^c \gamma_c + m) v(p, s) = 0
\]

which implies that a Dirac field obeys Dirac’s equation

\[
(\gamma^a \partial_a + m) \psi(x) = (\gamma^a \partial_a + m) \sum_s \int \frac{d^3p}{(2\pi)^3/2} \left[ u(\vec{p}, s) e^{i p^c x} a(\vec{p}, s) + v(\vec{p}, s) e^{-i p^c x} b^\dagger(\vec{p}, s) \right]
\]

\[
= \sum_s \int \frac{d^3p}{(2\pi)^3/2} \left[ (\gamma^a \partial_a + m) u(\vec{p}, s) e^{i p^c x} a(\vec{p}, s) \\
+ (\gamma^a \partial_a + m) v(\vec{p}, s) e^{-i p^c x} b^\dagger(\vec{p}, s) \right]
\]

\[
= \sum_s \int \frac{d^3p}{(2\pi)^3/2} \left[ (i \gamma^a p_a + m) u(\vec{p}, s) e^{i p^c x} a(\vec{p}, s) \\
+ (-i \gamma^a p_a + m) v(\vec{p}, s) e^{-i p^c x} b^\dagger(\vec{p}, s) \right] = 0.
\]

SW shows that under complex conjugation

\[
u^*(p, s) = -\beta C v(p, s) \quad \text{and} \quad u^*(p, s) = -\beta C u(p, s).
\]

So for a Dirac field to survive charge conjugation, the particle-antiparticle phases must be related

\[
\xi_b = \xi_a^*.
\]

Then under charge conjugation a Dirac field goes as

\[
C \psi(x) C^{-1} = -\xi^* \beta C \psi^*(x).
\]

If a Dirac particle is the same as its antiparticle, then \( \xi \) must be real (and
η imaginary), \( \xi = \pm 1 \), and must satisfy the reality condition
\[
\psi(x) = -\beta C \psi^*(x).
\] (1.268)

Suppose a particle and its antiparticle form a bound state
\[
|\Phi\rangle = \sum_{ss'} \int d^3p d^3p' \chi(p, s; p', s') a^\dagger(p, s) b^\dagger(p', s') |0\rangle.
\] (1.269)

Under charge conjugation
\[
C |\Phi\rangle = \xi_a \xi_b \sum_{ss'} \int d^3p d^3p' \chi(p, s; p', s') b^\dagger(p, s) a^\dagger(p', s') |0\rangle \tag{1.270}
\]
\[
= -\xi_a \xi_b \sum_{ss'} \int d^3p d^3p' \chi(p, s; p', s') a^\dagger(p', s') b^\dagger(p, s) |0\rangle
\]
\[
= -\xi_a \xi_b \sum_{ss'} \int d^3p d^3p' \chi(p', s'; p, s) a^\dagger(p, s) b^\dagger(p', s') |0\rangle \tag{1.270}
\]
\[
= -\xi_a \xi_b |\Phi\rangle = -\xi_a \xi^*_a |\Phi\rangle = -|\Phi\rangle.
\]

The intrinsic charge-conjugation parity of a bound state of a particle and its antiparticle is odd.

1.17 Expansion of massive and massless Dirac fields

A Dirac field is a sum over \( s = \pm 1/2 \) of
\[
\psi_a(x) = \sum_s \int \frac{d^3p}{(2\pi)^{3/2}} \left[ u_a(p, s) e^{ip \cdot x} a(p, s) + v_a(p, s) e^{-ip \cdot x} b^\dagger(p, s) \right] \tag{1.271}
\]
in which
\[
u_u(p, s) = \frac{m-i\gamma^0}{\sqrt{2p^0(p^0+m)}} u_0(s) \quad \text{and} \quad v_0(p, s) = \frac{m+i\gamma^0}{\sqrt{2p^0(p^0+m)}} v_0(s) \tag{1.272}
\]
and the zero-momentum spinors \([1.245]\) are
\[
u_u(0, s = \frac{1}{2}) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad u_0(0, s = -\frac{1}{2}) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \tag{1.273}
\]
\[
u_v(0, s = \frac{1}{2}) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \quad \text{and} \quad v_0(0, s = -\frac{1}{2}) = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}. \tag{1.273}
\]
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These static spinors and the matrix formulas (1.272) give explicit expressions for the spinors at arbitrary momentum $p$:

$$u(p, \frac{1}{2}) = \frac{1}{2 \sqrt{p^0(p^0 + m)}} \begin{pmatrix} m + p^0 - p_3 \\ -p_1 - ip_2 \\ m + p^0 + p_3 \\ p_1 + ip_2 \end{pmatrix}$$  \hspace{1cm} (1.274)

$$u(p, -\frac{1}{2}) = \frac{1}{2 \sqrt{p^0(p^0 + m)}} \begin{pmatrix} -p_1 + ip_2 \\ m + p^0 + p_3 \\ p_1 - ip_2 \\ m + p^0 - p_3 \end{pmatrix}$$  \hspace{1cm} (1.275)

$$v(p, \frac{1}{2}) = \frac{1}{2 \sqrt{p^0(p^0 + m)}} \begin{pmatrix} -p_1 + ip_2 \\ m + p^0 + p_3 \\ -p_1 + ip_2 \\ -m - p^0 + p_3 \end{pmatrix}$$  \hspace{1cm} (1.276)

$$v(p, -\frac{1}{2}) = \frac{1}{2 \sqrt{p^0(p^0 + m)}} \begin{pmatrix} -m - p^0 + p_3 \\ p_1 + ip_2 \\ m + p^0 + p_3 \\ p_1 + ip_2 \end{pmatrix}.$$  \hspace{1cm} (1.277)

In the $m \to 0$ limit with $p = p^0 \hat{z}$ in the 3-direction, they are

$$u(p^0 \hat{z}, \frac{1}{2}) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad u(p^0 \hat{z}, -\frac{1}{2}) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$  \hspace{1cm} (1.278)

$$v(p^0 \hat{z}, \frac{1}{2}) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad v(p^0 \hat{z}, -\frac{1}{2}) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$  \hspace{1cm} (1.279)

They tell us that the upper two components which are those of the left-handed Dirac field represent particles of negative helicity $u(p^0 \hat{z}, -\frac{1}{2})$ and antiparticles of positive helicity $v(p^0 \hat{z}, \frac{1}{2})$. They also say that the lower two components which are those of the right-handed field represent particles of positive helicity $u(p^0 \hat{z}, \frac{1}{2})$ and antiparticles of negative helicity $v(p^0 \hat{z}, -\frac{1}{2})$. 
In the $m \to 0$ limit, the spinors are for the particles

$$u(p, \frac{1}{2}) = \frac{1}{2p^0} \begin{pmatrix} 0 \\ 0 \\ p^0 + p_3 \\ p_1 + ip_2 \end{pmatrix} \quad \text{and} \quad u(p, -\frac{1}{2}) = \frac{1}{2p^0} \begin{pmatrix} -p_1 + ip_2 \\ p^0 + p_3 \\ 0 \\ 0 \end{pmatrix} \quad (1.280)$$

and for the antiparticles

$$v(p, \frac{1}{2}) = \frac{1}{2p^0} \begin{pmatrix} -p_1 + ip_2 \\ p^0 + p_3 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad v(p, -\frac{1}{2}) = \frac{1}{2p^0} \begin{pmatrix} 0 \\ 0 \\ p^0 + p_3 \\ p_1 + ip_2 \end{pmatrix}. \quad (1.281)$$

The massless Dirac field is

$$\psi_a(x) = \sum_{s = -\frac{1}{2}}^{\frac{1}{2}} \int \frac{d^3p}{(2\pi)^{3/2}} \left[ u_a(p, s) e^{ip \cdot x} a(p, s) + v_a(p, s) e^{-ip \cdot x} b^\dagger(p, s) \right]. \quad (1.282)$$

The left-handed field is

$$\psi_{\ell a}(x) = \int \frac{d^3p}{(2\pi)^{3/2}} \left[ e^{ip \cdot x} a(p, \frac{1}{2}) + e^{-ip \cdot x} b^\dagger(p, \frac{1}{2}) \right]. \quad (1.283)$$

The right-handed field is

$$\psi_{r a}(x) = \int \frac{d^3p}{(2\pi)^{3/2}} \left[ e^{ip \cdot x} a(p, \frac{1}{2}) + e^{-ip \cdot x} b^\dagger(p, -\frac{1}{2}) \right]. \quad (1.284)$$

The massless left-handed field and the massless right-handed field create and absorb different particles. These are different fields. That is why the unbroken gauge group $SU(2)_\ell$ can act on $\psi_\ell$ without affecting $\psi_r$. 
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1.18 Spinors and angular momentum

Under a Lorentz transformation a Dirac field goes as

\[ U(L)\psi_a(x)U^{-1}(L) = \sum_b D_{ab}(L^{-1})\psi_b(Lx) \] (1.285)

where \( D(L) \) is the \((\frac{1}{2},0) \oplus (0, \frac{1}{2})\) representation of the Lorentz group. If \( L = R \) is an active rotation of \( |\bar{\theta}| \) radians about the vector \( \bar{\theta} \), then the matrix is

\[ D(R^{-1}) = \begin{pmatrix} e^{i\bar{\theta} \sigma_3/2} & 0 \\ 0 & e^{-i\bar{\theta} \sigma_3/2} \end{pmatrix}. \] (1.286)

In particular, if \( \bar{\theta} = \theta \hat{z} \), then

\[ D(R^{-1}) = \begin{pmatrix} e^{i\theta \sigma_3/2} & 0 \\ 0 & e^{-i\theta \sigma_3/2} \end{pmatrix} = \begin{pmatrix} e^{i\theta/2} & 0 & 0 & 0 \\ 0 & e^{-i\theta/2} & 0 & 0 \\ 0 & 0 & e^{i\theta/2} & 0 \\ 0 & 0 & 0 & e^{-i\theta/2} \end{pmatrix}. \] (1.287)

So since the field is

\[ \psi_a(x) = \sum_s \int \frac{d^3p}{(2\pi)^{3/2}} \ u_a(\vec{p},s) e^{ip \cdot x} a(\vec{p},s) + v_a(\vec{p},s) e^{-ip \cdot x} a^\dagger(\vec{p},s), \] (1.288)

we need

\[ U(R)\psi_a(x)U^{-1}(R) = D(R^{-1})_{ab}\psi_b(Rx) \]

\[ = \sum_s \int \frac{d^3p}{(2\pi)^{3/2}} \left[ u_a(\vec{p},s) e^{ip \cdot x} U(R)a(\vec{p},s)U^{-1}(R) \\ + v_a(\vec{p},s) e^{-ip \cdot x} U(R)a^\dagger(\vec{p},s)U^{-1}(R) \right]. \] (1.289)

A state of momentum \( \vec{p} \), kind \( n \), and spin \( s = \pm 1/2 \) in the \( z \) direction transforms under a right-handed rotation \( R(\theta,z) \) of \( \theta \) radians about the \( z \) axis as

\[ U(R(\theta,z))|p\hat{z},s,n\rangle = e^{-i\theta J_z}|p\hat{z},s,n\rangle = e^{-i\theta s}|p\hat{z},s,n\rangle. \] (1.290)

The state \( |p\hat{z},s,n\rangle \) is made from the vacuum by the creation operator for a particle of momentum \( \vec{p} \), kind \( n \), and spin \( s = \pm 1/2 \) in the \( z \) direction

\[ |p\hat{z},s,n\rangle = a^\dagger(p\hat{z},s,n)|0\rangle. \] (1.291)

The creation operator transforms as

\[ U(R(z,\theta))a^\dagger(p\hat{z},s,n)U(R(z,\theta))^{-1} = e^{-i\theta s}a^\dagger(p\hat{z},s,n). \] (1.292)
The antiparticle creation operator goes as
\[ U(R) a_c^\dagger(p, s) U^{-1}(R) = \sum_{s'} \left( e^{-i\vec{\theta} \cdot \vec{\sigma}/2} \right)_{ss'} a_c^1(R\vec{p}, s'), \] (1.293)
and so the particle annihilation operator goes as
\[ U(R) a_c(p, s) U^{-1}(R) = \sum_{s'} \left( e^{-i\vec{\theta} \cdot \vec{\sigma}/2} \right)^*_{ss'} a_c(R\vec{p}, s'). \] (1.294)

Our requirement (1.289) is
\[ U(R) \psi_a(x) U^{-1}(R) = D(R^{-1})_{ab} \psi_b(Rx) \]
\[ = \sum_s \int \frac{d^3 p}{(2\pi)^{3/2}} \left[ u_a(\vec{p}, s) e^{ip \cdot x} \sum_{s'} \left( e^{i\vec{\theta} \cdot \vec{\sigma}/2} \right)_{s's} a_c(R\vec{p}, s') \right. \\
\left. + v_a(\vec{p}, s) e^{-ip \cdot x} \sum_{s'} \left( e^{-i\vec{\theta} \cdot \vec{\sigma}/2} \right)_{ss'} a_c^1(R\vec{p}, s') \right]. \] (1.295)

But
\[ D(R^{-1})_{ab} \psi_b(Rx) = \begin{pmatrix} e^{i\vec{\theta} \cdot \vec{\sigma}/2} & 0 \\ 0 & e^{i\vec{\theta} \cdot \vec{\sigma}/2} \end{pmatrix}_{ab} \psi_b(Rx) \]
\[ = \begin{pmatrix} e^{i\vec{\theta} \cdot \vec{\sigma}/2} & 0 \\ 0 & e^{i\vec{\theta} \cdot \vec{\sigma}/2} \end{pmatrix}_{ab} \sum_s \int \frac{d^3 p}{(2\pi)^{3/2}} \left[ u_b(\vec{p}, s) e^{ip \cdot x} a_c(R\vec{p}, s) \right. \\
\left. + v_b(\vec{p}, s) e^{-ip \cdot x} a_c^1(R\vec{p}, s) \right]. \] (1.296)

Equivalently since \( R p \cdot Rx = p \cdot x, \)
\[ D(R^{-1})_{ab} \psi_b(Rx) = \begin{pmatrix} e^{i\vec{\theta} \cdot \vec{\sigma}/2} & 0 \\ 0 & e^{i\vec{\theta} \cdot \vec{\sigma}/2} \end{pmatrix}_{ab} \psi_b(Rx) \]
\[ = \begin{pmatrix} e^{i\vec{\theta} \cdot \vec{\sigma}/2} & 0 \\ 0 & e^{i\vec{\theta} \cdot \vec{\sigma}/2} \end{pmatrix}_{ab} \sum_s \int \frac{d^3 p}{(2\pi)^{3/2}} \left[ u_b(\vec{R} p, s) e^{ip \cdot x} a_c(R\vec{p}, s) \right. \\
\left. + v_b(\vec{R} p, s) e^{-ip \cdot x} a_c^1(R\vec{p}, s) \right]. \] (1.297)

Let’s focus on the zero-momentum terms. We need
\[ \sum_{s'} \left( e^{i\vec{\theta} \cdot \vec{\sigma}/2} \right)_{ss'} u_a(\vec{0}, s') = \begin{pmatrix} e^{i\vec{\theta} \cdot \vec{\sigma}/2} & 0 \\ 0 & e^{i\vec{\theta} \cdot \vec{\sigma}/2} \end{pmatrix}_{ab} u_b(\vec{0}, s) \] (1.298)
and
\[ \sum_{s'} \left( e^{-i\vec{\theta} \cdot \vec{\sigma}/2} \right)_{s's} v_a(\vec{0}, s') = \begin{pmatrix} e^{i\vec{\theta} \cdot \vec{\sigma}/2} & 0 \\ 0 & e^{i\vec{\theta} \cdot \vec{\sigma}/2} \end{pmatrix}_{ab} v_b(\vec{0}, s). \] (1.299)
The spinors

\[ u(\vec{0}, \frac{1}{2}) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad u(\vec{0}, -\frac{1}{2}) = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \]  

(1.300)

satisfy the condition (1.298) on the spinor \( u_a(\vec{0}, s) \). Let’s look at the condition (1.299) on \( v_a(\vec{0}, s) \) for the simpler case of \( \vec{\theta} = \theta \hat{z} \). We set \( s = \frac{1}{2} \) or equivalently \( s = + \) and also \( a = 1 \). Since the rotation is about \( \hat{z} \),

\[
\left( e^{-i\vec{\theta} \vec{\sigma}/2} \right)_{s's} = \left( e^{-i\theta \sigma_3/2} \right)_{s's} = e^{-i\theta \delta_{s's}}.
\]

(1.301)

So the condition (1.299) says that

\[
e^{-i\theta/2} v_1(\vec{0}, \frac{1}{2}) = e^{i\theta/2} v_1(\vec{0}, \frac{1}{2}).
\]

(1.302)

So \( v_1(\vec{0}, \frac{1}{2}) = 0 \). Now we set \( a = 2 \). Then the condition says that

\[
e^{-i\theta/2} v_2(\vec{0}, \frac{1}{2}) = e^{-i\theta/2} v_2(\vec{0}, \frac{1}{2}).
\]

(1.303)

So \( v_2(\vec{0}, \frac{1}{2}) \) can be anything. Now we set \( a = 3 \). Then the same condition (1.299) says that

\[
e^{-i\theta/2} v_3(\vec{0}, \frac{1}{2}) = e^{i\theta/2} v_3(\vec{0}, \frac{1}{2})
\]

(1.304)

which implies that \( v_3(\vec{0}, \frac{1}{2}) = 0 \). Finally, we set \( a = 4 \). Now the same condition (1.299) says that

\[
e^{-i\theta/2} v_4(\vec{0}, \frac{1}{2}) = e^{-i\theta/2} v_4(\vec{0}, \frac{1}{2}).
\]

(1.305)

So \( v_4(\vec{0}, \frac{1}{2}) \) can be anything. That is, \( v(\vec{0}, \frac{1}{2}) \) must be of the form

\[
v(\vec{0}, \frac{1}{2}) = \begin{bmatrix} 0 \\ \alpha \\ 0 \\ \beta \end{bmatrix}
\]

(1.306)

in which \( \alpha \) and \( \beta \) are arbitrary.

Now we set \( s = -\frac{1}{2} \) or equivalently \( s = - \) and also \( a = 1 \). Then the condition (1.299) says that

\[
e^{i\theta/2} v_1(\vec{0}, -\frac{1}{2}) = e^{i\theta/2} v_1(\vec{0}, -\frac{1}{2}).
\]

(1.307)

So \( v_1(\vec{0}, -\frac{1}{2}) \) can be anything. But the condition (1.299) for \( a = 2 \) says

\[
e^{i\theta/2} v_2(\vec{0}, -\frac{1}{2}) = e^{-i\theta/2} v_2(\vec{0}, -\frac{1}{2})
\]

(1.308)
1.19 Charge conjugation

which means that \( v_2(\vec{0}, -\frac{1}{2}) = 0 \). For \( a = 3 \), we get

\[
e^{i\theta/2}v_3(\vec{0}, -\frac{1}{2}) = e^{i\theta/2}v_3(\vec{0}, -\frac{1}{2}),
\]

so \( v_3(\vec{0}, -\frac{1}{2}) \) is unrestricted. Finally, for \( a = 4 \), we have

\[
e^{i\theta/2}v_4(\vec{0}, -\frac{1}{2}) = e^{-i\theta/2}v_4(\vec{0}, -\frac{1}{2})
\]

which makes \( v_4(\vec{0}, -\frac{1}{2}) \) vanish. So

\[
v(\vec{0}, -\frac{1}{2}) = \begin{pmatrix} \gamma \\ 0 \\ \delta \\ 0 \end{pmatrix}
\]

in which \( \gamma \) and \( \delta \) are arbitrary. To make the \( v \)'s suitably normalized and orthogonal to the \( u \)'s, one may set

\[
v(\vec{0}, \frac{1}{2}) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \quad \text{and} \quad v(\vec{0}, -\frac{1}{2}) = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}.
\]

1.19 Charge conjugation

Under charge conjugation, a massive Dirac field

\[
\psi(x) = \sum_s \int \frac{d^3p}{(2\pi)^{3/2}} \ u(\vec{p}, s) e^{i\vec{p} \cdot \vec{x}} a(\vec{p}, s) + v(\vec{p}, s) e^{-i\vec{p} \cdot \vec{x}} a_c(\vec{p}, s)
\]

transforms as

\[
C\psi(x)C^{-1} = -\xi^* \beta \gamma_2 \beta \psi^*(x)
\]

in which \( \xi \) is the charge-conjugation parity of the particle annihilated by \( a \) (which is the complex conjugate of that of the particle annihilated by \( a_c \)) and

\[
\beta \gamma_2 \beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} (-i) \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = i \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix}.
\]

Under charge conjugation, the creation operators for particles and their antiparticles, transform as

\[
Ca^\dagger(\vec{p}, s)C^{-1} = \xi a_c^\dagger(\vec{p}, s) \quad \text{and} \quad Ca_c^\dagger(\vec{p}, s)C^{-1} = \xi^* a^\dagger(\vec{p}, s).
\]

So

\[
Ca(\vec{p}, s)C^{-1} = \xi^* a_c(\vec{p}, s) \quad \text{and} \quad Ca_c(\vec{p}, s)C^{-1} = \xi a(\vec{p}, s).
\]
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So, suppressing explicit mention of the sum over spins, we have

\[
C \psi(x)C^{-1} = \int \frac{d^3p}{(2\pi)^{3/2}} u(\vec{p}, s) e^{ip\cdot x} Ca(\vec{p}, s)C^{-1} + v(\vec{p}, s) e^{-ip\cdot x} Ca_c(\vec{p}, s)C^{-1}
\]

\[
= \int \frac{d^3p}{(2\pi)^{3/2}} u(\vec{p}, s) e^{ip\cdot x} \xi^* a_c(\vec{p}, s) + v(\vec{p}, s) e^{-ip\cdot x} \xi_c a_c^+(\vec{p}, s)C^{-1}.
\]

Under charge conjugation, the field changes to

\[
C \psi(x)C^{-1} = -i\xi^* \left( \begin{array}{c} 0 \\ -\sigma_2 \\ 0 \end{array} \right) \psi^*(x).
\]  \hspace{1cm} (1.318)

Explicitly and with \( \xi_c = \xi^* \), we get

\[
C \psi(x)C^{-1} = -i\xi^* \left( \begin{array}{c} 0 \\ -\sigma_2 \\ 0 \end{array} \right) \sum_s \int \frac{d^3p}{(2\pi)^{3/2}} \left[ u^*(\vec{p}, s) e^{-ip\cdot x} a_c^+(\vec{p}, s) + v^*(\vec{p}, s) e^{ip\cdot x} a_c(\vec{p}, s) \right] \]

\hspace{1cm} (1.319)

So leaving aside the factor \(-i\xi^*\), the spinors for the \( a_c \) particles (aka, antiparticles) are

\[
u_b(\vec{p}, s) = \left( \begin{array}{c} 0 \\ -\sigma_2 \\ 0 \end{array} \right) v^*(\vec{p}, s)
\]  \hspace{1cm} (1.320)

while the antispinors for the \( a \) particles are

\[
u_a(\vec{p}, s) = \left( \begin{array}{c} 0 \\ -\sigma_2 \\ 0 \end{array} \right) u^*(\vec{p}, s).
\]  \hspace{1cm} (1.321)

Let’s focus on the zero-momentum spinors. The zero-momentum spinors are

\[
u_b(\vec{0}, \frac{1}{2}) = \frac{1}{\sqrt{2}} \left[ \begin{array}{c} 1 \\ 0 \\ 0 \end{array} \right] \quad \text{and} \quad \nu_b(\vec{0}, -\frac{1}{2}) = \frac{1}{\sqrt{2}} \left[ \begin{array}{c} 0 \\ 1 \\ 0 \end{array} \right]
\]  \hspace{1cm} (1.322)

\[
u_c(\vec{0}, \frac{1}{2}) = \frac{1}{\sqrt{2}} \left[ \begin{array}{c} 0 \\ 1 \\ -1 \end{array} \right] \quad \text{and} \quad \nu_c(\vec{0}, -\frac{1}{2}) = \frac{1}{\sqrt{2}} \left[ \begin{array}{c} -1 \\ 0 \\ 1 \end{array} \right]
\]

The spinor \( \nu_b(0, +) \) is

\[
u_b(\vec{0}, +) = \left( \begin{array}{c} 0 \\ -\sigma_2 \\ 0 \end{array} \right) v^*(\vec{0}, +)
\]  \hspace{1cm} (1.323)
or in all its components,

\[
 u_b(\vec{0}, +) = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} = \frac{i}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = i u(\vec{0}, +) \quad (1.324)
\]

which is what we expect. Similarly,

\[
 u_b(\vec{0}, -) = \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix} \nu^*(\vec{0}, -) \quad (1.325)
\]

or in all its components,

\[
 u_b(\vec{0}, -) = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = i u(\vec{0}, -) \quad (1.326)
\]

which is what we expect. Similarly, the spinor \( v_a(0, +) \) is

\[
 v_a(\vec{0}, +) = \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix} u^*(\vec{0}, +) \quad (1.327)
\]

or in all its components,

\[
 v_a(\vec{0}, +) = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \frac{i}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = i v(\vec{0}, +) \quad (1.328)
\]

which is what we expect. Finally, the spinor \( v_a(0, -) \) is

\[
 v_a(\vec{0}, -) = \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix} u^*(\vec{0}, -) \quad (1.329)
\]

or in all its components,

\[
 v_a(\vec{0}, -) = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \frac{i}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \end{pmatrix} = i v(\vec{0}, -) \quad (1.330)
\]

which is what we expect.
1.20 Parity

Under parity the field $\psi(x)$ changes to

$$P\psi(x)P^{-1} = \eta^* \beta \psi(Px) \quad (1.313)$$

where $\eta$ is the parity of the particle annihilated by $a$, which is minus the complex conjugate of that of the particle annihilated by $b$. The annihilation and creation operators transform as

$$P a^\dagger(\vec{p}, s) P^{-1} = \eta a^\dagger(-\vec{p}, s) \quad \text{and} \quad P a^\dagger_c(\vec{p}, s) P^{-1} = \eta_c a^\dagger_c(-\vec{p}, s) \quad (1.332)$$

and

$$P a(\vec{p}, s) P^{-1} = \eta^* a(-\vec{p}, s) \quad \text{and} \quad P a_c(\vec{p}, s) P^{-1} = \eta^*_c a(-\vec{p}, s). \quad (1.333)$$

So

$$P\psi(x)P^{-1} = \int \frac{d^3p}{(2\pi)^{3/2}} u(\vec{p}, s) e^{ip\cdot x} P a(\vec{p}, s) P^{-1} + v(\vec{p}, s) e^{-ip\cdot x} P a^\dagger_c(\vec{p}, s) P^{-1}$$

$$= \int \frac{d^3p}{(2\pi)^{3/2}} u(\vec{p}, s) e^{ip\cdot x} \eta^* a(-\vec{p}, s) + v(\vec{p}, s) e^{-ip\cdot x} \eta_c a^\dagger_c(-\vec{p}, s)$$

But $\eta_c = -\eta^*$, so

$$P\psi(x)P^{-1} = \eta^* \int \frac{d^3p}{(2\pi)^{3/2}} u(\vec{p}, s) e^{ip\cdot x} a(-\vec{p}, s) - v(\vec{p}, s) e^{-ip\cdot x} a^\dagger_c(-\vec{p}, s).$$

Now

$$P\psi(x)P^{-1} = \eta^* \beta \psi(Px),$$

that is,

$$P\psi(x)P^{-1} = \eta^* \beta \int \frac{d^3p}{(2\pi)^{3/2}} u(\vec{p}, s) e^{ip\cdot x} a(\vec{p}, s) + v(\vec{p}, s) e^{-ip\cdot x} a^\dagger_c(\vec{p}, s)$$

$$= \eta^* \beta \int \frac{d^3p}{(2\pi)^{3/2}} u(-\vec{p}, s) e^{ip\cdot x} a(-\vec{p}, s) + v(-\vec{p}, s) e^{-ip\cdot x} a^\dagger_c(-\vec{p}, s)$$

So we need

$$\beta u(-\vec{p}, s) = u(\vec{p}, s) \quad \text{and} \quad \beta v(-\vec{p}, s) = -v(\vec{p}, s). \quad (1.334)$$

These rules are easy to check for $\vec{p} = 0$

$$\beta u(\vec{0}, s) = u(\vec{0}, s) \quad \text{and} \quad \beta v(\vec{0}, s) = -v(\vec{0}, s) \quad (1.335)$$

which the spinors (1.322) satisfy.
2

Feynman diagrams

2.1 Time-dependent perturbation theory

Most physics problems are insoluble, and we must approximate the unknown solution by numerical or analytic methods. The most common analytic method is called perturbation theory and is based on an assumption that something is small compared to something simple that we can analyze.

In time-dependent perturbation theory, we write the hamiltonian $H$ as the sum $H = H_0 + V$ of a simple hamiltonian $H_0$ and a small, complicated part $V$ which we give the time dependence induced by $H_0$

$$V(t) = e^{iH_0 t/\hbar} V e^{-iH_0 t/\hbar}. \quad (2.1)$$

We alter the time dependence of states by the same simple exponential (and set $\hbar = 1$)

$$|\psi, t\rangle = e^{iH_0 t} e^{-iHt} |\psi\rangle \quad (2.2)$$

and find as its time derivative

$$i \frac{d}{dt} |\psi, t\rangle = e^{iH_0 t} (H - H_0) e^{-iHt} |\psi\rangle = e^{iH_0 t} V e^{-iHt} e^{iH_0 t} e^{-iHt} |\psi\rangle = V(t) |\psi, t\rangle. \quad (2.3)$$

We can formally solve this differential equation by time-ordering factors of $V(t)$ in expansion of the exponential

$$|\psi, t\rangle = T[e^{-i \int_0^t V(t') dt'}] |\psi\rangle \quad (2.4)$$

so that $V$’s of later times occur to the left of $V$’s of earier times, that is, if $t > t'$ then $T[V(t)V(t')] = V(t)V(t')$ and so forth.
2.2 Dyson’s expansion of the S matrix

The time-evolution operator in the interaction picture is the time-ordered exponential of the integral over time of the interaction hamiltonian

\[ T \left[ e^{-i \int V(t) \, dt} \right]. \tag{2.5} \]

Time ordering has \( V(t_<) \) to the left of \( V(t_> \) if the time \( t_> \) is later than the time \( V(t_<). \) The interaction hamiltonian is

\[ V(t) = \int H(t, x) \, d^3x. \tag{2.6} \]

The density of the interaction hamiltonian is (or is taken to be) a sum of terms

\[ H(t, x) = \sum_i g_i H_i(t, x) \tag{2.7} \]

each of which is a monomial in the fields \( \psi_\ell(x) \) and their adjoints \( \psi_\ell^\dagger(x). \) A generic field is an integral over momentum

\[ \psi_\ell(x) = \sum_s \int \frac{d^3p}{(2\pi)^{3/2}} \left[ u_\ell(p, s, n) a(p, s, n) e^{ip \cdot x} + v_\ell(p, s, n) b_\ell^\dagger(p, s, n) e^{-ip \cdot x} \right] \tag{2.8} \]

in which \( n \) labels the kind of field.

The elements of the S matrix are amplitudes for an initial state \( |p_1, s_1, n_1; \ldots; p_k, s_k, n_k\rangle \) to evolve into a final state \( |p'_1, s'_1, n'_1; \ldots; p'_{k'}, s'_{k'}, n'_{k'}\rangle. \) The case considered most often is when \( k = 2 \) incoming particles and \( k' = 2 \) or 3 outgoing particles, but in high-energy collisions, \( k' \) can be much larger than 2 or 3. An S-matrix amplitude is

\[ S_{p_1', s_1', n_1'; \ldots; p_k', s_k', n_k'; \ldots; p_1, s_1, n_1; \ldots; p_k, s_k, n_k} = \langle p_1', s_1', n_1'; \ldots; p'_{k'}, s'_{k'}, n'_{k'} | T \left[ e^{-i \int H(x) \, dx} \right] | p_1, s_1, n_1; \ldots; p_k, s_k, n_k \rangle \]

\[ = \langle 0 | a(p_{k'}, s_{k'}, n_{k'}; \ldots; a(p_1, s_1, n_1) \]

\[ \times \sum_{N=0}^{\infty} \frac{(-i)^n}{N!} \int d^4x_1 \ldots d^4x_N T \left[ H(x_1) \ldots H(x_N) \right] \]

\[ \times a_1^\dagger(p_1, s_1, n_1; \ldots; a_1^\dagger(p_k, s_k, n_k)|0\rangle \tag{2.9} \]

in which \( |0\rangle \) is the vacuum state. It is the mean value in the vacuum state of an infinite polynomial in creation and annihilation operators.

To make sense of it, the first step is to normally order the time-evolution operator by moving all annihilation operators to the right of all creation operators. We assume for now that \( H(x) \) itself is already normally ordered.
To do that we use these commutation relations with plus signs for bosons and minus signs for fermions:

\[
a(p, s, n) a(p', s', n') = \pm a^\dagger(p', s', n') a(p, s, n) + \delta^3(p - p') \delta_{ss'} \delta_{nn}'.
\]

(2.10)

\[
a(p, s, n) a(p', s', n') = \pm a(p', s', n') a(p, s, n)
\]

Since

\[
a(p, s, n) |0\rangle = 0 \quad \text{and} \quad \langle 0 | a^\dagger(p, s, n) = 0,
\]

(2.11)

the S-matrix amplitude is just the left-over pieces proportional to products of delta functions multiplied by \(u\)'s and \(v\)'s and Fourier factors all divided by \(2\pi\)’s.

Each such piece arises from the pairing of one annihilation operator with one creation operator. Such pairs arise in six ways:

1. A particle \(p', s', n'\) in the final state can pair with a field adjoint \(\psi^\dagger_t(x)\) in \(H(x)\) and yield the factor

\[
[a(p', s', n'), \psi^\dagger_t(x)]_\mp = u^*_t(p', s', n') e^{-ip' \cdot x} (2\pi)^{-3/2}.
\]

(2.12)

2. An antiparticle \(p', s', n'\) in the final state can pair with a field \(\psi_t(x)\) in \(H(x)\) and yield the factor

\[
b(p', s', n'), \psi_t(x) = v_t(p', s', n') e^{-ip' \cdot x} (2\pi)^{-3/2}.
\]

(2.13)

3. A particle \(p, s, n\) in the initial state can pair with a field \(\psi_t(x)\) in \(H(x)\) and yield the factor

\[
[\psi_t(x), a^\dagger(p, s, n)]_\mp = u_t(p, s, n) e^{ip \cdot x} (2\pi)^{-3/2}.
\]

(2.14)

4. An antiparticle \(p, s, n\) in the initial state can pair with a field adjoint \(\psi^\dagger_t(x)\) in \(H(x)\) and yield the factor

\[
[\psi^\dagger_t(x), b^\dagger(p, s, n)]_\mp = v^*_t(p, s, n) e^{ip \cdot x} (2\pi)^{-3/2}.
\]

(2.15)

5. A particle (or antiparticle) \(p', s', n'\) in the final state can pair with a particle (or antiparticle) \(p, s, n\) in the initial state and yield

\[
[a(p', s', n'), a^\dagger(p, s, n)]_\mp = \delta^3(p - p') \delta_{ss'} \delta_{nn'}.
\]

(2.16)

6. A field \(\psi_t(x) = \psi^\dagger_t(x) + \psi^-_t(x)\) in \(H(x)\) can pair with a field adjoint in \(H'(y), \psi^\dagger_m(y) + \psi^-_m(y)\). But for \(\psi^\dagger_t(x)\) to cross \(\psi^-_m(y)\) the time \(x^0\) must be later than \(y^0\), and for \(\psi^-_m(y)\) to cross \(\psi^\dagger_t(x)\), the time \(y^0\) must
be later than \(x^0\). If for instance \(H(x) = \psi^\dagger(x)\psi(x)\phi(x)\), then in Dyson’s expansion these terms would appear

\[
\theta(x^0 - y^0) \psi^\dagger(x)\psi(x)\phi(x) \psi^\dagger(y)\psi(y)\phi(y) + \theta(y^0 - x^0) \psi^\dagger(y)\psi(y)\phi(y) \psi^\dagger(x)\psi(x)\phi(x).
\] (2.17)

The resulting pairings are the **propagator**

\[
\theta(x^0 - y^0) [\psi^\dagger(x), \psi^\dagger_m(y)]_\mp + \theta(y^0 - x^0) [\psi_m^\dagger(y), \psi(x)]_\mp \equiv -i\Delta_{\ell m}(x, y)
\] (2.18)

in which the ± signs will be explained later.

One then integrates over \(N\) spacetimes and the implicit momenta. The result is defined by a set of rules and **Feynman diagrams**. In general, the \(1/N!\) in Dyson’s expansion is cancelled by the \(N!\) ways of labelling the \(x_i\)’s. For example, in the term

\[
\frac{1}{2!} \int d^4x_1 d^4x_2 T[H(x_1)H(x_2)]
\] (2.19)

one can have \(H(x_1)\) absorb an incoming electron of momentum \(p\) and have \(H(x_2)\) absorb an incoming electron of momentum \(p'\) or the reverse.

But some processes require special combinatorics. So people often write

\[
H(x) = \frac{g}{3!}\phi^3(x) \quad \text{or} \quad H(x) = \frac{g}{4!}\phi^4(x)
\] (2.20)

to compensate for multiple possible pairings. But these factorials don’t always cancel. Fermions introduce minus signs. The surest way to check the signs and factorials in each process until one has gained sufficient experience.

### 2.3 The Feynman propagator for scalar fields

Adding \(\pm i\epsilon\) to the denominator of a pole term of an integral formula for a function \(f(x)\) can slightly shift the pole into the upper or lower half plane, causing the pole to contribute if a ghost contour goes around the upper half-plane or the lower half-plane. Such an \(i\epsilon\) can impose a boundary condition on a Green’s function.

The Feynman propagator \(\Delta_F(x)\) is a Green’s function for the Klein-Gordon differential operator \(^{\text{[Weinberg 1995]}}\) (274–280)

\[
(m^2 - \Box)\Delta_F(x) = \delta^4(x)
\] (2.21)

in which \(x = (x^0, \mathbf{x})\) and

\[
\Box = \Delta - \frac{\partial^2}{\partial t^2} = \Delta - \frac{\partial^2}{\partial (x^0)^2}
\] (2.22)
is the four-dimensional version of the laplacian $\triangle \equiv \nabla \cdot \nabla$. Here $\delta^4(x)$ is the four-dimensional Dirac delta function

$$\delta^4(x) = \int \frac{d^4q}{(2\pi)^4} \exp[i(q \cdot x - q^0x^0)] = \int \frac{d^4q}{(2\pi)^4} e^{iqx} \quad (2.23)$$

in which $qx = q \cdot x - q^0x^0$ is the Lorentz-invariant inner product of the 4-vectors $q$ and $x$. There are many Green’s functions that satisfy Eq.(2.21). Feynman’s propagator $\Delta_F(x)$

$$\Delta_F(x) = \int \frac{d^4q}{(2\pi)^4} \frac{\exp(iqx)}{q^2 + m^2 - i\epsilon} = \int \frac{d^3q}{(2\pi)^3} \int_0^\infty \frac{dq^0}{2\pi} e^{iq \cdot x} e^{-i(q^0 - q^0)x^0} \quad (2.24)$$

is the one that satisfies boundary conditions that will become evident when we analyze the effect of its $i\epsilon$. The quantity $E_q = \sqrt{q^2 + m^2}$ is the energy of a particle of mass $m$ and momentum $q$ in natural units with the speed of light $c = 1$. Using this abbreviation and setting $\epsilon' = \epsilon/2E_q$, we may write the denominator as

$$q^2 + m^2 - i\epsilon = q \cdot q - (q^0)^2 + m^2 - i\epsilon = (E_q - i\epsilon' - q^0)(E_q - i\epsilon' + q^0) + \epsilon'^2 \quad (2.25)$$

in which $\epsilon'^2$ is negligible. Dropping the prime on $\epsilon$, we do the $q^0$ integral

$$I(q) = -\int \frac{dq^0}{2\pi} e^{-iq^0x^0} \frac{1}{[q^0 - (E_q - i\epsilon)][q^0 - (-E_q + i\epsilon)]}. \quad (2.26)$$

As shown in Fig. 2.1 the integrand

$$e^{-iq^0x^0} \frac{1}{[q^0 - (E_q - i\epsilon)][q^0 - (-E_q + i\epsilon)]} \quad (2.27)$$

has poles at $E_q - i\epsilon$ and at $-E_q + i\epsilon$. When $x^0 > 0$, we can add a ghost contour that goes clockwise around the lower half-plane and get

$$I(q) = ie^{-iE_qx^0} \frac{1}{2E_q} x^0 > 0. \quad (2.28)$$

When $x^0 < 0$, our ghost contour goes counterclockwise around the upper half-plane, and we get

$$I(q) = ie^{iE_qx^0} \frac{1}{2E_q} x^0 < 0. \quad (2.29)$$

Using the step function $\theta(x) = (x + |x|)/2$, we combine (2.25) and (2.29) to get

$$-iI(q) = \frac{1}{2E_q} \left[ \theta(x^0) e^{-iE_qx^0} + \theta(-x^0) e^{iE_qx^0} \right]. \quad (2.30)$$
Figure 2.1 In equation (2.27), the function $f(q^0)$ has poles at $\pm(E_q - i\epsilon)$, and the function $\exp(-iq^0 x^0)$ is exponentially suppressed in the lower half plane if $x^0 > 0$ and in the upper half plane if $x^0 < 0$. So we can add a ghost contour (...) in the LHP if $x^0 > 0$ and in the UHP if $x^0 < 0$.

In terms of the Lorentz-invariant function

$$\Delta_\pm(x) = \frac{1}{(2\pi)^3} \int \frac{d^3q}{2E_q} \exp[i(q \cdot x - E_q x^0)]$$

(2.31)

and with a factor of $-i$, Feynman’s propagator (2.24) is

$$-i\Delta_F(x) = \theta(x^0) \Delta_+(x) + \theta(-x^0) \Delta_+(x,-x^0).$$

(2.32)
The integral (2.31) defining $\Delta_+(x)$ is insensitive to the sign of $q$, and so

$$
\Delta_+(-x) = \frac{1}{(2\pi)^3} \int \frac{d^3q}{2E_q} \exp[i(-q \cdot x + E_qx^0)]
$$

(2.33)

$$
= \frac{1}{(2\pi)^3} \int \frac{d^3q}{2E_q} \exp[i(q \cdot x + E_qx^0)] = \Delta_+(x, -x^0).
$$

Thus we arrive at the Standard form of the Feynman propagator

$$
-i\Delta_F(x) = \theta(x^0) \Delta_+(x) + \theta(-x^0) \Delta_+(-x).
$$

(2.34)

The annihilation operators $a(q)$ and the creation operators $a^\dagger(p)$ of a scalar field $\phi(x)$ satisfy the commutation relations

$$
[a(q), a^\dagger(p)] = \delta^3(q - p) \quad \text{and} \quad [a(q), a(p)] = [a^\dagger(q), a^\dagger(p)] = 0.
$$

(2.35)

Thus the commutator of the positive-frequency part $\phi^+(x) = \int \frac{d^3p}{\sqrt{(2\pi)^32p^0}} \exp[i(p \cdot x - p^0x^0)] a(p)$ of a scalar field $\phi = \phi^+ + \phi^-$ with its negative-frequency part $\phi^-(y) = \int \frac{d^3q}{\sqrt{(2\pi)^32q^0}} \exp[-i(q \cdot y - q^0y^0)] a^\dagger(q)$ is the Lorentz-invariant function $\Delta_+(x - y)$

$$
\{\phi^+(x), \phi^-(y)\} = \int \frac{d^3p}{(2\pi)^32p^0} \frac{d^3q}{(2\pi)^32q^0} \exp[ipx - iqy] [a(p), a^\dagger(q)]
$$

(2.36)

$$
= \int \frac{d^3p}{(2\pi)^32p^0} \exp[ip(x - y)] = \Delta_+(x - y)
$$

(2.37)

in which $p(x - y) = p \cdot (x - y) - p^0(x^0 - y^0)$.

At points $x$ that are space-like, that is, for which $x^2 = x^2 - (x^0)^2 \equiv r^2 > 0$, the Lorentz-invariant function $\Delta_+(x)$ depends only upon $r = +\sqrt{x^2}$ and has the value (Weinberg, 1995, p. 202)

$$
\Delta_+(x) = \frac{m}{4\pi^2r} K_1(mr)
$$

(2.38)

in which the Hankel function $K_1$ is

$$
K_1(z) = -\frac{\pi}{2} [J_1(iz) + iN_1(iz)] = \frac{1}{z} + \frac{z}{2} \left[ \ln \left( \frac{z}{2} \right) + \gamma - \frac{1}{2} \right] + \ldots
$$

(2.39)

where $J_1$ is the first Bessel function, $N_1$ is the first Neumann function, and $\gamma = 0.57721 \ldots$ is the Euler-Mascheroni constant.
The Feynman propagator arises most simply as the mean value in the vacuum of the **time-ordered product** of the fields $\phi(x)$ and $\phi(y)$

$$T\{\phi(x)\phi(y)\} \equiv \theta(x^0 - y^0)\phi(x)\phi(y) + \theta(y^0 - x^0)\phi(y)\phi(x). \tag{2.41}$$

The operators $a(p)$ and $a^\dagger(p)$ respectively annihilate the vacuum ket $a(p)|0\rangle = 0$ and bra $\langle 0|a^\dagger(p) = 0$, and so by (2.36 & 2.37) do the positive- and negative-frequency parts of the field $\phi^+(z)|0\rangle = 0$ and $\langle 0|\phi^-(z) = 0$. Thus the mean value in the vacuum of the time-ordered product is

$$\langle 0|T\{\phi(x)\phi(y)\}|0\rangle = \langle 0|\theta(x^0 - y^0)\phi(x)\phi(y) + \theta(y^0 - x^0)\phi(y)\phi(x)|0\rangle$$

$$= \langle 0|\theta(x^0 - y^0)\phi^+(x)\phi^-(y) + \theta(y^0 - x^0)\phi^+(y)\phi^-(x)|0\rangle$$

$$= \langle 0|\theta(x^0 - y^0)[\phi^+(x),\phi^-(y)]$$

$$+ \theta(y^0 - x^0)[\phi^+(y),\phi^-(x)]|0\rangle. \tag{2.42}$$

But by (2.38), these commutators are $\Delta^+(x - y)$ and $\Delta^-(y - x)$. Thus the mean value in the vacuum of the time-ordered product of two real scalar fields

$$\langle 0|T\{\phi(x)\phi(y)\}|0\rangle = \theta(x^0 - y^0)\Delta^+(x - y) + \theta(y^0 - x^0)\Delta^-(y - x)$$

$$= -i\Delta_F(x - y) = \int \frac{d^4q}{(2\pi)^4} e^{iq\cdot x} \frac{-i}{q^2 + m^2 - i\epsilon} \tag{2.43}$$

is the Feynman propagator (2.32) multiplied by $-i$.

### 2.4 Application to a cubic scalar field theory

The action density

$$L = -\frac{1}{2}\partial_\nu\phi \partial^\nu\phi - \frac{1}{2}m^2\phi^2 - \frac{g}{3!}\phi^3, \tag{2.44}$$

describes a scalar field with cubic interactions. We let

$$V(t) = \frac{g}{3!} \int \phi(x)^3 \, dx \tag{2.45}$$

in which the field $\phi$ has the free-field time dependence

$$\phi(x) = \int \frac{d^3p}{\sqrt{(2\pi)^32p^0}} [a(p)e^{ip\cdot x} + a^\dagger(p)e^{-ip\cdot x}] \tag{2.46}$$

in which $p \cdot x = \vec{p} \cdot \vec{x} - p^0t$ and $p^0 = \sqrt{\vec{p}^2 + m^2}$. The amplitude for the scattering of bosons with momenta $p$ and $k$ to momenta $p'$ and $k'$ is to
2.4 Application to a cubic scalar field theory

lowest order in the coupling constant $g$

$$A = \langle p', k' | - \frac{1}{2!} \int T[V(t')V(t'')] \, dt' \, dt'' \, |p, k)$$

$$= - \frac{g^2}{2! (3!)^2} \langle p', k' | \int T[\phi^3(x)\phi^3(y)] \, d^4xd^4y \, |p, k).$$

(2.47)

This process happens in three ways, so $A = A_1 + A_2 + A_3$.

The first way is for the incoming particles to be absorbed at the same vertex $x$ or $y$ and for the outgoing particles to be emitted at the other vertex $y$ or $x$. We cancel the $2!$ by choosing the initial particles to be absorbed at $y$ and the final particles to be emitted at $x$. Then using the expansion (2.46) of the field, we find

$$A_1 = - \frac{g^2}{4} \langle 0 | a(k') a(p') \int \frac{d^3p'' d^3k'' d^3p' d^3k'}{(2\pi)^6 \sqrt{2p'^0 k'^0 2p''0 k''0}} \, a^\dagger(p'') a^\dagger(k'')$$

$$\times e^{i(p'' + k'') \cdot x} T[\phi(x)\phi(y)] \, a(p'') a(k'') e^{i(p' + k') \cdot y} \, d^4xd^4y \, a^\dagger(p) a^\dagger(k) |0)$$

(2.48)

after canceling two factors of 3 because each of the 3 fields $\phi^3(x)$ and each of the 3 fields $\phi^3(y)$ could be the one to remain in the time-ordered product $T[\phi(x)\phi(y)]$. The commutation relations $[a(p), a^\dagger(k)] = \delta^3(p - k)$ now give

$$A_1 = - g^2 \int \frac{d^4xd^4y}{(2\pi)^6 \sqrt{2p^0 k^0 2p^0 k^0}} e^{i(p + k) \cdot y - i(p' + k') \cdot x} \langle 0 | T[\phi(x)\phi(y)] |0)$$

(2.49)

in which the mean value in the vacuum of the time-ordered product

$$\langle 0 | T[\phi(x)\phi(y)] |0) = - i \Delta_F(x - y) = - i \int \frac{d^4q}{(2\pi)^4} \frac{\exp(iqx)}{q^2 + m^2 - i\epsilon}$$

(2.50)

is Feynman’s propagator (2.24 & 2.43). Thus the amplitude $A_1$ is

$$A_1 = ig^2 \int \frac{d^4xd^4y \, d^4q}{(2\pi)^{10} \sqrt{2p^0 k^0 2p^0 k^0}} \frac{e^{i(p + k) \cdot y - i(p' + k') \cdot x + iq(x - y)}}{q^2 + m^2 - i\epsilon}.$$ 

(2.51)

Thanks to Dirac’s delta function, the integrals over $x, y,$ and $q$ are easy, and in the smooth $\epsilon \to 0$ limit $A_1$ is

$$A_1 = ig^2 \int \frac{d^4q}{(2\pi)^2 \sqrt{2p^0 k^0 2p^0 k^0}} \frac{\delta^4(p + k - q) \delta^4(q - p' - k')}{q^2 + m^2 - i\epsilon}$$

$$= \frac{\delta^4(p + k - p' - k')}{(2\pi)^2 \sqrt{2p^0 k^0 2p^0 k^0}} \frac{ig^2}{(p + k)^2 + m^2}.$$

(2.52)
The sum of the three amplitudes is

\[
A = \frac{\delta^4(p + k - p' - k')}{16\pi^2 \sqrt{p^0 k^0 p'^0 k'^0}} \left[ \frac{ig^2}{(p + k)^2 + m^2} + \frac{ig^2}{(p - k')^2 + m^2} + \frac{ig^2}{(p - p')^2 + m^2} \right]
\]

in which the delta function conserves energy and momentum.

### 2.5 Feynman’s propagator for fields with spin

The time-ordered product for fields with spin is defined so as to compensate for the minus sign that arises when a Fermi field is moved past a Fermi field. Thus the mean value in the vacuum of the time-ordered product \( T\{\psi_\ell(x)\psi^\dagger_m(y)\} \) is

\[
\langle 0 | T\{\psi_\ell(x)\psi^\dagger_m(y)\}|0 \rangle = \langle 0 | \theta(x^0 - y^0)\psi_\ell(x)\psi^\dagger_m(y) \pm \theta(y^0 - x^0)\psi^\dagger_m(y)\psi_\ell(x)|0 \rangle
\]

\[
= \langle 0 | \theta(x^0 - y^0)\psi^+_\ell(x)\psi^\dagger_m(y) \pm \theta(y^0 - x^0)\psi^\dagger_m(y)\psi^-_\ell(x)|0 \rangle
\]

\[
= \langle 0 | \theta(x^0 - y^0)[\psi^+_\ell(x), \psi^\dagger_m(y)] \pm \theta(y^0 - x^0)[\psi^\dagger_m(y), \psi^-_\ell(x)]|0 \rangle.
\]

In which the upper signs are used for bosons and the lower ones for fermions.

The expansions

\[
\psi^+_\ell(x) = (2\pi)^{-3/2} \sum_s \int d^3 p \ u_\ell(\vec{p}, s) e^{ip \cdot x} a(\vec{p}, s)
\]

\[
\psi^-_\ell(x) = (2\pi)^{-3/2} \sum_s \int d^3 p \ v_\ell(\vec{p}, s) e^{-ip \cdot x} b^\dagger(\vec{p}, s)
\]

Give for the (anti)commutators

\[
[\psi^+_\ell(x), \psi^\dagger_m(y)]_\mp = \left[ (2\pi)^{-3/2} \sum_s \int d^3 p \ u_\ell(\vec{p}, s) e^{ip \cdot x} a(\vec{p}, s),
\right.
\]

\[
\left. (2\pi)^{-3/2} \sum_r \int d^3 k \ u^*_m(\vec{k}, r) e^{-ik \cdot y} a^\dagger(\vec{k}, r) \right]_\mp
\]

\[
= \sum_s \int \frac{d^3 p}{(2\pi)^3} u_\ell(\vec{p}, s) u^*_m(\vec{p}, s) e^{ip \cdot (x - y)}
\]

(2.56)
and

\[
[\psi_\ell^+ (y), \psi_m^-(x)]_\mp = \left[ (2\pi)^{-3/2} \sum_s \int d^3p \ v_m^*(\vec{p}, s) e^{ip\cdot y} b(\vec{p}, s),
\right.
\]
\[
(2\pi)^{-3/2} \sum_r \int d^3k \ v_\ell(\vec{k}, r) e^{-ik\cdot x} b^+(\vec{k}, r) \left. \right]_\mp \tag{2.57}
\]
\[
= \sum_s \int \frac{d^3p}{(2\pi)^3} v_\ell(\vec{p}, s) v_m^*(\vec{p}, s) e^{ip\cdot (y-x)}.
\]

Putting these expansions into the formula (2.54) for the time-ordered product, we get for its mean value in the vacuum

\[
\langle 0| T \left\{ \psi_\ell(x) \psi_m^+(y) \right\} | 0 \rangle = \theta(x^0 - y^0) \sum_s \int \frac{d^3p}{(2\pi)^3} u_\ell(\vec{p}, s) u_m^*(\vec{p}, s) e^{ip\cdot (x-y)}
\]
\[
\pm \theta(y^0 - x^0) \sum_s \int \frac{d^3p}{(2\pi)^3} v_\ell(\vec{p}, s) v_m^*(\vec{p}, s) e^{ip\cdot (y-x)}.
\]
\[
\tag{2.58}
\]

### 2.6 Feynman’s propagator for spin-one-half fields

For fields of spin one half, the spin sums are

\[
[N(\vec{p})]_{\ell m} = \sum_s u_\ell(\vec{p}, s) u_m^*(\vec{p}, s) = \left[ \frac{1}{2p^0} (-i p^c \gamma_c + m) \right]_{\ell m},
\]
\[
[M(\vec{p})]_{\ell m} = \sum_s v_\ell(\vec{p}, s) v_m^*(\vec{p}, s) = \left[ \frac{1}{2p^0} (-i p^c \gamma_c - m) \right]_{\ell m},
\]
\[
\tag{2.59}
\]

so for spin-one-half fields Feynman’s propagator is

\[
\langle 0| T \left\{ \psi_\ell(x) \psi_m^+(y) \right\} | 0 \rangle = \theta(x^0 - y^0) \int \frac{d^3p}{(2\pi)^3} \left[ \frac{1}{2p^0} (-i p^c \gamma_c + m) \right]_{\ell m} e^{ip\cdot (x-y)}
\]
\[
- \theta(y^0 - x^0) \int \frac{d^3p}{(2\pi)^3} \left[ \frac{1}{2p^0} (-i p^c \gamma_c - m) \right]_{\ell m} e^{ip\cdot (y-x)}.
\]
\[
\tag{2.60}
\]
But at equal times $\Delta = $ so also as we can write Feynman’s propagator as

$$
\langle 0 | T \left\{ \psi(x) \psi_m^\dagger(y) \right\} | 0 \rangle = \theta(x^0 - y^0) \left[ \left( - \partial_c \gamma^c + m \right) \beta \right]_{\ell m} \int \frac{d^3p}{(2\pi)^3} 2p^0 e^{ip(x-y)} \\
- \theta(y^0 - x^0) \left[ (\partial_c \gamma^c - m) \beta \right]_{\ell m} \int \frac{d^3p}{(2\pi)^3} 2p^0 e^{ip(y-x)} \\
= \theta(x^0 - y^0) \left[ ( - \partial_c \gamma^c + m ) \beta \right]_{\ell m} \Delta_+(x - y) \\
+ \theta(y^0 - x^0) \left[ ( - \partial_c \gamma^c + m ) \beta \right]_{\ell m} \Delta_+(y - x).
$$

(2.61)

Since the derivative of the step function is a delta function

$$
\partial_0 \theta(x^0 - y^0) = \delta(x^0 - y^0),
$$

(2.62)

we can write Feynman’s propagator as

$$
\langle 0 | T \left\{ \psi(x) \psi_m^\dagger(y) \right\} | 0 \rangle = \left[ ( - \partial_c \gamma^c + m ) \beta \right]_{\ell m} \\
\times \left( \theta(x^0 - y^0) \Delta_+(x - y) + \theta(y^0 - x^0) \Delta_+(y - x) \right) \\
+ \partial_0 \gamma^0 \beta \theta(x^0 - y^0) \Delta_+(x - y) \\
- \partial_0 \gamma^0 \beta \theta(y^0 - x^0) \Delta_+(y - x)
$$

(2.63)

and so also as

$$
\langle 0 | T \left\{ \psi(x) \psi_m^\dagger(y) \right\} | 0 \rangle = \left[ ( - \partial_c \gamma^c + m ) \beta \right]_{\ell m} \\
\times \left( \theta(x^0 - y^0) \Delta_+(x - y) + \theta(y^0 - x^0) \Delta_+(y - x) \right) \\
+ \gamma^0 \beta \delta(x^0 - y^0) \Delta_+(x - y) - \gamma^0 \beta \delta(y^0 - x^0) \Delta_+(y - x) \\
= \left[ ( - \partial_c \gamma^c + m ) \beta \right]_{\ell m} \\
\times \left( \theta(x^0 - y^0) \Delta_+(x - y) - \theta(y^0 - x^0) \Delta_+(y - x) \right) \\
+ \gamma^0 \beta \delta(x^0 - y^0) (\Delta_+(x - y) - \Delta_+(y - x)).
$$

(2.64)

But at equal times $\Delta_+(x - y) = \Delta_+(y - x)$. So the ugly final term vanishes, and Feynman’s propagator for spin-one-half fields is

$$
\langle 0 | T \left\{ \psi(x) \psi_m^\dagger(y) \right\} | 0 \rangle = \left[ ( - \partial_c \gamma^c + m ) \beta \right]_{\ell m} \\
\times \left( \theta(x^0 - y^0) \Delta_+(x - y) + \theta(y^0 - x^0) \Delta_+(y - x) \right).
$$

(2.65)
But this last piece is Feynman’s propagator for real scalar fields

\[ \langle 0|T\{\phi(x)\phi(y)\}|0\rangle = \theta(x^0 - y^0)\Delta_+(x - y) + \theta(y^0 - x^0)\Delta_+(y - x) \]

\[ = -i\Delta_F(x - y) = \int \frac{d^4q}{(2\pi)^4} \frac{-i}{q^2 + m^2 - i\epsilon} e^{iq(x - y)}. \]

so Feynman’s propagator for spin-one-half fields is

\[ \langle 0|T\{\psi_\ell(x)\psi_m^\dagger(y)\}|0\rangle \equiv -i\Delta_{\ell m}(x - y) \]

\[ = \left[ (-\partial_c\gamma^c + m)\beta \right]_{\ell m} \int \frac{d^4q}{(2\pi)^4} \frac{-i}{q^2 + m^2 - i\epsilon} e^{iq(x - y)}. \]

(2.66)

Letting the derivatives act on the exponential, we get

\[ \langle 0|T\{\psi_\ell(x)\psi_m^\dagger(y)\}|0\rangle \equiv -i\Delta_{\ell m}(x - y) \]

\[ = -i \int d^4q \left[ (-iq_a\gamma^a + m)\beta \right]_{\ell m} \frac{-i}{q^2 + m^2 - i\epsilon} e^{iq(x - y)}. \]

(2.67)

Since

\[ (-iq_a\gamma^a + m)(iq_a\gamma^a + m) = q^2 + m^2, \]

people often write this as

\[ \langle 0|T\{\psi_\ell(x)\psi_m^\dagger(y)\}|0\rangle \equiv -i\Delta_{\ell m}(x - y) \]

\[ = -i \int d^4q \left[ \frac{1}{iq_a\gamma^a + m - i\epsilon} \right]_{\ell m} e^{iq(x - y)}. \]

(2.69)

Feynman was a master of notation (and of everything else). He set \( \not{p} = p_\alpha\gamma^\alpha \)

and wrote

\[ \langle 0|T\{\psi_\ell(x)\psi_m^\dagger(y)\}|0\rangle \equiv -i\Delta_{\ell m}(x - y) \]

\[ = -i \int d^4q \left[ \frac{1}{iq + m - i\epsilon} \right]_{\ell m} e^{iq(x - y)}. \]

(2.70)
In the common notation $\bar{\psi} = \psi^\dagger \beta$, his propagator is

$$\langle 0 | T \{ \psi(x) \bar{\psi}(y) \} | 0 \rangle = -i \Delta_{\ell m}(x - y) \beta$$

$$= -i \int \frac{d^4q}{(2\pi)^4} \frac{\left( -i \not{q} + m \right)_{\ell m}}{q^2 + m^2 - i\epsilon} \, e^{iq(x - y)}$$

$$= -i \int \frac{d^4q}{(2\pi)^4} \left( \frac{1}{i \not{q} + m - i\epsilon} \right)_{\ell m} \, e^{iq(x - y)}. \tag{2.71}$$

### 2.7 Application to a theory of fermions and bosons

Let us consider first the theory

$$H(x) = g_{\ell m} \psi^\dagger_\ell (x) \psi_m(x) \phi(x) \tag{2.72}$$

where $g_{\ell m}$ is a coupling constant, $\psi(x)$ is the field of a fermion $f$, and $\phi(x) = \phi^\dagger(x)$ is a real boson $b$. Let's compute the amplitude $A$ for $f + b \rightarrow f' + b'$. The lowest-order term is

$$A = -\frac{1}{2!} \int d^4x d^4y \langle 0 | a(p', s') b(k') T \left[ H(x) \, H(y) \right] b^\dagger(k) a^\dagger(p, s) | 0 \rangle \tag{2.73}$$

$$= -\frac{g_{\ell m} g_{\ell' m'}}{2!} \int d^4x d^4y \langle 0 | a(p', s') b(k') T \left[ \psi^\dagger_\ell(x) \, \psi_m(x) \phi(x) \psi^\dagger_{\ell'}(y) \, \psi_{m'}(y) \phi(y) \right]$$

$$\times b^\dagger(k) a^\dagger(p, s) | 0 \rangle.$$

Here the operators $b(k')$ and $b^\dagger(k)$ are the boson deletion and addition operators. Either the boson field $\phi(y)$ deletes the boson from the initial state and the boson field $\phi(x)$ deletes the boson from the final state or the boson field $\phi(x)$ deletes the boson from the initial state and the boson field $\phi(y)$ deletes the boson from the final state. These give the same result. So we cancel the $2!$ and let $\phi(y)$ delete the boson from the initial state and have
\( \phi(x) \) add the boson of the final state. We then get

\[
A = - g \epsilon m g' m' \int d^4 x d^4 y \langle 0 | a(p', s') b(k') T \left[ \psi^\dagger_\ell (x) \psi_m (x) \phi^- (x) \psi^\dagger_\ell (y) \psi_{m'} (y) \phi^+ (y) \right] \\
\times b^\dagger (k) a^\dagger (p, s) |0 \rangle
\]

\[
= - g \epsilon m g' m' \int d^4 x d^4 y \langle 0 | a(p', s') b(k') T \left[ \psi^\dagger_\ell (x) \psi_m (x) \\
\times \frac{d^3 k''}{\sqrt{(2\pi)^3 2k''}} b^\dagger (k'') e^{-ik'' \cdot x} \psi^\dagger_\ell (y) \psi_{m'} (y) \phi^+ (y) \right] b^\dagger (k) a^\dagger (p, s) |0 \rangle
\]

\[
= - g \epsilon m g' m' \int d^4 x d^4 y \langle 0 | a(p', s') T \left[ \psi^\dagger_\ell (x) \psi_m (x) \frac{1}{\sqrt{(2\pi)^3 2k''}} e^{-ik'' \cdot x} \\
\times \psi^\dagger_\ell (y) \psi_{m'} (y) \phi^+ (y) \right] b^\dagger (k) a^\dagger (p, s) |0 \rangle.
\]

We now let \( \phi^+ (y) \) delete the boson from the initial state.

\[
A = - g^2 \int d^4 x d^4 y \langle 0 | a(p', s') T \left[ \psi^\dagger_\ell (x) \psi_m (x) \frac{1}{\sqrt{(2\pi)^3 2k''}} e^{-ik'' \cdot x} \\
\times \psi^\dagger_\ell (y) \psi_{m'} (y) \int \frac{d^3 k''}{\sqrt{(2\pi)^3 2k''}} b(k'') e^{ik'' \cdot y} \right] b^\dagger (k) a^\dagger (p, s) |0 \rangle
\]

\[
= - g \epsilon m g' m' \int d^4 x d^4 y \langle 0 | a(p', s') T \left[ \psi^\dagger_\ell (x) \psi_m (x) \frac{1}{\sqrt{(2\pi)^3 2k''}} e^{-ik'' \cdot y} \\
\times \psi^\dagger_\ell (y) \psi_{m'} (y) \int \frac{d^3 k''}{\sqrt{(2\pi)^3 2k''}} \delta^3 (k'' - k) e^{ik'' \cdot y} \right] a^\dagger (p, s) |0 \rangle
\]

\[
= - g \epsilon m g' m' \int d^4 x d^4 y \langle 0 | a(p', s') T \left[ \psi^\dagger_\ell (x) \psi_m (x) \frac{1}{\sqrt{(2\pi)^3 2k''}} e^{-ik'' \cdot x} \\
\times \psi^\dagger_\ell (y) \psi_{m'} (y) \frac{1}{\sqrt{(2\pi)^3 2k''}} e^{ik'' \cdot y} \right] a^\dagger (p, s) |0 \rangle
\]

\[
= - g \epsilon m g' m' \int \frac{d^4 x d^4 y}{\sqrt{(2\pi)^3 2k''}} e^{-ik'' \cdot y -ik'' \cdot x} \langle 0 | a(p', s') T \left[ \psi^\dagger_\ell (x) \psi_m (x) \psi^\dagger_\ell (y) \psi_{m'} (y) \right] a^\dagger (p, s) |0 \rangle.
\]
Now there are two terms. In one the initial fermion is deleted at $y$ and the final fermion is added at $x$

$$A_1 = -g_{\ell m}g'_{\ell' m'} \int \frac{d^4 x d^4 y}{(2\pi)^{12} 2^{16} k_0^4 k_0^4} e^{ik_{y} - ik'_{x}} \langle 0 | a(p', s') T \left[ \psi^+_{\ell}(x) \psi_m(x) \psi^+_{\ell'}(y) \psi^+_{m'}(y) \right] a^+(p, s) | 0 \rangle$$

$$= -g_{\ell m}g'_{\ell' m'} \int \frac{d^4 x d^4 y}{(2\pi)^{12} 2^{16} k_0^4 k_0^4} e^{ik_{y} - ik'_{x}} \langle 0 | a(p', s') T \left[ \int \frac{d^3 p''}{(2\pi)^3} e^{-ip''_{x} a^+(p'', s') u^+_{\ell'}(p'', s'')} \right. \psi_m(x) \psi^+_\ell(y) \int \frac{d^3 p'''}{(2\pi)^3} e^{ip'''_{y}} u_{m'}(p'''_{m}, s''') a^+(p, s) | 0 \rangle$$

$$= -g_{\ell m}g'_{\ell' m'} \int \frac{d^4 x d^4 y}{(2\pi)^{12} 2^{16} k_0^4 k_0^4} e^{ik_{y} - ik'_{x}} \langle 0 | T \left[ \int \frac{d^3 p''}{(2\pi)^3} e^{-ip_{x}'} u_{\ell'}(p', s) \delta^3(p' - p'') \delta_{s s'} \right] \psi_m(x) \psi^+_\ell(y) \int \frac{d^3 p''}{(2\pi)^3} e^{ip''_{y}} u_{m'}(p'''_{m}, s''') \delta^3(p'' - p''') \delta_{s s'''} a^+(p, s) | 0 \rangle$$

$$= -g_{\ell m}g'_{\ell' m'} u^+_{\ell'}(p', s') u_{m'}(p, s) \int \frac{d^4 x d^4 y}{(2\pi)^{12} 2^{16} k_0^4 k_0^4} e^{ik_{y} - ik'_{x}} e^{-ip_{x}'} e^{ip''_{y}} \langle 0 | T \left[ \psi_m(x) \psi^+_\ell(y) \right] 0 \rangle$$

In terms of SW's definition (6.2.31) of the fermion propagator, $A_1$ is

$$A_1 = -g_{\ell m}g'_{\ell' m'} u^+_{\ell'}(p', s') u_{m'}(p, s) \int \frac{d^4 x d^4 y}{(2\pi)^{12} 2^{16} k_0^4 k_0^4} e^{i(k+p)_{y} - i(k'+p')_{x}} (-i\Delta_{m'\ell'}(x, y)).$$

(2.77)

In the other term, the initial fermion is deleted at $x$ and the final fermion is added at $y$

$$A_2 = -g_{\ell m}g'_{\ell' m'} \int \frac{d^4 x d^4 y}{(2\pi)^{12} 2^{16} k_0^4 k_0^4} e^{ik_{y} - ik'_{x}} \langle 0 | a(p', s') T \left[ \psi^+_{\ell'}(y) \psi^+_{\ell}(x) \psi^+_m(y) \psi^+_m(y) \right] a^+(p, s) | 0 \rangle$$

$$= -g_{\ell m}g'_{\ell' m'} \int \frac{d^4 x d^4 y}{(2\pi)^{12} 2^{16} k_0^4 k_0^4} e^{ik_{y} - ik'_{x}} \langle 0 | a(p', s') T \left[ \psi^+_{\ell}(y) \psi^+(y) \psi^+_m(y) \psi^+_m(y) \right] a^+(p, s) | 0 \rangle$$

(2.78)

$$= g_{\ell m}g'_{\ell' m'} \int \frac{d^4 x d^4 y}{(2\pi)^{12} 2^{16} k_0^4 k_0^4} e^{ik_{y} - ik'_{x}} \langle 0 | a(p', s') T \left[ \psi^+_{\ell}(y) \psi^+_m(y) \psi^+_m(y) \psi^+_m(y) \right] a^+(p, s) | 0 \rangle$$

in which the minus sign arises from the transposition $\psi^+_{\ell}(x) \psi^+_m(y) \rightarrow \psi^+_m(y) \psi^+_\ell(x)$. The earlier transposition $\psi^+_{\ell}(x) \psi^+_m(y) \psi^+_m(y) \rightarrow \psi^+_m(y) \psi^+_m(y) \psi^+_\ell(x)$ produced two minus signs or one plus sign. Inserting the expansions of $\psi^+_m(y)$
and $\psi^+(x)$, we have

$$A_2 = g_{\ell m}g_{\ell' m'} \int \frac{d^4x d^4y}{(2\pi)^{12}k^{\alpha_0}2k'^{\alpha_0}} e^{ik_y-ik'_{-x}} \langle 0|a(p', s')T \left[ \int \frac{d^3p''}{(2\pi)^3} u_{\ell'}(p'', s') a_{\ell'}(p'', s'') e^{-ip''_{-y}} \right. $$

$$\left. \times \psi^+_1(x) \psi_{m'}(y) \int \frac{d^3p'^m}{(2\pi)^3} u_m(p'', s'')(a^{\dagger}(p'', s'') e^{ip'_{-x}}) a_{\ell'}(p, s)|0 \rangle \right)$$

$$= g_{\ell m}g_{\ell' m'} u_\ell^*(p', s') u_m(p, s) \int \frac{d^4x d^4y}{(2\pi)^{12}k^{\alpha_0}2k'^{\alpha_0}} e^{i(k'_{-y} - i(k'-p)_{-x})} (0|T \left[ \psi^+_1(x) \psi_{m'}(y) \right] |0 \rangle. \quad (2.79)$$

In terms of SW’s definition (6.2.31) of the fermion propagator, $A_2$ is

$$A_2 = -g_{\ell m}g_{\ell' m'} u_\ell^*(p'', s'') u_{m'}(p, s) \int \frac{d^4x d^4y}{(2\pi)^{12}k^{\alpha_0}2k'^{\alpha_0}} e^{i(k'_{-y} - i(k'-p)_{-x})} (0|T \left[ \psi_{m'}(y) \psi^+_1(x) \right] |0 \rangle$$

$$= -g_{\ell m}g_{\ell' m'} u_\ell^*(p', s') u_{m'}(p, s) \int \frac{d^4x d^4y}{(2\pi)^{12}k^{\alpha_0}2k'^{\alpha_0}} e^{i(k'_{-y} - i(k'-p)_{-x})} \left( -i\Delta_{m'}(y, x) \right). \quad (2.80)$$

The full amplitude for $f + b \rightarrow f' + b'$ is the sum $A = A_1 + A_2$ of the two amplitudes

$$A = -g_{\ell m}g_{\ell' m'} u_\ell^*(p', s') u_{m'}(p, s) \int \frac{d^4x d^4y}{(2\pi)^{12}k^{\alpha_0}2k'^{\alpha_0}} e^{i(k_+p_{-y} - i(k'+p')_{-x})} \left( -i\Delta_{m'}(x, y) \right)$$

$$-g_{\ell m}g_{\ell' m'} u_\ell^*(p', s') u_{m'}(p, s) \int \frac{d^4x d^4y}{(2\pi)^{12}k^{\alpha_0}2k'^{\alpha_0}} e^{i(k'_{-y} - i(k'-p)_{-x})} \left( -i\Delta_{m'}(y, x) \right). \quad (2.81)$$

Interchanging $x$ and $y$, $m$ and $m'$, and $\ell$ and $\ell'$ in $A_1$, we get

$$A = -g_{\ell m}g_{\ell' m'} u_\ell^*(p', s') u_{m'}(p, s) \int \frac{d^4x d^4y}{(2\pi)^{12}k^{\alpha_0}2k'^{\alpha_0}} e^{i(k_+p_{-y} - i(k'+p')_{-x})} \left( -i\Delta_{m'}(y, x) \right)$$

$$-g_{\ell m}g_{\ell' m'} u_\ell^*(p', s') u_{m'}(p, s) \int \frac{d^4x d^4y}{(2\pi)^{12}k^{\alpha_0}2k'^{\alpha_0}} e^{i(k'_{-y} - i(k'-p)_{-x})} \left( -i\Delta_{m'}(y, x) \right). \quad (2.82)$$

Combining terms, we get

$$A = -g_{\ell m}g_{\ell' m'} u_\ell^*(p', s') u_{m'}(p, s) \int \frac{d^4x d^4y}{(2\pi)^{12}k^{\alpha_0}2k'^{\alpha_0}} e^{ip_{-x} - ip'_{-y}} \left( -i\Delta_{m'}(y, x) \right)$$

$$\times \left( e^{ik'_{-y} + i(k'-p')_{-x}} \right) \quad (2.83)$$

which agrees with SW’s (6.1.27) when his boson is restricted to a single scalar field.
We now replace mean value in the vacuum of the time-ordered product by its value \[ (2.70) \]

\[
\langle 0 | T \{ \psi(x) \psi^\dagger_m(y) \} | 0 \rangle \equiv -i \Delta_{\ell m}(x-y) = -i \int \frac{d^4q}{(2\pi)^4} \frac{\left[ \left( -i q_\alpha \gamma^\alpha + m \beta \right) \delta_{\ell m} \right]}{q^2 + m^2 - i\epsilon} e^{iq.x}. \tag{2.84}
\]

Replacing \( \ell, m, x, y \) by \( m', \ell, y, x \), we get

\[
\langle 0 | T \{ \psi_{m'}(y) \psi^\dagger_{m}(x) \} | 0 \rangle \equiv -i \Delta_{m'\ell}(y-x) = -i \int \frac{d^4q}{(2\pi)^4} \frac{\left[ \left( -i q_\alpha \gamma^\alpha + m \beta \right) \right]}{m' \ell} e^{iq.(y-x)}. \tag{2.85}
\]

We then find

\[
A = -g_{m'm} g_{\ell\ell} u^p_{p'} (p', s') u_m (p, s) \int \frac{d^4x d^4y}{(2\pi)^6 \sqrt{2\hbar^0 2\hbar^0}} e^{ipx-ip'y} \times \left( e^{ikx-ik'y} + e^{ik'y-ik'x} \right) (-i) \int \frac{d^4q}{(2\pi)^4} \frac{\left[ \left( -i q + m \beta \right) \right]}{q^2 + m^2 - i\epsilon} e^{iq(y-x)}. \tag{2.86}
\]

In matrix notation, this is

\[
A = -\int \frac{d^4x d^4y}{(2\pi)^6 \sqrt{2\hbar^0 2\hbar^0}} \left( e^{i(p+k)x-i(p'+k')y} + e^{i(k-p')y-i(p-k')x} \right) \times (-i) \int \frac{d^4q}{(2\pi)^4} \frac{u^p(p', s') g \left[ \left( -i q + m \beta \right) \right] u(p, s)}{q^2 + m^2 - i\epsilon} e^{iq(y-x)}. \tag{2.87}
\]

The \( d^4x \) and \( d^4y \) integrations give

\[
A = i \int \frac{d^4q}{(2\pi)^2 \sqrt{2\hbar^0 2\hbar^0}} \left( \delta(q-k'-p')\delta(p+k-q) + \delta(q+k-p')\delta(p-k'-q) \right) \times \frac{u^p(p', s') g \left[ \left( -i q + m \beta \right) \right] u(p, s)}{q^2 + m^2 - i\epsilon} \tag{2.88}
\]

which is

\[
A = \frac{i \delta(p'+ k' - p - k)}{8\pi^2 \sqrt{2\hbar^0 2\hbar^0}} \frac{u^p(p', s') g}{\left( p+k \right)^2 + m^2} \left[ \frac{-i(p + k') + m}{\left( p+k' \right)^2 + m^2} + \frac{-i(p + k') + m}{\left( p-k' \right)^2 + m^2} \right] \beta g u(p, s). \tag{2.89}
\]
2.8 Feynman propagator for spin-one fields

The general form of the propagator is

$$\langle 0 | T [\psi_a(x)\psi_b^\dagger(y)] | 0 \rangle = \theta(x^0 - y^0) \sum_s \int \frac{d^3 p}{(2\pi)^3} u_a(\vec{p}, s) u_b^\dagger(\vec{p}, s) e^{ip \cdot (x-y)}$$

$$\pm \theta(y^0 - x^0) \sum_s \int \frac{d^3 p}{(2\pi)^3} v_a(\vec{p}, s) v_b^\dagger(\vec{p}, s) e^{ip \cdot (y-x)}.$$  \hspace{1cm} (2.90)

We use the upper (+) sign for spin-one fields because they are bosons. For single real massive vector field

$$\psi^a(x) = \sum_{s=-1}^1 \int \frac{d^3 p}{\sqrt{(2\pi)^3 2p^0}} \left[ c^a(\vec{p}, s) a(\vec{p}, s) e^{ip \cdot x} + e^{*a}(\vec{p}, s) a^\dagger(\vec{p}, s) e^{-ip \cdot x} \right],$$  \hspace{1cm} (2.91)

the mean value in the vacuum of its time-ordered product is

$$\langle 0 | T [\psi_a(x)\psi_b^\dagger(y)] | 0 \rangle = \theta(x^0 - y^0) \sum_s \int \frac{d^3 p}{(2\pi)^3 2p^0} c_a(\vec{p}, s) c_b^\dagger(\vec{p}, s) e^{ip \cdot (x-y)}$$

$$+ \theta(y^0 - x^0) \sum_s \int \frac{d^3 p}{(2\pi)^3 2p^0} c_a(\vec{p}, s) c_b^\dagger(\vec{p}, s) e^{ip \cdot (y-x)}.$$  \hspace{1cm} (2.92)

The spin-one spin sum is

$$\sum_s c_a(\vec{p}, s) c_b^\dagger(\vec{p}, s) = \eta_{ab} + p_a p_b / m^2.$$  \hspace{1cm} (2.93)

So the mean value \(2.58\) is

$$\langle 0 | T [\psi_a(x)\psi_b^\dagger(y)] | 0 \rangle = \theta(x^0 - y^0) \int \frac{d^3 p}{(2\pi)^3 2p^0} \left( \eta_{ab} + \frac{p_a p_b}{m^2} \right) e^{ip \cdot (x-y)}$$

$$+ \theta(y^0 - x^0) \int \frac{d^3 p}{(2\pi)^3 2p^0} \left( \eta_{ab} + \frac{p_a p_b}{m^2} \right) e^{ip \cdot (y-x)}$$  \hspace{1cm} (2.94)
in which the momentum \( p_a \) is physical (aka, on the mass shell) in that \( p^0 = \sqrt{p^2 + m^2} \) and \( p_0^2 = p^2 + m^2 \). In terms of derivatives, we have

\[
\langle 0 | T [\psi_a(x)\psi_b^\dagger(y)] | 0 \rangle = \theta(x^0 - y^0) \left( \eta_{ab} - \frac{\partial_a \partial_b}{m^2} \right) \int \frac{d^3 p}{(2\pi)^3 2p^0} e^{ip(x-y)}
\]

\[
+ \theta(y^0 - x^0) \left( \eta_{ab} - \frac{\partial_a \partial_b}{m^2} \right) \int \frac{d^3 p}{(2\pi)^3 2p^0} e^{ip(y-x)}.
\]

\[\text{(2.95)}\]

In this formula, we may interpret the double time derivative as \( \partial^2_0 = \nabla^2 - m^2 \).

For spatial values of \( a \) and \( b \), we can move the derivatives to the left of the step functions. And we can move the product \( \partial_0 \partial_1 \) of one spatial and one time derivative by the argument we used for spin one-half. We find

\[
\langle 0 | T [\psi_a(x)\psi_b^\dagger(y)] | 0 \rangle = \left( \eta_{ab} - \frac{\partial_a \partial_b}{m^2} \right) \theta(x^0 - y^0) \int \frac{d^3 p}{(2\pi)^3 2p^0} e^{ip(x-y)}
\]

\[
+ \theta(y^0 - x^0) \int \frac{d^3 p}{(2\pi)^3 2p^0} e^{ip(y-x)}
\]

\[= \left( \eta_{ab} - \frac{\partial_a \partial_b}{m^2} \right) \left[ \theta(x^0 - y^0) \Delta_+(x - y) + \theta(y^0 - x^0) \Delta_+(y - x) \right]
\]

\[= \left( \eta_{ab} - \frac{\partial_a \partial_b}{m^2} \right) \left[ -i \Delta_F(x - y) \right]
\]

\[= -i \left( \eta_{ab} - \frac{\partial_a \partial_b}{m^2} \right) \int \frac{d^4 q}{(2\pi)^4} \frac{e^{iq(x-y)}}{q^2 + m^2 - i\epsilon} \]

\[\text{(2.96)}\]

in which \( \partial^2_0 = \nabla^2 - m^2 \).

We can relax the rule \( \partial^2_0 = \nabla^2 - m^2 \) if we add an extra term to the propagator. The extra term is

\[
\frac{i}{m^2} (\nabla^2 - m^2 - \partial^2_0) \int \frac{d^4 q}{(2\pi)^4} \frac{e^{iq(x-y)}}{q^2 + m^2 - i\epsilon} = \frac{i}{m^2} \int \frac{d^4 q}{(2\pi)^4} \frac{-q^2 - m^2 + p_0^2}{q^2 + m^2 - i\epsilon} e^{iq(x-y)}
\]

\[= \frac{i}{m^2} \delta^4(x - y). \]

\[\text{(2.97)}\]

So the Feynman propagator for spin-one fields is

\[
\langle 0 | T [\psi_a(x)\psi_b^\dagger(y)] | 0 \rangle = -i \int \frac{d^4 q}{(2\pi)^4} \frac{(\eta_{ab} + p_a p_b/m^2)}{q^2 + m^2 - i\epsilon} e^{iq(x-y)} - \frac{i}{m^2} \delta^4_0 \delta^4_0 \delta^4(x - y).
\]

\[\text{(2.98)}\]

Suppose the vector field \( \psi_a(x) \) interacts with a current \( j^a(x) \) thru a term \( \psi_a(x)j^a(x) \) in \( \mathcal{H}(x) \). Then in Dyson’s expansion, the awkward second term
in the Feynman propagator (2.98) contributes the term
\[
- i \mathcal{H}_2(x) = \frac{1}{2} \int \left[ -i j^a(x) \right] \left[ -i j^b(y) \right] \left[ - \frac{i}{m^2} \delta^0_\alpha \delta^0_\beta \delta^4(x - y) \right] d^4y = \frac{i [j^0(x)]^2}{2m^2}.
\]

So if \( \mathcal{H}(x) \) also contains the term
\[
\mathcal{H}_C(x) = \frac{[j^0(x)]^2}{2m^2},
\]
then the effect of the awkward second term in the Feynman propagator is cancelled. This actually happens in a natural way as SW explains in chapter 7.

### 2.9 The Feynman rules

1) Draw all diagrams of the order you are working at. Label each internal line with its (unphysical) 4-momentum considered to flow in the direction of the arrow or in either direction if the particle is uncharged.

2) For each vertex of type \( i \), include the factor
\[
- i (2\pi)^4 g_i \delta^4 \left( \sum p + q - p' - q' \right)
\]
which makes the sum of the 4-momenta \( p + q \) entering each vertex add up to sum of the 4-momenta \( p' + q' \) leaving each vertex.

For each outgoing line include the factor
\[
\frac{u^*_\ell(\vec{p}', s', n')}{(2\pi)^{3/2}} \quad \text{or} \quad \frac{v_\ell(\vec{p}', s', n')}{(2\pi)^{3/2}}
\]
for arrows pointing out \( (u^*_\ell(\vec{p}', s', n')) \) or pointing in \( (v_\ell(\vec{p}', s', n')) \).

For each incoming line include the factor
\[
\frac{u_\ell(\vec{p}, s, n)}{(2\pi)^{3/2}} \quad \text{or} \quad \frac{v^*_\ell(\vec{p}, s, n)}{(2\pi)^{3/2}}
\]
for arrows pointing in \( (u_\ell(\vec{p}, s, n)) \) or pointing out \( (v^*_\ell(\vec{p}, s, n)) \).

For each internal line of a spin-zero particle carrying momentum \( q^a \) include the factor
\[
- \frac{i}{(2\pi)^4} \left( \frac{1}{q^2 + m^2 - i\epsilon} \right).
\]

For each internal line of a spin-one-half particle with ends labelled by \( \ell \) and \( m \) and carrying momentum \( q^a \) include the factor
\[
- \frac{i}{(2\pi)^4} \left( \frac{[(-ig + m)\beta]}{q^2 + m^2 - i\epsilon} \right)_{\ell m}.
\]
For each internal line of a spin-one particle with ends labelled by $\ell$ and $m$ and carrying momentum $q^a$ include the factor

$$\frac{-i}{(2\pi)^4} \frac{\eta_{\ell m} + q\epsilon q\epsilon}{q^2 + m^2}$$

and keep in mind the delta-function term in (2.98).

3). Integrate the product of all these factors over all the internal momenta and sum over $\ell$ and $m$, etc.

4) Add the results of all the Feynman diagrams.

### 2.10 Fermion-antifermion scattering

We can watch Feynman’s rules emerge in fermion-antifermion scattering. We consider a fermion interacting with a neutral scalar boson (2.72)

$$H(x) = g_{\ell m} \psi^\dagger(x) \psi(x) \phi(x).$$

The initial state is $|p, s; q, t\rangle = a^\dagger(p, s) b^\dagger(q, t)|0\rangle$ and the final state is $|p', s'; q', t'\rangle = a^\dagger(p', s') b^\dagger(q', t')|0\rangle$. The lowest-order term is

$$A = -\frac{1}{2!} \int d^4x d^4y \langle 0| b(q', t') a(p', s') T[H(x) H(y)] a^\dagger(p, s) b^\dagger(q, t)|0\rangle$$

$$= -\frac{g_{\ell m} \epsilon_{\ell' m'}}{2!} \int d^4x d^4y \langle 0| b(q', t') a(p', s') T[\psi^\dagger_\ell(x) \psi_m(x) \phi(x) \psi^\dagger_{\ell'}(y) \psi_{m'}(y) \phi(y)]$$

in which the operators $a$ and $b$ delete fermions and antifermions. There is only one Feynman diagram

![Feynman diagram](image)

which appears in the conventional way with incoming particles on the left and outgoing particles on the right. TikZ-Feynman does not easily use SW’s vertical flow.

We cancel the $2!$ by choosing to absorb the incoming fermion-antifermion
pair at vertex $y$ and to add the outgoing fermion-antifermion pair at vertex $x$. We are left with

$$A = -\frac{g_{\ell m} g_{\ell' m'}}{(2\pi)^6} \int d^4x d^4y \langle 0 | u_\ell^*(p', s') v_m(q', t') e^{-ix(p'+q')} T[\phi(x) \phi(y)]$$

$$\times u_{m'}(p, s) v_{\ell'}^*(q, t) e^{iy(p+q)} |0 \rangle.$$  \hfill (2.109)

Adding in the scalar propagator (2.50), we get

$$A = -\frac{g_{\ell m} g_{\ell' m'}}{(2\pi)^6} \int d^4x d^4y u_\ell^*(p', s') v_{m'}^*(q', t') e^{-ix(p'+q')}
\times u_{m'}(p, s) v_{\ell'}^*(q, t) e^{iy(p+q)} \int \frac{d^4k}{(2\pi)^4} \frac{-ie^{ik(x-y)}}{k^2 + m^2 - i\epsilon}. \hfill (2.110)$$

The integration over $y$ conserves 4-momentum at vertex $y$, and the integration over $x$ conserves 4-momentum at vertex $x$. We then have

$$A = i \frac{g_{\ell m} g_{\ell' m'}}{(2\pi)^2} \int d^4k u_\ell^*(p', s') v_{m}(q', t') \delta(k - p' - q') \delta(p + q - k)$$
$$\times u_{m'}(p, s) v_{\ell'}^*(q, t) \frac{1}{k^2 + m^2 - i\epsilon} \hfill (2.111)$$

or

$$A = i \delta(p + q - k' - q') \frac{g_{\ell m} g_{\ell' m'}}{(2\pi)^2} \frac{u_\ell^*(p', s') v_{m}(q', t') u_{m'}(p, s) v_{\ell'}^*(q, t)}{(p + k)^2 + m^2}. \hfill (2.112)$$
3

Action

3.1 Lagrangians and hamiltonians

A transformation is a symmetry of a theory if the action is invariant or changes by a surface term. So we choose to work with actions that are symmetrical. The action is normally an integral over spacetime of an action density often called a lagrangian. Often the action density itself is invariant under the transformation of the symmetry.

There are procedures, sometimes clumsy procedures, for computing the hamiltonian from the lagrangian. The hamiltonian often is not invariant under the transformation of the symmetry. So it’s very hard to find a suitably symmetrical theory by starting with a hamiltonian. But once one has a hamiltonian, one can compute scattering amplitudes energies, and states with these energies.

3.2 Canonical variables

In quantum mechanics, we use the equal-time commutation relations

\[[q_i, p_k] = i\delta_{ik}, \quad [q_i, q_k] = 0, \quad \text{and} \quad [p_i, p_k] = 0. (3.1)\]

In general, the operators \(q_i\) and \(q_i(t) = e^{iHt}q_i e^{-iHt}\) do not commute. Quantum field theory promotes these equal-time commutation relations to ones in which the indexes \(i\) and \(k\) denote different points of space

\[[q^n(x, t), p_m(y, t)]_\mp = i\delta(x - y)\delta^n_m, \quad [q^n(x, t), q^m(y, t)]_\mp = 0, \quad \text{and} \quad [p_n(x, t), p_m(y, t)]_\mp = 0. (3.2)\]

The commutator of a real scalar field

\[\phi(x) = \int \frac{d^3p}{\sqrt{(2\pi)^32p^0}} \left[ a(p) e^{ip\cdot x} + a^\dagger(\bar{p}) e^{-ip\cdot x} \right] (3.3)\]
3.2 Canonical variables

is

\[ [\phi(x), \phi(y)] = \Delta(x - y) = \int \frac{d^3p}{(2\pi)^3 2p^0} \left( e^{ip \cdot (x-y)} - e^{-ip \cdot (x-y)} \right). \]  

(3.4)

At equal times, one has

\[ \Delta(\vec{x} - \vec{y}, 0) = 0, \quad \frac{\partial}{\partial x^0} \Delta(x - y)|_{x^0=y^0} = -i \delta^3(\vec{x} - \vec{y}), \quad \text{and} \quad \frac{\partial^2}{\partial x^0 \partial y^0} \Delta(x - y)|_{x^0=y^0} = 0. \]  

(3.5)

So a real field \( \phi \) and its time derivative \( \dot{\phi} \) satisfy the equal-time commutation relations (3.2)

\[ [\phi(\vec{x}, t), \dot{\phi}(\vec{y}, t)]_- = i \delta^3(\vec{x} - \vec{y}), \quad [\phi(\vec{x}, t), \phi(\vec{y}, t)]_- = 0 \quad \text{and} \quad [\dot{\phi}(\vec{x}, t), \dot{\phi}(\vec{y}, t)]_- = 0. \]  

(3.6)

A complex scalar field

\[ \phi(x) = \frac{1}{\sqrt{2}} [\phi_1(x) + i\phi_2(x)] \]  

(3.7)

obeys the commutation relations

\[ [\phi(x), \phi(y)]_- = \frac{1}{2} [\phi_1(x) + i\phi_2(x), \phi_1(x) - i\phi_2(x)] \]

\[ = \frac{1}{2} ([\phi_1(x), \phi_1(x)] + [\phi_2(x), \phi_2(x)]) = \Delta(x - y) \]  

and

\[ [\phi(x), \phi(y)]_- = \frac{1}{2} [\phi_1(x) + i\phi_2(x), \phi_1(x) + i\phi_2(x)] \]

\[ = \frac{1}{2} ([\phi_1(x), \phi_1(x)] - [\phi_2(x), \phi_2(x)]) = 0. \]  

(3.8)

(3.9)

So the complex scalar fields \( \phi(x) \) and \( \dot{\phi}(x) \) obey the equal-time commutation relations (3.6)

\[ [\phi(\vec{x}, t), \dot{\phi}(\vec{y}, t)]_- = i \delta^3(\vec{x} - \vec{y}) \quad \text{and} \quad [\phi(\vec{x}, t), \phi(\vec{y}, t)]_- = 0 \]  

(3.10)

\[ [\dot{\phi}(\vec{x}, t), \dot{\phi}(\vec{y}, t)] = 0 \quad \text{and} \quad [\dot{\phi}(\vec{x}, t), \phi(\vec{y}, t)] = 0 \]  

(3.11)

\[ [\phi(\vec{x}, t), \phi(\vec{y}, t)] = 0 \quad \text{and} \quad [\phi(\vec{x}, t), \phi(\vec{y}, t)] = 0. \]  

(3.12)
3.3 Principle of stationary action in field theory

If $\phi(x)$ is a scalar field, and $L(\phi)$ is its action density, then its action $S[\phi]$ is the integral over all of spacetime

$$S[\phi] = \int L(\phi(x)) \, d^4x. \quad (3.13)$$

The principle of least (or stationary) action says that the field $\phi(x)$ that satisfies the classical equation of motion is the one for which the first-order change in the action due to any tiny variation $\delta \phi(x)$ in the field vanishes, $\delta S[\phi] = 0$. And to keep things simple, we’ll assume that the action (or Lagrange) density $L(\phi)$ is a function only of the field $\phi$ and its first derivatives $\partial_a \phi = \partial \phi / \partial x^a$. The first-order change in the action then is

$$\delta S[\phi] = \int \left[ \frac{\partial L}{\partial \phi} \delta \phi + \frac{\partial L}{\partial (\partial_a \phi)} \delta (\partial_a \phi) \right] \, d^4x \quad (3.14)$$

in which we sum over the repeated index $a$ from 0 to 3. Now $\delta (\partial_a \phi) = \partial_a (\phi + \delta \phi) - \partial_a \phi = \partial_a \delta \phi$. So we may integrate by parts and drop the surface terms because we set $\delta \phi = 0$ on the surface at infinity

$$\delta S[\phi] = \int \left[ \frac{\partial L}{\partial \phi} \delta \phi + \frac{\partial L}{\partial (\partial_a \phi)} \partial_a (\delta \phi) \right] \, d^4x \quad (3.15)$$

This first-order variation is zero, for arbitrary $\delta \phi$ only if the field $\phi(x)$ satisfies Lagrange’s equation

$$\partial_a \left( \frac{\partial L}{\partial (\partial_a \phi)} \right) = \frac{\partial}{\partial x^a} \left[ \frac{\partial L}{\partial (\partial \phi / \partial x^a)} \right] = \frac{\partial L}{\partial \phi} \quad (3.16)$$

which is the classical equation of motion.

**Example 3.1** (Theory of a scalar field) The action density of a scalar field $\phi$ of mass $m$ is

$$L = \frac{1}{2} (\dot{\phi})^2 - \frac{1}{2} (\nabla \phi)^2 - \frac{1}{2} m^2 \phi^2 = -\frac{1}{2} \partial_a \phi \partial^a \phi - \frac{1}{2} m^2 \phi^2. \quad (3.17)$$

Lagrange’s equation (3.16) for this action density is

$$\nabla^2 \phi - \ddot{\phi} = \partial_a \partial^a \phi = m^2 \phi \quad (3.18)$$

which is the Klein-Gordon equation (1.56). \qed
In a theory of several fields \( \phi_1, \ldots, \phi_n \) with action density \( L(\phi_k, \partial_a \phi_k) \), the fields obey \( n \) copies of Lagrange’s equation

\[
\frac{\partial}{\partial x^a} \left( \frac{\partial L}{\partial (\partial_a \phi_k)} \right) = \frac{\partial L}{\partial \phi_k}
\]

one for each \( k \).

### 3.4 Symmetries and conserved quantities in field theory

An action density \( L(\phi_i, \partial_a \phi_i) \) that is invariant under a transformation of the coordinates \( x^a \) or of the fields \( \phi_i \) and their derivatives \( \partial_a \phi_i \) is a symmetry of the action density. Such a symmetry implies that something is conserved or time independent.

Suppose that an action density \( L(\phi_i, \partial_a \phi_i) \) is unchanged when the fields \( \phi_i \) and their derivatives \( \partial_a \phi_i \) change by \( \delta \phi_i \) and by \( \delta (\partial_a \phi_i) \)

\[
0 = \delta L = \sum_i \frac{\partial L}{\partial \phi_i} \delta \phi_i + \frac{\partial L}{\partial \partial_a \phi_i} \delta (\partial_a \phi_i).
\]

For many transformations (but not for Lorentz transformations), we can set \( \delta (\partial_a \phi_i) = \partial_a (\delta \phi_i) \). In such cases by using Lagrange’s equations (3.19) to rewrite \( \partial L/\partial \phi_i \), we find

\[
0 = \sum_i \left( \partial_a \frac{\partial L}{\partial \partial_a \phi_i} \right) \delta \phi_i + \frac{\partial L}{\partial \partial_a \phi_i} \partial_a \delta \phi_i = \partial_a \sum_i \frac{\partial L}{\partial \partial_a \phi_i} \delta \phi_i
\]

which says that the current

\[
J^a = \sum_i \frac{\partial L}{\partial \partial_a \phi_i} \delta \phi_i
\]

has zero divergence

\[
\partial_a J^a = 0.
\]

Thus the time derivative of the volume integral of the charge density \( J^0 \)

\[
Q_V = \int_V J^0 \, d^3x
\]

is the flux of current \( \vec{J} \) entering through the boundary \( S \) of the volume \( V \)

\[
\dot{Q}_V = \int_V (\partial_0 J^0) \, d^3x = - \int_V (\partial_0 J^k) \, d^3x = - \int_S J^k \, d^2S_k.
\]
If no current enters \( V \), then the charge \( Q \) inside \( V \) is conserved. When the volume \( V \) is the whole universe, the charge is the integral over all of space

\[
Q = \int J^0 \, d^3 x = \int \sum_i \frac{\partial L}{\partial \dot{\phi}_i} \, \delta \phi_i \, d^3 x = \int \sum_i \pi_i \, \delta \phi_i \, d^3 x \tag{3.26}
\]

in which \( \pi_i \) is the momentum conjugate to the field \( \phi_i \)

\[
\pi_i = \frac{\partial L}{\partial \dot{\phi}_i} \tag{3.27}
\]

**Example 3.2** \((O(n) \text{ symmetry and its charge})\) Suppose the action density \( L \) is the sum of \( n \) copies of the quadratic action density (3.17)

\[
L = \sum_{i=1}^{n} \left( \frac{1}{2} \phi_i^2 - \frac{1}{2} (\nabla \phi_i)^2 - \frac{1}{2} m^2 \phi_i^2 \right) = -\frac{1}{2} \partial_a \phi \partial^a \phi - \frac{1}{2} m^2 \phi^2 \tag{3.28}
\]

and \( A_{ij} \) is any constant antisymmetric matrix, \( A_{ij} = -A_{ji} \). Then if the fields change by \( \delta \phi_i = \epsilon \sum_j A_{ij} \phi_j \), the change (3.20) in the action density

\[
\delta L = -\epsilon \sum_{i,j=1}^{n} \left[ m^2 \phi_i A_{ij} \phi_j + \partial^a \phi_i A_{ij} \partial_a \phi_j \right] = 0 \tag{3.29}
\]

vanishes. Thus the charge (3.26) associated with the matrix \( A \)

\[
Q_A = \int \sum_i \pi_i \delta \phi_i \, d^3 x = \epsilon \int \sum_i \pi_i A_{ij} \phi_j \, d^3 x \tag{3.30}
\]

is conserved. There are \( n(n-1)/2 \) antisymmetric \( n \times n \) imaginary matrices; they generate the group \( O(n) \) of \( n \times n \) orthogonal matrices.

An action density \( L(\phi_i, \partial_a \phi_i) \) that is invariant under a spacetime translation, \( x'^a = x^a + \delta x^a \), depends upon \( x^a \) only through the fields \( \phi_i \) and their derivatives \( \partial_a \phi_i \)

\[
\frac{\partial L}{\partial x^a} = \sum_i \left( \frac{\partial L}{\partial \phi_i} \frac{\partial \phi_i}{\partial x^a} + \frac{\partial L}{\partial \partial_b \phi_i} \frac{\partial^2 \phi_i}{\partial x^b \partial x^a} \right) \tag{3.31}
\]

Using Lagrange’s equations (3.19) to rewrite \( \partial L/\partial \phi_i \), we find

\[
0 = \left( \sum_i \partial_b \left( \frac{\partial L}{\partial \partial_b \phi_i} \right) \partial_a \phi_i + \frac{\partial L}{\partial \partial_b \phi_i} \frac{\partial^2 \phi_i}{\partial x^b \partial x^a} \right) - \frac{\partial L}{\partial x^a} \tag{3.32}
\]

\[
0 = \partial_b \left[ \sum_i \left( \frac{\partial L}{\partial \partial_b \phi_i} \frac{\partial \phi_i}{\partial x^a} - \delta_b^a L \right) \right]
\]
that the **energy-momentum tensor**

\[ T^b_a = \sum_i \frac{\partial L}{\partial \dot{\phi}_i} \frac{\partial \phi_i}{\partial x^a} - \delta^b_a L \]  

(3.33)

has zero divergence, \( \partial_b T^b_a = 0 \).

Thus the time derivative of the 4-momentum \( P_aV \) inside a volume \( V \)

\[ P_aV = \int_V \left( \sum_i \frac{\partial L}{\partial \dot{\phi}_i} \frac{\partial \phi_i}{\partial x^a} - \delta^0_a L \right) d^3x = \int_V T^0_a d^3x \]  

(3.34)

is equal to the flux entering through \( V \)’s boundary \( S \)

\[ \partial_b P_aV = \int_V \partial_b T^0_a d^3x = - \int_V \partial_k T^k_a d^3x = - \int_S T^k_a d^2S_k. \]  

(3.35)

The invariance of the action density \( L \) under spacetime translations implies the **conservation of energy** \( P_0 \) and **momentum** \( \vec{P} \).

The momentum \( \pi_i(x) \) that is canonically conjugate to the field \( \phi_i(x) \) is the derivative of the action density \( L \) with respect to the time derivative of the field

\[ \pi_i = \frac{\partial L}{\partial \dot{\phi}_i}. \]  

(3.36)

If one can express the time derivatives \( \dot{\phi}_i \) of the fields in terms of the fields \( \phi_i \) and their momenta \( \pi_i \), then hamiltonian of the theory is the spatial integral of

\[ H = P_0 = T^0_0 = \left( \sum_{i=1}^n \pi_i \dot{\phi}_i \right) - L \]  

(3.37)

in which \( \dot{\phi}_i = \dot{\phi}_i(\phi, \pi) \).

**Example 3.3** (Hamiltonian of a scalar field) The hamiltonian density \( H \) of the theory \( L \) is

\[ H = \frac{1}{2} \pi^2 + \frac{1}{2} (\nabla \phi)^2 + \frac{1}{2} m^2 \phi^2. \]  

(3.38)

A Lorentz transformation \( U(\Lambda, a) \psi_{\ell}(x)U^{-1}(\Lambda, a) = \sum_{\ell} D_{\ell\ell}(\Lambda^{-1}) \psi_{\ell}(\Lambda x + a) \).  

(3.39)

The action is invariant under Lorentz transformations. The action density
is constructed so as to be invariant when the fields transform this way:

$$\psi^\ell(x) = \sum_\ell D^\ell_\ell(A^{-1})\psi^\ell(x)$$  \hfill (3.40)

The action density is constructed so as to be invariant when the fields and their derivatives of the fields transform under infinitesimal Lorentz transformations as

$$\delta\psi^\ell(x) = \frac{i}{2} \omega^{ab}(J_{ab})^\ell_m \psi^m(x)$$  \hfill (3.41)

$$\delta(\partial_k \psi^\ell)(x) = \frac{i}{2} \omega^{ab}(J_{ab})^\ell_m \partial_k \psi^m + \omega^j_k \partial_j \psi^\ell(x).$$

The invariance of the action density says that

$$0 = \partial L/\partial \psi^\ell \delta\psi^\ell + \partial L/\partial(\partial_k \psi^\ell) \delta(\partial_k \psi^\ell)$$  \hfill (3.42)

$$= \partial L/\partial \psi^\ell \frac{i}{2} \omega^{ab}(J_{ab})^\ell_m \psi^m + \partial L/\partial(\partial_k \psi^\ell) \left[ \frac{i}{2} \omega^{ab}(J_{ab})^\ell_m \partial_k \psi^m + \omega^j_k \partial_j \psi^\ell \right].$$

Remembering that $\omega_{ab} = -\omega_{ba}$ and $\omega^{ab} = -\omega^{ba}$, we rewrite the last bit as

$$\omega^b_k \partial_b \psi^\ell = \eta_{ka} \omega^{ab} \partial_b = \frac{1}{2} (\eta_{ka} \omega^{ab} \partial_b - \eta_{kb} \omega^{ab} \partial_a) \psi^\ell.$$  \hfill (3.43)

We then have

$$0 = \partial L/\partial \psi^\ell \frac{i}{2} \omega^{ab}(J_{ab})^\ell_m \psi^m$$

$$+ \partial L/\partial(\partial_k \psi^\ell) \left[ \frac{i}{2} \omega^{ab}(J_{ab})^\ell_m \partial_k \psi^m + \frac{1}{2} (\eta_{ka} \omega^{ab} \partial_b - \eta_{kb} \omega^{ab} \partial_a) \psi^\ell \right]$$  \hfill (3.44)

or

$$0 = \partial L/\partial \psi^\ell \frac{i}{2} (J_{ab})^\ell_m \psi^m$$

$$+ \partial L/\partial(\partial_k \psi^\ell) \left[ \frac{i}{2} (J_{ab})^\ell_m \partial_k \psi^m + \frac{1}{2} (\eta_{ka} \partial_b - \eta_{kb} \partial_a) \psi^\ell \right].$$  \hfill (3.45)

The equations of motion now give

$$0 = \left( \partial_k \partial L/\partial(\partial_k \psi^\ell) \right) \frac{i}{2} (J_{ab})^\ell_m \psi^m$$

$$+ \partial L/\partial(\partial_k \psi^\ell) \left[ \frac{i}{2} (J_{ab})^\ell_m \partial_k \psi^m + \frac{1}{2} (\eta_{ka} \partial_b - \eta_{kb} \partial_a) \psi^\ell \right]$$  \hfill (3.46)

or

$$0 = \partial_k \left( \frac{i}{2} (J_{ab})^\ell_m \psi^m \right) + \partial L/\partial(\partial_k \psi^\ell) \frac{1}{2} (\eta_{ka} \partial_b - \eta_{kb} \partial_a) \psi^\ell$$
or
\[ 0 = \partial_k \left( \frac{\partial L}{\partial k \psi^\ell} \frac{i}{2} (J_{ab})^\ell_m \psi^m \right) + \frac{1}{2} \left( \frac{\partial L}{\partial \psi^\ell} \partial_b - \frac{\partial L}{\partial \psi^b} \partial_\ell \psi^\ell \right). \]

We recall (3.33) the energy-momentum tensor
\[ T^b_a = \sum_i \frac{\partial L}{\partial \partial_b \phi_i} \partial \phi_i \partial_a x^a - \delta^b_a L \tag{3.47} \]
or
\[ T_{ba} = \frac{\partial L}{\partial \partial^b \psi^\ell} \partial \psi^\ell \partial_a x^a - \eta_{ba} L \tag{3.48} \]
which has zero divergence, \( \partial_b T^b_a = 0 \). In terms of it, we have
\[ 0 = \partial_k \left( \frac{\partial L}{\partial k \psi^\ell} \frac{i}{2} (J_{ab})^\ell_m \psi^m \right) - \frac{1}{2} (T_{ab} - T_{ba}). \]

So Belinfante defined the symmetric energy-momentum tensor as
\[ \Theta^{ab} = T^{ab} - \frac{i}{2} \partial_k \left[ \frac{\partial L}{\partial \partial_k \psi^\ell} (J^{ab})^\ell_m \psi^m - \frac{\partial L}{\partial \partial_a \psi^\ell} (J^{ka})^\ell_m \psi^m \right]. \tag{3.49} \]
The quantity in the square brackets is antisymmetric in \( a, k \). So
\[ \partial_a \Theta^{ab} = \partial_a T^{ab} - \frac{i}{2} \partial_a \partial_k \left[ \frac{\partial L}{\partial \partial_k \psi^\ell} (J^{ab})^\ell_m \psi^m - \frac{\partial L}{\partial \partial_a \psi^\ell} (J^{ka})^\ell_m \psi^m \right] \]
\[ = \partial_a T^{ab} = 0. \tag{3.50} \]
The Belinfante energy-momentum tensor is symmetric
\[ \Theta^{ab} = \Theta^{ba} \tag{3.51} \]
and so it is the one to use when gravity is involved.

The quantity
\[ M^{abc} = x^b \Theta^{ac} - x^c \Theta^{ab} \tag{3.52} \]
is a conserved current
\[ \partial_a M^{abc} = \partial_a \left( x^b \Theta^{ac} - x^c \Theta^{ab} \right) = \Theta^{bc} - \Theta^{cb} = 0. \tag{3.53} \]
So the angular-momentum operators
\[ J^{bc} = \int M^{0bc} d^3 x = \int \left( x^b \Theta^{0c} - x^c \Theta^{0b} \right) d^3 x \tag{3.54} \]
are conserved.
4
Quantum electrodynamics

4.1 Global $U(1)$ symmetry
In most theories of charged fields, the action density is invariant when the charged fields change by a phase transformation

$$\psi'_\ell(x) = e^{i\theta_\ell} \psi_\ell(x).$$ (4.1)

If the phase $\theta$ is independent of $x$, then the symmetry is called a **global $U(1)$ symmetry**.

Section 3.4 describes Noether’s theorem according to which a current

$$J^a = \sum_\ell \frac{\partial L}{\partial \partial_a \psi_\ell} \delta \psi_\ell$$ (4.2)

has zero divergence

$$\partial_a J^a = 0.$$ (4.3)

The charge

$$Q = \int J^0 d^3x$$ (4.4)

is conserved due to the global $U(1)$ symmetry. Here for tiny $\theta_\ell$, $\delta \psi_\ell = i\theta_\ell \psi_\ell$, and the charge density $J^0$ is

$$J^0 = \sum_\ell \frac{\partial L}{\partial \partial_0 \psi_\ell} \delta \psi_\ell = \sum_\ell \frac{\partial L}{\partial \partial_0 \psi_\ell} i\theta_\ell \psi_\ell = \sum_\ell \pi_\ell i\theta_\ell \psi_\ell.$$ (4.5)

One imagines that the angle is proportional to the charge of the field $\psi_\ell$, $\theta_\ell = q_\ell \theta$. 
4.2 Abelian gauge invariance

Quantum electrodynamics is a theory in which the action density is invariant when the charged fields change by a phase transformation that varies with the spacetime point $x$

$$\psi'_\ell(x) = e^{i\theta(x)}\psi_\ell(x). \quad (4.6)$$

Such a symmetry is called a local $U(1)$ symmetry. A theory with a local $U(1)$ symmetry also has a global $U(1)$ symmetry and so it conserves charge.

Quantities like $\psi_\ell^\dagger(x)\psi_\ell(x)$ are intrinsically invariant under local $U(1)$ symmetries because

$$\left(\psi_\ell^\dagger(x)\right)'\psi_\ell'(x) = \psi_\ell^\dagger(x)e^{-i\theta(x)}e^{i\theta(x)}\psi_\ell(x) = \psi_\ell^\dagger(x)\psi_\ell(x). \quad (4.7)$$

Derivatives of fields present problems, however, because

$$\partial_a\left(\psi_\ell^\dagger(x)e^{i\theta(x)}\psi_\ell(x)\right) = e^{i\theta(x)}\left[\partial_a\left(\psi_\ell(x)\right) + i(\delta_a\theta(x))\psi_\ell(x)\right] \neq e^{i\theta(x)}\partial_a\left(\psi_\ell(x)\right) \quad (4.8)$$

so things like $\psi_\ell^\dagger(x)\partial_a\psi_\ell(x)$ and like $(\partial^a\psi_\ell^\dagger(x))\partial_a\psi_\ell(x)$ are not invariant under local phase transformations.

The trick is to introduce a field $A_a(x)$ that transforms so as to cancel the awkward term $e^{i\theta(x)}(\partial_a\theta)\psi_\ell$ in the derivative (4.8). We want a new derivative $D_a\psi_\ell$ that transforms like $\psi_\ell$.

$$\left(D_a\psi_\ell\right)' = e^{i\theta}D_a\psi_\ell. \quad (4.9)$$

So we set

$$D_a = \partial_a + iA_a \quad (4.10)$$

and require that

$$\left(D_a\psi_\ell\right)' = (\partial_a + iA'_a)\psi_\ell' = e^{i\theta}D_a\psi_\ell = e^{i\theta}(\partial_a + iA_a)\psi_\ell. \quad (4.11)$$

That is, we insist that

$$\left(\partial_a + iA'_a\right)\psi_\ell' = (\partial_a + iA'_a)e^{i\theta}\psi_\ell = e^{i\theta}(\partial_a + iA_a)\psi_\ell. \quad (4.12)$$

So we need

$$i\partial_a\theta + iA'_a = iA_a \quad (4.13)$$

or

$$A'_a(x) = A_a(x) - i\partial_a\theta. \quad (4.14)$$

This is what we need except that the field $A_a$ does not carry the index $\ell$. 
4.2 Abelian gauge invariance

The solution to this problem is to define the symmetry transformation

\[ \psi'_{\ell}(x) = e^{iq_\ell \theta(x)} \psi_{\ell}(x). \]  

(4.15)

This definition has the advantage that the charge density (4.5) becomes

\[ J^0 = i\theta \sum_{\ell} q_\ell \pi_{\ell} \psi_{\ell} \]  

(4.16)

which makes more sense than the old formula (4.5). More importantly, the definition (4.15) means that equations (4.9–4.22) change to

\[ (D_a \psi_{\ell})' = e^{iq_\ell \theta} D_a \psi_{\ell}. \]  

(4.17)

So we make the covariant derivative

\[ D_{a} \psi_{\ell} = (\partial_{a} + iq_\ell A_{a}) \psi_{\ell} \]  

(4.18)

depend upon the field \( \psi_{\ell} \) and require that

\[ (D_{a} \psi_{\ell})' = (\partial_{a} + iq_\ell A_{a})' \psi_{\ell}' = e^{iq_\ell \theta} D_{a} \psi_{\ell} = e^{iq_\ell \theta} (\partial_{a} + iq_\ell A_{a}) \psi_{\ell}. \]  

(4.19)

That is, we insist that

\[ \psi_{\ell}'(x) = e^{iq_\ell \theta(x)} \psi_{\ell}(x) \]  

(4.23)

couple to the electromagnetic field \( A_{a} \) in a way

\[ q_\ell A_{a} \psi_{\ell} \]  

is proportional to its charge \( q_\ell \).

To insure that the right charge \( q_\ell \) appears in the right place, we can introduce a charge operator \( q \) such that \( q \psi_{\ell} = q_\ell \psi_{\ell} \) and redefine the abelian gauge transformation (4.23) as

\[ q_{\ell} A_{a} \psi_{\ell} = q_{\ell} A_{a} \psi_{\ell} \]  

(4.24)

constitute an abelian gauge transformation.

Note that a field \( \psi_{\ell} \) couples to the electromagnetic field \( A_{a} \) in a way

\[ q_{\ell} A_{a} \psi_{\ell} \]  

that is proportional to its charge \( q_\ell \).
So
\[ \partial_b A'_a = \partial_b A_a - i \partial_b \partial_a \theta \] (4.25)
is not invariant, but the antisymmetric combination
\[ \partial_b A'_a - \partial_a A'_b = \partial_b A_a - i \partial_b \partial_a \theta - \partial_a A_b + i \partial_a \partial_b \theta = \partial_b A_a - \partial_a A_b \] (4.26)
is invariant. Maxwell introduced this combination
\[ F_{ba} = \partial_b A_a - \partial_a A_b. \] (4.27)
Thus
\[ L = -\frac{1}{4} F_{ba} F^{ba} \] (4.28)
is a Lorentz-invariant, gauge-invariant action density for the electromagnetic field.

### 4.3 Coulomb-gauge quantization

The first step in the canonical quantization of a gauge theory is to pick a gauge. The most physical gauge for electrodynamics is the Coulomb gauge defined by the gauge condition
\[ 0 = \nabla \cdot \vec{A}. \] (4.29)

If the action density (4.28) is modified by an interaction with a current \( J^a \)
\[ L = -\frac{1}{4} F_{ba} F^{ba} + A_a J^a \] (4.30)
then the equation of motion is
\[ \partial_b F^{ba} = - J^a \] (4.31)
while the homogeneous equations
\[ 0 = \partial_a F_{bc} + \partial_b F_{ca} + \partial_c F_{ab} \] (4.32)
follow from the antisymmetry of \( F_{ab} \).

Because of the antisymmetry of \( F_{ab} \), the derivative \( \dot{A}_0 \) does not appear in the action density (4.30). So for \( a = 0 \), the equation of motion (4.31) actually is a constraint
\[ \partial_i F^{i0} = - \partial_i E^i = - J^0 \] (4.33)
known as Gauss’s law
\[ \nabla \cdot \vec{E} = \rho = J^0. \] (4.34)
This constraint together with the Coulomb gauge condition (4.29) lets us express \( A^0 \) in terms of the charge density \( \rho = J^0 \). We find
\[
\nabla^2 A_0 - \partial_0 \nabla \cdot \vec{A} = \nabla \cdot E = J^0
\]
(4.35)
or
\[
\nabla^2 A_0 = J^0.
\]
(4.36)
The solution is
\[
A^0(\vec{x}, t) = \int \frac{J^0(\vec{y}, t) d^3 y}{4\pi |\vec{x} - \vec{y}|}.
\]
(4.37)

The quantum fields in this gauge are the transverse parts of \( \vec{A} \) and their conjugate momentum \( \vec{E} \).

### 4.4 QED in the interaction picture

The Coulomb-gauge hamiltonian in the interaction picture is
\[
H = H_0 + V
\]
\[
H_0 = \frac{1}{2} \int \vec{E}^2 + \vec{B}^2 \, d^3 x + H_{\psi,0}
\]
\[
V = -\int \vec{J} \cdot \vec{A} \, d^3 x + V_C + V_\psi
\]
(4.38)
in which \( \vec{B} = \nabla \times \vec{A} \) and both \( \vec{E} \) and \( \vec{A} \) are transverse, that is
\[
\nabla \cdot \vec{E} = 0 \quad \text{and} \quad \nabla \cdot \vec{A} = 0
\]
(4.39)
and the Coulomb potential energy is
\[
V_C = \frac{1}{2} \int \frac{J^0(\vec{x},0)J^0(\vec{y},0) d^3 x d^3 y}{4\pi |\vec{x} - \vec{y}|}.
\]
(4.40)
The electromagnetic field is
\[
A^b(x) = \int \frac{d^3 p}{\sqrt{(2\pi)^3 2p^0}} \sum_s \left[ e^{ip \cdot x} e^b(p,s)a(p,s) + e^{-ip \cdot x} e^{b*}(p,s)a^\dagger(p,s) \right].
\]
(4.41)
The polarization vectors may be chosen to be
\[
e^b(p, \pm 1) = R(\hat{p}) \frac{1}{\sqrt{2}} \begin{pmatrix}
0 \\
\pm i \\
0
\end{pmatrix}
\]
(4.42)
where $R(\hat{p})$ rotates $\hat{z}$ into $\hat{p}$. Thus
\begin{equation}
\vec{p} \cdot \vec{e}(p, s) = 0
\end{equation}
(4.43)
and
\begin{equation}
e^0(p, s) = 0
\end{equation}
(4.44)
because $A^0$ is a dependent variable \[4.37\]. The commutation relations are
\begin{equation}
[a(p, s), a^\dagger(p', s')] = \delta_{ss'} \delta^3(\vec{p} - \vec{p}') \quad \text{and} \quad [a(p, s), a(p', s')] = 0.
\end{equation}
(4.45)

The term $H_0$ in the hamiltonian \[4.38\] is
\begin{equation}
H_0 = \int \sum_s \frac{1}{2} p^0 [a(p, s), a^\dagger(p, s)] d^3p
= \int \sum_s p^0 \left( a^\dagger(p, s) a(p, s) + \frac{1}{2} \delta^3(\vec{p} - \vec{p}) \right) d^3p.
\end{equation}
(4.46)

The interaction is
\begin{equation}
V(t) = e^{iH_0 t} \left[ -\int \vec{J}(\vec{x}, 0) \cdot \vec{A}(\vec{x}, 0) d^3x + \int \frac{J^0(\vec{x}, 0) J^0(\vec{y}, 0)}{8\pi |\vec{x} - \vec{y}|} d^3xd^3y + V_m(0) \right] e^{-iH_0 t}
\end{equation}
(4.47)
in which $V_m(0)$ is the non-electromagnetic part of the matter interaction. Since $A^0 = 0$, $\vec{J} \cdot \vec{A} = J \cdot \vec{A}$.

4.5 Photon propagator

The photon propagator is
\begin{equation}
- i\Delta_{ab}(x - y) = \langle 0 | T[A_a(x), A_b(y)] | 0 \rangle.
\end{equation}
(4.48)

Inserting the formula for the electromagnetic field \[4.41\], we get
\begin{equation}
- i\Delta_{ij}(x - y) = \int \frac{d^4p}{(2\pi)^3 2p^0} \left( \delta_{ij} - \frac{n_i n_j}{\vec{p}^2} \right) \times e^{ip \cdot (x - y)} \theta(x^0 - y^0) + e^{-ip \cdot (x - y)} \theta(y^0 - x^0)
= \int \frac{d^4q}{(2\pi)^4} \frac{-i}{q^2 - i\epsilon} \left( \delta_{ij} - \frac{q_i q_j}{q^2} \right) e^{i_q \cdot (x - y)}
\end{equation}
(4.49)
with the understanding that this propagator vanishes when $a$ or $b$ is zero. If $n^c = (1, 0, 0, 0)$ has only a time component, one can write the $\delta_{ij} - q_i q_j/q^2$ as
\begin{equation}
\left( \delta_{ab} - \frac{q_a q_b}{q^2} \right) \theta(ab) = \eta_{ab} + q^0 (q_a n_b + q_b n_a) - q_a q_b + q^2 n_a n_b
\end{equation}
(4.50)
in which \( q^0 \) is arbitrary and may be chosen to be determined by conservation of energy. The terms \( q_a n_b \), \( q_b n_a \), and \( q_a q_b \) act like \( \partial_a n_b \), \( \partial_b n_a \), and \( \partial_a \partial_b \) so they appear in the S-matrix as \( \partial_a J^a n_b \) and \( \partial_b J^b n_a \), etc., which vanish because of current conservation. The remaining term

\[
\frac{q^2 n_a n_b}{q^2} \frac{-i}{q^2 - i\epsilon} = -i n_a n_b
\]

(4.51)
gives in the S-matrix a term

\[
T = \frac{1}{2} \int d^4 x d^4 y [-i J^0(x)][-i J^0(y)] \frac{-i}{(2\pi)^4} \int \frac{d^4 q}{q^2} e^{i q \cdot (x-y)}
\]

\[
= \frac{1}{2} \int d^4 x d^4 y J^0(x) J^0(y) \frac{d^3 q}{(2\pi)^3} \frac{\delta(x^0 - y^0)}{q^2} e^{i \vec{q} \cdot (\vec{x} - \vec{y})}
\]

(4.52)

\[
= \frac{i}{2} \int d^3 x d^3 y dt \frac{J^0(\vec{x}, t) J^0(\vec{y}, t)}{4\pi |\vec{x} - \vec{y}|}
\]

(4.53)

which cancels the Coulomb term

\[
T_C = -i \frac{1}{2} \int d^3 x d^3 y dt \frac{J^0(\vec{x}, t) J^0(\vec{y}, t)}{4\pi |\vec{x} - \vec{y}|}.
\]

(4.54)
The bottom line is that the effective photon propagator is

\[
- i \Delta_{ab}(x - y) = \int \frac{d^4 q}{(2\pi)^4} \frac{-i \eta_{ab}}{q^2 - i\epsilon} e^{i q \cdot (x-y)} = -i \Delta_{ab}(y - x).
\]

(4.55)

Dirac’s action density is

\[
L_{\psi} = - \bar{\psi} (\gamma^a \partial_a + m) \psi
\]

(4.56)

with \( \bar{\psi} = i \psi^\dagger \gamma^0 \). The conjugate momentum is

\[
\pi = \frac{\partial L}{\partial \dot{\psi}} = -\bar{\psi} \gamma^0 = -i \psi^\dagger \gamma^0 \gamma^0 = i \psi^\dagger.
\]

(4.57)
The hamiltonian is

\[
H = \int (\pi \dot{\psi} - L) d^3 x = \int [i \psi^\dagger \dot{\psi} + \bar{\psi} (\gamma^a \partial_a + m) \psi] d^3 x
\]

\[
= \int \bar{\psi} (\vec{\gamma} \cdot \vec{\nabla} + m) \psi d^3 x.
\]

(4.58)
The free Dirac field aka the Dirac field in the interaction picture is

\[
\psi(x) = \int \frac{d^3 p}{(2\pi)^3/2} \sum_s \left[ u(p, s) e^{ipx} a(p, s) + v(p, s) e^{-ipx} b^\dagger(p, s) \right]
\]

(4.59)
Quantum electrodynamics

The addition and deletion operators obey the anticommutation relations

\[ [a(p, s), a^\dagger(p', s')]_+ = [b(p, s), b^\dagger(p', s')]_+ = \sigma_{ss'} \delta^3(\vec{p} - \vec{p}') \quad (4.60) \]

with the other anticommutators equal to zero. Putting \( \psi(x) \) into \( H \) gives

\[ H = \int \sum_s p^0 \left[ a^\dagger(p, s)a(p, s) + b^\dagger(p, s)b(p, s) - \delta^3(\vec{p} - \vec{p}') \right] d^3p. \quad (4.61) \]

4.6 Feynman's rules for QED

The action density is

\[ L = -\frac{1}{4} F_{ab} F^{ab} - \bar{\psi}[\gamma^a(\partial_a + ieA_a) + m]\psi. \quad (4.62) \]

The electric current is

\[ J^a = \frac{\partial L}{\partial A_a} = -ie\bar{\psi}\gamma^a\psi. \quad (4.63) \]

The interaction is

\[ V(t) = ie \int \bar{\psi}(\vec{x}, t)\gamma^a\psi(\vec{x}, t)A_a(\vec{x}, t) d^3x + V_C(t) \quad (4.64) \]

Draw all appropriate diagrams.

Label each vertex with \( \alpha = 1, 2, 3, \) or 4 on an electron line of momentum \( p \) entering the vertex and \( \beta = 1, \ldots, 4 \) on an electron line of momentum \( p' \) leaving the vertex, and an index \( a \) on the photon line of momentum \( q \). The vertex carries the factor

\[ (2\pi)^4 e\gamma^a_{\beta\alpha} \delta^4(p - p' + q). \quad (4.65) \]

An outgoing electron line gives

\[ \frac{\bar{u}_\beta(p, s)}{(2\pi)^{3/2}}. \quad (4.66) \]

An outgoing positron line gives

\[ \frac{v_\alpha(p, s)}{(2\pi)^{3/2}}. \quad (4.67) \]

An incoming electron line gives

\[ \frac{u_\alpha(p, s)}{(2\pi)^{3/2}}. \quad (4.68) \]
4.7 Electron-positron scattering

An incoming positron line gives
\[ \bar{v}_\beta(p, s) \frac{\sqrt{(2\pi)^3}}{2p^0}. \]  
\[ (4.69) \]

An outgoing photon gives
\[ e^*_a(p, s) \frac{\sqrt{(2\pi)^3}}{2p^0}. \]  
\[ (4.70) \]

An incoming photon gives
\[ e_a(p, s) \frac{\sqrt{(2\pi)^3}}{2p^0}. \]  
\[ (4.71) \]

An internal electron line of momentum \( p \) from vertex \( \beta \) to vertex \( \alpha \) gives
\[ -i \frac{(-ip + m)_{\alpha\beta}}{(2\pi)^4 \frac{1}{p^2 + m^2 - i\epsilon}}. \]  
\[ (4.72) \]

An internal photon line of momentum \( q \) linking vertexes \( a \) and \( b \) gives
\[ -i \frac{\eta_{ab}}{(2\pi)^4 \frac{1}{q^2 - i\epsilon}}. \]  
\[ (4.73) \]

Integrate over all momenta, sum over all indexes.
Add up all terms and get the combinatorics and minus signs right.

4.7 Electron-positron scattering

There are two Feynman diagrams for electron-positron scattering at order \( e^2 \) where \( \alpha = e^2/(4\pi\epsilon_0hc) = 0.00729735256 \approx 1/137 \) is the fine-structure constant. The annihilation diagram is represented in Fig. 4.2.
Feynman’s rules give for the annihilation diagram

\[
A_s = \int \, d^4 q \, (2\pi)^4 \epsilon_\gamma^{\mu} \delta^4(p + k - q) (2\pi)^4 e_\gamma^{\nu} \delta^4(-p' - k' + q) \\
\times \bar{u}_\alpha(p', s') v_\delta(k', t') u_\alpha(p, s) \bar{v}_\beta(k, t) - i \frac{\eta_{ab}}{q^2 - i\epsilon} \eta_{ab}
\]

\[
= (2\pi)^4 e_\gamma^{\mu} \delta^4(p + k - p' - k') \\
\times \bar{u}_\alpha(p', s') v_\delta(k', t') u_\alpha(p, s) \bar{v}_\beta(k, t) - i \frac{\eta_{ab}}{(p + k)^2}
\]

\[
= e_\gamma^{\mu} \delta^4(p + k - p' - k') \\
\times \bar{u}_\alpha(p', s') v_\delta(k', t') u_\alpha(p, s) \bar{v}_\beta(k, t) - i \frac{\eta_{ab}}{(p + k)^2}
\]

Suppressing indexes, we get

\[
A_s = - i \frac{e^2}{(2\pi)^2} \delta^4(p + k - p' - k') \frac{\bar{v}^{\gamma a} u \bar{v}^{\gamma a} v'}{(p + k)^2}.
\]

The exchange diagram is

\[
A_t = \pm i \frac{e^2}{(2\pi)^2} \delta^4(p + k - p' - k') \frac{\bar{u}^{\gamma a} u \bar{v}^{\gamma a} v'}{(p - p')^2},
\]

and we must check our minus signs.

The initial state is \(|p, s; k, t\rangle = a^\dagger(p, s) b^\dagger(k, t) |0\rangle\) and the final state is
$|p', s'; k', t'| = a^\dagger(p', s')b^\dagger(k', t')|0\rangle$. The direct diagram comes from
\[
T[\bar{\psi}(x)\gamma^a\psi(x)A_a(x)\bar{\psi}(y)\gamma^b\psi(y)A_b(y)]
\]
\[=
(\bar{\psi})^- (x)\gamma^a\psi^- (x)T[A_a(x)A_b(y)](\bar{\psi})^+(y)\gamma^b\psi^+(y)
\] (4.77)
\[
\sim a^\dagger(p, s')\gamma^a\gamma^b(k', t')T[A_a(x)A_b(y)]b(k, s)\gamma^b a(p, s),
\]
while the exchange diagram comes from
\[
T[\bar{\psi}(x)\gamma^a\psi(x)A_a(x)\bar{\psi}(y)\gamma^b\psi(y)A_b(y)]
\]
\[=
(\bar{\psi})^+(x)\gamma^a\psi^-(x)T[A_a(x)A_b(y)](\bar{\psi})^-(y)\gamma^b\psi^+(y)
\] (4.78)
\[
\sim a^\dagger(p', s)\gamma^a\gamma^b(k', t')T[A_a(x)A_b(y)]\gamma^b a(p, s)
\]
\[
\sim -a^\dagger(p', s')\gamma^a\gamma^b(k', t')T[A_a(x)A_b(y)]b(k, t)\gamma^b a(p, s)
\]
which differs by a minus sign. So the total amplitude is
\[
A = A_s + A_t
\]
\[
= -i \frac{e^2}{(2\pi)^2} \delta^4(p + k - p' - k') \frac{\bar{v}\gamma^a u \bar{u}' \gamma_b v'}{(p + k)^2}
\]
\[- \left[ -i \frac{e^2}{(2\pi)^2} \delta^4(p + k - p' - k') \frac{\bar{u}' \gamma^b u \bar{v} \gamma_b v'}{(p - p')^2} \right]
\] (4.79)
\[
= -i \frac{e^2}{(2\pi)^2} \delta^4(p + k - p' - k') \left[ \frac{\bar{v} \gamma^a u \bar{u}' \gamma_a v'}{(p + k)^2} - \frac{\bar{u}' \gamma^b u \bar{v} \gamma_b v'}{(p - p')^2} \right].
\]
The probability is the square $|A|^2$. To compute such things, we need to know some trace identities.

### 4.8 Trace identities

Since $\gamma^5$ is a fourth spatial gamma matrix (1.173), its square is unity, $\gamma^5 \gamma^5 = 1$, and it anticommutes with the ordinary gammas. So since the trace is cyclic,
\[
\text{Tr}(\gamma^a) = \text{Tr}(\gamma^5 \gamma^5 \gamma^a) = -\text{Tr}(\gamma^5 \gamma^a \gamma^5) = -\text{Tr}(\gamma^5 \gamma^5 \gamma^a) = -\text{Tr}(\gamma^a)
\] (4.80)
the trace of a single gamma matrix is zero.

The trace of two gammas
\[
\text{Tr}(\gamma^a \gamma^b) = \text{Tr}(2\eta^{ab} - \gamma^h \gamma^a)
\] (4.81)
is
\[
\text{Tr}(\gamma^a \gamma^b) = \eta^{ab} \text{Tr}(1) = 4\eta^{ab}.
\] (4.82)
The trace of an odd number of gammas vanishes because
\[
\text{Tr} (\gamma_1 \ldots \gamma_{2n+1}) = \text{Tr} (\gamma_1 \gamma_2 \gamma_3 \ldots \gamma_{2n+1}) = - \text{Tr} (\gamma_1 \gamma_2 \gamma_3 \ldots \gamma_{2n+1} \gamma_5)
\]
\[
= - \text{Tr} (\gamma_1 \gamma_2 \gamma_3 \ldots \gamma_{2n+1}) = - \text{Tr} (\gamma_1 \ldots \gamma_{2n+1})
\] (4.83)
which implies that
\[
\text{Tr} (\gamma_1 \ldots \gamma_{2n+1}) = 0.
\] (4.84)

To find the trace of four gammas, we use repeatedly the fundamental rule
\[
\gamma_1 \gamma_2 = 2 \eta_1^2 - \gamma_2 \gamma_1.
\] (4.85)

We thus find
\[
\text{Tr} (\gamma_1 \gamma_2 \gamma_3 \gamma_4) = \text{Tr} [2 \eta_1^2 - \gamma_2 \gamma_1 \gamma_3 \gamma_4]
\]
\[
= \text{Tr} [2 \eta_1^2 \gamma_3 \gamma_4 - \gamma_3 (2 \eta_2 \gamma_4 - \gamma_4 \gamma_2) \gamma_3]
\]
\[
= \text{Tr} [2 \eta_1^2 \gamma_3 \gamma_4 - 2 \eta_2 \gamma_4 + \gamma_4 \gamma_2 \gamma_3]
\] (4.86)

or
\[
\text{Tr} (\gamma_1 \gamma_2 \gamma_3 \gamma_4) = \text{Tr} [2 \eta_1^2 \gamma_3 \gamma_4 - 2 \eta_2 \gamma_4 + \gamma_4 \gamma_2 \gamma_3 - \gamma_2 \gamma_3 \gamma_4 \gamma_1]
\]
\[
= 4 \eta_1^2 \eta_2 - \eta_2 \eta_3 + \eta_3 \eta_4
\] (4.87)

Reversing completely the order of the gammas changes nothing:
\[
\text{Tr} (\gamma_1 \gamma_2 \ldots \gamma_5) = \text{Tr} (\gamma_5 \gamma_4 \ldots \gamma_2 \gamma_1)
\] (4.88)

4.9 Electron-positron to muon-antimuon

To avoid the extra complications that the sum of two amplitudes \(A_s + A_t\) entail, let’s consider the process \(e^- (p) + e^+ (p') \rightarrow \mu^- (k) + \mu^+ (k')\) which requires only the direct amplitude
\[
A = - \frac{e^2}{(2\pi)^2} \delta^4 (p + p' - k - k') \ \frac{\bar{v}(p', s') \gamma_a u(p, s) \ \bar{u}(k, t) \gamma_a v(k', t')}{(p + p')^2}
\] (4.89)
\[
= - 2\pi i \delta^4 (p + p' - k - k') \ M
\]
where
\[
M = \frac{e^2}{(2\pi)^2} \ \frac{\bar{v}(p', s') \gamma_a u(p, s) \ \bar{u}(k, t) \gamma_a v(k', t')}{(p + p')^2}
\] (4.90)
in which I have relabeled the momenta so that \(k \& k'\) belong to the muons.
The probability $|A|^2$ includes the term
\[ \bar{v}_\gamma u \bar{\mu}_\gamma v_\mu (\bar{v}_\gamma b u \bar{\mu}_\gamma b v_\mu)^* \] (4.91)
in which $\mu$'s label muons. Since $\bar{v} = iv^\dagger \gamma^0 = v^\dagger \beta$ as well as $\beta^2 = (i\gamma^0)^2 = 1$, and since $\beta^{-1} \beta = -\gamma^0$, we get
\[(\bar{v}_\gamma a u)^* = (v^\dagger \beta a u)^* = u^\dagger \beta a \beta v = u^\dagger \beta a v = \bar{\mu}_\gamma a v. \] (4.92)
So the electron part of the term (4.91) is
\[(\bar{v}_\gamma a u (\bar{v}_\gamma b u)^* = \bar{v}_\gamma a u \bar{\mu}_\gamma b v. \] (4.93)
The spin sums are
\[ \sum_s u_\ell (\vec{p}, s) u^*_m (\vec{p}, s) = \left[ \frac{1}{2p^\dagger} (-i p^\dagger \gamma_e + m) \beta \right]_{\ell m} \] (4.94)
equivalently and more simply,
\[ \sum_s v_\ell (\vec{p}, s) v^*_m (\vec{p}, s) = \left[ \frac{1}{2p^\dagger} (-i p^\dagger \gamma_e - m) \beta \right]_{\ell m}. \] (4.95)
So
\[ \sum_{s,s'} \bar{v}^\gamma a u \bar{w}^\gamma b v = \sum_{s',j,\ell,m,n} \bar{v}_j (p', s') \gamma^a_{j\ell} u_\ell (p, s) \bar{w}_m (p, s) \gamma^b_{m\ell} v_n (p', s') \]
\[ = \frac{1}{2p^\dagger} \sum_{s',j,\ell,m,n} \bar{v}_j (p', s') \gamma^a_{j\ell} (-i \gamma^c + m)_{\ell m} \gamma^b_{m\ell} v_n (p', s') \] (4.96)
\[ = \frac{1}{2p^\dagger} \sum_{s',j,\ell,m,n} v_n (p', s') \bar{v}_j (p', s') \gamma^a_{j\ell} (-i \gamma^c + m)_{\ell m} \gamma^b_{m\ell} \]
\[ = \frac{1}{2p^\dagger 2p^\dagger} \sum_{s,j,\ell,m,n} (-i \gamma^c - m)_{j\ell} \gamma^a_{j\ell} (-i \gamma^c + m)_{\ell m} \gamma^b_{m\ell} \]
\[ = \frac{1}{2p^\dagger 2p^\dagger} \text{Tr} [(-i \gamma^c - m)_{j\ell} \gamma^a_{j\ell} (-i \gamma^c + m)_{\ell m} \gamma^b_{m\ell}]. \]
The muon part of the term (4.91) is
\[ \bar{\mu}_\gamma a v (\bar{\mu}_\gamma b v)^* = \bar{\mu}_\gamma a v \bar{\mu}_\gamma b u = \bar{v}_\gamma b u \bar{\mu}_\gamma a v. \] (4.97)
So the sum over all spins gives for the muons
\[
\sum_{t,t'} \bar{u} \gamma_a \nu \gamma_b u = \frac{1}{2k^0 t^0} \text{Tr}([-i \slashed{p} - m_\mu] \gamma_b(-i \slashed{k} + m_\mu) \gamma_a). \tag{4.98}
\]

Using the formula (4.86) for the trace of 4 gammas, we evaluate the trace for the electrons in stages:
\[
\text{Tr}([-i \slashed{p} - m_e] \gamma^a(-i \slashed{p} + m_e) \gamma^b] = - \text{Tr}(\slashed{p} \gamma^a \gamma^b) - m_e^2 \text{Tr}(\gamma^a \gamma^b). \tag{4.99}
\]
The 4-gamma term is
\[
\text{Tr}([-i \slashed{p} - m_e] \gamma^a(-i \slashed{p} + m_e) \gamma^b] = - \text{Tr}[p^c \gamma^c p_d \gamma^d] = - 4 (p^a p^b - p^i p^j \eta^{ij}). \tag{4.100}
\]
The 2-gamma term is
\[
- m_e^2 \text{Tr}(\gamma^a \gamma^b) = - 4 m_e^2 \eta^{ab}. \tag{4.101}
\]
So their sum is
\[
\text{Tr}([-i \slashed{p} - m_e] \gamma^a(-i \slashed{p} + m_e) \gamma^b] = - 4 [p^a p^b + p^b p^a + \eta^{ab}(m_e^2 - p^i p^i)]. \tag{4.102}
\]
The similar term for muons is
\[
\text{Tr}([-i \slashed{p} - m_e] \gamma^a(-i \slashed{k} + m_\mu) \gamma^b] = - 4 \left[ k^a _a k^b + k^b _a k^a + \eta_{ab}(m_\mu^2 - k^i k^i) \right]. \tag{4.103}
\]
Their product is
\[
\text{Tr}([-i \slashed{p} - m_e] \gamma^a(-i \slashed{p} + m_e) \gamma^b] \text{Tr}([-i \slashed{k} - m_\mu] \gamma_a(-i \slashed{k} + m_\mu) \gamma_b) \tag{4.104}
\]
\[
= 16 \left[ p^a p^b + p^b p^a + \eta^{ab}(m_e^2 - p^i p^i) \right] \left[ k^a _a k^b + k^b _a k^a + \eta_{ab}(m_\mu^2 - k^i k^i) \right]
\]
\[
= 32 \left[ (p' k')(pk) + (p' k)(pk') + m_e^2 (pp') + m_\mu^2 (k' k') \right].
\]
So apart from the delta function and the $2\pi$'s, the squared amplitude summed over final spins and averaged over initial spins is
\[
\frac{1}{4} \sum_{s,s'} |M|^2 = \frac{e^4}{(2\pi)^6 p^0 k^0 k^0} \left[ (p' k')(pk) + (p' k)(pk') + m_e^2 (pp') + m_\mu^2 (k' k') \right] \tag{4.105}
\]
where $(p + p')^4 = [(p + p')^2]^2$.

In the rest frame of a collider, $q^2 = (p + p')^2 = 4E_0^2 = 4p^0$. The cosine of the scattering angle $\theta$ is $\cos \theta = \hat{p} \cdot \hat{k} / (|\vec{p}| |\vec{k}|)$.

Since $m_e \approx m_\mu / 200$, I will neglect $m_e$ in what follows. One then gets
\( p \cdot p' = -2E^2 \), \( p \cdot k = E(|\vec{k}| \cos \theta - E) \), and \( p' \cdot k = -E(|\vec{k}| \cos \theta + E) \). Thus

\[
\frac{1}{4} \sum_{s,s'} |M|^2 = e^4 \left[ \frac{E^2(E - k \cos \theta)^2 + E^2(E + k \cos \theta)^2 + 2m_{\mu}^2E^2}{2(2\pi)^6E^416E^4} \right]
\]

\[
= \frac{e^4 \left[ (E - k \cos \theta)^2 + (E + k \cos \theta)^2 + 2m_{\mu}^2 \right]}{32(2\pi)^6E^6}
\]

\[
= \frac{e^4 \left[ 2E^2 + 2k^2 \cos^2 \theta + 2m_{\mu}^2 \right]}{32(2\pi)^6E^6} = \frac{e^4 \left[ E^2 + k^2 \cos^2 \theta + m_{\mu}^2 \right]}{16(2\pi)^6E^6}
\]

(4.106)

The squared amplitude is

\[
|A|^2 = \frac{(2\pi)^2}{V} \delta^4(p + p' - k - k') \left( \frac{1}{4} \sum_{s,s'} |M|^2 + \frac{1}{4} \sum_{ss'} |M|^2 \right)
\]

(4.107)

\[
= \frac{VT}{(2\pi)^3} \delta^4(p + p' - k - k') \left( \frac{1}{4} \sum_{s,s'} |M|^2 \right)
\]

(4.108)

since

\[
\delta^4(0) = \int \frac{d^4x}{(2\pi)^4} = \frac{VT}{(2\pi)^4}
\]

(4.109)

in which \( V \) is the volume of the universe and \( T \) is its infinite time. We also need to switch from delta-function normalization of states to unit normalization. The relation is that between a continuum delta function and a Kronecker delta

\[
\delta^3(\vec{p}' - \vec{p}) = \frac{V}{(2\pi)^3} \delta_{pp'}.
\]

(4.110)

Since there are two particles in the initial and final states, the probability is

\[
P = \left[ \frac{(2\pi)^3}{V} \right]^4 |A|^2 N_f
\]

(4.111)

where \( N_f \) is the number of final states. For a range of momenta \( d^3kd^3k' \), the number of final states is

\[
N_f = \left[ \frac{V}{(2\pi)^3} \right]^{2} d^3kd^3k'.
\]

(4.112)
So the rate $P/T$ is

$$R = \left[ \frac{(2\pi)^3}{V} \right] 4 |A|^2 N_f = \left[ \frac{(2\pi)^3}{V} \right] N_f \left[ \frac{V}{(2\pi)^3} \right]^2 d^3k d^3k'$$

$$= \left[ \frac{(2\pi)^3}{V} \right] 4 |A|^2 N_f = \left[ \frac{(2\pi)^3}{V} \right]^2 |A|^2 d^3k d^3k'$$

$$= \frac{(2\pi)^4}{V} \delta^4(p + p' - k - k') \sum_{ss'} |M|^2 d^3k d^3k'$$

(4.113)

The flux of incoming particles is $u/V$ where $u = |\vec{v}_1 - \vec{v}_2|$ is the relative velocity, which with $c = 1$ for massless electrons is $u = 2$. So the differential cross-section is

$$d\sigma = \frac{1}{2} (2\pi)^4 \delta^4(p + p' - k - k') \sum_{ss'} |M|^2 d^3k d^3k'.$$

(4.114)

Integrating over $d^3k'$ sets $\vec{k}' = -\vec{k}$, and so

$$d\sigma = \frac{1}{2} (2\pi)^4 \delta(p^0 + p^0' - 2k^0) \sum_{ss'} |M|^2 d^3k$$

$$= \frac{1}{2} (2\pi)^4 \delta \left( p^0 + p^0' - 2\sqrt{k^2 + m^2} \right) \frac{1}{4} \sum_{ss'} |M|^2 k^2 dk d\Omega$$

(4.115)

where $k = |\vec{k}|$. The derivative of the delta function is $k2p^0/k^2$, so

$$d\sigma = \frac{1}{2} (2\pi)^4 \frac{1}{4} \sum_{ss'} |M|^2 k^2 d\Omega \frac{k^2}{2kp^0} = 2\pi^4 \frac{1}{4} \sum_{ss'} |M|^2 \frac{k^2}{p^0} d\Omega$$

$$= 4\pi^4 \frac{1}{4} \sum_{ss'} |M|^2 \frac{k^2}{p^0} d\Omega.$$  

(4.116)
So then

\[ d\sigma = 4\pi^4 \frac{e^4 \left[ E^2 + k^2 \cos^2 \theta + m_\mu^2 \right]}{4(2\pi)^6 E^6} \frac{k k^\prime}{p^\prime} d\Omega \]

\[ = \frac{e^4}{4(2\pi)^2 E^5} \left[ 1 + \frac{k^2}{E^2} \cos^2 \theta + \frac{m_\mu^2}{E^2} \right] k^\prime d\Omega \]

\[ = \frac{e^4}{4(2\pi)^2 E^5} \left[ 1 + \frac{E^2 - m_\mu^2}{E^2} \cos^2 \theta + \frac{m_\mu^2}{E^2} \right] k^\prime d\Omega \] (4.117)

\[ = \frac{e^4}{4(2\pi)^2 E^5} \sqrt{1 - \frac{m_\mu^2}{E^2}} \left[ 1 + \frac{m_\mu^2}{E^2} + \left( 1 - \frac{m_\mu^2}{E^2} \right) \cos^2 \theta \right] d\Omega \]

\[ = \frac{\alpha^2}{16 E^2} \sqrt{1 - \frac{m_\mu^2}{E^2}} \left[ 1 + \frac{m_\mu^2}{E^2} + \left( 1 - \frac{m_\mu^2}{E^2} \right) \cos^2 \theta \right] d\Omega. \]

The total cross-section is the integral over solid angle

\[ \sigma = \frac{\alpha^2}{16 E^2} \sqrt{1 - \frac{m_\mu^2}{E^2}} \left[ (1 + \frac{m_\mu^2}{E^2}) \int \cos^2 \theta \ d\Omega \right] \]

\[ = \frac{\alpha^2}{16 E^2} \sqrt{1 - \frac{m_\mu^2}{E^2}} \left[ (1 + \frac{m_\mu^2}{E^2}) \int \cos^2 \theta \ 2\pi \ d\cos \theta \right] \]

\[ = \frac{\pi \alpha^2}{4 E^2} \sqrt{1 - \frac{m_\mu^2}{E^2}} \left[ (1 + \frac{m_\mu^2}{E^2}) + \frac{1}{3} \left( 1 - \frac{m_\mu^2}{E^2} \right) \right] \]

\[ = \frac{\pi \alpha^2}{3 E^2} \left( 1 - \frac{m_\mu^2}{E^2} \right) \left( 1 + \frac{1}{2} \frac{m_\mu^2}{E^2} \right) \] (4.118)

where \( E \) is the energy of each electron in the lab frame of the collider. The energy of the collider is \( E_{\text{cm}} = 2E \). At very high energies, the x-section is \( \sigma = \pi \alpha^2/(3E^2) \).
4.10 Electron-muon scattering

For the process $e^-(p_1) + \mu^- (p_2) \rightarrow e^-(p'_1) + \mu^- (p'_2)$, the Feynman rules give

$$A = -i \frac{e^2}{(2\pi)^2} \delta^4(p_1 + p_2 - p'_1 - p'_2) \frac{\bar{u}(p'_1, s') \gamma^a u(p_1, s) \bar{u}(p'_2, t') \gamma_a u(p_2, t)}{(p_1 - p'_1)^2}$$

$$= -2\pi i \delta^4(p_1 + p_2 - p'_1 - p'_2) M. \quad (4.119)$$

The product of the traces over $q^2 = (p_1 - p'_1)^2$ is

$$\text{Tr}[(-i\gamma^\mu + m_e)\gamma^\alpha(-i\gamma^\mu + m_e)\gamma^b] \text{Tr}[(-i\gamma^\mu + m_\mu)\gamma_a(-i\gamma^\mu + m_\mu)\gamma_b]$$

$$= (p_1 - p'_1)^2. \quad (4.120)$$

For $e^+ - e^- \rightarrow \mu^+ + \mu^-$ we get

$$\text{Tr}[(-i\gamma^\mu - m_e)\gamma^\alpha(-i\gamma^\mu + m_e)\gamma^b] \text{Tr}[(-i\gamma^\mu - m_\mu)\gamma_a(-i\gamma^\mu + m_\mu)\gamma_b]$$

$$= (p + p')^2. \quad (4.121)$$

If in this amplitude for $e^+ - e^- \rightarrow \mu^+ + \mu^-$ we make these replacements

$$p \rightarrow p_1, \quad p' \rightarrow -p'_1, \quad k \rightarrow p'_2, \quad \text{and} \quad k' \rightarrow -p_2, \quad (4.122)$$

it becomes

$$\text{Tr}[(i\gamma^\mu - m_e)\gamma^\alpha(-i\gamma^\mu + m_e)\gamma^b] \text{Tr}[(i\gamma^\mu - m_\mu)\gamma_a(-i\gamma^\mu + m_\mu)\gamma_b]$$

$$= \frac{\text{Tr}[(i\gamma^\mu + m_e)\gamma^\alpha(-i\gamma^\mu + m_e)\gamma^b] \text{Tr}[(i\gamma^\mu + m_\mu)\gamma_a(-i\gamma^\mu + m_\mu)\gamma_b]}{(p_1 - p'_1)^2} \quad (4.123)$$

which is exactly the same as the amplitude (4.120) for $e^- (p_1) + \mu^- (p_2) \rightarrow e^- (p'_1) + \mu^- (p'_2)$ scattering. So apart from the delta function and the $(2\pi)^2$s, the squared amplitude summed over final spins and averaged over initial spins is the one for $e^+ - e^- \rightarrow \mu^+ + \mu^-$ but with the substitutions (4.122).

That is, for $e^- (p_1) + \mu^- (p_2) \rightarrow e^- (p'_1) + \mu^- (p'_2)$ scattering, the summed amplitudes are

$$\frac{1}{4} \sum_{s, s', t, t'} |M|^2 = \frac{e^4 (p'_1 p_2)(p_1 p'_2) + (p'_1 p'_2)(p_1 p_2) - m_e^2 (p_1 p'_1) - m_\mu^2 (p_2 p'_2)}{2(2\pi)^6 p_1^0 p'_1 p_2^0 p'_2 (p_1 - p'_1)^4}. \quad (4.124)$$

The equality of the two amplitudes (4.120) and (4.121) under the correspondence (4.122) is an example of crossing symmetry. More generally, the scattering amplitude for a process in which an incoming particle of 4-momentum $p$ that interacts with other particles is equal to the scattering
amplitude for the process in which the antiparticle of the incoming particle leaves with 4-momentum $-p$ after interacting with the same other particles

$$S(-p, n_c; \cdots | \cdots ) = S(\cdots | p, n; \cdots )$$  \hspace{1cm} (4.125)

in which type $n_c$ is labels the antiparticle of type $n$. Since $-p^0 + p^0 = 0$, this is not the amplitude for a physical process in which each particle has its appropriate energy, $p_i^0 = \sqrt{m_i^2 + \vec{p}_i^2}$. This symmetry caused huge excitement on the West Coast during the 1960s.

Let $k = |\vec{p}_1'| = p_0'$ be energy or equivalently the modulus (magnitude) of the 3-momentum of the (massless) final electron. Let $E = p_0'$ be the energy of the initial muon. Then the differential x-section in the lab frame of the collider is

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{2k^2(E + k)^2(1 - \cos \theta)^2} \left[ (E + k)^2 + (E + k \cos \theta)^2 - m_{\mu}^2 (1 - \cos \theta) \right]$$  \hspace{1cm} (4.126)

in which the cosine of the scattering angle of the electron is

$$\cos \theta = \frac{\vec{p}_1 \cdot \vec{p}_1'}{|\vec{p}_1||\vec{p}_1'|}.$$  \hspace{1cm} (4.127)

The total x-section $\sigma$ is infinite.

### 4.11 One-loop QED

The action density is

$$L = -\frac{1}{4} F^{ab}_b F_{ab} - \bar{\psi}_b \left[ \gamma_\alpha (\partial^\alpha + ie_b A^\mu_b) + m_b \right] \psi_b$$  \hspace{1cm} (4.128)

in which the subscript $b$ means that the values of these “bare” terms include infinite quantities that may be used to cancel the infinities that arise in perturbation theory. The actual physical terms are defined as

$$\psi \equiv Z_2^{-1/2} \psi_b$$  \hspace{1cm} (4.129)

$$A^a \equiv Z_3^{-1/2} A^a_b$$  \hspace{1cm} (4.130)

$$e \equiv Z_3^{1/2} e_b$$  \hspace{1cm} (4.131)

$$m = m_b + \delta m$$  \hspace{1cm} (4.132)

so that in particular $eA^a = e_b A^a_b$. In these terms, the action density is

$$L = L_0 + L_1 + L_2$$  \hspace{1cm} (4.133)
where $L_0$ is the physical or naive free action density

$$L_0 = -\frac{1}{4} F^{ab} F_{ab} - \bar{\psi} (\gamma_a \partial^a + m) \psi,$$

(4.134)

$L_1$ is the interaction

$$L_1 = -ie A_a \bar{\psi} \gamma^a \psi,$$

(4.135)

and $L_2$ is a sum of infinite “counterterms”

$$L_2 = -\frac{1}{4} (Z_3 - 1) F^{ab} F_{ab} - (Z_2 - 1) \bar{\psi} [\gamma_a \partial^a + m] \psi$$

$$+ Z_2 \delta m \bar{\psi} \psi - ie (Z_2 - 1) A_a \bar{\psi} \gamma^a \psi.$$

(4.136)

Vacuum polarization is a one-loop correction to the photon propagator. Its Feynman diagram is in figure 4.3. Let’s use $L_1$ to compute it. The amplitude $S$ is

$$S = \langle q', s' | T \left[ e^{i \int L_1 d^4 x} \right] | q, s \rangle = \langle q, s | T \left[ e^{i \int A_a(x) \bar{\psi}(x) \gamma^a \psi(x) d^4 x} \right] | q, s \rangle.$$

(4.137)

The one-loop correction is due to the order-$e^2$ term

$$S_2 = \frac{1}{2} \langle q', s' | T \left[ e^{2 \int A_a(x) \bar{\psi}(x) \gamma^a \psi(x) d^4 x} \int A_b(y) \bar{\psi}(y) \gamma^b \psi(y) d^4 y \right] | q, s \rangle.$$

(4.138)

We choose to absorb the photon at $y$ and cancel the factor of one-half. The electromagnetic field is

$$A_b(y) = \int \frac{d^4 p}{\sqrt{(2\pi)^3 2p^0}} \sum_{s'} \left[ e^{ip \cdot y} e_b(p, s') a(p, s') + e^{-ip \cdot y} e_b^*(p, s') a^\dagger(p, s') \right].$$

(4.139)
So $S_2$ is
\[
S_2 = \frac{e^2}{(2\pi)^6 2q^0} \langle 0 | T \left[ \int e^{-i\mathbf{q} \cdot \mathbf{x}} e_a^*(\mathbf{q}, \mathbf{s}') \tilde{\psi}(\mathbf{x}) \gamma^a \psi(\mathbf{x}) \, d^4x \right. \\
\left. \times \int e^{i\mathbf{q} \cdot \mathbf{y}} e_b(q, s) \tilde{\psi}(\mathbf{y}) \gamma^b \psi(\mathbf{y}) \, d^4y \right] | 0 \rangle.
\] (4.140)

The term $L_1$ is normally ordered (it should be surrounded by colons), so fermions emitted by $\tilde{\psi}(\mathbf{y})$ and $\psi(\mathbf{y})$ must be absorbed by $\psi(\mathbf{x})$ and $\tilde{\psi}(\mathbf{x})$. Thus $S_2$ is
\[
S_2 = \frac{e^2 e_a^*(\mathbf{q}, \mathbf{s}') e_b(q, s)}{(2\pi)^6 2q^0} \langle 0 | T [\psi_\alpha(x) \gamma_\alpha \psi_\beta(y)]| 0 \rangle \\
\times \langle 0 | T [\psi_\beta(x) \tilde{\psi}_\gamma(y) \gamma_\gamma \psi_\delta(y)]| 0 \rangle \, d^4x \, d^4y.
\] (4.141)

Moving $\psi_\delta(y)$ to the left of $\tilde{\psi}_\alpha(x)$ adds a minus sign, so
\[
S_2 = -\frac{e^2 e_a^*(\mathbf{q}, \mathbf{s}') e_b(q, s)}{(2\pi)^6 2q^0} \langle 0 | T [\psi_\beta(x) \tilde{\psi}_\gamma(y) \gamma_\gamma \psi_\delta(y)]| 0 \rangle \\
\times \langle 0 | T [\psi_\beta(x) \psi_\alpha(x) \gamma_\alpha]| 0 \rangle \, d^4x \, d^4y.
\] (4.142)

This minus sign is an instance of Feynman’s rule that closed fermion loops make minus signs. In the common notation $\bar{\psi} = \psi^\dagger \beta$, his propagator is
\[
\langle 0 | T \{ \psi_\mathbf{k}(x) \tilde{\psi}_\mathbf{m}(y) \} | 0 \rangle \equiv -i \Delta_\mathbf{m}(x - y) \beta \\
= -i \int \frac{d^4q}{(2\pi)^4} \left( -i \mathbf{q} + m \right)_\mathbf{m} e^{i\mathbf{q} \cdot (x - y)} \\
= -i \int \frac{d^4q}{(2\pi)^4} \left( \frac{1}{-i \mathbf{q} + m - i\epsilon} \right)_\mathbf{m} e^{i\mathbf{q} \cdot (x - y)}.
\] (4.143)

Putting all this together, we get
\[
S_2 = \frac{\epsilon^2 e_a^*(\mathbf{q}, \mathbf{s}') e_b(q, s)}{(2\pi)^6 2q^0} \int e^{-i\mathbf{q} \cdot \mathbf{x}} e^{i\mathbf{q} \cdot \mathbf{y}} \int \frac{d^4k}{(2\pi)^4} \left( -i \mathbf{k} + m \right)_\mathbf{k} e^{i\mathbf{k} \cdot (y - x)} \gamma_\alpha \\
\times \int \frac{d^4p}{(2\pi)^4} \left( -i \mathbf{p} + m \right)_\mathbf{p} e^{i\mathbf{p} \cdot (x - y)} \gamma_\gamma \gamma_\delta \, d^4x \, d^4y.
\] (4.144)

Dirac’s delta functions simplify this 16-dimensional integral to
\[
S_2 = \frac{\epsilon^2 e_a^*(\mathbf{q}, \mathbf{s}') e_b(q, s) \delta^4(\mathbf{q} - \mathbf{q}' - \mathbf{q})}{(2\pi)^6 2q^0} \int \frac{d^4p}{(2\pi)^4} \left[ -i (\mathbf{p} - \mathbf{q} + m) \right]_\mathbf{p} \\
\times \gamma_\alpha \gamma_\beta \delta_\mathbf{p} \left( \frac{-i \mathbf{p} + m}{p^2 + m^2 - i\epsilon} \right) e^{i\mathbf{p} \cdot (x - y)} \gamma_\gamma \gamma_\delta.
\] (4.145)
The product of matrices is a trace

\[ S_2 = \frac{e^2 e_A^*(q', s') e_b(q, s) \delta^4(q' - q)}{(2\pi)^3 2q^0} \int d^4p \frac{\text{Tr}\{[−i(p - q) + m] \gamma^a(−i\psi + m) \gamma^b\}}{[(p - q)^2 + m^2 - i\epsilon](p^2 + m^2 - i\epsilon)}. \]  

(4.146)

 equivalently, since \( \text{Tr}AB = \text{Tr}BA \), we have

\[ S_2 = \frac{e^2 e_A^*(q', s') e_b(q, s) \delta^4(q' - q)}{(2\pi)^3 2q^0} \int d^4p \frac{\text{Tr}\{[−i\psi + m] \gamma^b[−i(p - q) + m] \gamma^a\}}{(p^2 + m^2 - i\epsilon)(p^2 + m^2 - i\epsilon)}. \]  

(4.147)

or interchanging \( a \) and \( b \)

\[ S_2 = \frac{e^2 e_a(q, s) e_b^*(q', s') \delta^4(q' - q)}{(2\pi)^3 2q^0} \int d^4p \frac{\text{Tr}\{[−i\psi + m] \gamma^a[−i(p - q) + m] \gamma^b\}}{(p^2 + m^2 - i\epsilon)(p^2 + m^2 - i\epsilon)}. \]  

(4.148)

SW extracts from this the definition

\[ \Pi^{\alpha\beta}(q) = -\frac{i e^2}{(2\pi)^4} \int d^4p \frac{\text{Tr}\{[−i\psi + m] \gamma^\alpha[−i(p - q) + m] \gamma^\beta\}}{(p^2 + m^2 - i\epsilon)(p^2 + m^2 - i\epsilon)}. \]  

(4.149)

We use the Feynman trick

\[ \frac{1}{AB} = \int_0^1 \frac{dx}{[(1 - x)A + xB]^2}. \]  

(4.150)

to write

\[ \frac{1}{(p^2 + m^2 - i\epsilon)(p^2 - q^2 + m^2 - i\epsilon)} = \int_0^1 \frac{dx}{\left[(p^2 + m^2 - i\epsilon)(1 - x) + [(p^2 - q^2 + m^2 - i\epsilon)x]^2\right]^2} \]

\[ = \int_0^1 \frac{dx}{\left[p^2 + m^2 - i\epsilon - 2p \cdot q x + q^2 x^2\right]^2}. \]  

(4.151)

The next step is to make the integral finite somehow. One can use the counterterms, one can regard quantum field theory as merely a way to interpolate between values taken from experiment, or one can impose dimensional regularization as we’ll see momentarily. If we assume that we have made the integral finite one way or another, then we can shift \( p \) to \( p + xq \).
After this shift, the quantity (4.152) becomes

\[
\Pi^{ab}(q) = \frac{-ie^2}{(2\pi)^4} \int_0^1 dx \int d^4p \frac{\text{Tr}\{(-i(p + xq) + m)\gamma^a[-i(p - q(1 - x)) + m]\gamma^b\}}{[p^2 + m^2 - i\epsilon + x(1 - x)q^2]^2}.
\]

(4.152)

The trace identities of section 4.8 give

\[
\text{Tr}\{(-i(p + xq) + m)\gamma^a[-i(p - q(1 - x)) + m]\gamma^b\} = 4\left[ - (p + xq)^a(p - (1 - x)q)^b + (p + xq) \cdot (p - (1 - x)q) \eta^{ab} \right.
\]

\[
- (p + xq)^b(p - (1 - x)q)^a + m^2q^{ab}\right].
\]

(4.153)

The variable \(p^0\) is being integrated along the real axis of the complex \(p^0\) plane from \(-\infty\) to \(+\infty\). We now imagine that we have made the \(p\)-integral sufficiently finite that we can rotate the \(p^0\) contour by 90 degrees so that it runs up the imaginary axis in the complex \(p^0\) plane from \(p^0 = -i\infty\) to \(p^0 = i\infty\). This is a Wick rotation. We set \(p^0 = ip^4\) and integrate over \(p^4 = -ip^0\), which is real, from \(-\infty\) to \(+\infty\). We also set \(q^0 = iq^4\) and \(q^4 = -iq^0\). So now

\[
p \cdot q = \vec{p} \cdot \vec{q} - p^0q^0 = \bar{p} \cdot \bar{q} + p^4q^4.
\]

(4.154)

All the Lorentz signs are now positive and

\[
p^0p^0\eta^{00} = ip^4ip^4(-1) = p^4p^4 = p^4p^4\delta^{44}
\]

(4.155)

as well as

\[
q^0q^0\eta^{00} = iq^4iq^4(-1) = q^4q^4 = q^4q^4\delta^{44}.
\]

(4.156)

The quantity \(q^2\) is real whether \(q^0\) is real or imaginary; it is positive when \(q^0 < q^2\) whether or not \(q^0\) is imaginary.

We now can drop \(i\epsilon\) and express the quantity

\[
\Pi^{ab}(q) = \Pi^{ab}(\vec{q}, q^0) = \Pi^{ab}(\vec{q}, iq^4)
\]

(4.157)

either as

\[
\Pi^{ab}(q) = \frac{-4ie^2}{(2\pi)^4} \int_0^1 dx \int d^4p \frac{1}{[p^2 + m^2 - i\epsilon + x(1 - x)q^2]^2} \left[ - (p + xq)^a(p - (1 - x)q)^b + (p + xq) \cdot (p - (1 - x)q) \eta^{ab} \right.
\]

\[
- (p + xq)^b(p - (1 - x)q)^a + m^2q^{ab}\right].
\]

(4.158)
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or with \( p^4 = -ip^0 \) real and \( d^4p_e = dp^1dp^2dp^3dp^4 \) as

\[
\Pi^{ab}(q) = \frac{4e^2}{(2\pi)^4} \int_0^1 dx \int d^4p_e \frac{1}{[p^2 + m^2 + x(1-x)q^2]^2} \\
\times \left[ - (p + xq)^a(p - (1-x)q)^b + (p + xq) \cdot (p - (1-x)q) \delta^{ab} \\
- (p + xq)^b(p - (1-x)q)^a + m^2 \delta^{ab} \right] \tag{4.159}
\]

in which all Lorentz signs are +1 and \( q^4 = -iq^0 \) is imaginary if \( q \) is real.

The next step is to say how we regularize the integral over \( d^4p_e \). Since the 1970s, the method of choice has been \textit{dimensional regularization} because it preserves Lorentz and gauge invariance. In dimensional regularization, the \( p \) integration is taken to be over an arbitrary number \( d \) of spacetime dimensions.

The denominator of the integral \((4.159)\) is a function of \( p^2 \) which is taken to be the square of the radius of a sphere in \( d \) dimensions. Terms in the numerator that are odd powers of \( p^a \) vanish. We set

\[
p^a p^b = \frac{p^2 \delta^{ab}}{d}. \tag{4.160}
\]

The area of a sphere of unit radius in \( d \) dimensions is

\[
A_d = \frac{2\pi^{d/2}}{\Gamma(d/2)} \tag{4.161}
\]

as we can verify by computing an integral of the gaussian \( \exp(-x^2) \) in \( d \) dimensions using both rectangular and spherical coordinates.

\[
\int e^{-x^2} d^d x = \left( \int_{-\infty}^{\infty} e^{-x^2} dx \right)^d = \pi^{d/2} \\
= A_d \int_0^{\infty} r^{d-1} e^{-r^2} dr = A_d \frac{1}{2} \Gamma(d/2). \tag{4.162}
\]

The integral \((4.159)\) converges as long as \( d \) is complex or a fraction or anything but an even integer. Nonetheless, the divergencies in that integral reappear as powers of \( 1/(d-4) \) as \( d \to 4 \). We’ll use the counterterms \((4.136)\) in \( L_2 \) to cancel them.

The area of a sphere of radius \( k = \sqrt{p^2} \) in \( d \) dimensions is \( A_d k^{d-1} \). So using the formulas \((4.160)\) and \((4.161)\) of dimensional regularization, we can...
write the integral (4.159) as
\[ \Pi^{ab}(q) = \frac{4e^2 A_d}{(2\pi)^4} \int_0^1 dx \int_0^\infty dk \frac{1}{k^2 + m^2 + x(1-x)q^2} \]
\[ \times \left[ -2 \frac{k^2 \delta^{ab}}{d} + 2q^a q^b x(1-x) + [k^2 - x(1-x)q^2] \delta^{ab} + m^2 \delta^{ab} \right]. \]

The needed integrals are
\[ \int_0^\infty \frac{k^{d-1}}{(k^2 + c^2)^2} dk = \frac{1}{2} (c^2)^{d/2-2} \Gamma(d/2) \Gamma(2-d/2) \] (4.164)
\[ \int_0^\infty \frac{k^{d+1}}{(k^2 + c^2)^2} dk = \frac{1}{2} (c^2)^{d/2-1} \Gamma(1+d/2) \Gamma(1-d/2). \] (4.165)

Using these integrals, we get
\[ \Pi^{ab}(q) = \frac{2e^2 A_d}{(2\pi)^4} \int_0^1 dx \left[ \left( 1 - \frac{2}{d} \right) \delta^{ab}(c^2)^{d/2-1} \Gamma(1+d/2) \Gamma(1-d/2) \right. \]
\[ \left. + \left[ (2q^a q^b - q^2 \delta^{ab}) x(1-x) + m^2 \delta^{ab} \right] (c^2)^{d/2-2} \Gamma(d/2) \Gamma(2-d/2) \right] \]
in which
\[ c^2 = m^2 + x(1-x)q^2. \] (4.167)

The identity
\[ \Gamma(z+1) = z \Gamma(z) \] (4.168)
implies that
\[ \Gamma(2-d/2) = (1-d/2) \Gamma(1-d/2), \] (4.169)
and that
\[ \Gamma(1+d/2) = (d/2) \Gamma(d/2), \] (4.170)
so
\[ \Gamma(1+d/2) \Gamma(1-d/2) = (2/d-1)^{-1} \Gamma(d/2) \Gamma(2-d/2). \] (4.171)

So we can combine terms in the integral (4.172)
\[ \Pi^{ab}(q) = \frac{2e^2 A_d}{(2\pi)^4} \Gamma(d/2) \Gamma(2-d/2) \int_0^1 dx \left[ -\delta^{ab} \right. \]
\[ \left. + \left[ (2q^a q^b - q^2 \delta^{ab}) x(1-x) + m^2 \delta^{ab} \right] (c^2)^{d/2-2}. \right] \]
Using the value \(4.167\) of \(c^2\), we have
\[
-\delta^{ab}c^2 + (2q^aq^b - q^2\delta^{ab})x(1-x) + m^2\delta^{ab} = 2(q^aq^b - q^2\delta^{ab})x(1-x). \tag{4.173}
\]
Thus we find
\[
\Pi^\ast_{ab}(q) = \frac{4e^2A_d}{(2\pi)^4}\Gamma(d/2)\Gamma(2 - d/2)(q^aq^b - q^2\delta^{ab})
\times \int_0^1 \! dx(x(1-x)[m^2 + x(1-x)q^2]^{d/2-2}.
\tag{4.174}
\]
Current conservation and gauge invariance imply that
\[
q_a \Pi^\ast_{ab}(q) = 0. \tag{4.175}
\]
One of the advantages of dimensional regularization is that it preserves the vanishing of this quantity
\[
q_a \Pi^\ast_{ab}(q) \propto q_a(q^aq^b - q^2\delta^{ab}) = q^2q^b - q^2q^b = 0. \tag{4.176}
\]
This works because current conservation and the dimension of spacetime have nothing to do with each other:

The vanishing of \(q_a \Pi^\ast_{ab}(q)\) follows from the conservation
\[
\partial_a J^a(x) = \partial_a[\psi(x)\gamma^a\psi(x)] = 0 \tag{4.177}
\]
of the current
\[
J^a(x) = \psi(x)\gamma^a\psi(x) \tag{4.178}
\]
and from the neutrality of the current which is due to the fact that \(\psi\) lowers the charge while \(\bar{\psi}\) raises it.

Apart from a constant \(f\), the quantity \(\Pi^\ast_{ab}(q)\) is by \(4.142\)
\[
\Pi^\ast_{ab}(q) = f \int d^4xd^4ye^{iq(y-x)}\langle0|T[J^a(x),J^b(y)]|0\rangle. \tag{4.179}
\]
So
\[
q_a \Pi^\ast_{ab}(q) = f \int d^4xd^4y q_a e^{iq(y-x)}\langle0|T[J^a(x),J^b(y)]|0\rangle
\]
\[
= f \int d^4xd^4y e^{iq(y-x)}(i\partial_a) \langle0|T[J^a(x),J^b(y)]|0\rangle
\]
\[
= f \int d^4xd^4y e^{iq(y-x)}\langle0|T[i\partial_a J^a(x),J^b(y)]|0\rangle
\tag{4.180}
\]
\[
+ f \int d^4xd^4y i \delta(x^0 - y^0) e^{iq(y-x)}\langle0|[J^0(x),J^b(y)]|0\rangle
\]
\[
= f \int d^4xd^4y i \delta(x^0 - y^0) e^{iq(y-x)}\langle0|[J^0(x),J^b(y)]|0\rangle
\]
since the current is conserved. But the equal-time commutator of the charge $J^0(x)$ with any operator is proportional to the charge of that operator because $J^0(x)$ generates the $U(1)$ rotation of the symmetry that conserves charge. So since currents are neutral, the commutator vanishes

$$
\delta(x^0-y^0) e^{iq(y-x)} [J^0(x), J^b(y)] = 0,
$$

and we have

$$
q_a \Pi^{*ab}(q) = 0.
$$

The gamma function $\Gamma(2-d/2)$ in $\Pi^{*ab}(q)$ (4.174) blows up as $d \to 4$. So we bring in the contribution to $\Pi^{*ab}(q)$ arising from the counterterm $-(Z_3-1)F_{ab}F^{ab}$. This quadratic form in derivatives of $A_a$ and $A_b$ adds to $\Pi^{*ab}(q)$ the term

$$
\Pi^{*ab}(q) Z_3 = -(Z_3-1)(q^2\delta^{ab} - q^a q^b)
$$

in euclidean space. So to order $e^2$, $\Pi^{*ab}(q)$ is

$$
\Pi^{*ab}(q) = (q^2\delta^{ab} - q^a q^b) \pi(q^2)
$$

where

$$
\pi(q) = -\frac{4e^2 A_d}{(2\pi)^4} \Gamma(d/2)\Gamma(2-d/2) \int_0^1 dx x(1-x)[m^2 + x(1-x)q^2]^{d/2-2}
$$

$$
- (Z_3 - 1).
$$

So we can use $Z_3$ to cancel the poles in the gamma functions.

The quantity $\Pi^{*ab}(q)$ is a correction to the photon propagator. And $\Pi^{*ab}(0)$ is a correction to the mass of the photon which we want to remain at zero. So we set

$$
\Pi^{*ab}(0) = 0.
$$

This means that set set

$$
Z_3 = 1 - \frac{4e^2 A_d}{(2\pi)^4} \Gamma(d/2)\Gamma(2-d/2) (m^2)^{d/2-2} \int_0^1 dx x(1-x),
$$

so to order $e^2$

$$
\pi(q) = -\frac{4e^2 A_d}{(2\pi)^4} \Gamma(d/2)\Gamma(2-d/2) \int_0^1 dx x(1-x)
$$

$$
\times [m^2 + x(1-x)q^2]^{d/2-2} - (m^2)^{d/2-2}.
$$
As we let $d \to 4$, the tricky term is

$$[m^2 + x(1 - x)q^2]^{d/2-2} - (m^2)^{d/2-2} = \exp \left[ (d/2 - 2) \ln[m^2 + x(1 - x)q^2] \right]$$
$$- \exp \left[ (d/2 - 2) \ln(m^2) \right]$$
$$= (d/2 - 2) \ln[1 + x(1 - x)q^2/m^2].$$

(4.189)

The final result is

$$\pi(q) = \frac{\epsilon^2}{2\pi^2} \int_0^1 x(1 - x) \ln\left[1 + x(1 - x)q^2/m^2\right] dx.$$

(4.190)
5

Nonabelian gauge theory

5.1 Yang and Mills invent local nonabelian symmetry

The action of a Yang-Mills theory is unchanged when a spacetime-dependent unitary matrix $U(x) = \exp(-it_a \theta^a(x))$ maps a vector $\psi(x)$ of matter fields to $\psi_i'(x) = U_{ij}(x)\psi_j(x)$. The symmetry $\psi_i'(x)U^\dagger(x)U(x)\psi(x) = \psi_i(x)\psi(x)$ is obvious, but how can kinetic terms like $\partial_i\psi^\dagger \partial_i\psi$ be made invariant?

Yang and Mills introduced matrices $A_i = t_a A_i^a$ of gauge fields in which the hermitian matrices $t_a$ are the generators of the Lie algebra of a compact gauge group and so obey the commutation relation

$$[t_a, t_b] = i f_{abc} t_c \tag{5.1}$$

in which the $f_{abc}$ are real, totally antisymmetric structure constants. More importantly, they replaced ordinary derivatives $\partial_i$ by covariant derivatives

$$D_{\alpha\beta} = \partial_i \delta_{\alpha\beta} + A_{i\alpha\beta} \equiv \partial_i \delta_{\alpha\beta} + t_{\alpha\beta} A_i^a \tag{5.2}$$

and required that covariant derivatives of fields transform like fields so that

$$(D\psi)' = UD\psi \quad \text{or} \quad D'_\psi = (\partial_i + A_i') U \psi = (\partial_i U + U \partial_i + A_i' U) \psi = U (\partial_i + A_i) \psi. \tag{5.3}$$

The nonabelian gauge field transforms as

$$A_i'(x) = U(x)A_i(x)U^\dagger(x) - (\partial_i U(x)) U^\dagger(x). \tag{5.4}$$

In full detail, this is

$$A_{i\alpha\beta}'(x) = U_{\alpha\beta}(x)A_{i\gamma\gamma}(x)(U^\dagger)_{\gamma\beta}(x) - (\partial_i U_{\alpha\beta}(x)) (U^\dagger)_{\beta\delta}(x). \tag{5.5}$$

The nonabelian Faraday tensor is

$$F_{ik} = [D_i, D_k] = \partial_i A_k - \partial_k A_i + [A_i, A_k] \tag{5.6}$$
in which both $F$ and $A$ are matrices in the Lie algebra. The nonabelian Faraday tensor transforms as

$$F'_{ik}(x) = U(x)F_{ik}U^{-1}(x) = U(x)[D_i, D_k]U^{-1}(x).$$

The trace $\text{Tr}[F_{ik}F^{ik}]$ of the Lorentz-invariant product $F_{ik}F^{ik}$ is both Lorentz invariant and gauge invariant. The nonabelian generalization of the abelian action density

$$L = -\frac{1}{4}F_{jk}F^{jk} - \bar{\psi}[\gamma^k(\partial_k + ieA_k) + m]\psi$$

is

$$L = -\frac{1}{4}\text{Tr}[F_{jk}F^{jk}] - \bar{\psi}[\gamma^k(\partial_k + ieA_k) + m]\psi.$$ 

In more detail, the Fermi action is

$$\bar{\psi}[\gamma^k(\partial_k + ieA_k) + m]\psi = \bar{\psi}_{\ell\alpha}\left[\gamma^k(\partial_k\delta_{\alpha\beta} + ie\epsilon_{\alpha\beta}\lambda^k) + m\delta_{\ell\ell'}\delta_{\alpha\beta}\right]\psi_{\ell'}\beta.$$ 

Use of matrix notation and of summation conventions is necessary.

### 5.2 $SU(3)$

The gauge group of quantum chromodynamics is $SU(3)$ which has eight generators. The Gell-Mann matrices are

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ i & 0 & 0 \end{pmatrix}, \quad \text{and} \quad \lambda_8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$ 

The generators $t_a$ of the $3 \times 3$ defining representation of $SU(3)$ are these Gell-Mann matrices divided by 2

$$t_a = \lambda_a/2$$

(Murray Gell-Mann, 1929–).
The eight generators $t_a$ are orthogonal with $k = 1/2$

$$\text{Tr} \left( t_a t_b \right) = \frac{1}{2} \delta_{ab}$$

and satisfy the commutation relation

$$[t_a, t_b] = i f_{abc} t_c.$$  \hspace{1cm} (5.14)

The trace formula

$$f^c_{ab} = -\frac{i}{k} \text{Tr} \left( [t_a, t_b] t_c^\dagger \right) .$$ \hspace{1cm} (5.15)

gives us the $SU(3)$ structure constants as

$$f_{abc} = -2i \text{Tr} ([t_a, t_b] t_c) .$$ \hspace{1cm} (5.16)

They are real and totally antisymmetric with $f_{123} = 1$, $f_{458} = f_{678} = \sqrt{3}/2$, and $f_{147} = -f_{156} = f_{246} = f_{257} = f_{345} = -f_{367} = 1/2$.

While no two generators of $SU(2)$ commute, two generators of $SU(3)$ do. In the representation (5.11,5.12), $t_3$ and $t_8$ are diagonal and so commute

$$[t_3, t_8] = 0.$$ \hspace{1cm} (5.17)

They generate the Cartan subalgebra of $SU(3)$. The generators defined by Eqs.(5.12 & 5.11) give us the $3 \times 3$ representation

$$D(\alpha) = \exp \left( i \alpha_a t_a \right)$$ \hspace{1cm} (5.18)

in which the sum $a = 1, 2, \ldots 8$ is over the eight generators $t_a$. This representation acts on complex 3-vectors and is called the 3.

Note that if

$$D(\alpha_1) D(\alpha_2) = D(\alpha_3)$$ \hspace{1cm} (5.19)

then the complex conjugates of these matrices obey the same multiplication rule

$$D^*(\alpha_1) D^*(\alpha_2) = D^*(\alpha_3)$$ \hspace{1cm} (5.20)

and so form another representation of $SU(3)$. It turns out that (unlike in $SU(2)$) this representation is inequivalent to the 3; it is the $\overline{3}$.

There are three quarks with masses less than about 100 MeV/c$^2$—the u, d, and s quarks. The other three quarks c, b, and t are more massive; $m_c = 1.28$ GeV, $m_b = 4.18$ GeV, and $m_t = 173.1$ GeV. Nobody knows why. Gell-Mann and Zweig suggested that the low-energy strong interactions were approximately invariant under unitary transformations of the three light quarks, which they represented by a 3, and of the three light antiquarks, which they represented by a $\overline{3}$. They imagined that the eight
light pseudoscalar mesons, that is, the three pions $\pi^-, \pi^0, \pi^+$, the neutral $\eta$, and the four kaons $K^0, K^+, K^- \bar{K}^0$, were composed of a quark and an antiquark. So they should transform as the tensor product
\[ 3 \otimes 3 = 8 \oplus 1. \quad (5.21) \]

They put the eight pseudoscalar mesons into an $8$.

They imagined that the eight light baryons — the two nucleons $N$ and $P$, the three sigmas $\Sigma^-, \Sigma^0, \Sigma^+$, the neutral lambda $\Lambda$, and the two cascades $\Xi^-$ and $\Xi^0$ were each made of three quarks. They should transform as the tensor product
\[ 3 \otimes 3 \otimes 3 = 10 \oplus 8 \oplus 8 \oplus 1. \quad (5.22) \]

They put the eight light baryons into one of these $8$’s. When they were writing these papers, there were nine spin-3/2 resonances with masses somewhat heavier than 1200 MeV/c$^2$ — four $\Delta$’s, three $\Sigma^*$’s, and two $\Xi^*$’s. They put these into the $10$ and predicted the tenth and its mass. In 1964, a tenth spin-3/2 resonance, the $\Omega^-$, was found with a mass close to their prediction of 1680 MeV/c$^2$, and by 1973 an MIT-SLAC team had discovered quarks inside protons and neutrons. (George Zweig, 1937–)

For a given quark, say the up quark, the action of quantum chromodynamics is
\[ L = -\frac{1}{4} \text{Tr} \left[ F_{jk} \bar{F}^{jk} \right] - \bar{\psi} \gamma^k \left( \partial_k + ieA_k \right) \psi \quad (5.23) \]

in which the index $c$ on $\psi_c$ takes on the values 1, 2, 3 which we may think of as red, green, and blue. The Faraday tensor is a $3 \times 3$ matrix
\[ F_{ik} = \left[ D_i, D_k \right] = \partial_i A_k - \partial_k A_i + [A_i, A_k]. \quad (5.24) \]

The trace term in the QCD action (5.23) is
\[ \text{Tr} \left[ F_{jk} \bar{F}^{jk} \right] = \text{Tr} \left[ \left( \partial_i A_k - \partial_k A_i + [A_i, A_k] \right) \left( \partial^i A^k - \partial^k A^i + [A^i, A^k] \right) \right] \quad (5.25) \]

Quarks and gluons are confined in color-singlet particles. This effect is robust and mysterious.
6.1 Quantum chromodynamics

If to a pure SU(3) gauge theory we add massless quarks in the fundamental or defining representation, then we get a theory of the strong interactions called quantum chromodynamics or QCD. Thus, let \( \psi \) be a complex 3-vector of Dirac fields

\[
\psi = \begin{pmatrix} \psi_r \\ \psi_g \\ \psi_b \end{pmatrix}
\] (6.1)

(so 12 fields in all). This complex 12-vector could represent \( u \) or “up” quarks.

In terms of the covariant derivative

\[
D_\mu = i \partial_\mu + ig \sum_{b=1}^{8} t^b A^b_\mu(x),
\] (6.2)

the action density is

\[
L = \frac{1}{2g^2} \text{Tr} [F_{\mu\nu} F^{\mu\nu}] - \bar{\psi} (\gamma^\mu D_\mu + m) \psi
\] (6.3)

where

\[
F_{\mu\nu} = [D_\mu, D_\nu].
\] (6.4)

Nonperturbative effects are supposed to “confine” the quarks and massless gluons. There are 6 known “flavors” of quarks—\( u, d, c, s, t, b \).

6.2 Abelian Higgs mechanism

Suppose \( \phi = (\phi_1 + i\phi_2)/\sqrt{2} \) is a complex scalar field with action density

\[
L = -\partial_\mu \phi^* \partial^\mu \phi + \mu^2 \phi^* \phi - \lambda (\phi^* \phi)^2.
\] (6.5)
The minimum of the potential $V = -\mu^2 \phi^* \phi + \lambda (\phi^* \phi)^2$ is given by the equation

$$0 = \frac{\partial V}{\partial |\phi|^2} = -\mu^2 + 2\lambda |\phi|^2.$$ (6.6)

It is a circle with

$$|\phi|^2 = \frac{\mu^2}{2\lambda} = \frac{v^2}{2}.$$ (6.7)

As is customary, we pick one asymmetrical minimum at $\phi = \mu/\sqrt{2\lambda}$, so the mean values in the vacuum are

$$\langle 0 | \phi_1 | 0 \rangle = v = \sqrt{\frac{\mu^2}{\lambda}} \quad \text{and} \quad \langle 0 | \phi_2 | 0 \rangle = 0.$$ (6.8)

The $U(1)$ symmetry has been spontaneously broken. So

$$\phi = \frac{v}{\sqrt{2}} + \frac{\phi_1}{\sqrt{2}} + i \frac{\phi_2}{\sqrt{2}}$$ (6.9)

where now $\phi_1$ is the departure of the real part of the field from its mean value $v$. So the potential $V$ is

$$V = -\mu^2 \phi^* \phi + \lambda (\phi^* \phi)^2$$
$$= -\frac{1}{2} \mu^2 \left[ (v + \phi_1)^2 + (\phi_2)^2 \right] + \frac{\lambda}{4} \left[ (v + \phi_1)^2 + (\phi_2)^2 \right]^2$$ (6.10)

where the dots denote cubic and quartic in $\phi_1$ and $\phi_2$. The quadratic part of the action determines the properties of the particles of the physical fields. The cubic and higher-order terms tell us how the particles interact with each other.

The first part of the last homework problem is to find the masses of the particles of this theory.

Scalar particles of zero mass that arise from spontaneous symmetry breaking are called Goldstone bosons.

Now we give the theory a local $U(1)$ symmetry by adding an abelian gauge field to the action density

$$L = -(D_a \phi)^* D^a \phi + \mu^2 \phi^* \phi - \lambda (\phi^* \phi)^2 - \frac{1}{4} F_{ab} F^{ab}$$ (6.11)

in which

$$D_a = \partial_a + ie A_a \quad \text{and} \quad F_{ab} = [D_a, D_b].$$ (6.12)
Once again the field assumes a mean value in the vacuum, let us say (6.9). But we write the field \( \phi \) as

\[
\phi = \frac{1}{\sqrt{2}} (v + \phi_1 + i \phi_2).
\]

(6.13)

The term \(- (D_a \phi)^* D^a \phi \) then is

\[
-(D_a \phi)^* D^a \phi = - (\partial_a \phi^* - ieA_a \phi^*) (\partial^a \phi + ieA^a \phi)
\]

\[
= - \frac{1}{2} (\partial_a \phi_1 - i \partial_a \phi_2 - ieA_a (v + \phi_1 - i \phi_2)) \times [(\partial^a \phi_1 + i \partial^a \phi_2 + ieA^a (v + \phi_1 + i \phi_2)]
\]

\[
= - \frac{1}{2} (\partial^a \phi_1 - eA^a \phi_2) (\partial_a \phi_1 - eA_a \phi_2)
\]

\[
- \frac{1}{2} (\partial^a \phi_2 + eA^a (v + \phi_1)) (\partial_a \phi_2 + eA_a (v + \phi_1)).
\]

(6.14)

So we set

\[
B_a = A_a + \frac{1}{ev} \partial_a \phi_2 \quad \text{or} \quad A_a = B_a - \frac{1}{ev} \partial_a \phi_2
\]

(6.15)

and find that

\[
-(D_a \phi)^* D^a \phi = - \frac{1}{2} (\partial^a \phi_1 - eB^a \phi_2 + \frac{\phi_2}{v} \partial^a \phi_2) (\partial_a \phi_1 - eB_a \phi_2 + \frac{\phi_2}{v} \partial_a \phi_2)
\]

\[
- \frac{1}{2} (evB^a + eB^a \phi_1 - \frac{\phi_1}{v} \partial^a \phi_2) (evB_a + eB_a \phi_1 - \frac{\phi_1}{v} \partial_a \phi_2)
\]

\[
- \frac{1}{2} (\partial^a \phi_1 \partial_a \phi_1 + e^2 v^2 B^a B_a) + \ldots
\]

(6.16)

in which the dots denote cubic and quartic terms.

The second part of the last homework problem is to find the masses of the particles of this theory.

### 6.3 Higgs’s mechanism

The local gauge group of the Glashow-Salam-Weinberg electroweak theory is \( SU(2)_L \times U(1) \). What’s weird is that the \( SU(2)_L \) symmetry applies only to the left-handed quarks and leptons and to the Higgs boson, a complex doublet (or 2-vector) of scalar fields \( H \).

Let’s leave out the fermions for the moment, and focus just on the Higgs and the gauge fields. The gauge transformation is

\[
H'(x) = U(x)H(x)
\]

\[
A'_\mu(x) = U(x)A_\mu(x)U^\dagger(x) + (\partial_\mu U(x))U^\dagger(x)
\]

(6.17)
in which the $2 \times 2$ unitary matrix $U(x)$ is

$$U(x) = \exp \left[ i g \frac{\sigma^a}{2} \alpha^a(x) + i g' \frac{Y}{2} \beta(x) \right]. \quad (6.18)$$

The generators here are the 3 Pauli matrices and the matrix $I/2$, where $I$ is the $2 \times 2$ identity matrix.

The action density of the theory (without the fermions) is

$$L = -(D_\mu H)^\dagger D^\mu H + \frac{1}{4k} \text{Tr}(F_{\mu\nu}F^{\mu\nu}) + m^2 |H|^2 - \lambda |H|^4 \quad (6.19)$$
in which the covariant derivative for the Higgs doublet is

$$D_\mu H = \left( I \partial_\mu + i g \frac{\sigma^a}{2} A_\mu^a + i g' \frac{Y I}{2} B_\mu \right) H. \quad (6.20)$$

The potential energy of the Higgs field is

$$V = -m^2 |H|^2 + \lambda |H|^4. \quad (6.21)$$

Its minimum is where

$$0 = \frac{\partial V}{\partial |H|^2} = 2\lambda |H|^2 - m^2. \quad (6.22)$$

So

$$|H| = \frac{m}{\sqrt{2\lambda}} = \frac{v}{\sqrt{2}}. \quad (6.23)$$

By making an $SU(2)_L \times U(1)$ gauge transformation, we can transform this mean value to

$$H_0 = \langle 0 | H(x) | 0 \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}. \quad (6.24)$$

This transformation is called “going to unitary gauge.”

In this gauge, the Higgs potential is

$$V(v) = -\frac{1}{2} m^2 v^2 + \frac{1}{4} \lambda v^4, \quad (6.25)$$

and its second derivative is

$$V''(v) = 3 \lambda v^2 - m^2 = 2m^2 = m_H^2. \quad (6.26)$$

The mass of the Higgs then is

$$m_H = \sqrt{2} m = \sqrt{2 \lambda} v. \quad (6.27)$$

Experiments at LEP2 and at the LHC have revealed value of $v$ to be

$$v = 246.22 \text{ GeV}. \quad (6.28)$$
In 2012, the Higgs boson was discovered at the LHC. Its mass has been measured to be

\[ m_H = 125.18 \pm 0.16 \text{ GeV}. \]  

(6.29)

The self coupling \( \lambda \) therefore is

\[ \lambda = \frac{m_H^2}{2v^2} = \frac{1}{2} \left( \frac{125.18}{246.22} \right)^2 = 0.12924. \]  

(6.30)

After spontaneous symmetry breaking, mass terms for the gauge bosons emerge from the kinetic action of the Higgs doublet \( -(D_\mu H)\dagger D^\mu H \). Since the generators of a compact group like \( SU(2)_L \times U(1) \) are hermitian, the part of the kinetic action that contains the mass terms is

\[ L_m = -H\dagger \left( -i g \frac{\sigma^a}{2} A^a_\mu - i g' Y I \frac{1}{2} B_\mu \right) \left( i g \frac{\sigma^a}{2} A^a_\mu + i g' Y I B_\mu \right) H. \]  

(6.31)

In unitary gauge \( \text{(6.24)} \), these mass terms are

\[ L_m = -\frac{1}{2} (0, v) \begin{pmatrix} g \frac{\sigma^a}{2} A^a_\mu + g' I B_\mu \\ g \frac{\sigma^a}{2} A^a_\mu + g' Y I B_\mu \end{pmatrix} \begin{pmatrix} 0 \\ v \end{pmatrix} \]

(6.32)

\[ = -\frac{1}{8} (0, v) \begin{pmatrix} gA_3^a + g' B_\mu \\ g(A_1^a + iA_2^a) \\ g(A_1^a - iA_2^a) \\ -gA_3^a + g' B_\mu \end{pmatrix} \begin{pmatrix} 0 \\ v \end{pmatrix} \]

\[ = -\frac{v^2}{8} \begin{pmatrix} g(A_1^a + iA_2^a), & -gA_3^a + g' B_\mu \end{pmatrix} \begin{pmatrix} g(A_1^a + iA_2^a) \\ -gA_3^a + g' B_\mu \end{pmatrix} \]

\[ = -\frac{v^2}{8} \left[ g^2 (A_1^aA_1^a + A_2^aA_2^a) + (-gA_3^a + g' B_\mu) (-gA_3^a + g' B_\mu) \right]. \]

The normalized complex, charged gauge bosons are

\[ W_\mu^\pm = \frac{1}{\sqrt{2}} (A^a_1 \mp iA^a_2) \]  

(6.33)

and the normalized neutral ones are

\[ Z_\mu = \frac{1}{\sqrt{g^2 + g'^2}} (gA_3^a - g' B_\mu) \]  

(6.34)

and the photon, which is the normalized linear combination orthogonal to \( Z_\mu \)

\[ A_\mu = \frac{1}{\sqrt{g^2 + g'^2}} (g' A_3^a + g B_\mu). \]  

(6.35)
Standard model

It remains massless. In terms of these properly normalized fields, the mass terms are

$$L_M = \frac{-g^2 v^2}{4} W^\mu_- W^{\mu+} - \frac{(g^2 + g'^2) v^2}{8} Z^\mu Z^\mu.$$  (6.36)

So the $W^+$ and the $W^-$ get the same mass

$$M_W = g \frac{v}{2} = 80.370 \pm 0.012 \text{ GeV}/c^2.$$  (6.37)

while the $Z$ (also called the $Z^0$) has mass

$$M_Z = \sqrt{g^2 + g'^2} \frac{v}{2} = 91.1876 \pm 0.0021 \text{ GeV}/c^2.$$  (6.38)

The accuracy and precision of the mass of the $Z$ boson is an order of magnitude higher than that of the $W$ boson because of measurements made in the Large Electron Collider.

It is the photon. The fine-structure constant is

$$\alpha = \frac{e^2}{4\pi \hbar c} = 1/137.035 \, 999 \, 139(31) \approx 1/137.$$  (6.39)

Why do three gauge bosons become massive? Because there are three Goldstone bosons corresponding to three ways of moving $\langle 0 | H | 0 \rangle$ without changing the Higgs potential. Why does one gauge boson stay massless? Because one linear combination of the generators of $SU_L(2) \otimes U(1)$ maps $\langle 0 | H | 0 \rangle$ to zero.

In terms of these mass eigenstates, the original gauge bosons are

$$A^1_\mu = \frac{1}{\sqrt{2}} (W^+ \mu + W^- \mu)$$
$$A^2_\mu = \frac{1}{i \sqrt{2}} (W^- \mu - W^+ \mu)$$
$$A^3_\mu = \frac{1}{\sqrt{g^2 + g'^2}} (g' A_\mu + g Z_\mu)$$
$$B_\mu = \frac{1}{\sqrt{g^2 + g'^2}} (g A_\mu - g' Z_\mu).$$  (6.40)

The $4 \times 4$ matrices

$$P_\ell = \frac{1}{2} (1 + \gamma^5) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad P_r = \frac{1}{2} (1 - \gamma^5) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$  (6.41)

project out the upper and lower two components of a 4-component Dirac field. The $SU(2)_\ell$ gauge fields $A^\mu_\ell$ interact only with the upper two components $\psi_\ell = P_\ell \psi$, while the $U(1)_Y$ gauge field $B_\mu$ interacts with all four components of a Dirac field.
Thus the covariant derivative for a fermion of coupling $g'$ to the $U(1)$ gauge field $B_\mu$ with hypercharge $Y$ and coupling $g$ to the $SU(2)_\ell$ gauge fields is

$$D_\mu = i \partial_\mu + i g \frac{\sigma^a}{2} A^a_\mu \gamma_\mu + i g' Y I B_\mu$$

$$= i \partial_\mu + i g \left[ \frac{\sigma_1}{2} \frac{1}{\sqrt{2}} (W^\mu_+ + W^-_\mu) + \frac{\sigma_2}{2} \frac{1}{i \sqrt{2}} (W^-_\mu - W^\mu_+) \right] + \frac{\sigma_3}{2} \frac{1}{\sqrt{g^2 + g'^2}} (g'A_\mu + gZ_\mu)$$

$$+ i g' Y \frac{I}{\sqrt{g^2 + g'^2}} (gA_\mu - g'Z_\mu).$$

(6.42)

The $SU(2)_\ell$ generators are

$$\bar{T} = \frac{1}{2} \sigma, \quad T^\pm = \frac{1}{2} (\sigma_1 \pm i \sigma_2) \quad \text{and} \quad T^3 = \frac{\sigma_3}{2}. \quad (6.43)$$

In terms of them, the covariant derivative is

$$D_\mu = i \partial_\mu + i g \sqrt{2} \left( W^\mu_+ T^+ + W^-_\mu T^- \right) P_\ell + i \frac{1}{\sqrt{g^2 + g'^2}} Z_\mu (g^2 T^3 P_\ell - g'^2 Y I)$$

$$+ i g g' \frac{1}{\sqrt{g^2 + g'^2}} A_\mu (T^3 P_\ell + Y I) \quad (6.44)$$

Equivalently, the left and right covariant derivatives are

$$D_{\mu \ell} = \left[ i \partial_\mu + i g \sqrt{2} \left( W^\mu_+ T^+ + W^-_\mu T^- \right) + i \frac{1}{\sqrt{g^2 + g'^2}} Z_\mu (g^2 T^3 - g'^2 Y I) \right] P_\ell$$

$$+ i \frac{g g'}{\sqrt{g^2 + g'^2}} A_\mu (T^3 + Y I) \quad (6.45)$$

and

$$D_{\mu r} = \left[ i \partial_\mu - i \frac{g^2 Y I}{\sqrt{g^2 + g'^2}} Z_\mu + i \frac{g g' Y I}{\sqrt{g^2 + g'^2}} A_\mu \right] P_\ell. \quad (6.46)$$

The interaction strength or coupling constant of the photon $A_\mu$ is

$$e = \frac{gg'}{\sqrt{g^2 + g'^2}} > 0. \quad (6.47)$$

The charge operator is

$$Q = T^3_\ell + Y I. \quad (6.48)$$

When acting on the doublet

$$E_\ell = \left( \frac{\nu_\ell}{e} \right) \quad (6.49)$$
to which we assign $Y = -1/2$, the charge $Q$ gives 0 as the charge of the neutrino and $-1$ as the charge of the electron. The photon-lepton term then is

$$
\frac{gg'}{\sqrt{g^2 + g'^2}} A_{\mu} (T^3 + Y I) E_{\ell} = e A_{\mu} (T^3 + Y I) E_{\ell}
= e A_{\mu} \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \nu_e \\ e \end{pmatrix} = -e A_{\mu} e
$$

(6.50)

in which the first $e$ is the absolute value of the charge (6.47) of the electron and the second is the field of the electron.

The **weak mixing angle** $\theta_w$ is defined by

$$
\begin{pmatrix} Z \\ A \end{pmatrix} = \begin{pmatrix} \cos \theta_w & -\sin \theta_w \\ \sin \theta_w & \cos \theta_w \end{pmatrix} \begin{pmatrix} A^3 \\ B \end{pmatrix},
$$

(6.51)

Equations (6.34 and 6.35) identify these trigonometric values as

$$
\cos \theta_w = \frac{g}{\sqrt{g^2 + g'^2}} \quad \text{and} \quad \sin \theta_w = \frac{g'}{\sqrt{g^2 + g'^2}}.
$$

(6.52)

Since the charge is $Q = T^3 + Y I$, the hypercharge $Y I = Q - T^3$, and so fields couple to the $Z$ as

$$
\frac{1}{\sqrt{g^2 + g'^2}} Z_{\mu} \left[ g^2 T^3 - g'^2 Y \right] = \frac{1}{\sqrt{g^2 + g'^2}} Z_{\mu} \left[ (g^2 + g'^2) T^3 - g'^2 Q \right]
$$

(6.53)

and to the photon as

$$
\frac{gg'}{\sqrt{g^2 + g'^2}} A_{\mu} (T^3 + Y) = \frac{gg'}{\sqrt{g^2 + g'^2}} A_{\mu} Q.
$$

(6.54)

We also have

$$
\frac{g^2 + g'^2}{\sqrt{g^2 + g'^2}} = \sqrt{g^2 + g'^2} = \frac{g}{\cos \theta_w}
$$

(6.55)

and

$$
\frac{g'^2}{\sqrt{g^2 + g'^2}} = \sqrt{g^2 + g'^2} \frac{g'^2}{g^2 + g'^2} = \frac{g}{\cos \theta_w} \sin^2 \theta_w.
$$

(6.56)

So the coupling to the $Z$ is

$$
\frac{1}{\sqrt{g^2 + g'^2}} Z_{\mu} \left[ (g^2 + g'^2) T^3 - g'^2 Q \right] = \frac{g}{\cos \theta_w} Z_{\mu} (T^3 - \sin^2 \theta_w Q)
$$

(6.57)

and, if we set

$$
e = g \sin \theta_w,
$$

(6.58)
then the coupling to the photon \(A\) is
\[
\frac{g g'}{\sqrt{g^2 + g'^2}} A_\mu Q = g \sin \theta_w A_\mu Q = e A_\mu Q.
\]
(6.59)

In these terms, the covariant derivative (6.44) is
\[
D_\mu = i \partial_\mu + i g \frac{1}{\sqrt{2}} (W_\mu^+ T^+ + W_\mu^- T^-) P_\ell + i \frac{g}{\sqrt{g^2 + g'^2}} Z_\mu (g^2 T^3 P_\ell - g'^2 Y)
\]
\[
+ i g' \frac{1}{\sqrt{g^2 + g'^2}} A_\mu (T^3 P_\ell + Y)
\]
\[
= i \partial_\mu + i \frac{g}{\sqrt{2}} (W_\mu^+ T^+ + W_\mu^- T^-) P_\ell + i \frac{g}{\cos \theta_w} Z_\mu (T^3 P_\ell - \sin^2 \theta_w Q)
\]
\[
+ i e A_\mu Q
\]
(6.60)
in which the generators \(T^\pm\) and \(T^3\) are those of the representation to which the fields they act on belong. When acting on left-handed fermions, they are half the Pauli matrices, \(T = \frac{1}{2} \sigma\). When acting on right-handed fermions, they are zero, \(T = 0\), and so the explicit appearance of \(P_\ell\) is unnecessary. Since \(g = e/\sin \theta_w\), the couplings involve one new parameter \(\theta_w\).

Our mass formulas (6.37 and 6.38) for the \(W\) and the \(Z\) show that their masses are related by
\[
M_W = g \frac{v}{2} = \frac{g}{\sqrt{g^2 + g'^2}} \sqrt{g^2 + g'^2} \frac{v}{2} = \cos \theta_w \sqrt{g^2 + g'^2} \frac{v}{2} = \cos \theta_w M_Z.
\]
(6.61)

Experiments have determined the masses and shown that at the mass of the \(Z\) and in the \(\overline{\text{MS}}\) convention
\[
\sin^2 \theta_w = 0.23122(4) \quad \text{or} \quad \theta_w = 0.50163
\]
(6.62)
and that
\[
v = 246.22 \text{ GeV}.
\]
(6.63)

### 6.4 Quark and lepton interactions

The right-handed fermions \(u_r, d_r, e_r, \) and \(\nu_{e,r}\) are singlets under \( SU_L(2) \otimes U_Y(1) \). So they have \(T^3 = 0\). The definition (6.48) of the charge \(Q\)
\[
Q = T^3 + Y I
\]
(6.64)
then implies that
\[
Y_r = Q_r.
\]
(6.65)
That is, $Y_{\nu_e,r} = 0$, $Y_{e,r} = -1$, $Y_{u,r} = 2/3$, and $Y_{d,r} = -1/3$.

The left-handed fermions are in doublets

$$E_\ell = \begin{pmatrix} \nu_e \\ e^- \end{pmatrix} \quad \text{and} \quad Q_\ell = \begin{pmatrix} u \\ d \end{pmatrix}$$

(6.66)

with $T^3 = \pm 1/2$. So the choices $Y_E = -1/2$ and $Y_Q = 1/6$ and the definition (6.48) of the charge $Q$ give the right charges:

$$QE_\ell = Q \begin{pmatrix} \nu_e \\ e^- \end{pmatrix} = \begin{pmatrix} 0 \\ -e^- \end{pmatrix} \quad \text{and} \quad QQ_\ell = Q \begin{pmatrix} u \\ d \end{pmatrix} = \begin{pmatrix} 2u/3 \\ -d/3 \end{pmatrix}.$$  

(6.67)

Fermion-gauge-boson interactions are due to the covariant derivative (6.60) acting on either the left- or right-handed fields. On right-handed fermions, the covariant derivative is just

$$D^r_\mu = I \partial_\mu + i \frac{g}{\cos \theta_w} Z_\mu \left(- \sin^2 \theta_w Q\right) + ie A_\mu Q$$

$$= I \partial_\mu - ig \frac{\sin^2 \theta_w}{\cos \theta_w} Z_\mu Q + ie A_\mu Q$$

$$= I \partial_\mu - ie \frac{\sin \theta_w}{\cos \theta_w} Z_\mu Q + ie A_\mu Q = I \partial_\mu - ie \tan \theta_w Z_\mu Q + ie A_\mu Q.$$  

(6.68)

So the covariant derivative of a neutral right-handed fermion is just the ordinary derivative.

On left-handed fermions, the covariant derivative is

$$D^\ell_\mu = I \partial_\mu + i \frac{g}{\sqrt{2}} \left(W^+_\mu T^+ + W^-_\mu T^-\right) + i \frac{g}{\cos \theta_w} Z_\mu \left(T^3 - \sin^2 \theta_w Q\right) + ie A_\mu Q$$

$$= I \partial_\mu + i \frac{e}{\sqrt{2} \sin \theta_w} \left(W^+_\mu T^+ + W^-_\mu T^-\right) + i \frac{e}{\cos \theta_w} Z_\mu \left(\frac{T^3}{\sin \theta_w} - \sin \theta_w Q\right)$$

$$+ ie A_\mu Q.$$  

(6.69)

For the first family or generation of quarks and leptons, the kinetic action density is

$$L_k = -\overline{E}_\ell \gamma^\ell E_\ell - \overline{E}_r \gamma^r E_r - \overline{Q}_\ell \gamma^\ell Q_\ell - Q_r \gamma^r Q_r$$

(6.70)

in which $\gamma^\mu \equiv \gamma^a D_\mu$. The $4 \times 4$ matrix $\gamma_5 = \gamma^5$ plays the role of a fifth (spatial) gamma matrix $\gamma^4 = \gamma_5$ in 5-dimensional space-time in the sense that the anticommutator

$$\{\gamma^a, \gamma^b\} = 2\eta^{ab}.$$  

(6.71)
6.5 Quark and Lepton Masses

in which \( \eta \) is the \( 5 \times 5 \) diagonal matrix with \( \eta^{00} = -1 \) and \( \eta^{aa} = 1 \) for \( a = 1, 2, 3, 4 \). In Weinberg’s notation, \( \gamma_5 \) is

\[
\gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\] (6.72)

The combinations

\[
P_\ell = \frac{1}{2} \left( 1 + \gamma_5 \right) \quad \text{and} \quad P_r = \frac{1}{2} \left( 1 - \gamma_5 \right)
\] (6.73)

are projection operators onto the left- and right-handed fields. That is,

\[
P_\ell Q = Q P_\ell \quad \text{and} \quad P_\ell Q_\ell = Q_\ell
\] (6.74)

with a similar equation for \( P_r \). We can write \( L_k \) as

\[
L_k = - \mathcal{E} \Phi^\dagger P_\ell E - \mathcal{E} \Phi^\dagger P_r E - \mathcal{Q} \Phi^\dagger P_\ell Q - \mathcal{Q} \Phi^\dagger P_r Q = - \left[ \mathcal{E} \Phi^\dagger P_\ell E + \mathcal{E} \Phi^\dagger P_r E + \mathcal{Q} \Phi^\dagger P_\ell Q + \mathcal{Q} \Phi^\dagger P_r Q \right].
\] (6.75)

Homework set 4, problem 1: Show that

\[
E_\ell P_\ell = \frac{1}{2} \left( 1 + \gamma_5 \right) E = \frac{1}{2} \left( 1 + \gamma_5 \right) E
\] (6.76)

Recall that in Weinberg’s notation

\[
\bar{\psi} = \psi^\dagger i \gamma^0 = \psi^\dagger \beta = \psi^\dagger \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}
\] (6.77)

in which \( I \) is the \( 2 \times 2 \) identity matrix.

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The Higgs mechanism also gives masses to the fermions, but somewhat arbitrarily. Dirac’s action density (??) has as its mass term

\[
-m \bar{\psi} \psi = -im \psi^\dagger \gamma^0 \psi = -im \psi^\dagger \gamma^0 (P_\ell + P_r) \psi = -im \psi^\dagger \gamma^0 (P_\ell^2 + P_r^2) \psi.
\] (6.78)

Since \( \{ \gamma^0, \gamma^5 \} = 0 \), this mass term is

\[
-m \bar{\psi} \psi = -im \psi^\dagger P_r \gamma^0 P_\ell \psi - im \psi^\dagger P_\ell \gamma^0 P_r \psi
\]

\[
= -im (P_r \psi)^\dagger \gamma^0 P_\ell \psi - im (P_\ell \psi)^\dagger \gamma^0 P_r \psi
\]

\[
= -im \psi_\ell^\dagger \gamma^0 \psi_\ell - im \psi_r^\dagger \gamma^0 \psi_r = -m \bar{\psi}_r \psi_\ell - m \bar{\psi}_\ell \psi_r.
\] (6.79)

Incidentally, because the fields \( \psi_\ell \) and \( \psi_r \) are independent, we can redefine them as

\[
\psi_\ell' = e^{i\theta} \psi_\ell \quad \text{and} \quad \psi_r' = e^{i\phi} \psi_r,
\] (6.80)
at will. Such a redefinition changes the mass term to
\[-m' \bar{\psi}_r \psi_\ell - m'' \bar{\psi}_\ell \psi_r = -m e^{i(\theta - \phi)} \bar{\psi}_r \psi_\ell - m e^{-i(\theta - \phi)} \bar{\psi}_\ell \psi_r. \quad (6.81)\]
So the phase of a Dirac mass term has no significance.

The definition (6.77) of $\bar{\psi}$ shows that the Dirac mass term is
\[-m \bar{\psi} \psi = -m \bar{\psi}^\dagger \beta \psi = -m \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} \psi_\ell \\ \psi_r \end{pmatrix} = -m \left( \psi_\ell^\dagger \psi_r + \psi_r^\dagger \psi_\ell \right). \quad (6.82)\]
These mass terms are invariant under the Lorentz transformations
\[
\psi_\ell' = \exp(-z \cdot \sigma) \psi_\ell \quad \text{and} \quad \psi_r' = \exp(z^* \cdot \sigma) \psi_r \quad (6.83)
\]
because
\[
\psi_\ell'^\dagger \psi_r' = \psi_\ell^\dagger \exp(-z^* \cdot \sigma) \exp(z \cdot \sigma) \psi_r = \psi_\ell^\dagger \psi_r, \quad (6.84)
\]
They are not invariant under rigid, let alone local, $SU(2)_\ell$ transformations. But we can make them invariant by using the Higgs field $H(x)$. For instance, the quantity $Q_\ell^\dagger H d_r$ is invariant under local $SU(2)_\ell$ transformations. In unitary gauge, its mean value in the vacuum is
\[
\langle 0 | Q_\ell^\dagger H d_r | 0 \rangle = \frac{1}{\sqrt{2}} v \langle 0 | d_\ell^\dagger d_r | 0 \rangle. \quad (6.85)
\]
So the term
\[-c_d Q_\ell^\dagger H d_r - c_d^* d_\ell^\dagger H^\dagger Q_\ell \quad (6.86)\]
is invariant, and in the vacuum it is
\[
\langle 0 | -c_d Q_\ell^\dagger H d_r - c_d^* d_\ell^\dagger H^\dagger Q_\ell | 0 \rangle = \frac{1}{\sqrt{2}} v \langle 0 | -c_d d_\ell^\dagger d_r - c_d^* d_r^\dagger d_\ell | 0 \rangle \quad (6.87)
\]
which gives to the $d$ quark the mass
\[
m_d = \frac{|c_d|}{\sqrt{2}} v. \quad (6.88)
\]
Note that we must add one new parameter $c_d$ to get one new mass $m_d$. This parameter $c_d$ is complex in general, but the mass $m_d$ depends only upon the absolute value and not upon its phase of $c_d$.

Similarly, the term
\[-c_e E_\ell^\dagger H e_r - c_e^* e_r^\dagger H^\dagger E_\ell \quad (6.89)\]
6.5 Quark and Lepton Masses

is invariant, and in the vacuum it is

$$\langle 0 | - c_e E^\dagger_\ell H e_r - c^*_e e^\dagger_r H^T Q_\ell | 0 \rangle = \frac{1}{\sqrt{2}} v \langle 0 | - c_e e^\dagger_\ell e_r - c^*_e e^\dagger_r | 0 \rangle$$  \hspace{1cm} (6.90)$$

which gives to the electron the mass

$$m_e = \frac{|c_e|}{\sqrt{2}} v.$$  \hspace{1cm} (6.91)

Again, we must add one new (complex) parameter $c_e$ to get one new mass $m_e$.

The mass of the up quark requires a new trick. The Higgs field $H$ transforms under $SU(2)_\ell \times U(1)$ as

$$H'(x) = \exp \left[ i g \frac{\sigma^a}{2} \alpha^a(x) + i g' \frac{I}{2} \beta(x) \right] H(x).$$  \hspace{1cm} (6.92)

If for clarity, we leave aside the $U(1)$ part for the moment, then the Higgs field $H$ transforms under $SU(2)_\ell$ as

$$H'(x) = \exp \left[ i g \frac{\sigma^a}{2} \alpha^a(x) \right] H(x).$$  \hspace{1cm} (6.93)

Let us use $H^*$ to be the complex column vector whose components are $H^\dagger_1$ and $H^\dagger_2$. How does $\sigma_2 H^*$ transform under $SU(2)_\ell$? Suppressing our explicit mention of the space-time dependence and using the asterisk to mean hermitian conjugation when applied to operators but complex conjugation when applied to matrices and vectors, we have, since $\sigma_2$ is imaginary with $\sigma_2^2 = I$ while $\sigma_1$ and $\sigma_3$ are real,

$$(\sigma_2 H^*)' = \sigma_2 \left[ \exp \left( i g \frac{\sigma^a}{2} \alpha^a \right) H \right]^* = \sigma_2 \exp \left( -i g \frac{\sigma^a}{2} \alpha^a \right) H^*$$

$$= \sigma_2 \exp \left( -i g \frac{\sigma^*_a}{2} \alpha^a \right) \sigma_2 \sigma_2 H^* = \exp \left( i g \frac{\sigma^a}{2} \alpha^a \right) \sigma_2 H^*.$$  \hspace{1cm} (6.94)

Thus, the term

$$- c_u Q^\dagger_\ell \sigma_2 H^* u_r - c^*_u u^\dagger_r H^T \sigma_2 Q_\ell$$  \hspace{1cm} (6.95)

is invariant under $SU(2)_\ell$. In the vacuum of the unitary gauge, it is

$$\langle 0 | - c_u Q^\dagger_\ell \sigma_2 H^* u_r - c^*_u u^\dagger_r H^T \sigma_2 Q_\ell | 0 \rangle = \frac{1}{\sqrt{2}} v \langle 0 | i c_u u^\dagger_\ell u_r - i c^*_u u^\dagger_r u_\ell | 0 \rangle$$

$$= \frac{1}{\sqrt{2}} v \langle 0 | i c_u u^\dagger_\ell u_r - i c^*_u u^\dagger_r u_\ell | 0 \rangle$$  \hspace{1cm} (6.96)
which gives the up quark the mass\[m_u = \frac{|c_u|}{\sqrt{2}} v. \quad (6.97)\]

But there are three families of generations of quarks and leptons on which the gauge fields act simply:

\[
F_1 = \begin{pmatrix} u \\ d \\ \nu_e \end{pmatrix}', \quad F_2 = \begin{pmatrix} c \\ s \\ \nu_\mu \end{pmatrix}', \quad \text{and} \quad F_3 = \begin{pmatrix} t \\ b \\ \nu_\tau \end{pmatrix}'. \quad (6.98)
\]

The quark and lepton flavor families are

\[
Q'_1 = \begin{pmatrix} u \\ d \end{pmatrix}', \quad Q'_2 = \begin{pmatrix} c \\ s \end{pmatrix}', \quad \text{and} \quad Q'_3 = \begin{pmatrix} t \\ b \end{pmatrix}';
\]

\[
E'_1 = \begin{pmatrix} \nu_e \end{pmatrix}', \quad E'_2 = \begin{pmatrix} \nu_\mu \end{pmatrix}', \quad \text{and} \quad E'_3 = \begin{pmatrix} \nu_\tau \end{pmatrix}'. \quad (6.99)
\]

These are called the flavor eigenstates or more properly flavor eigenfields, designated here with primes. They are the ones on which the $W^\pm$ act simply.

The weak interactions use $W^-_{\mu} T^-_{\mu}$ to map the flavor up fields $u'_1 = u', \ u'_2 = c', \ u'_3 = t'$ into the flavor down fields $d'_1 = d', \ d'_2 = s', \ d'_3 = b'$, and $W^+_{\mu} T^+$ to map the flavor down fields $d'_i$ into the flavor up fields $u'_i$.

The action density

\[
\sum_{i,j=1}^{3} -c_{dij} Q'_{\ell i} H d'_{r j} - c^{*}_{dij} d'_{r j} H^\dagger Q'_{\ell i} \quad (6.100)
\]

gives for the $d', s'$, and $b'$ quarks the mixed mass terms

\[
\frac{v}{\sqrt{2}} \sum_{i,j=1}^{3} -c_{dij} d'^\dagger_{\ell i} d'_{r j} - c^{*}_{dij} d'_{r j} d'^\dagger_{\ell i} = \frac{v}{\sqrt{2}} \sum_{i,j=1}^{3} -c_{dij} d'^\dagger_{\ell i} d'_{r j} - c^{*}_{dij} d'_{r j} d'^\dagger_{\ell i} \quad (6.101)
\]

The $3 \times 3$ mass matrix $M_d$ with entries

\[
[M_d]_{ij} = \frac{v}{\sqrt{2}} c_{dij} \quad (6.102)
\]

need have no special properties. It need not be hermitian because for each $i$ and $j$, the term $[6.101]$ is hermitian. But every $3 \times 3$ complex matrix has a singular-value decomposition

\[
M_d = L_d \Sigma_d R_d^\dagger 
\quad (6.103)
\]
in which $L_d$ and $R_d$ are $3 \times 3$ unitary matrices, and $\Sigma_d$ is a $3 \times 3$ diagonal matrix with nonincreasing positive singular values on its main diagonal.

The singular value decomposition works for any $N \times M$ (real or) complex matrix. Every complex $M \times N$ rectangular matrix $A$ is the product of an $M \times M$ unitary matrix $U$, an $M \times N$ rectangular matrix $\Sigma$ that is zero except on its main diagonal which consists of its nonnegative singular values $S_k$, and an $N \times N$ unitary matrix $V^\dagger$

$$A = U \Sigma V^\dagger.$$  \hfill (6.104)

This singular-value decomposition is a key theorem of matrix algebra. One can use the Matlab command “[U,S,V] = svd(A)” to perform the svd $A = USV^\dagger$.

The singular values of $\Sigma_d$ are the masses $m_b$, $m_s$, and $m_d$:

$$\Sigma_d = \begin{pmatrix} m_b & 0 & 0 \\ 0 & m_s & 0 \\ 0 & 0 & m_d \end{pmatrix}.$$  \hfill (6.105)

So the mass eigenfields of the left and right down-quark fields are

$$d_{ri} = R^\dagger_{dij} d^r_{rj} \quad \text{and} \quad d^\dagger_{\ell i} = d^\dagger_{\ell j} L_{dij} \quad \text{or} \quad d_{\ell i} = L_{dij}^\dagger d_{\ell j}. \hfill (6.106)$$

The inverse relations are

$$d^r_{rj} = R_{dij} d_{rj} \quad \text{and} \quad d^\dagger_{\ell i} = d^\dagger_{\ell j} L_{dji} \quad \text{or} \quad d^r_{\ell i} = L_{dij} d_{\ell j} \hfill (6.107)$$

or in matrix notation

$$d^r = R d, \quad d^\dagger = d^\dagger R^\dagger, \quad d^\dagger = d^\dagger L^\dagger, \quad \text{and} \quad d^r = L d \hfill (6.108)$$

in which

$$d_\ell = \begin{pmatrix} b \\ s \\ d \end{pmatrix}_\ell \hfill (6.109)$$

are the down-quark fields of definite masses.

Similarly, the up quark action density

$$\sum_{i,j=1}^{3} -c_{uij} Q^\dagger_{\ell i} \sigma_2 H^* u^r_{rj} - c_{uij}^* u^r_{rj} H^T \sigma_2 Q^\dagger_{\ell i} \hfill (6.110)$$

gives for the three known families the mixed mass terms

$$\sum_{i,j=1}^{3} c_{uij} u^r_{\ell i} u^r_{rj} - c_{uij}^* u^r_{rj} u^r_{\ell i} = \frac{iv}{\sqrt{2}} \sum_{i,j=1}^{3} c_{uij} u^r_{\ell i} u^r_{rj} - c_{uij}^* u^r_{rj} u^r_{\ell i}. \hfill (6.111)$$
The $3 \times 3$ mass matrix $M_u$ with entries

$$[M_u]_{ij} = \frac{iv}{\sqrt{2}} c_{uij} \tag{6.112}$$

need have no special properties. It need not be hermitian because for each $i$ and $j$, the term (6.111) is hermitian. But every $3 \times 3$ complex matrix $M_u$ has a singular value decomposition

$$M_u = L_u \Sigma_u R_u^\dagger \tag{6.113}$$

in which $L_u$ and $R_u$ are $3 \times 3$ unitary matrices, and $\Sigma_u$ is a $3 \times 3$ diagonal matrix with nonincreasing positive singular values on its main diagonal. These singular values are the masses $m_t$, $m_c$, and $m_u$:

$$\Sigma_u = \begin{pmatrix} m_t & 0 & 0 \\ 0 & m_c & 0 \\ 0 & 0 & m_u \end{pmatrix}. \tag{6.114}$$

So the mass eigenfields of the left and right up-quark fields are

$$u_{ri} = R_{uij}^\dagger u'_{rj} \quad \text{and} \quad u_{\ell i}^\dagger = u_{\ell j}^\dagger L_{uij} \quad \text{or} \quad u_{ri} = L_{uij} u'_{rj}. \tag{6.115}$$

The inverse relations are

$$u_{ri}' = R_{uij} u_{rj} \quad \text{and} \quad u_{\ell i}' = u_{\ell j}^\dagger L_{uij}^\dagger \quad \text{or} \quad u_{ri}' = L_{uij} u_{rj}. \tag{6.116}$$

or in matrix notation

$$u_r' = R_u u_r, \quad u_r'^\dagger = u_r^\dagger R_u^\dagger \quad u_\ell'^\dagger = u_\ell^\dagger L_u^\dagger, \quad \text{and} \quad u_\ell' = L_u u_\ell \tag{6.117}$$

in which

$$u_\ell = \begin{pmatrix} t \\ c \\ u_\ell \end{pmatrix} \tag{6.118}$$

are the up-quark fields of definite masses.

### 6.6 CKM matrix

We will use the labels $u$, $c$, $t$ and $d$, $s$, $b$ for the states that are eigenstates of the quadratic part of the hamiltonian after the Higgs mechanism has given a mean value to the real part of the neutral Higgs boson in the unitary gauge. The $u$, $c$, $t$ quarks have the same charge $2e/3 > 0$ and the same $T^3 = 1/2$, so they all have the same electroweak interactions. Similarly, the $d$, $s$, $b$ quarks have the same charge $-e/3 < 0$ and the same $T^3 = -1/2$, so they also all have the same electroweak interactions.
The right-handed covariant derivative \[\tag{6.119}\]
\[D_r^\mu = I \partial_\mu - ie \tan \theta_w Z_\mu Q + ie A_\mu Q\]
just sends the fields of these mass eigenstates into themselves multiplied by their charge and either a \(Z\) or a photon. That is,
\[\tag{6.120}\]
\[u_r' D_r^\mu u_r = u_r^\dagger R_u^\dagger D_\mu R_u u_r\]
\[d_r' D_r^\mu d_r = d_r^\dagger R_d^\dagger D_\mu R_d d_r\]
In these terms, the interactions of the \(Z\) and the photon with the right-handed fields are diagonal both in mass and in flavor.

But the left-handed covariant derivative \[\tag{6.69}\]
\[D_\ell^\mu = I \partial_\mu + \frac{e}{\sqrt{2} \sin \theta_w} (W^+_\mu T^+ + W^-_\mu T^-)\]
\[+ i \frac{e}{\cos \theta_w} Z_\mu \left( \frac{T^3}{\sin \theta_w} - \sin \theta_w Q \right) + ie A_\mu Q.\]
So we have
\[\tag{6.122}\]
\[\left( u_\ell^\dagger \quad d_\ell^\dagger \right) D_\mu^\ell \left( u_\ell \quad d_\ell \right) = \left( u_\ell^\dagger L_u^\dagger \quad d_\ell^\dagger L_d^\dagger \right) D_\mu^\ell \left( L_u u_\ell \quad L_d d_\ell \right).\]
Some of the unitary matrices just give unity, \(L_u^\dagger L_u = I\) and \(L_d^\dagger L_d = I\) like \(R_u^\dagger R_u = I\) and \(R_d^\dagger R_d = I\) in the right-handed covariant derivatives \[\tag{6.120}\].
Thus the interactions of the \(Z\) and the photon with both the right-handed fields and with the left-handed fields are diagonal both in mass and in flavor. The \(Z\) and the photon do not mediate top-to-charm or charm-to-up or \(\mu^- \to e^- + \gamma\) decays. Also, the Higgs mass terms are diagonal, so the neutral Higgs boson can’t mediate such processes. **Thus, in the standard model, there are no flavor-changing neutral-currents.**

The only changes are in the nonzero parts of \(T^\pm\) which become
\[\tag{6.123}\]
\[T^+_\text{ckm} = \begin{pmatrix} 0 & L_u^\dagger L_d \\ 0 & 0 \end{pmatrix} \equiv \begin{pmatrix} 0 & V \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad T^-_\text{ckm} = \begin{pmatrix} 0 & L_u^\dagger L_d \\ 0 & 0 \end{pmatrix} \equiv \begin{pmatrix} 0 & V^\dagger \\ 0 & 0 \end{pmatrix}\]
in which the unitary matrix \(V = L_u^\dagger L_d\) is the CKM matrix (Nicola Cabibbo, Makoto Kobayashi, and Toshihide Maskawa). The left-handed covariant derivative on the mass eigenfields then is
\[\tag{6.124}\]
\[D_\mu^\ell = I \partial_\mu + \frac{e}{\sqrt{2} \sin \theta_w} (W^+_\mu T^+_{\text{ckm}} + W^-_\mu T^-_{\text{ckm}})\]
\[+ i \frac{e}{\cos \theta_w} Z_\mu \left( \frac{T^3}{\sin \theta_w} - \sin \theta_w Q \right) + ie A_\mu Q.\]
It has a second part that acts more or less like the right-handed covariant derivative, but the first part uses $W^\mu_T$ to map the up fields $u, c, t$ into linear combinations of the down fields $d, s, b$ and $W^{\mu}_T$ to map the down fields into linear combinations of the up fields. The $W^\pm$ terms are sensitive to the CKM matrix $V = L_u^\dagger L_d$. We write them suggestively as

$$
(u \ c \ t \ d \ s \ b)^\dagger \left( \begin{array}{c} 0 \\ W^-_\mu \\ 0 \\ 0 
\end{array} \right) V W^+ \mu 
$$

(6.125)

$$
= \left( \begin{array}{c} u \\ c \\ t \\ s \\ d \\ b 
\end{array} \right)^\dagger V \left( \begin{array}{c} d \\ s \\ b \\ t \\ c \\ u 
\end{array} \right) 
$$

(6.126)

By choosing the phases of the six quark fields, that is, $u(x) \rightarrow e^{i\theta_u} u(x)$ . . . $b(x) \rightarrow e^{i\theta_b} b(x)$, one may make the CKM matrix $L_u^\dagger L_d$ real apart from a single phase. The existence of that phase probably is the cause of most of the breakdown of $CP$ invariance that Fitch and Cronin and others have observed since 1964. The magnitudes of the elements of the CKM matrix $V$ are

$$
V = \left( \begin{array}{ccc} |V_{ud}| & |V_{us}| & |V_{ub}| \\ |V_{cd}| & |V_{cs}| & |V_{cb}| \\ |V_{td}| & |V_{ts}| & |V_{tb}| 
\end{array} \right) = \left( \begin{array}{ccc} 0.97427 & 0.22536 & 0.00355 \\ 0.22522 & 0.97343 & 0.0414 \\ 0.00886 & 0.0405 & 0.99914 
\end{array} \right). 
$$

(6.127)

Although there is only one phase $e^{i\delta}$ in the CKM matrix $V$, the experimental constraints on this phase often are expressed in terms of the angles $\alpha, \beta, \gamma$ defined as

$$
\alpha = \arg \left[ -V_{td}^* V_{tb}^* / (V_{ud} V_{ub}^*) \right] \\
\beta = \arg \left[ -V_{cd}^* V_{cb}^* / (V_{td} V_{tb}^*) \right] \\
\gamma = \arg \left[ -V_{ad} V_{ub}^* / (V_{cd} V_{cb}^*) \right]. 
$$

(6.128)

If $V$ is unitary, then $\alpha + \beta + \gamma = 180^\circ$. From $B \rightarrow \pi\pi, \rho\pi, \rho\rho$ decays, the limits on the angle $\alpha$ are roughly

$$
\alpha = (85.4 \pm 4)^{\circ}. 
$$

(6.129)
From $B^\pm \rightarrow D K^\pm$ decays, the limits on the angle $\gamma$ are roughly
\[ \gamma = (68.0 \pm 8)^\circ. \] (6.130)

So the angle $\beta$ is about 26.6°.

One of the quark-Higgs interactions is
\[
-c_{dij} Q_d^{|i} H d^{|j}_r = -\frac{\sqrt{2}}{v} Q_d^{|i} M_d d^{|j}_r H = -\frac{\sqrt{2}}{v} Q_d^{|i} L_d \Sigma_d R^{|j}_d H \\
= -\frac{\sqrt{2}}{v} Q_d^{|i} \sum_d d^{|j}_r = -\frac{\sqrt{2}}{v} Q_d^{|i} \left( \frac{0}{(v + h)/\sqrt{2}} \right) \sum_d d^|r,
\] (6.131)

A similar term describes the coupling of the up quarks to the Higgs
\[
-m_{ui} u^{|i}_r \left( 1 + \frac{h}{v} \right) u^{|r}_i.
\] (6.132)

Thus, the rate of quark-antiquark to Higgs is proportional to the mass of the quark in the standard model.

### 6.7 Lepton Masses

We can treat the leptons just like the quarks. The up leptons are the flavor neutrinos $\nu'_e$, $\nu'_\mu$, and $\nu'_\tau$, and the down leptons are the flavor charged leptons $e'$, $\mu'$, and $\tau'$. The action density
\[
\sum_{i,j=1}^{3} -c_{eij} E^{|i}_e H e^{|j}_r - c^*_{eij} E^{|j}_e H^{|i}_e
\] (6.133)
gives for the $e'$, $\mu'$, and $\tau'$ the mixed mass terms
\[
\frac{v}{\sqrt{2}} \sum_{i,j=1}^{3} -c_{eij} E^{|i}_e e^{|j}_r - c^*_{eij} E^{|j}_e e^{|i}_e.
\] (6.134)

The $3 \times 3$ mass matrix $M_e$ with entries
\[
[M_e]_{ij} = \frac{v}{\sqrt{2}} c_{eij}
\] (6.135)
has a singular value decomposition
\[
M_e = L_e \Sigma_e R^\dagger_e
\] (6.136)
in which $L_e$ and $R_e$ are $3 \times 3$ unitary matrices, and $\Sigma_e$ is a $3 \times 3$ diagonal matrix with nonincreasing positive singular values $m_\tau$, $m_\mu$, and $m_e$ on its main diagonal.
6.8 Before and after symmetry breaking

Before spontaneous symmetry breaking, all the fields of the standard model are massless, and the local symmetry under $SU(2)_L \otimes U(1)$ is exact. Under these gauge transformations, the left-handed electron and neutrino fields are rotated among themselves. If $e'_L$ is a linear combination of itself and of $\nu'_{\ell L}$, then these two fields, $e'_L$ and $\nu'_{\ell L}$, must be of the same kind. The left-handed electron field is a Dirac field. Thus, the left-handed neutrino field also must be a Dirac field. This makes sense because before symmetry breaking, all the fields are massless, and so there is no problem combining two Majorana fields of the same mass, namely zero, into one Dirac field. Thus, there are three flavor Dirac neutrino fields $\nu'_{eL}, \nu'_{\mu L},$ and $\nu'_{\tau L}$.

A massless left-handed neutrino field $\nu'_{\ell}$ satisfies the two-component Dirac equation

$$\left(\partial_0 I - \nabla \cdot \sigma\right) \nu'_{\ell}(x) = 0$$

which in momentum space is

$$(E + p \cdot \sigma) \nu'_{\ell}(p) = 0.$$  

Since the angular momentum is $J = \sigma/2$, and $E = |p|$, we have

$$\hat{p} \cdot J \nu'_{\ell}(p) = -\frac{1}{2} \nu'_{\ell}(p).$$

The left-handed neutrino field $\nu'_{\ell}$ annihilates neutrinos of negative helicity and creates antineutrinos of positive helicity.

Since the neutrinos are massive, there may be right-handed neutrino fields. As for the up quarks, we can use them to make an action density

$$3 \sum_{i,j=1}^{3} -c_{\nu ij} E'_{\ell_i} \sigma_2 H^* \nu'_{\ell_j} - c_{\nu ij}^* \nu'_{\ell_j} H^T \sigma_2 E'_{\ell_i}$$

that is invariant under $SU(2)_L \otimes U(1)$ and that gives for the neutrinos the mixed mass terms

$$3 \sum_{i,j=1}^{3} \frac{i}{\sqrt{2}} (c_{\nu ij} \nu'_{\ell_i} \nu'_{\ell_j} - c_{\nu ij}^* \nu'_{\ell_j} \nu'_{\ell_i}).$$

The $3 \times 3$ mass matrix $M_\nu$ with entries

$$[M_\nu]_{ij} = \frac{i}{\sqrt{2}} c_{\nu ij}$$

has a singular value decomposition

$$M_\nu = L_\nu \Sigma_\nu R_\nu^T.$$
in which \( L_\nu \) and \( R_\nu \) are 3 \( \times \) 3 unitary matrices, and \( \Sigma_\nu \) is a 3 \( \times \) 3 diagonal matrix with nonincreasing positive singular values \( m_\nu, m_\nu, \) and \( m_\nu \) on its main diagonal (here, I have assumed that the neutrino masses mimic those of the charged leptons and quarks, rising with family number). The neutrino CKM matrix then would be \( L_\nu^\dagger L_e \), but since we are accustomed to treating the charged leptons as flavor and mass eigenfields, we apply the neutrino CKM matrix to the neutrinos rather than to the charged leptons. Thus the neutrino CKM matrix is

\[
V_\nu = L_e^\dagger L_\nu.
\]

By choosing the phases of the six lepton fields, we can make the neutrino CKM matrix real except for \( CP \)-breaking phases. If the neutrinos are Dirac fields, then there is one such phase; if not, there are three.

So far, I have assumed that the mass terms for the neutrinos are the usual Dirac mass terms. However, the right-handed Majorana neutrino fields \( \nu'_r \) are not affected by the \( SU(2)_\ell \otimes U(1) \).

Note that a gauge transformation between \( e \) and \( \nu_e \) rotates the operators \( a(p, s, e) \) and \( a(p, s, \nu_e) \) into each other. This rotation makes sense only when the two particles have the same mass. In the standard model, such a gauge transformation makes sense only before symmetry breaking when all the particles are massless. Moreover, only when the particles are massless can one say that they are left- or right-handed. While the particles are massless, the operator \( a(p, -) \) annihilates a particle of negative helicity and occurs only in a left-handed field, while the operator \( a(p, +) \) annihilates a particle of positive helicity and occurs only in a right-handed field. But when the particles are massive, the operator \( a(p, \frac{1}{2}) \) annihilates a particle that is spin up and occurs in both left-handed and right-handed fields. So a symmetry transformation that acted on the operator \( a(p, \frac{1}{2}) \) would change both left-handed and right-handed fields.

The left-handed fields of the neutrino and electron are

\[
\nu_{e, l}(x) = \int u(p, -) \frac{a_1(p, -, \nu_e) + ia_2(p, -, \nu_e)}{\sqrt{2}} e^{ipx} \frac{d^3p}{(2\pi)^{3/2}}
\]

(6.145)

\[
+ v(p, +) \frac{a_1^\dagger(p, +, \nu_e) + ia_2^\dagger(p, +, \nu_e)}{\sqrt{2}} e^{-ipx} \frac{d^3p}{(2\pi)^{3/2}}
\]

(6.146)

\[
e_{e}(x) = \int u(p, -) \frac{a_1(p, -, e) + ia_2(p, -, e)}{\sqrt{2}} e^{ipx} \frac{d^3p}{(2\pi)^{3/2}}
\]

(6.147)

\[
+ v(p, +) \frac{a_1^\dagger(p, +, e) + ia_2^\dagger(p, +, e)}{\sqrt{2}} e^{-ipx} \frac{d^3p}{(2\pi)^{3/2}}
\]

(6.148)

where \((p, -, \nu_e)\) means momentum \( p \), spin down, and electron flavor, and
Standard model

$(p, +, \nu_e)$ means momentum $p$, spin up, and electron flavor. These fields satisfy equations like (6.137–6.139) apart from their interactions with other fields. Since a gauge transformation maps the fields $\nu_{e, \ell}(x)$ and $e_\ell(x)$ into each other, we know that when all the fields are massless, before symmetry breaking, there are (for each momentum) at least two neutrino and antineutrino states

$$\frac{1}{\sqrt{2}} \left[ a_1^\dagger (p, -, \nu_e) - ia_2^\dagger (p, -, \nu_e) \right] |0\rangle$$

(6.149)

$$\frac{1}{\sqrt{2}} \left[ a_1^\dagger (p, +, \nu_e) + ia_2^\dagger (p, +, \nu_e) \right] |0\rangle$$

(6.150)

for each of the three flavors, $f = e, \mu, \tau$. So there are at least six neutrino (and antineutrino) states.

The right-handed electron field exists and interacts with gauge bosons and other fields. So there are 12 electron states $a_i^\dagger (p, \pm, e_f) |0\rangle$ for $i = 1$ and 2 and for the three flavors, $f = e, \mu, \tau$. We don’t know yet whether a right-handed neutrino field exists or interacts with other fields. So there may be only 6 neutrino states or as many as 12.

Neutrino oscillations tell us that neutrinos have masses. If there are 12 neutrino states, then there can be three massive Dirac neutrinos analogous to the $e$, $\mu$, and $\tau$ or six massive Majorana neutrinos or some intermediate combination. If there are only 6, then there can be 3 massive Majorana neutrinos.

The Majorana mass terms for the right-handed neutrino fields are

$$\sum_{ij=1}^{6} \frac{1}{2} \left[ \text{im}_{ij} \nu'^T r_i \sigma_2 \nu'^j r_j + (\text{im}_{ij} \nu''^T r_i \sigma_2 \nu''^j r_j) \right] .$$

(6.151)

They are Lorentz invariant because under the Lorentz transformations (6.83)

$$\nu'^T r \sigma_2 \nu'' = \nu'^T r \exp(\mathbf{z}^* \cdot \sigma^T) \sigma_2 \exp(\mathbf{z}^* \cdot \sigma) \nu'' r = \nu'^T r \sigma_2 \exp(-\mathbf{z}^* \cdot \sigma) \exp(\mathbf{z}^* \cdot \sigma) \nu'' r = \nu'^T, \sigma_2 \nu'' r .$$

(6.152)

The Majorana mass terms (6.151) are unrelated to the scale $v$ of the Higgs field’s mean value. One can show that the complex matrix $m_{ij}$ is symmetric. One then must combine the mass matrix in (6.151) with the mass matrix $M_\nu$ in (6.142). The resulting mass matrix will have a singular-value decomposition with six singular values that would be the masses of the “physical” neutrinos. If these six masses are equal in pairs, then the three pairs would form three Dirac neutrinos.

Whether or not there are right-handed neutrinos, we can make Majorana mass terms like $\nu^T \sigma_2 \nu_\ell$, which are Lorentz invariant but not invariant under
6.8 Before and after symmetry breaking

SU$_{\ell}(2)$ or U$_Y(1)$. We can make them gauge invariant by using a triplet $\vec{\phi} = \sigma_i \phi_i$ of Higgs fields that transforms as $\vec{\phi} \cdot \vec{\sigma} = g (\vec{\phi} \cdot \vec{\sigma}) g^\dagger$ for $g \in SU_{\ell}(2)$ and that carries a value of $Y = -1$. Then if $\sigma_2$ has Lorentz indices and $\sigma'_2$ has $SU_{\ell}(2)$ indices, the term

$$E_\ell^T \sigma_2 \sigma'_2 (\vec{\phi} \cdot \vec{\sigma}) E_\ell$$

is both Lorentz invariant and gauge invariant. If the potential $V(\vec{\phi})$ has minima at $\vec{\phi} \neq 0$, then this term violates lepton number and gives a Majorana mass to the neutrino. Another possibility is to say that at higher energies a theory with new fields of very high mass $\Lambda$ plays a role, and that when one path-integrates over these heavy fields, one is left with an effective, nonrenormalizable term in the action

$$- \frac{(H^\dagger E_\ell)^2}{\Lambda}$$

which gives a Majorana mass to $\nu_\ell$.

Models with both right-handed and left-handed neutrinos are easier to think about, but only experiments can tell us whether right-handed neutrinos exist.

What is known experimentally is that there are at least three masses that satisfy

$$|\Delta m^2_{21}| \equiv |m^2_2 - m^2_1| = (7.53 \pm 0.18) \times 10^{-5} \text{eV}^2$$
$$|\Delta m^2_{32}| \equiv |m^2_3 - m^2_2| = (2.44 \pm 0.06) \times 10^{-3} \text{eV}^2$$

(normal mass hierarchy)

$$|\Delta m^2_{32}| \equiv |m^2_3 - m^2_2| = (2.52 \pm 0.07) \times 10^{-3} \text{eV}^2$$

(inverted mass hierarchy).

If the neutrinos are Dirac particles, then they have a CKM matrix like that of the quarks with one $CP$-violating phase. But whereas one chooses to make the mass and flavor eigenfields the same for the up quarks $u,c,t$, for the leptons one makes the mass and flavor eigenfields the same for the down or charged leptons $e,\mu,\tau$. So the neutrino CKM matrix actually is $V = L^\dagger_\nu L_\nu$. If they are three Majorana particles, then their CKM matrix has two extra $CP$-violating phases $\alpha_{12}$ and $\alpha_{31}$. A common convention for the
neutrino CKM matrix is
\[
V = \begin{pmatrix}
1 & 0 & 0 \\
0 & \cos \theta_{23} & \sin \theta_{23} \\
0 & -\sin \theta_{23} & \cos \theta_{23}
\end{pmatrix}
\begin{pmatrix}
\cos \theta_{13} & 0 & \sin \theta_{13} e^{-i\delta} \\
0 & 1 & 0 \\
-\sin \theta_{13} e^{i\delta} & 0 & \cos \theta_{13}
\end{pmatrix}
\times
\begin{pmatrix}
\cos \theta_{12} & \sin \theta_{12} & 0 \\
-\sin \theta_{12} & \cos \theta_{12} & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & e^{i\alpha_{12}/2} & 0 \\
0 & e^{i\alpha_{31}/2} & 0 & e^{i\alpha_{32}/2}
\end{pmatrix}
\] (6.156)

This convention without the last $3 \times 3$ matrix also is used for the quark CKM matrix. The current estimates are
\[
sin^2(2\theta_{12}) = 0.846 \pm 0.021 \quad (6.158)
\]
\[
sin^2(2\theta_{23}) = 0.999 +0.001 \quad -0.018 \quad \text{normal mass hierarchy} \quad (6.159)
\]
\[
sin^2(2\theta_{23}) = 1.000 +0.000 \quad -0.017 \quad \text{inverted mass hierarchy} \quad (6.160)
\]
\[
sin^2(2\theta_{13}) = 0.093 \pm 0.008. \quad (6.161)
\]

Two of these are big angles: $2\theta_{12} \approx 2\theta_{23} = \pi/2 \pm n\pi$. In the normal hierarchy, the lightest neutrino is about 2/3 electron, 1/6 muon, and 1/6 tau; the very slightly heavier neutrino is about 1/3 electron, 1/3 muon, and 1/3 tau; and the much heavier heavier neutrino is about 1/6 electron, 5/12 muon, and 5/12 tau.

### 6.9 The Seesaw Mechanism

Why are the neutrino masses so light? Suppose we wish to find the eigenvalues of the real, symmetric mass matrix
\[
\mathcal{M} = \begin{pmatrix}
0 & m \\
m & M
\end{pmatrix}
\] (6.162)

in which $m$ is an ordinary mass and $M$ is a huge mass. The eigenvalues $\mu$ of this hermitian mass matrix satisfy $\det(\mathcal{M} - \mu I) = \mu(\mu - M) - m^2 = 0$ with solutions $\mu_{\pm} = (M \pm \sqrt{M^2 + 4m^2})/2$. The larger mass $\mu_+ \approx M + m^2/M$ is approximately the huge mass $M$ and the smaller mass $\mu_- \approx -m^2/M$ is tiny.

The physical mass of a fermion is the absolute value of its mass parameter, here $m^2/M$.

The product of the two eigenvalues is the constant $\mu_+ \mu_- = \det \mathcal{M} = -m^2$ so as $\mu_-$ goes down, $\mu_+$ must go up. In 1975, Gell-Mann, Ramond, Slansky, and Jerry Stephenson invented this "seesaw" mechanism as an explanation
of why neutrinos have such small masses, less than 1 eV/c². If \( mc^2 = 10 \text{ MeV} \), and \( \mu_c^2 \approx 0.01 \text{ eV} \), which is a plausible light-neutrino mass, then the rest energy of the huge mass would be \( Mc^2 = 10^7 \text{ GeV} \). This huge mass would be one of the six neutrino masses and would point at new physics, beyond the standard model. Yet the small masses of the neutrinos may be related to the weakness of their interactions.

Before leaving the subject of fermion masses, let’s look more closely at Dirac and Majorana mass terms. A Dirac field is a linear combination of two Majorana fields of the same mass

\[
\psi = \frac{1}{\sqrt{2}} \begin{pmatrix} L + i\ell \\ R + ir \end{pmatrix}
\]  

(6.163)

in which \( L \) and \( \ell \) are two-component left-handed spinors, and \( R \) and \( r \) are two-component right-handed spinors. The Dirac mass term

\[
m \bar{\psi} \psi = i m \psi^\dagger \gamma^0 \psi = m \psi^\dagger \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \psi
\]

\[
= m \frac{1}{2} \begin{pmatrix} L - i\ell^\dagger, & R^\dagger - ir^\dagger \end{pmatrix} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} L + i\ell \\ R + ir \end{pmatrix}
\]

\[
= m \frac{1}{2} \begin{pmatrix} L - i\ell^\dagger, & R^\dagger - ir^\dagger \end{pmatrix} \begin{pmatrix} R + ir \\ L + i\ell \end{pmatrix}
\]

\[
= m \frac{1}{2} \left[ (L^\dagger - i\ell^\dagger) (R + ir) + (R^\dagger - ir^\dagger) (L + i\ell) \right]
\]

(6.164)

\[
= m \frac{1}{2} \left( R^\dagger - ir^\dagger \right) (L + i\ell) + \text{h.c.}
\]

We may set

\[
R = i\sigma_2 L^* \iff L = -i\sigma_2 R^*
\]

(6.165)

\[
r = i\sigma_2 \ell^* \iff \ell = -i\sigma_2 r^*
\]

(6.166)

or equivalently

\[
\begin{pmatrix} R_1 \\ R_2 \end{pmatrix} = \begin{pmatrix} L_2^\dagger \\ -L_1^\dagger \end{pmatrix} \iff \begin{pmatrix} L_1 \\ L_2 \end{pmatrix} = \begin{pmatrix} -R_2^\dagger \\ R_1^\dagger \end{pmatrix}
\]

(6.167)

\[
\begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = \begin{pmatrix} \ell_2^\dagger \\ -\ell_1^\dagger \end{pmatrix} \iff \begin{pmatrix} \ell_1 \\ \ell_2 \end{pmatrix} = \begin{pmatrix} -r_2^\dagger \\ r_1^\dagger \end{pmatrix}
\]

(6.168)

which are the Majorana conditions. Since \( R^\dagger = -iL^T \sigma_2 \), we can write the
Dirac mass term (6.164) in terms of left-handed fields as

\[ m \bar{\psi} \psi = \frac{1}{2} m \left( -iL^T - i\ell^T \right) \sigma_2 (L + i\ell) + \text{h.c.} \]  \hspace{1cm} (6.169)

\[ = \frac{1}{2} m \left( L^T - i\ell^T \right) (-i \sigma_2) (L + i\ell) + \text{h.c.} \]  \hspace{1cm} (6.170)

\[ = \frac{1}{2} m \left( L_1 - i\ell_1, \ L_2 - i\ell_2 \right) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} L_1 + i\ell_1 \\ L_2 + i\ell_2 \end{pmatrix} + \text{h.c.} \]  \hspace{1cm} (6.171)

\[ = \frac{1}{2} m \left( L_1 - i\ell_1 \right) \left( -L_2 - i\ell_2 \right) + \text{h.c.} \]  \hspace{1cm} (6.172)

\[ = \frac{1}{2} m \left( L_1 - i\ell_1 \right) \left( -L_2 - i\ell_2 \right) + \left( L_2 - i\ell_2 \right) \left( L_1 + i\ell_1 \right) + \text{h.c.} \]  \hspace{1cm} (6.173)

The fermion fields anticommute, so the Dirac mass term is

\[ m \bar{\psi} \psi = \frac{1}{2} m \left( -2L_1L_2 - 2\ell_1\ell_2 \right) + \text{h.c.} = -m \left( L_1L_2 + \ell_1\ell_2 \right) + \text{h.c.,} \]  \hspace{1cm} (6.174)

and it says that the fields \( L \) and \( \ell \) have the same mass \( m \), as they must if they are to form a Dirac field.

Since \( L^\dagger = iR^T \sigma_2 \), we also can write the Dirac mass term in terms of the right-handed fields as

\[ m \bar{\psi} \psi = \frac{1}{2} m \left( -2L_1L_2 - 2\ell_1\ell_2 \right) + \text{h.c.} = -m \left( L_1L_2 + \ell_1\ell_2 \right) + \text{h.c.} \]  \hspace{1cm} (6.175)

\[ = \frac{1}{2} m \left( R_1 - i\ell_1 \right) \left( -R_2 - i\ell_2 \right) + \left( R_2 - i\ell_2 \right) \left( R_1 + i\ell_1 \right) + \text{h.c.} \]  \hspace{1cm} (6.176)

\[ = m \left( R_1R_2 + r_1r_2 \right) + \text{h.c.} \]  \hspace{1cm} (6.177)

So the fields \( R \) and \( r \) have the same mass \( m \), as they must if they are to form a Dirac field.

The Majorana mass term for a right-handed field \( r \) of mass \( m \) evidently is

\[ m r_1 r_2 + \text{h.c.} \]  \hspace{1cm} (6.178)

### 6.10 Neutrino Oscillations

The phase difference \( \Delta \phi \) between two highly relativistic neutrinos of momentum \( p \) going a distance \( L \) in a time \( t \approx L \) varies with their masses \( m_1 \) and \( m_2 \) as

\[ \Delta \phi = t \Delta E = \frac{LE}{p} \Delta E = \frac{LE}{p} \left( \sqrt{p^2 + m_1^2} - \sqrt{p^2 + m_2^2} \right) \]  \hspace{1cm} (6.179)
in natural units. We can approximate this phase by using the first two terms of the binomial expansion of the square roots with \( y = 1 \) and \( x = m_i^2/p^2 \)

\[
\Delta \phi = LE \left( \sqrt{1 + m_1^2/p^2} - \sqrt{1 + m_2^2/p^2} \right) \approx \frac{LE \Delta m^2}{p^2} \approx \frac{L \Delta m^2}{E} \quad (6.180)
\]

or in ordinary units \( \Delta \phi \approx L \Delta m^2 c^3/(hE) \).
7

Path integrals

7.1 Path integrals and Richard Feynman
Since Richard Feynman invented them over 70 years ago, path integrals have been used with increasing frequency in high-energy and condensed-matter physics, in optics and biophysics, and even in finance. Feynman used them to express the amplitude for a process as a sum of all the ways the process could occur each weighted by an exponential of its classical action \( \exp(iS/\hbar) \). Others have used them to compute partition functions and to study the QCD vacuum. (Richard Feynman, 1918–1988)

7.2 Gaussian integrals and Trotter’s formula
Path integrals are based upon the gaussian integral (7.1) which holds for real \( a \neq 0 \) and real \( b \)

\[
\int_{-\infty}^{\infty} e^{iax^2+2ibx} \, dx = \sqrt{\frac{i\pi}{a}} e^{-ib^2/a} \quad (7.1)
\]

and upon the gaussian integral (7.2)

\[
\int_{-\infty}^{\infty} e^{-ax^2+2ibx} \, dx = \sqrt{\frac{\pi}{a}} e^{-b^2/a} \quad (7.2)
\]

which holds both for \( \text{Re} \, a > 0 \) and also for \( \text{Re} \, a = 0 \) with \( b \) real and \( \text{Im} \, a \neq 0 \).

The extension of the integral formula (7.1) to any \( n \times n \) real symmetric nonsingular matrix \( s_{jk} \) and any real vector \( c_j \) is (exercises ?? & ??)

\[
\int_{-\infty}^{\infty} e^{is_{jk}x_jx_k+2ic_jx_j} \, dx_1 \ldots dx_n = \sqrt{\frac{(i\pi)^n}{\det s}} e^{-ic_j(s^{-1})_{jk}c_k} \quad (7.3)
\]
in which \( \det a \) is the determinant of the matrix \( a \), \( a^{-1} \) is its inverse, and sums over the repeated indices \( j \) and \( k \) from 1 to \( n \) are understood. One may similarly extend the gaussian integral (7.2) to any positive symmetric \( n \times n \) matrix \( s_{jk} \) and any vector \( c_j \) (exercises ?? & ??)

\[
\int_{-\infty}^{\infty} e^{-s_{jk}x_jx_k + 2ic_jx_j} dx_1 \ldots dx_n = \sqrt{\frac{\pi^n}{\det s}} e^{-c_j(s^{-1})_{jk}c_k}.
\] (7.4)

Path integrals also are based upon Trotter’s product formula (Trotter, 1959; Kato, 1978)

\[
e^{a+b} = \lim_{n \to \infty} \left( e^{a/n} e^{b/n} \right)^n
\] (7.5)

both sides of which are symmetrically ordered and obviously equal when \( ab = ba \).

Separating a given hamiltonian \( H = K + V \) into a kinetic part \( K \) and a potential part \( V \), we can use Trotter’s formula to write the time-evolution operator \( e^{-itH/\hbar} \) as

\[
e^{-it(K+V)/\hbar} = \lim_{n \to \infty} \left( e^{-itK/(n\hbar)} e^{-itV/(n\hbar)} \right)^n
\] (7.6)

and the Boltzmann operator \( e^{-\beta H} \) as

\[
e^{-\beta(K+V)} = \lim_{n \to \infty} \left( e^{-\beta K/n} e^{-\beta V/n} \right)^n.
\] (7.7)

7.3 Path integrals in quantum mechanics

Path integrals can represent matrix elements of the time-evolution operator \( \exp(-i(t_b - t_a)H/\hbar) \) in which \( H \) is the hamiltonian. For a particle of mass \( m \) moving nonrelativistically in one dimension in a potential \( V(q) \), the hamiltonian is

\[
H = \frac{p^2}{2m} + V(q).
\] (7.8)

The position and momentum operators \( q \) and \( p \) obey the commutation relation \([q, p] = i\hbar\). Their eigenstates \(|q'\rangle \) and \(|p'\rangle \) have eigenvalues \( q' \) and \( p' \) for all real numbers \( q' \) and \( p' \)

\[
q |q'\rangle = q' |q'\rangle \quad \text{and} \quad p |p'\rangle = p' |p'\rangle.
\] (7.9)

These eigenstates are complete. Their outer products \(|q'\rangle \langle q'| \) and \(|p'\rangle \langle p'| \)
provide expansions for the identity operator $I$ and have inner products (??) that are phases

$$I = \int_{-\infty}^{\infty} |q'| \langle q' | dq' = \int_{-\infty}^{\infty} |p'| \langle p' | dp'$$

and $\langle q' | p' \rangle = \frac{e^{iq'p'/\hbar}}{\sqrt{2\pi\hbar}}$. (7.10)

Setting $\epsilon = (t_b - t_a)/n$ and writing the hamiltonian (7.8) over $\hbar$ as $H/\hbar = p^2/(2m\hbar) + V/\hbar = k + v$, we can write Trotter’s formula (7.6) for the time-evolution operator as the limit as $n \to \infty$ of $n$ factors of $e^{-i\epsilon k e^{-i\epsilon v}}$

$$e^{-i(t_b-t_a)(k+v)} = e^{-i\epsilon k} e^{-i\epsilon v} e^{-i\epsilon k} e^{-i\epsilon v} \ldots e^{-i\epsilon k} e^{-i\epsilon v}. \quad (7.11)$$

The advantage of using Trotter’s formula is that we now can evaluate the matrix element $\langle q_1 | e^{-i\epsilon k} e^{-i\epsilon v} | q_a \rangle$ between eigenstates $| q_a \rangle$ and $| q_1 \rangle$ of the position operator $q$ by inserting the momentum-state expansion (7.10) of the identity operator $I$ between these two exponentials

$$\langle q_1 | e^{-i\epsilon k} e^{-i\epsilon v} | q_a \rangle = \langle q_1 | e^{-i\epsilon p^2/(2m\hbar)} \int_{-\infty}^{\infty} |p'\rangle \langle p'| dp' e^{-i\epsilon V(q)/\hbar} | q_a \rangle \quad (7.12)$$

and using the eigenvalue formulas (7.9)

$$\langle q_1 | e^{-i\epsilon k} e^{-i\epsilon v} | q_a \rangle = \int_{-\infty}^{\infty} e^{-i\epsilon p^2/(2m\hbar)} \langle q_1 | p' \rangle e^{-i\epsilon V(q)/\hbar} \langle p' | q_a \rangle dp'. \quad (7.13)$$

Now using the formula (7.10) for the inner product $\langle q_1 | p' \rangle$ and the complex conjugate of that formula for $\langle p' | q_a \rangle$, we get

$$\langle q_1 | e^{-i\epsilon k} e^{-i\epsilon v} | q_a \rangle = e^{-i\epsilon V(q_a)/\hbar} \int_{-\infty}^{\infty} e^{-i\epsilon p^2/(2m\hbar)} e^{i(q_1-q_a)p'/\hbar} \frac{dp'}{2\pi\hbar}. \quad (7.14)$$

In this integral, the momenta that are important are very high, being of order $\sqrt{m\hbar/\epsilon}$ which diverges as $\epsilon \to 0$; nonetheless, the integral converges.

If we adopt the suggestive notation $q_1 - q_a = \epsilon \dot{q}_a$ and use the gaussian integral (7.11) with $a = -\epsilon/(2m\hbar)$, $x = p$, and $b = \epsilon \dot{q}_a/(2\hbar)$

$$\int_{-\infty}^{\infty} \exp \left( -i \frac{\epsilon p^2}{2m\hbar} + i \epsilon \frac{\dot{q}_a p}{\hbar} \right) \frac{dp}{2\pi\hbar} = \sqrt{\frac{m}{2\pi\epsilon \hbar}} \exp \left( i \frac{\epsilon m \dot{q}_a^2}{2} \right), \quad (7.15)$$

then we find

$$\langle q_1 | e^{-i\epsilon k} e^{-i\epsilon v} | q_a \rangle = \frac{1}{2\pi\hbar} e^{-\epsilon V(q_a)} \int_{-\infty}^{\infty} \exp \left( -i \frac{\epsilon p^2}{2m\hbar} + i \epsilon \frac{\dot{q}_a p}{\hbar} \right) dp'$$

$$= \left( \frac{m}{2\pi\epsilon \hbar} \right)^{1/2} \exp \left[ i \frac{\epsilon}{\hbar} \left( \frac{m \dot{q}_a^2}{2} - V(q_a) \right) \right]. \quad (7.16)$$

The dependence of the amplitude $\langle q_1 | e^{-i\epsilon k} e^{-i\epsilon v} | q_a \rangle$ upon $q_1$ is hidden in the formula $\dot{q}_a = (q_1 - q_a)/\epsilon$. 

Path integrals
The next step is to use the position-state expansion (7.10) of the identity operator to link two of these matrix elements together

\[
\langle q_2 | (e^{-i\epsilon k} e^{-i\epsilon v})^2 | q_a \rangle = \int_{-\infty}^{\infty} \langle q_2 | e^{-i\epsilon k} e^{-i\epsilon v} | q_1 \rangle \langle q_1 | e^{-i\epsilon k} e^{-i\epsilon v} | q_a \rangle \, dq_1
\]

\[
= \frac{m}{2\pi i\hbar} \int_{-\infty}^{\infty} \exp \left[ i \frac{\epsilon}{\hbar} \left( \frac{mq_1^2}{2} - V(q_1) + \frac{mq_a^2}{2} - V(q_a) \right) \right] \, dq_1
\]

(7.17)

where now \( \dot{q}_1 = (q_2 - q_1)/\epsilon \).

By stitching together \( n = (t_b - t_a)/\epsilon \) time intervals each of length \( \epsilon \) and letting \( n \to \infty \), we get

\[
\langle q_b | e^{-ni\epsilon H/\hbar} | q_a \rangle = \int \langle q_b | e^{-i\epsilon k} e^{-i\epsilon v} | q_{n-1} \rangle \cdots \langle q_1 | e^{-i\epsilon k} e^{-i\epsilon v} | q_a \rangle \, dq_{n-1} \cdots dq_1
\]

\[
= \left( \frac{m}{2\pi i\hbar} \right)^{n/2} \int \exp \left[ i \frac{n}{\hbar} \sum_{j=0}^{n-1} \frac{mq_j^2}{2} - V(q_j) \right] \, dq_{n-1} \cdots dq_1
\]

\[
= \left( \frac{m}{2\pi i\hbar} \right)^{n/2} \int \exp \left( i \frac{n}{\hbar} \sum_{j=0}^{n-1} L_j \right) \, dq_{n-1} \cdots dq_1
\]

(7.18)

in which \( L_j = mq_j^2/2 - V(q_j) \) is the lagrangian of the \( j \)th interval, and the \( q_j \) integrals run from \(-\infty\) to \( \infty \). In the limit \( \epsilon \to 0 \) with \( n\epsilon = (t_b - t_a)/\epsilon \), this multiple integral is an integral over all paths \( q(t) \) that go from \( q_a, t_a \) to \( q_b, t_b \)

\[
\langle q_b | e^{-i(t_b-t_a)H/\hbar} | q_a \rangle = \int \exp[iS[q]/\hbar] \, Dq
\]

(7.19)

in which each path is weighted by the phase of its classical action

\[
S[q] = \int_{t_a}^{t_b} L(q, \dot{q}) \, dt = \int_{t_a}^{t_b} \left( \frac{\dot{q}^2}{2} - V(q(t)) \right) \, dt
\]

(7.20)

in units of \( \hbar \) and \( Dq = (mn/(2\pi i\hbar(t_b - t_a)))^{n/2} \, dq_{n-1} \cdots dq_1 \).

If we multiply the path-integral (7.19) for \( \langle q_b | e^{-i(t_b-t_a)H/\hbar} | q_a \rangle \) from the left by \( |q_b\rangle \) and from the right by \( \langle q_a| \) and integrate over \( q_a \) and \( q_b \) as in the resolution (7.10) of the identity operator, then we can write the time-evolution operator as an integral over all paths from \( t_a \) to \( t_b \)

\[
e^{-i(t_b-t_a)H/\hbar} = \int |q_b\rangle \exp[iS[q]/\hbar] \langle q_a| \, Dq \, dq_a \, dq_b
\]

(7.21)

with \( Dq = (mn/(2\pi i\hbar(t_b - t_a)))^{n/2} \, dq_{n-1} \cdots dq_1 \) and \( S[q] \) the action (7.20).
Path integrals

The path integral for a particle moving in three-dimensional space is

$$\langle q_b | e^{-i(t_b - t_a)H/\hbar} | q_a \rangle = \int \exp \left[ \frac{i}{\hbar} \int_{t_a}^{t_b} \frac{1}{2} m \dot{q}^2(t) - V(q(t)) \, dt \right] Dq$$  \hspace{1cm} (7.22)

where \( Dq = (\pi m / (2\pi i \hbar(t_b - t_a)))^{3n/2} dq_{n-1} \cdots dq_1 \).

Let us first consider macroscopic processes whose actions are large compared to \( \hbar \). Apart from the factor \( Dq \), the amplitude (7.22) is a sum of phases \( e^{iS[q]}/\hbar \) one for each path from \( q_a, t_a \) to \( q_b, t_b \). When is this amplitude big? When is it small? Suppose there is a path \( q_c(t) \) from \( q_a, t_a \) to \( q_b, t_b \) that obeys the classical equation of motion (7.23)

\[
\frac{\delta S[q_c]}{\delta q_{jc}} = m \ddot{q}_{jc} + V'(q_c) = 0.
\]  \hspace{1cm} (7.23)

Its action may be minimal. It certainly is stationary: a path \( q_c(t) + \delta q(t) \) that differs from \( q_c(t) \) by a small detour \( \delta q(t) \) has an action \( S[q_c + \delta q] \) that differs from \( S[q_c] \) only by terms of second order and higher in \( \delta q \). Thus a classical path has infinitely many neighboring paths whose actions differ only by integrals of \( (\delta q)^n, n \geq 2 \), and so have the same action to within a small fraction of \( \hbar \). These paths add with nearly the same phase to the path integral (7.22) and so make a huge contribution to the amplitude \( \langle q_b | e^{-i(t_b - t_a)H/\hbar} | q_a \rangle \). But if no classical path goes from \( q_a, t_a \) to \( q_b, t_b \), then the nonclassical, nonstationary paths that go from \( q_a, t_a \) to \( q_b, t_b \) have actions that differ from each other by large multiples of \( \hbar \). These amplitudes cancel each other, and their sum, which is amplitude for going from \( q_a, t_a \) to \( q_b, t_b \), is small. Thus the path-integral formula for an amplitude in quantum mechanics explains why macroscopic processes are described by the principle of stationary action (section 7.2).

What about microscopic processes whose actions are tiny compared to \( \hbar \)? The path integral (7.22) gives large amplitudes for all microscopic processes. On very small scales, anything can happen that doesn’t break a conservation law.

The path integral for two or more particles \( \{q\} = \{q_1, \ldots, q_k\} \) interacting with a potential \( V(\{q\}) \) is

\[
\langle \{q\}_b | e^{-i(t_b - t_a)H/\hbar} | \{q\}_a \rangle = \int e^{iS[\{q\}]/\hbar} D\{q\}
\]  \hspace{1cm} (7.24)

where

\[
S[\{q\}] = \int_{t_a}^{t_b} \left[ \frac{m_1 \dot{q}_1^2(t)}{2} + \cdots + \frac{m_k \dot{q}_k^2(t)}{2} - V(\{q(t)\}) \right] dt
\]  \hspace{1cm} (7.25)

and \( D\{q\} = Dq_1 \cdots Dq_k \).
Example 7.1 (A free particle) For a free particle, the potential is zero, and the path integral (7.19, 7.20) is the $\epsilon \to 0, n \to \infty$ limit of

$$\langle q_b | e^{-i t H/\hbar} | q_a \rangle = \left( \frac{m}{2\pi i \hbar \epsilon} \right)^{n/2} \times \exp \left[ \frac{im}{2\hbar} \left( \frac{(q_n - q_{n-1})^2}{\epsilon^2} + \cdots + \frac{(q_1 - q_0)^2}{\epsilon^2} \right) \right] dq_{n-1} \cdots dq_1. \quad (7.26)$$

The $q_1$ integral is by the gaussian formula (7.1)

$$\frac{m}{2\pi i \hbar \epsilon} \int e^{im[(q_2 - q_1)^2 + (q_1 - q_0)^2]/(2\hbar \epsilon)} dq_1 = \sqrt{\frac{m}{2\pi i \hbar 2 \epsilon} e^{im(q_2 - q_0)^2/(2\hbar \epsilon)}}. \quad (7.27)$$

The $q_2$ integral is (exercise ??)

$$\frac{m}{2\sqrt{2} \pi i \hbar \epsilon} \int e^{im[(q_3 - q_2)^2 + (q_2 - q_1)^2]/(2\hbar \epsilon)} dq_2 = \sqrt{\frac{m}{2\pi i \hbar 3 \epsilon} e^{im(q_3 - q_0)^2/(2\hbar \epsilon)}}. \quad (7.28)$$

Doing all $n-1$ integrals (7.26) in this way and setting $n \epsilon = t_b - t_a$, we get

$$\langle q_b | e^{-i(t_b - t_a) H/\hbar} | q_a \rangle = \sqrt{\frac{m}{2\pi i \hbar \epsilon} \exp \left[ \frac{im(q_b - q_a)^2}{2\hbar \epsilon} \right]} \quad (7.29)$$

The path integral (7.26) is perfectly convergent even though the velocities $\dot{q}_j = (q_{j+1} - q_j)/\epsilon$ that are important are very high, being of order $\sqrt{\hbar/(me)}$.

It is easier to compute this amplitude (7.29) by using the outer products (7.10) (exercise ??).

In three dimensions, the amplitude to go from $q_a, t_a$ to $q_b, t_b$ is

$$\langle q_b | e^{-i(t_b - t_a) H/\hbar} | q_0 \rangle = \left( \frac{m}{2\pi i \hbar (t_b - t_a)} \right)^{3/2} \exp \left[ \frac{im(q_b - q_a)^2}{2\hbar (t_b - t_a)} \right]. \quad (7.30)$$

7.4 Path integrals for quadratic actions

If a path $q(t) = q_c(t) + x(t)$ differs from a classical path $q_c(t)$ by a detour $x(t)$ that vanishes at the endpoints $x(t_a) = 0 = x(t_b)$ so that both paths go from $q_a, t_a$ to $q_b, t_b$, then the difference $S[q_c + x] - S[q_c]$ in their actions vanishes to first order in the detour $x(t)$ (section ??). Thus the actions of
the two paths differ by a time integral of quadratic and higher powers of the detour $x(t)$

\[
S[q_c + x] = \int_{t_a}^{t_b} \frac{1}{2} m \dot{q}(t)^2 - V(q(t)) \, dt \\
= \int_{t_a}^{t_b} \frac{1}{2} m (\dot{q}_c(t) + \dot{x}(t))^2 - V(q_c(t) + x(t)) \, dt \\
= \int_{t_a}^{t_b} \left[ \frac{m}{2} \dot{q}_c^2 + m \dot{q}_c \dot{x} + \frac{m}{2} \dot{x}^2 - V(q_c) - V'(q_c) x - \frac{V''(q_c)}{2} x^2 \\
- \frac{V'''(q_c)}{6} x^3 - \frac{V''''(q_c)}{24} x^4 - \ldots \right] \, dt \\
= \int_{t_a}^{t_b} \left[ \frac{m}{2} \dot{q}_c^2 - V(q_c) \right] \, dt + \int_{t_a}^{t_b} \left[ \frac{m}{2} \dot{x}^2 - \frac{V''(q_c)}{2} x^2 \\
- \frac{V'''(q_c)}{6} x^3 - \frac{V''''(q_c)}{24} x^4 - \ldots \right] \, dt \\
= S[q_c] + \Delta S[q_c, x]
\]

in which $S[q_c]$ is the action of the classical path, and the detour $x(t)$ is a loop that goes from $x(t_a) = 0$ to $x(t_b) = 0$.

If the potential $V(q)$ is quadratic in the position $q$, then the third $V'''$ and higher derivatives of the potential vanish, and the second derivative is a constant $V''(q_c(t)) = V''$. In this quadratic case, the correction $\Delta S[q_c, x]$ depends only on the time interval $t_b - t_a$ and on $\hbar, m, \text{ and } V''$

\[
\Delta S[q_c, x] = \Delta S[x] = \int_{t_a}^{t_b} \left[ \frac{1}{2} m \ddot{x}(t) - \frac{1}{2} V'' x^2(t) \right] \, dt. \tag{7.32}
\]

It is independent of the classical path.

Thus for quadratic actions, the path integral (7.19) is an exponential of the action $S[q_c]$ of the classical path multiplied by a function $f(t_b - t_a, h, m, V'')$ of the time interval $t_b - t_a$ and of $h, m, \text{ and } V''$

\[
\langle q_b | e^{-i(t_b-t_a)H/\hbar} | q_a \rangle = \int e^{iS[q]/\hbar} Dq = \int e^{i(S[q_c]+\Delta S[x]/\hbar)} Dq \\
= e^{iS[q_c]/\hbar} \int e^{i\Delta S[x]/\hbar} Dx \\
= f(t_b - t_a, h, m, V'') e^{iS[q_c]/\hbar}. \tag{7.33}
\]

The function $f = f(t_b - t_a, h, m, V'')$ is the limit as $n \to \infty$ of the $(n - 1)$-
dimensional integral
\[ f = \left[ \frac{mn}{2\pi i\hbar(t_b - t_a)} \right]^{n/2} \int e^{i\Delta S[x]/\hbar} dx_{n-1} \ldots dx_1 \] (7.34)

where
\[ \Delta S[x] = \frac{t_b - t_a}{n} \sum_{j=1}^{n} \frac{1}{2} m \left( x_j - x_{j-1} \right)^2 - \frac{1}{2} V'' x_j^2 \] (7.35)

and \( x_n = 0 = x_0 \).

More generally, the path integral for any quadratic action of the form
\[ S[q] = \int_{t_a}^{t_b} \left[ \frac{1}{2} m \dot{q}^2(t) + v q(t) \dot{q}(t) + w q^2(t) + s(t) \dot{q}(t) + j(t) q(t) \right] dt \] (7.36)
is (exercise ??)
\[ \langle q_b | e^{-i(t_b-t_a)H/\hbar} | q_a \rangle = f(t_a, t_b, \hbar, u, v, w) e^{iS[q_c]/\hbar}. \] (7.37)

The dependence of the amplitude upon \( s(t) \) and \( j(t) \) is contained in the classical action \( S[q_c] \).

These formulas (7.33–7.37) may be generalized to any number of particles with coordinates \( \{q\} = \{q^1, \ldots, q^k\} \) moving nonrelativistically in a space of multiple dimensions as long as the action is quadratic in the \( \{q\}'s \) and their velocities \( \{\dot{q}\} \). The amplitude is then an exponential of the action \( S[\{q\}_c] \) of the classical path multiplied by a function \( f(t_a, t_b, \hbar, \ldots) \) that is independent of the classical path \( q_c \)
\[ \langle \{q\}_b | e^{-i(t_b-t_a)H/\hbar} | \{q\}_a \rangle = f(t_a, t_b, \hbar, \ldots) e^{iS[\{q\}_c]/\hbar}. \] (7.38)

**Example 7.2** (A free particle) The classical path of a free particle going from \( q_a \) at time \( t_a \) to \( q_b \) at time \( t_b \) is
\[ q_c(t) = q_a + \frac{t - t_a}{t_b - t_a} (q_b - q_a). \] (7.39)

Its action is
\[ S[q_c] = \int_{t_a}^{t_b} \frac{1}{2} m \dot{q}_c^2 dt = \frac{m(q_b - q_a)^2}{2(t_b - t_a)} \] (7.40)

and for this case our quadratic-potential formula (7.38) is
\[ \langle q_b | e^{-i(t_b-t_a)H/\hbar} | q_a \rangle = f(t_b - t_a, \hbar, m) \exp \left[ i \frac{m(q_b - q_a)^2}{2\hbar(t_b - t_a)} \right] \] (7.41)

which agrees with our explicit calculation (7.30) when \( f(t_b - t_a, \hbar, m) = [m/(2\pi i\hbar(t_b - t_a))]^{3/2} \). \( \square \)
Example 7.3 (Bohm-Aharonov effect) From the formula (??) for the action of a relativistic particle of mass \( m \) and charge \( e \), it follows (exercise ??) that the action a nonrelativistic particle in an electromagnetic field with no scalar potential is

\[
S = \int_{t_a}^{t_b} \left[ \frac{1}{2} m \dot{q}^2 + e A \cdot \dot{q} \right] dt
\]

Since this action is quadratic in \( \dot{q} \), the amplitude for a particle to go from \( q_a \) at \( t_a \) to \( q_b \) at \( t_b \) is an exponential of the classical action

\[
\langle q_b | e^{-i \frac{(t_b - t_a)H}{\hbar}} | q_a \rangle = f(t_b - t_a, \hbar, m, e) e^{i S[q_c]/\hbar}
\]

multiplied by a function \( f(t_b - t_a, \hbar, m, e) \) that is independent of the path \( q_c \). A beam of such particles goes horizontally past but not through a vertical pipe in which a vertical magnetic field is confined. The particles can go both ways around the pipe of cross-sectional area \( S \) but do not enter it. The difference in the phases of the amplitudes for the two paths is a loop integral

\[
\oint \left[ \frac{m \dot{q}^2}{2} + e A \cdot dq \right] \frac{\hbar}{2} = \oint \frac{m \dot{q} \cdot dq}{2} \frac{\hbar}{2} + e \int_S B \cdot dS = \oint \frac{m \dot{q} \cdot dq}{2} \frac{\hbar}{2} + e \Phi \frac{\hbar}{2}
\]

in which \( \Phi \) is the magnetic flux through the cylinder. \( \Box \)

Example 7.4 (Harmonic oscillator) The action

\[
S = \int_{t_a}^{t_b} \frac{1}{2} m q^2(t) - \frac{1}{2} m \omega^2 q^2(t) dt
\]

of a harmonic oscillator is quadratic in \( q \) and \( \dot{q} \). So apart from a factor \( f \), its path integral (7.33–7.35) is an exponential

\[
\langle q_b | e^{-i \frac{(t_b - t_a)H}{\hbar}} | q_a \rangle = f(t_b - t_a, \hbar, m, m\omega^2) e^{i S[q_c]/\hbar}
\]

of the action \( S[q_c] \) (exercise ??)

\[
S[q_c] = m \omega \left[ (q_a^2 + q_b^2) \cos(\omega(t_b - t_a)) - 2q_aq_b \right] \frac{2}{2 \sin(\omega(t_b - t_a))}
\]

of the classical path

\[
q_c(t) = q_a \cos(\omega(t - t_a)) + \frac{q_b - q_a \cos(\omega(t_b - t_a))}{\sin(\omega(t_b - t_a))} \sin(\omega(t - t_a))
\]

that runs from \( q_a, t_a \) to \( q_b, t_b \) and obeys the classical equation of motion

\[
m \ddot{q}_c(t) = -\omega^2 q_c(t).
\]

The factor \( f \) is a function \( f(t_b - t_a, \hbar, m, m\omega^2) \) of the time interval and the
parameters of the oscillator. It is the \( n \to \infty \) limit of the \((n-1)\)-dimensional integral \([7.34]\)

\[
f = \left[ \frac{mn}{2\pi i \hbar (t_b - t_a)} \right]^{n/2} \int e^{i\Delta S[x]/\hbar} dx_{n-1} \ldots dx_1
\]

(7.49)

over all loops that run from 0 to 0 in time \( t_b - t_a \) in which the quadratic correction to the classical action is \([7.35]\)

\[
\Delta S[x] = \frac{t_b - t_a}{n} \sum_{j=1}^{n} \frac{1}{2} m \left( \frac{x_j - x_{j-1}}{n} \right)^2 - \frac{1}{2} m \omega^2 x_j^2;
\]

(7.50)

and \( x_n = 0 = x_0 \).

Setting \( t_b - t_a = T \), we use the many-variable imaginary gaussian integral \([7.3]\) to write \( f \) as

\[
f = \left[ \frac{mn}{2\pi i \hbar T} \right]^{n/2} \int e^{ia_{jk}x_jx_k} dx_{n-1} \ldots dx_1 = \left[ \frac{mn}{2\pi i \hbar T} \right]^{n/2} \sqrt{\frac{(i\pi)^{n-1}}{\det a}}
\]

(7.51)

in which the quadratic form \( a_{jk}x_jx_k \) is

\[
\frac{nm}{\hbar T} \sum_{j=1}^{n} \left[ -y_j x_{j-1} + \frac{1}{2} (x_j^2 + x_{j-1}^2) - \frac{(\omega T)^2}{2n^2} x_j^2 \right]
\]

(7.52)

which has no linear term because \( x_0 = x_n = 0 \).

The \((n-1)\)-dimensional square matrix \( a \) is a tridiagonal Toeplitz matrix

\[
a = \frac{nm}{2\hbar T} \begin{pmatrix}
y & -1 & 0 & 0 & \cdots \\
-1 & y & -1 & 0 & \cdots \\
0 & -1 & y & -1 & \cdots \\
0 & 0 & -1 & y & \cdots \\
\vdots & \vdots & \vdots & \ddots & \ddots
\end{pmatrix}
\]

(7.53)

Apart from the factor \( nm/(2\hbar T) \), the matrix \( a = (nm/(2\hbar T)) C_{n-1}(y) \) is a tridiagonal matrix \( C_{n-1}(y) \) whose off-diagonal elements are \(-1\) and whose diagonal elements are

\[
y = 2 - \frac{(\omega T)^2}{n^2}.
\]

(7.54)

Their determinants \( |C_n(y)| = \det C_n(y) \) obey (exercise ??) the recursion relation

\[
|C_{n+1}(y)| = y |C_n(y)| - |C_{n-1}(y)|
\]

(7.55)
Path integrals

and have the initial values $|C_1(y)| = y$ and $|C_2(y)| = y^2 - 1$. The trigonometric functions $U_n(y) = \sin[(n+1)\theta]/\sin \theta$ with $y = 2 \cos \theta$ obey the same recursion relation and have the same initial values (exercise ??), so

$$|C_n(y)| = \frac{\sin(n+1)\theta}{\sin \theta}. \quad (7.56)$$

Since for large $n$

$$\theta = \arccos(y/2) = \arccos \left(1 - \frac{\omega^2 t^2}{2n^2}\right) \approx \frac{\omega T}{n}, \quad (7.57)$$

the determinant of the matrix $a$ is

$$\det a = \left(\frac{nm}{2\hbar T}\right)^{n-1} \frac{|C_{n-1}(y)|}{\sin n\theta} = \left(\frac{nm}{2\hbar T}\right)^{n-1} \frac{\sin n\theta}{\sin(\omega T/ n)} \approx \left(\frac{nm}{2\hbar T}\right)^{n-1} \frac{n \sin \omega T}{\omega T}. \quad (7.58)$$

Thus the factor $f$ is

$$f = \sqrt{\frac{m \omega}{2\pi i \hbar \sin \omega T}}. \quad (7.59)$$

The amplitude (7.46) is then an exponential of the action $S[q_c] \quad (7.47)$ of the classical path (7.48) multiplied by this factor $f$

$$\langle q_b | e^{-i(t_b-t_a)H/\hbar} | q_a \rangle = \sqrt{\frac{m \omega}{2\pi i \hbar \sin \omega(t_b-t_a)}} \exp \left\{ i \frac{m \omega}{\hbar} \left(\frac{q_a^2 + q_b^2}{2} \cos(\omega(t_b-t_a)) - 2q_a q_b \right) \right\}.$$ \quad (7.60)

As these examples (7.2 & 7.4) suggest, path integrals are as mathematically well defined as ordinary integrals.

7.5 Path integrals in statistical mechanics

At the imaginary time $t = -i\hbar \beta = -i\hbar/(kT)$, the time-evolution operator $e^{-itH/\hbar}$ becomes the Boltzmann operator $e^{-\beta H}$ whose trace is the
partition function $Z(\beta)$ at inverse energy $\beta = 1/(kT)$

$$Z(\beta) = \text{Tr}(e^{-\beta H}) = \sum_n \langle n|e^{-\beta H}|n\rangle$$

(7.61)

in which the states $|n\rangle$ form a complete orthonormal set, $k = 8.617 \times 10^{-5}$ eV/K is Boltzmann’s constant, and $T$ is the absolute temperature. Partition functions are important in statistical mechanics and quantum field theory.

Since the Boltzmann operator $e^{-\beta H}$ is the time-evolution operator $e^{-itH/\hbar}$ at the imaginary time $t = -i\hbar\beta$, we can write it as a path integral by imitating the derivation of the preceding section (7.5). We will use the same hamiltonian $H = p^2/(2m) + V(q)$ and the operators $q$ and $p$ which have complete sets of eigenstates (7.6) that satisfy (7.10).

Changing our definitions of $\epsilon$, $k$, and $v$ to $\epsilon = \beta/n$, $k = \beta p^2/(2m)$, and $v = \beta V(q)$, we can write Trotter’s formula (7.7) for the Boltzmann operator as the $n \to \infty$ limit of $n$ factors of $e^{-\epsilon k} e^{-\epsilon v}$

$$e^{-\beta H} = e^{-\epsilon k} e^{-\epsilon v} e^{-\epsilon k} e^{-\epsilon v} \cdots e^{-\epsilon k} e^{-\epsilon v}.$$  

(7.62)

To evaluate the matrix element $\langle q_1|e^{-\epsilon k} e^{-\epsilon v}|q_a\rangle$, we insert the identity operator $\langle q_1|e^{-\epsilon k} I e^{-\epsilon v}|q_a\rangle$ as an integral (7.10) over outer products $|p\rangle \langle p'|$ of momentum eigenstates and use the inner products $\langle q_1|p\rangle = e^{i\epsilon q_1 p/\hbar}/\sqrt{2\pi\hbar}$ and $\langle p'|q_a\rangle = e^{-i\epsilon q_a p'/\hbar}/\sqrt{2\pi\hbar}$

$$\langle q_1|e^{-\epsilon k} e^{-\epsilon v}|q_a\rangle = \int_{-\infty}^{\infty} \langle q_1|e^{-\epsilon p^2/(2m)}|p\rangle \langle p|e^{-\epsilon V(q)}|q_a\rangle \, dp'$$

$$= e^{-\epsilon V(q_a)} \int_{-\infty}^{\infty} e^{-\epsilon p^2/(2m)} e^{i\epsilon (q_1 - q_a)/\hbar} \frac{dp'}{2\pi\hbar}. \tag{7.63}$$

If we adopt the suggestive notation $q_1 - q_a = \hbar \dot{q}_a$ and use the gaussian integral (7.2) with $a = \epsilon/(2m)$, $x = p$, and $b = \epsilon \dot{q}_a/2$

$$\int_{-\infty}^{\infty} \exp\left(-\frac{\epsilon p^2}{2m} + i\epsilon \dot{q}_a p\right) \frac{dp}{2\pi\hbar} = \sqrt{\frac{m}{2\pi\epsilon \hbar^2}} \exp\left(-\epsilon \frac{m \dot{q}_a^2}{2}\right), \tag{7.64}$$

then we find

$$\langle q_1|e^{-\epsilon k} e^{-\epsilon v}|q_a\rangle = e^{-\epsilon V(q_a)} \int_{-\infty}^{\infty} \exp\left(-\frac{\epsilon p^2}{2m} + i\epsilon p' \dot{q}_a\right) \frac{dp'}{2\pi\hbar}$$

$$= \left(\frac{m}{2\pi\hbar^2 \epsilon}\right)^{1/2} \exp\left[-\epsilon \left(\frac{m \dot{q}_a^2}{2} + V(q_a)\right)\right] \tag{7.65}$$

in which $q_1$ is hidden in the formula $q_1 - q_a = \hbar \epsilon \dot{q}_a$. 


The next step is to link two of these matrix elements together

\[ \langle q_2 | (e^{-\epsilon k} e^{-\epsilon v})^2 | q_a \rangle = \int_{-\infty}^{\infty} \exp \left\{ -\epsilon \left[ \frac{m q_1^2}{2} + V(q_1) + \frac{m q_0^2}{2} + V(q_a) \right] \right\} dq_1. \] (7.66)

Passing from 2 to \( n \) and suppressing some integral signs, we get

\[ \langle q_b | e^{-n\epsilon H} | q_a \rangle = \int_{-\infty}^{\infty} \exp \left\{ -\epsilon \sum_{j=0}^{n-1} \left( \frac{m q_j^2}{2} + V(q_j) \right) \right\} dq_1 \ldots dq_1. \] (7.67)

in which each path is weighted by its euclidian action

\[ S_e[q] = \int_0^{h \beta} \frac{m \dot{q}(u)^2}{2} + V(q(u)) \ du, \] (7.68)

\( \dot{q} \) is the derivative of the coordinate \( q(u) \) with respect to euclidian time \( u = \hbar \beta \), and \( Dq \equiv (n m/2 \pi \hbar^2 \beta)^{n/2} dq_{n-1} \ldots dq_1 \).

A derivation identical to the one that led from (7.62) to (7.68) leads in a more elaborate notation to

\[ \langle q_b | e^{-(-\beta_b - \beta_a) H} | q_a \rangle = \int e^{-S_e[q]/\hbar} Dq \] (7.69)

in which each path is weighted by its euclidian action

\[ S_e[q] = \int_{h \beta_a}^{h \beta_b} \frac{m \dot{q}(u)^2}{2} + V(q(u)) \ du, \] (7.70)

and \( \dot{q} \) and \( Dq \) are the same as in (7.68).

If we multiply the path integral (7.69) from the left by \( |q_b\rangle \) and from the right by \( \langle q_a| \) and integrate over \( q_a \) and \( q_b \) as in the resolution (7.10) of the identity operator, then we can write the Boltzmann operator as an integral over all paths from \( t_a \) to \( t_b \)

\[ e^{-(\beta_b - \beta_a) H} = \int |q_b\rangle e^{-S_e[q]/\hbar} \langle q_a| \ Dq \ dq_a \ dq_b \] (7.71)

with \( Dq = (mn/(2\pi i\hbar(t_b - t_a)))^{n/2} dq_{n-1} \ldots dq_1 \) and \( S_e[q] \) the action (7.70).
To get the partition function $Z(\beta)$, we set $q_b = q_a \equiv q_n$ and integrate over all $n$ $q$’s letting $n \rightarrow \infty$

$$Z(\beta) = \text{Tr} \, e^{-\beta H} = \int \langle q_n | e^{-\beta H} | q_n \rangle \, dq_n$$

$$= \int \exp \left[ -\frac{1}{\hbar} \int_0^{\hbar \beta} \frac{m \dot{q}^2(u)}{2} + V(q(u)) \, du \right] \, Dq \quad (7.72)$$

where $Dq \equiv (n m / 2 \pi \hbar^2 \beta)^{n/2} dq_n \cdots dq_1$. We sum over all loops $q(u)$ that go from $q(0) = q_n$ at euclidian time 0 to $q(h \beta) = q_n$ at euclidian time $\hbar \beta$.

In the low-temperature limit, $T \rightarrow 0$ and $\beta \rightarrow \infty$, the Boltzmann operator $\exp(-\beta H)$ projects out the ground state $|E_0\rangle$ of the system

$$\lim_{\beta \rightarrow \infty} e^{-\beta H} = \lim_{\beta \rightarrow \infty} \sum_n e^{-\beta E_n} |E_n\rangle \langle E_n| = e^{-\beta E_0} |E_0\rangle \langle E_0| \quad (7.73)$$

The maximum-entropy density operator (section ???, example ???) is the Boltzmann operator $e^{-\beta H}$ divided by its trace $Z(\beta)$

$$\rho = e^{-\beta H} / \text{Tr}(e^{-\beta H}) = e^{-\beta H} / Z(\beta) \quad (7.74)$$

Its matrix elements are matrix elements of Boltzmann operator (7.68) divided by the partition function (7.72)

$$\langle q_b | \rho | q_a \rangle = \frac{\langle q_b | e^{-\beta H} | q_a \rangle}{Z(\beta)} \quad (7.75)$$

In three dimensions with $\dot{q}(u) = dq(u)/du$, the $q_a, q_b$ matrix element of the Boltzmann operator is the analog of equation (7.68) (exercise ???)

$$\langle q_b | e^{-\beta H} | q_a \rangle = \int \exp \left[ -\frac{1}{\hbar} \int_0^{\hbar \beta} \frac{m \dot{q}^2(u)}{2} + V(q(u)) \, du \right] \, Dq \quad (7.76)$$

where $Dq \equiv (n m / 2 \pi \hbar^2 \beta)^{3n/2} dq_{n-1} \cdots dq_1$, and the partition function is the integral over all loops that go from $q_0 = q_n$ to $q_n$ in time $\hbar \beta$

$$Z(\beta) = \int \exp \left[ -\frac{1}{\hbar} \int_0^{\hbar \beta} \frac{m \dot{q}^2(u)}{2} + V(q(u)) \, du \right] \, Dq \quad (7.77)$$

where now $Dq \equiv (n m / 2 \pi \hbar^2 \beta)^{3n/2} dq_{n} \cdots dq_1$.

Because the Boltzmann operator $e^{-\beta H}$ is the time-evolution operator $e^{-itH/\hbar}$ at the imaginary time $t = -iu = -i\hbar \beta = -i\hbar/(kT)$, the path integrals of statistical mechanics are called euclidian path integrals.
Example 7.5 (Density operator for a free particle) For a free particle, the matrix element of the Boltzmann operator $e^{-\beta H}$ is the $n = \beta/\epsilon \to \infty$ limit of the integral
\[ \langle q_b | e^{-\beta H} | q_a \rangle = \left( \frac{m}{2\pi \hbar^2 \epsilon} \right)^{n/2} \times \exp \left[ -\frac{m(q_b - q_{n-1})^2}{2\hbar^2 \epsilon} \cdots - \frac{(q_1 - q_0)^2}{2\hbar^2 \epsilon} \right] dq_{n-1} \cdots dq_1. \] (7.78)

The formula (7.2) gives for the $q_1$ integral
\[ \left( \frac{m}{2\pi \hbar^2 \epsilon} \right)^{1/2} \int e^{-m(q_2 - q_1)^2/(2\hbar^2 \epsilon)} dq_1 = \frac{e^{-m(q_2 - q_0)^2/(2\hbar^2 \epsilon)}}{\sqrt{2}}. \] (7.79)

The $q_2$ integral is (exercise ??)
\[ \left( \frac{m}{4\pi \hbar^2 \epsilon} \right)^{1/2} \int e^{-m(q_3 - q_2)^2/(4\hbar^2 \epsilon)} dq_2 = \frac{e^{-m(q_3 - q_0)^2/(2\hbar^2 \epsilon)}}{\sqrt{3}}. \] (7.80)

All $n - 1$ integrations give
\[ \langle q_b | e^{-\beta H} | q_a \rangle = \sqrt{\frac{m}{2\pi \hbar^2 \epsilon}} \frac{e^{-m(q_b - q_a)^2/(2\hbar^2 \epsilon)}}{\sqrt{n}} = \sqrt{\frac{m}{2\pi \hbar^2 \beta}} e^{-m(q_b - q_a)^2/(2\hbar^2 \beta)}. \] (7.81)

The partition function is the integral of this matrix element over $q_a = q_b$
\[ Z(\beta) = \left( \frac{m}{2\pi \hbar^2 \beta} \right)^{1/2} \int dq_a = \left( \frac{m}{2\pi \hbar^2 \beta} \right)^{1/2} L \] (7.82)

where $L$ is the (infinite) 1-dimensional volume of the system. The $q_b, q_a$ matrix element of the maximum-entropy density operator is
\[ \langle q_b | \rho | q_a \rangle = \frac{e^{-m(q_b - q_a)^2/(2\hbar^2 \beta)}}{L}. \] (7.83)

In 3 dimensions, equations (7.82 & 7.83) are
\[ \langle q_b | e^{-\beta H} | q_a \rangle = \left( \frac{mkT}{2\pi \hbar^2} \right)^{3/2} e^{-m(q_b - q_a)^2/(2\hbar^2 \beta)} \] and
\[ Z(\beta) = \left( \frac{mkT}{2\pi \hbar^2} \right)^{3/2} L^3. \] (7.84)

Example 7.6 (Partition function at high temperatures) At high temperatures, the time $\hbar \beta = \hbar/(kT)$ is very short, and the density operator
\[ \boxed{\text{(7.85)}} \]
for a free particle shows that free paths are damped and limited to distances of order $\hbar/\sqrt{mkT}$. We thus can approximate the path integral (7.77) for the partition function by replacing the potential $V(q(u))$ by $V(q_n)$ and then using the free-particle matrix element (7.85)

$$Z(\beta) \approx \int e^{-\beta V(q_n)} \exp \left[ -\frac{1}{\hbar} \int_0^{\hbar \beta} \frac{m q^2(u)}{2} \right] Dq$$

(7.86)

$$= \int e^{-\beta V(q_n)} \langle q_n | e^{-\beta H} | q_n \rangle dq_n = \left( \frac{mkT}{2\pi \hbar^2} \right)^{3/2} \int e^{-\beta V(q_n)} dq_n.$$

7.6 Boltzmann path integrals for quadratic actions

Apart from the factor $Dq \equiv \left( \frac{nm/2\pi \hbar^2 \beta^{n/2}}{dq_{n-1} \ldots dq_1} \right)$, the euclidian path integral

$$\langle q_b | e^{-\beta H} | q_a \rangle = \int \exp \left[ -\frac{1}{\hbar} \int_0^{\hbar \beta} \frac{m q^2(u)}{2} + V(q(u)) \right] Dq$$

(7.87)

is a sum of positive terms $e^{-S_e[q]/\hbar}$ one for each path from $q_a, 0$ to $q_b, \beta$. If a path from $q_a, 0$ to $q_b, \beta$ obeys the classical euclidian equation of motion

$$m \frac{d^2 q_{ce}}{du^2} = m \ddot{q}_{ce} = V'(q_{ce})$$

(7.88)

then its euclidian action

$$S_e[q] = \int_0^{\hbar \beta} \frac{m q^2(u)}{2} + V(u) \ du$$

(7.89)

is stationary and may be minimal. So we can approximate the euclidian action $S_e[q_{ce} + x]$ as we approximated the action $S[q_c + x]$ in section 7.4.

The euclidian action $S_e[q_{ce} + x]$ of an arbitrary path from $q_a, 0$ to $q_b, \beta$ is the stationary euclidian action $S_e[q_{ce}]$ plus a $u$-integral of quadratic and higher powers of the detour $x$ which goes from $x(0) = 0$ to $x(\hbar \beta) = 0$

$$S_e[q_{ce} + x] = \int_0^{\hbar \beta} \left[ \frac{m q^2_{ce} + V(q_{ce})}{2} \right] du + \int_0^{\hbar \beta} \left[ \frac{m x^2}{2} + \frac{V'(q_{ce})}{2} x^2 + \frac{V''(q_{ce})}{6} x^3 + \frac{V'''(q_{ce})}{24} x^4 + \ldots \right] du$$

$$= S_e[q_{ce}] + \Delta S_e[q_{ce}, x],$$

(7.90)
and the path integral for the matrix element $\langle q_b | e^{-\beta H} | q_a \rangle$ is
\begin{equation}
\langle q_b | e^{-\beta H} | q_a \rangle = e^{-S_{e}[q_{ce}]/\hbar} \int e^{-\Delta S_{e}[x]/\hbar} Dx \tag{7.91}
\end{equation}
as $n \to \infty$ where $Dx = (nm/2\pi \hbar^2 \beta)^{n/2} dq_{n-1} \ldots dq_1$ in the limit $n \to \infty$.

If the action is quadratic in $q$ and $\dot{q}$, then the integral $\Delta S_{e}[q_{ce}, x]$ over the detour $x$ is a gaussian path integral that is independent of the path $q_{ce}$ and so is a function $f$ only of the parameters $\beta$, $m$, $\hbar$, and $V''$
\begin{equation}
\langle q_b | e^{-\beta H} | q_a \rangle = f(\beta, \hbar, m, V'') e^{-S_{e}[q_{ce}]/\hbar} \tag{7.92}
\end{equation}
where
\begin{equation}
f(\beta, \hbar, m, V'') = \left[ \frac{mn^2}{2\pi \hbar^2 \beta} \right]^{n/2} \int e^{-\Delta S_{e}[x]/\hbar} dx_{n-1} \ldots dx_1,
\end{equation}

\begin{equation}
\Delta S_{e}[x] = \frac{\hbar \beta}{n} \sum_{j=1}^{n} \frac{m (x_j - x_{j-1})^2}{2\hbar^2 (\beta/n)^2} + \frac{1}{2} V'' x_j^2, \tag{7.93}
\end{equation}
and $x_n = 0 = x_0$.

**Example 7.7** (Density operator for the harmonic oscillator) The path $q_{ce}(\beta)$ that satisfies the classical euclidian equation of motion (7.88)
\begin{equation}
\ddot{q}_{ce}(u) = \frac{d^2 q_{ce}(u)}{du^2} = \omega^2 q_{ce}(u) \tag{7.94}
\end{equation}
and goes from $q_a, 0$ to $q_b, \hbar \beta$ is
\begin{equation}
q_{ce}(u) = \frac{\sinh(\omega u) q_b + \sinh[\omega(\hbar \beta - u)] q_a}{\sinh(\hbar \omega \beta)}. \tag{7.95}
\end{equation}
Its euclidian action is (exercise ??)
\begin{equation}
S_{e}[q_{ce}] = \int_{0}^{h \beta} \frac{\hbar q_{ce}^2(u)}{2} + \frac{m \omega^2 q_{ce}^2(u)}{2} du = \frac{m \omega}{2\hbar \sinh(\hbar \omega \beta)} \left\{ \cosh(\hbar \omega \beta)(q_a^2 + q_b^2) - 2q_a q_b \right\}. \tag{7.96}
\end{equation}
Since $V'' = m \omega^2$, our formulas (7.92 & 7.93) for quadratic actions give as the matrix element
\begin{equation}
\langle q_b | e^{-\beta H} | q_a \rangle = f(\beta, \hbar, m, m \omega^2) e^{-S_{e}[q_{ce}]/\hbar} \tag{7.97}
\end{equation}
The matrix elements of the maximum-entropy density operator (7.74) are
\[
\langle q_\theta | x_n \rangle = \theta / \sinh(C_n) \text{ with } \sinh(C_n) = \sqrt{2} \pi \beta / \hbar \]
whose diagonal elements are
\[
y_n = 2 \cosh(\beta) \sinh(\omega \beta) / \hbar
\]
in which the positive quadratic form \( a_{jk} x_j x_k \) is
\[
\frac{mn \beta}{2\pi \hbar^2} \sum_{j=1}^{n} \left[ -2 x_j x_{j-1} + x_j^2 + x_{j-1}^2 + \frac{(\hbar \omega \beta)^2}{n^2} x_j \right]
\]
which has no linear term because \( x_0 = x_n = 0 \).

The matrix \( a \) is \((nm/(2\hbar^2 B)) C_{n-1}(y)\) in which \( C_{n-1}(y) \) is a square, tridiagonal, \((n-1)\)-dimensional matrix whose off-diagonal elements are \(-1\) and whose diagonal elements are \( y = 2 + (\hbar \omega \beta)^2 / n^2 \). The determinants \( |C_n(y)| \) obey the recursion relation \( |C_{n+1}(y)| = y |C_n(y)| - |C_{n-1}(y)| \) and have the initial values \( C_1(y) = y \) and \( C_2(y) = y^2 - 1 \). So do the hyperbolic functions \( \sinh(n+1) \theta / \sinh \theta \) with \( y = 2 \cosh \theta \). So we set \( C_n(y) = \sinh(n+1) \theta / \sinh \theta \) with \( \theta = \text{arccosh}(y/2) \). We then get as the matrix element (7.97)
\[
\langle q_b | e^{-\beta H} | q_a \rangle = \frac{m \omega}{2 \pi \hbar \sinh(\hbar \omega \beta)} \exp \left[ - \frac{m \omega [\cosh(\hbar \omega \beta) (q_a^2 + q_b^2) - 2 q_a q_b]}{2 \hbar \sinh(\hbar \omega \beta)} \right].
\]

The partition function is the integral over \( q_a \) of this matrix element for \( q_b = q_a \)
\[
Z(\beta) = \frac{m \omega}{2 \pi \hbar \sinh(\hbar \omega \beta)} \int \exp \left[ - \frac{m \omega [\cosh(\hbar \omega \beta) - 1] q_a^2}{\hbar \sinh(\hbar \omega \beta)} \right] dq_a
\]
\[
= \frac{1}{\sqrt{[\cosh(\hbar \omega \beta) - 1]}}.
\]

The matrix elements of the maximum-entropy density operator (7.74) are
\[
\langle q_\rho | q_a \rangle = \frac{\langle q_b | e^{-\beta H} | q_a \rangle}{Z(\beta)}
\]
\[
= \sqrt{\frac{m \omega [\cosh(\hbar \omega \beta) - 1]}{\pi \hbar \sinh(\hbar \omega \beta)}} \exp \left[ - \frac{m \omega [\cosh(\hbar \omega \beta) (q_a^2 + q_b^2) - 2 q_a q_b]}{2 \hbar \sinh(\hbar \omega \beta)} \right].
\]
which reveals the ground-state wave functions

$$\lim_{\beta \to \infty} \langle q_b | \rho | q_a \rangle = \langle q_b | 0 \rangle \langle 0 | q_a \rangle = \sqrt{\frac{m \omega}{\pi \hbar}} e^{-m \omega (q_a^2 + q_b^2)/(2 \hbar)}.$$

(7.104)

The partition function gives us the ground-state energy

$$\lim_{\beta \to \infty} Z(\beta) = \lim_{\beta \to \infty} \frac{1}{\sqrt{2 \cosh(\hbar \omega \beta) - 1}} = e^{-\beta E_0} = e^{-\beta \hbar \omega/2}.$$

(7.105)

### 7.7 Mean values of time-ordered products

In the Heisenberg picture, the position operator at time $t$ is

$$q(t) = e^{itH/\hbar} q e^{-itH/\hbar}$$

(7.106)

in which $q = q(0)$ is the position operator at time $t = 0$ or equivalently the position operator in the Schrödinger picture. The position operator $q$ at the imaginary time $t = -iu = -i \hbar \beta = -i \hbar/(kT)$ is the euclidian position operator

$$q_e(u) = q_e(h\beta) = e^{uH/\hbar} q e^{-uH/\hbar}.$$

(7.107)

The time-ordered product of two position operators is

$$\mathcal{T}[q(t_1)q(t_2)] = \begin{cases} q(t_1) q(t_2) & \text{if } t_1 \geq t_2 \\ q(t_2) q(t_1) & \text{if } t_2 \geq t_1 \end{cases} = q(t_>) q(t_<)$$

(7.108)

in which $t_>$ is the later and $t_<$ the earlier of the two times $t_1$ and $t_2$. Similarly, the time-ordered product of two euclidian position operators at euclidian times $u_1 = h\beta_1$ and $u_2 = h\beta_2$ is

$$\mathcal{T}[q_e(u_1)q_e(u_2)] = \begin{cases} q_e(u_1) q_e(u_2) & \text{if } u_1 \geq u_2 \\ q_e(u_2) q_e(u_1) & \text{if } u_2 \geq u_1 \end{cases} = q_e(u_>) q_e(u_<).$$

(7.109)

The matrix element of the time-ordered product (7.108) of two position operators and two exponentials $e^{-itH/\hbar}$ between states $|a\rangle$ and $|b\rangle$ is

$$\langle b | e^{-itH/\hbar} \mathcal{T}[q(t_1)q(t_2)] e^{-itH/\hbar} | a \rangle = \langle b | e^{-itH/\hbar} q(t_<) q(t_>) e^{-itH/\hbar} | a \rangle$$

$$= \langle b | e^{-i(t-t_>)H/\hbar} q e^{-i(t_>-t_<)H/\hbar} q e^{-i(t+t_<)H/\hbar} | a \rangle.$$
We use the path-integral formula (7.21) for each of the exponentials on the right-hand side of this equation and find (exercise ??)

\[
\langle b| e^{-itH/\hbar}T[q(t_1)q(t_2)]e^{-itH/\hbar}|a\rangle = \int \langle b|q_b\rangle q(t_1)q(t_2)e^{iS[q]/\hbar}\langle q_a|a\rangle Dq
\]

(7.111)

in which the integral is over all paths that run from \(-t\) to \(t\). This equation simplifies if the states \(|a\rangle\) and \(|b\rangle\) are eigenstates of \(H\) with eigenvalues \(E_m\) and \(E_n\)

\[
e^{-it(E_n+E_m)/\hbar}\langle n|T[q(t_1)q(t_2)]|m\rangle = \int \langle n|q_b\rangle q(t_1)q(t_2)e^{iS[q]/\hbar}\langle q_a|m\rangle Dq.
\]

(7.112)

By setting \(n = m\) and omitting the time-ordered product, we get

\[
e^{-2itE_n/\hbar} = \int \langle n|q_b\rangle e^{iS[q]/\hbar}\langle q_a|n\rangle Dq.
\]

(7.113)

The ratio of (7.112) with \(n = m\) to (7.113) is

\[
\langle n|T[q(t_1)q(t_2)]|n\rangle = \frac{\int \langle n|q_b\rangle q(t_1)q(t_2)e^{iS[q]/\hbar}\langle q_a|n\rangle Dq}{\int \langle n|q_b\rangle e^{iS[q]/\hbar}\langle q_a|n\rangle Dq}
\]

(7.114)

in which the integrations are over all paths that go from \(-t \leq t_\prec \) to \(t \geq t_\succ\). The mean value of the time-ordered product of \(k\) position operators is

\[
\langle n|T[q(t_1)\cdots q(t_k)]|n\rangle = \frac{\int \langle n|q_b\rangle q(t_1)\cdots q(t_k)e^{iS[q]/\hbar}\langle q_a|n\rangle Dq}{\int \langle n|q_b\rangle e^{iS[q]/\hbar}\langle q_a|n\rangle Dq}
\]

(7.115)

in which the integrations are over all paths that go from some time before \(t_1, \ldots, t_k\) to some time them.

We may perform the same operations on the euclidian position operators by replacing \(t\) by \(-iu = -i\hbar\beta\). A matrix element of the euclidian time-ordered product (7.109) between two states is

\[
\langle b|e^{-uH/\hbar}T[q_e(u_1)q_e(u_2)]e^{-uH/\hbar}|a\rangle = \langle b|e^{-uH/\hbar}q_e(u_>)q_e(u_\prec)e^{-uH/\hbar}|a\rangle
\]

(7.116)

\[
= \langle b|e^{-(u-u_>)H/\hbar}q_e^{-(u_>-u_\prec)H/\hbar}q_e^{-(u+u_\prec)H/\hbar}|a\rangle.
\]

As \(u \to \infty\), the exponential \(e^{-uH/\hbar}\) projects (7.73) states in onto the ground state \(|0\rangle\) which is an eigenstate of \(H\) with energy \(E_0\). So we replace the
arbitrary states in \(\text{(7.116)}\) with the ground state and use the path-integral formula \(\text{(7.71)}\) for the last three exponentials of \(\text{(7.116)}\)

\[
e^{-2uE_0/\hbar}\langle 0|T[q_e(u_1)q_e(u_2)]|0\rangle = \int \langle 0|q_b\rangle q(u_1)q(u_2)e^{-S_e[q]/\hbar}\langle q_a|0\rangle Dq.
\]

(7.117)

The same equation without the time-ordered product is

\[
e^{-2uE_0/\hbar}\langle 0|0\rangle = e^{-2uE_0/\hbar} = \int \langle 0|q_b\rangle e^{-S_e[q]/\hbar}\langle q_a|0\rangle Dq.
\]

(7.118)

The ratio of the last two equations is

\[
\langle 0|T[q_e(u_1)q_e(u_2)]|0\rangle = \frac{\int \langle 0|q_b\rangle q(u_1)q(u_2)e^{-S_e[q]/\hbar}\langle q_a|0\rangle Dq}{\int \langle 0|q_b\rangle e^{-S_e[q]/\hbar}\langle q_a|0\rangle Dq}
\]

(7.119)

in which the integration is over all paths from \(u = -\infty\) to \(u = \infty\). The mean value in the ground state of the time-ordered product of \(k\) euclidian position operators is

\[
\langle 0|T[q_e(u_1)\cdots q_e(u_k)]|0\rangle = \frac{\int \langle 0|q_b\rangle q(u_1)\cdots q(u_k)e^{-S_e[q]/\hbar}\langle q_a|0\rangle Dq}{\int \langle 0|q_b\rangle e^{-S_e[q]/\hbar}\langle q_a|0\rangle Dq}.
\]

(7.120)

### 7.8 Quantum field theory

Quantum mechanics imposes upon \(n\) coordinates \(q_i\) and conjugate momenta \(p_k\) the equal-time commutation relations

\[
[q_i, p_k] = i\hbar \delta_{i,k} \quad \text{and} \quad [q_i, q_k] = [p_i, p_k] = 0.
\]

(7.121)

In the theory of a single spinless quantum field, a coordinate \(q_x \equiv \phi(x)\) and a conjugate momentum \(p_x \equiv \pi(x)\) are associated with each point \(x\) of space. The operators \(\phi(x)\) and \(\pi(x)\) obey the commutation relations

\[
[\phi(x), \pi(x')] = i\hbar \delta(x-x')
\]

\[
[\phi(x), \phi(x')] = [\pi(x), \pi(x')] = 0
\]

(7.122)

inherited from quantum mechanics.

To make path integrals, we will replace space by a 3-dimensional lattice of
points \( \mathbf{x} = a(i, j, k) = (ai, aj, ak) \) and eventually let the distance \( a \) between adjacent points go to zero. On this lattice and at equal times \( t = 0 \), the field operators obey discrete forms of the commutation relations (7.122)

\[
[\phi(a(i, j, k)), \pi(a(\ell, m, n))] = \frac{\hbar a^3}{2} \delta_{i,\ell} \delta_{j,m} \delta_{k,n}
\]

(7.123)

The vanishing commutators imply that the field and the momenta have “simultaneous” eigenvalues

\[
\phi(a(i, j, k)) |\phi'\rangle = \phi'(a(i, j, k)) |\phi'\rangle \quad \text{and} \quad \pi(a(i, j, k)) |\pi'\rangle = \pi'(a(i, j, k)) |\pi'\rangle
\]

(7.124)

for all lattice points \( a(i, j, k) \). Their inner products are

\[
\langle \phi'|\phi''\rangle = \prod_{i,j,k} \delta(\phi'(a(i, j, k)) - \phi''(a(i, j, k)))
\]

(7.127)

with a similar equation for \( \langle \pi'|\pi''\rangle \).

The Hamiltonian for a free field of mass \( m \) is

\[
H = \frac{1}{2} \int (\pi^2 + c^2(\nabla \phi)^2) \, d^3x = \frac{a^3}{2} \sum_v \pi_v^2 + c^2(\nabla \phi_v)^2 + \frac{m^2 c^4}{2} \phi_v^2
\]

(7.128)

where \( v = a(i, j, k) \), \( \pi_v = \pi(a(i, j, k)) \), \( \phi_v = \phi(a(i, j, k)) \), and the square of the lattice gradient \( (\nabla \phi_v)^2 \) is

\[
[(\phi(a(i+1, j, k)) - \phi(a(i, j, k)))^2 + (\phi(a(i, j+1, k)) - \phi(a(i, j, k)))^2 + (\phi(a(i, j, k+1)) - \phi(a(i, j, k)))^2]/a^2.
\]

(7.129)

Other fields or terms, such as \( c^3 \phi^4/\hbar \), can be added to this Hamiltonian.

To simplify the appearance of the equations in the rest of this chapter, I will mostly use **natural units** in which \( \hbar = c = 1 \). To convert the value of a physical quantity from natural units to universal units, one multiplies or divides its natural-unit value by suitable factors of \( \hbar \) and \( c \) until one gets the right dimensions. For instance, if \( V = 1/m \) is the value of a time in
natural units, where \( m \) is a mass, then the time you want is \( T = \hbar/(mc^2) \). If \( V = 1/m \) is supposed to be a length, then the needed length is \( L = \hbar/(mc) \).

We set \( K = a^3 \sum_v \pi_v^2/2 \) and \( V = (a^3/2) \sum_v (\nabla \phi_v)^2 + m^2 \phi_v^2 + P(\phi_v) \) in which \( P(\phi_v) \) represents the self-interactions of the field. With \( \epsilon = (t_b - t_a)/n \), Trotter’s product formula (7.6) is the \( n \to \infty \) limit of

\[
e^{-i(t_b-t_a)(K+V)} = \left( e^{-i(t_b-t_a)K/n} e^{-i(t_b-t_a)V/n} \right)^n = (e^{-ieK} e^{-ieV})^n. \tag{7.130}
\]

We insert \( I \) in the form (7.126) between \( e^{-ieK} \) and \( e^{-ieV} \)

\[
\langle \phi_1 | e^{-ieK} e^{-ieV} | \phi_a \rangle = \langle \phi_1 | e^{-ieK} \int |\pi\rangle \langle \pi' | \prod_v d\pi_v e^{-ieV} | \phi_a \rangle \tag{7.131}
\]

and use the eigenstate formula (7.124)

\[
\langle \phi_1 | e^{-ieK} e^{-ieV} | \phi_a \rangle = e^{-ieV} \langle \phi_a | f_{\phi_1} e^{-ieK(\pi')} \langle \phi_1 | \pi' \rangle \prod_v d\pi_v'. \tag{7.132}
\]

The inner product formula (7.125) now gives

\[
\langle \phi_1 | e^{-ieK} e^{-ieV} | \phi_a \rangle = e^{-ieV} \prod_v \left[ \int \frac{a^3 d\pi_v'}{2\pi} e^{ia^3(\pi_v'^2/2 + i(\phi_{1v} - \phi_{av})\pi_v')} \right]. \tag{7.133}
\]

We again adopt the suggestive notation \( \dot{\phi}_a = (\phi_1 - \phi_a)/\epsilon \) and use the gaussian integral (7.1) to find

\[
\langle \phi_1 | e^{-ieK} e^{-ieV} | \phi_a \rangle = \prod_v \left[ \left( \frac{a^3}{2\pi i(\epsilon)} \right)^{1/2} e^{ia^3(\dot{\phi}_a - (\nabla \phi_a)^2 - m^2 \phi_a^2 - P(\phi_a))/2} \right]. \tag{7.134}
\]

The product of \( n = (t_b - t_a)/\epsilon \) such time intervals is

\[
\langle \phi_b | e^{-i(t_b-t_a)H} | \phi_a \rangle = \prod_v \left[ \left( \frac{a^3}{2\pi i(t_b - t_a)} \right)^{n/2} \int e^{iS_v} D\phi_v \right] \tag{7.135}
\]

in which

\[
S_v = \frac{t_b - t_a}{n} a^3 \sum_{j=0}^{n-1} \left[ \dot{\phi}_{jv}^2 - (\nabla \phi_{jv})^2 - m^2 \phi_{jv}^2 - P(\phi_v) \right] \tag{7.136}
\]

\( \dot{\phi}_{jv} = n(\phi_{j+1,v} - \phi_{j,v})/(t_b - t_a) \), and \( D\phi_v = d\phi_{n-1,v} \cdots d\phi_1,v. \)

The amplitude \( \langle \phi_b | e^{-i(t_b-t_a)H} | \phi_a \rangle \) is the integral over all fields that go from \( \phi_a(\mathbf{x}) \) at \( t_a \) to \( \phi_b(\mathbf{x}) \) at \( t_b \) each weighted by an exponential

\[
\langle \phi_b | e^{-i(t_b-t_a)H} | \phi_a \rangle = \int e^{iS[\phi]} D\phi \tag{7.137}
\]
of its action
\[ S[\phi] = \int_{t_a}^{t_b} dt \int d^3x \frac{1}{2} \left[ \dot{\phi}^2 - (\nabla \phi)^2 - m^2 \phi^2 - P(\phi) \right] \quad (7.138) \]
in which \( D\phi \) is the \( n \to \infty \) limit of the product over all spatial vertices \( v \)
\[ D\phi = \prod_v \left[ \left( \frac{a^3 n}{2 \pi i (t_b - t_a)} \right)^{n/2} d\phi_{n-1,v} \cdots d\phi_{1,v} \right]. \quad (7.139) \]

Equivalently, the time-evolution operator is
\[ e^{-i(t_b-t_a)H} = \int |\phi_b\rangle e^{iS[\phi]} \langle \phi_a| D\phi D\phi_a D\phi_b \quad (7.140) \]
in which \( D\phi_a D\phi_b = \prod_v d\phi_{a,v} d\phi_{b,v} \) is an integral over the initial and final states.

As in quantum mechanics (section 7.4), the path integral for an action that is quadratic in the fields is an exponential of the action of a stationary process times a function of the times and of the other parameters in the action
\[ \langle \phi_b| e^{-i(t_b-t_a)H} |\phi_a\rangle = \int e^{iS[\phi]} D\phi = f(t_a, t_b, \ldots) e^{iS[\phi_c]} \quad (7.141) \]
in which \( S[\phi_c] \) is the action of the process that goes from \( \phi(x, t_a) = \phi_a(x) \) to \( \phi(x, t_b) = \phi_b(x) \) and obeys the classical equations of motion, and the function \( f \) is a path integral over all fields that go from \( \phi(x, t_a) = 0 \) to \( \phi(x, t_b) = 0 \).

Example 7.8 (A stationary process) The field
\[ \phi(x, t) = \int e^{i k \cdot x} [a(k) \cos \omega t + b(k) \sin \omega t] \, d^3k \quad (7.142) \]
with \( \omega = \sqrt{k^2 + m^2} \) makes the action \( (7.138) \) for \( P = 0 \) stationary because it is a solution of the equation of motion \( \nabla^2 \phi - \ddot{\phi} - m^2 \phi = 0 \). In terms of the Fourier transforms
\[ \tilde{\phi}(k, t_a) = \int e^{-i k \cdot x} \phi(x, t_a) \frac{d^3x}{(2\pi)^3} \quad \text{and} \quad \tilde{\phi}(k, t_b) = \int e^{-i k \cdot x} \phi(x, t_b) \frac{d^3x}{(2\pi)^3}, \quad (7.143) \]
the solution that goes from \( \phi(x, t_a) \) to \( \phi(x, t_b) \) is
\[ \phi(x, t) = \int e^{i k \cdot x} \sin \omega (t_b - t) \tilde{\phi}(k, t_a) + \sin \omega (t - t_a) \tilde{\phi}(k, t_b) \frac{d^3k}{\sin \omega (t_b - t_a)}. \quad (7.144) \]
The solution that evolves from $\phi(x, t_a)$ and $\dot{\phi}(x, t_a)$ is

$$\phi(x, t) = \int e^{ik\cdot x} \left[ \cos \omega(t - t_a) \tilde{\phi}(k, t_a) + \frac{\sin \omega(t - t_a)}{\omega} \tilde{\phi}(k, t_a) \right] d^3k$$

(7.145)

in which the Fourier transform $\tilde{\phi}(k, t_a)$ is defined as in (7.143).

Like a position operator (7.106), a field at time $t$ is defined as

$$\phi(x, t) = e^{itH/\hbar}\phi(x) e^{-itH/\hbar}$$

(7.146)

in which $\phi(x) = \phi(x, 0)$ is the field at time zero, which obeys the commutation relations (7.122). The time-ordered product of several fields is their product with newer (later time) fields standing to the left of older (earlier time) fields as in the definition (7.108). The logic (7.110–7.115) of the derivation of the path-formulas for time-ordered products of position operators applies directly to field operators. One finds (exercise ??) for the mean value of the time-ordered product of two fields in an energy eigenstate $|n\rangle$

$$\langle n| T[\phi(x_1)\phi(x_2)]|n\rangle = \frac{\int \langle n|\phi_b\rangle \phi(x_1)\phi(x_2)e^{iS[\phi]/\hbar}\langle \phi_a|n\rangle D\phi}{\int \langle n|\phi_b\rangle e^{iS[\phi]/\hbar}\langle \phi_a|n\rangle D\phi}$$

(7.147)

in which the integrations are over all paths that go from before $t_1$ and $t_2$ to after both times. The analogous result for several fields is (exercise ??)

$$\langle n| T[\phi(x_1)\cdots\phi(x_k)]|n\rangle = \frac{\int \langle n|\phi_b\rangle \phi(x_1)\cdots\phi(x_k)e^{iS[\phi]/\hbar}\langle \phi_a|n\rangle D\phi}{\int \langle n|\phi_b\rangle e^{iS[\phi]/\hbar}\langle \phi_a|n\rangle D\phi}$$

(7.148)

in which the integrations are over all paths that go from before the times $t_1, \ldots, t_k$ to after them.

### 7.9 Finite-temperature field theory

Since the Boltzmann operator $e^{-\beta H} = e^{-H/(kT)}$ is the time evolution operator $e^{-itH/\hbar}$ at the imaginary time $t = -i\hbar\beta = -i\hbar/(kT)$, the formulas of finite-temperature field theory are those of quantum field theory with $t$ replaced by $-iu = -i\hbar\beta = -i\hbar/(kT)$. 
If as in section 7.8 we use as our hamiltonian \( H = K + V \) where \( K \) and \( V \) are sums over all lattice vertices \( v = a(i,j,k) = (ai,aj,ak) \) of the cubes of volume \( a^3 \) times the squared momentum and potential terms

\[
H = K + V = \frac{a^3}{2} \sum_v \pi_v^2 + \frac{a^3}{2} \sum_v (\nabla \phi_v)^2 + m^2 \phi_v^2 + P(\phi_v).
\]

(7.149)

A matrix element of the first term of the Trotter product formula (7.7)

\[
e^{-\beta(K+V)} = \lim_{n \to \infty} \left( e^{-\beta K/n} e^{-\beta V/n} \right)^n
\]

(7.150)
is the imaginary-time version of (7.133) with \( \epsilon = \hbar \beta/n \)

\[
\langle \phi_1 | e^{-\epsilon K} e^{-\epsilon V} | \phi_a \rangle = e^{-\epsilon V(\phi_a)} \prod_v \left[ \int \frac{a^3 d\pi'_v}{2\pi} e^{-a^3 (\epsilon^2/2 + i(\phi_{1v}-\phi_{av}) \pi'_v)} \right].
\]

(7.151)

Setting \( \dot{\phi}_{av} = (\phi_{1v} - \phi_{av})/\epsilon \), we find, instead of (7.134)

\[
\langle \phi_1 | e^{-\epsilon K} e^{-\epsilon V} | \phi_a \rangle = \prod_v \left[ \left( \frac{a^3}{2\pi \epsilon} \right)^{1/2} e^{-\epsilon a^3 (\dot{\phi}_{av}^2 + (\nabla \phi_{av})^2 + m^2 \phi_{av}^2 + P(\phi_v))/2} \right].
\]

(7.152)
The product of \( n = \hbar \beta/\epsilon \) such inverse-temperature intervals is

\[
\langle \phi_b | e^{-\beta(H-H_a)} | \phi_a \rangle = \prod_v \left[ \left( \frac{a^3 \beta}{2\pi \epsilon} \right)^{n/2} \int e^{-S_{ev}} D\phi_v \right]
\]

(7.153)
in which the euclidian action is

\[
S_{ev} = \frac{\beta a^3}{n} \sum_{j=0}^{n-1} \left[ \dot{\phi}_{jv}^2 + (\nabla \phi_{jv})^2 + m^2 \phi_{jv}^2 + P(\phi_v) \right]
\]

(7.154)

\( \dot{\phi}_{jv} = n(\phi_{j+1,v} - \phi_{j,v})/\beta \), and \( D\phi_v = d\phi_{n-1,v} \cdots d\phi_{1,v} \).

The amplitude \( \langle \phi_b | e^{-(\beta_b - \beta_a)H} | \phi_a \rangle \) is the integral over all fields that go from \( \phi_a(x) \) at \( \beta_a \) to \( \phi_b(x) \) at \( \beta_b \) each weighted by an exponential

\[
\langle \phi_b | e^{-(\beta_b - \beta_a)H} | \phi_a \rangle = \int e^{-S_e(\phi)} D\phi
\]

(7.155)
of its euclidian action

\[
S_e[\phi] = \int_{\beta_a}^{\beta_b} du \int d^3 x \frac{1}{2} \left[ \dot{\phi}^2 + (\nabla \phi)^2 + m^2 \phi^2 + P(\phi) \right]
\]

(7.156)
in which \( D\phi \) is the \( n \to \infty \) limit of the product over all spatial vertices \( v \)

\[
D\phi = \prod_v \left[ \frac{a^3 n}{2\pi(\beta_b - \beta_a)} \right]^{n/2} d\phi_{n-1,v} \cdots d\phi_{1,v} . \tag{7.157}
\]

Equivalently, the Boltzmann operator is

\[
e^{-\beta_b - \beta_a)H} = \int \langle \phi_b \rangle e^{-S_c[\phi]} \langle \phi_a \rangle D\phi D\phi_a D\phi_b \tag{7.158}
\]

in which \( D\phi_a D\phi_b = \prod_v d\phi_{a,v} d\phi_{b,v} \) is an integral over the initial and final states.

The trace of the Boltzmann operator is the partition function

\[
Z(\beta) = \text{Tr}(e^{-\beta H}) = \int e^{-S_c[\phi]} \langle \phi_a \rangle D\phi D\phi_a D\phi_b = \int e^{-S_c[\phi]} D\phi D\phi_a
\]

which is an integral over all fields that go back to themselves in euclidian time \( \beta \).

Like a position operator \((7.107)\), a field at an imaginary time \( t = -iu = -i\hbar \beta \) is defined as

\[
\phi_e(x, u) = \phi_e(x, \hbar \beta) = e^{uH/\hbar} \phi(x) e^{-uH/\hbar} . \tag{7.160}
\]

in which \( \phi(x) = \phi(x, 0) = \phi_e(x, 0) \) is the field at time zero, which obeys the commutation relations \((7.122)\). The euclidian-time-ordered product of several fields is their product with newer (higher \( u = \hbar \beta \)) fields standing to the left of older (lower \( u = \hbar \beta \)) fields as in the definition \((7.109)\).

The euclidian path integrals for the mean values of euclidian-time-ordered-products of fields are similar to those \((7.161) \& (7.148)\) for ordinary time-ordered-products. The euclidian-time-ordered-product of the fields \( \phi(x_j) = \phi(x_j, u_j) \) is the path integral

\[
\langle n|T[\phi_e(x_1) \phi_e(x_2)]|n\rangle = \frac{\int \langle n|\phi_b\rangle \phi(x_1) \phi(x_2) e^{-S_c[\phi]/\hbar} \langle \phi_a|n\rangle D\phi}{\int \langle n|\phi_b\rangle e^{-S_c[\phi]/\hbar} \langle \phi_a|n\rangle D\phi} \tag{7.161}
\]

in which the integrations are over all paths that go from before \( u_1 \) and \( u_2 \) to after both euclidian times. The analogous result for several fields is

\[
\langle n|T[\phi_e(x_1) \cdots \phi_e(x_k)]|n\rangle = \frac{\int \langle n|\phi_b\rangle \phi(x_1) \cdots \phi(x_k) e^{-S_c[\phi]/\hbar} \langle \phi_a|n\rangle D\phi}{\int \langle n|\phi_b\rangle e^{-S_c[\phi]/\hbar} \langle \phi_a|n\rangle D\phi} \tag{7.162}
\]
7.10 Perturbation theory

in which the integrations are over all paths that go from before the times
$u_1,\ldots,u_k$ to after them.

A distinctive feature of these formulas is that in the low-temperature
$\beta = 1/(kT) \to \infty$ limit, the Boltzmann operator is a multiple of an outer
product $|0\rangle\langle 0|$ of the ground-state kets, $e^{-\beta H} \to e^{-\beta E_0}|0\rangle\langle 0|$. In this limit,
the integrations are over all fields that run from $u = -\infty$ to $u = \infty$ and the
energy eigenstates are the ground state of the theory

$$\langle 0|T[\phi_\epsilon(x_1)\cdots\phi_\epsilon(x_k)]|0\rangle = \frac{\int \langle 0|\phi_b(x_1)\cdots\phi(x_k) e^{-S_\epsilon[\phi]/\hbar}\langle \phi_a|0\rangle D\phi}{\int \langle 0|\phi_b e^{-S_\epsilon[\phi]/\hbar}\langle \phi_a|0\rangle D\phi}.$$  \hspace{1cm} (7.163)

Formulas like this one are used in lattice gauge theory.

7.10 Perturbation theory

Field theories with hamiltonians that are quadratic in their fields like

$$H_0 = \int \frac{1}{2} \left[ \pi^2(x) + (\nabla \phi(x))^2 + m^2 \phi^2(x) \right] d^4x$$  \hspace{1cm} (7.164)

are soluble. Their fields evolve in time as

$$\phi(x,t) = e^{itH_0}\phi(x,0)e^{-itH_0}.$$  \hspace{1cm} (7.165)

The mean value in the ground state of $H_0$ of a time-ordered product of these
fields is a ratio \hspace{1cm} (7.148) of path integrals

$$\langle 0|T [\phi(x_1)\ldots\phi(x_k)]|0\rangle = \frac{\int \langle 0|\phi_b(x_1)\cdots\phi(x_n) e^{iS_0[\phi]}\langle \phi_a|0\rangle D\phi}{\int \langle 0|\phi_b e^{iS_0[\phi]}\langle \phi_a|0\rangle D\phi}$$  \hspace{1cm} (7.166)

in which the action $S_0[\phi]$ is quadratic in the field $\phi$

$$S_0[\phi] = \int \frac{1}{2} \left[ \phi^2(x) - (\nabla \phi(x))^2 - m^2 \phi^2(x) \right] d^4x = \int \frac{1}{2} \left[ -\partial_\alpha \phi(x)\partial^\alpha \phi(x) - m^2 \phi^2(x) \right] d^4x$$  \hspace{1cm} (7.167)

and the integrations are over all fields that run from $\phi_a$ at a time before
the times $t_1,\ldots,t_k$ to $\phi_b$ at a time after $t_1,\ldots,t_k$. The path integrals in the
ratio \hspace{1cm} (7.166) are gaussian and doable.
The Fourier transforms

\[ \tilde{\phi}(p) = \int e^{-ipx} \phi(x) \, d^4x \quad \text{and} \quad \phi(x) = \int e^{ipx} \tilde{\phi}(p) \frac{d^4p}{(2\pi)^4} \]  

(7.168)

turn the spacetime derivatives in the action into a quadratic form

\[ S_0[\phi] = -\frac{1}{2} \int |\tilde{\phi}(p)|^2 (p^2 + m^2) \frac{d^4p}{(2\pi)^4} \]  

(7.169)
in which \( p^2 = p^2 - p^0 \) and \( \tilde{\phi}(-p) = \tilde{\phi}^*(p) \) by (??) since the field \( \phi \) is real.

The initial \( \langle \phi_a | 0 \rangle \) and final \( \langle 0 | \phi_b \rangle \) wave functions produce the \( i\epsilon \) in the Feynman propagator [2.24]. Although its exact form doesn't matter here, the wave function \( \langle \phi | 0 \rangle \) of the ground state of \( H_0 \) is the exponential (??)

\[ \langle \phi | 0 \rangle = c \exp \left[ -\frac{1}{2} \int |\tilde{\phi}(p)|^2 \sqrt{p^2 + m^2} \frac{d^4p}{(2\pi)^3} \right] \]  

(7.170)
in which \( \tilde{\phi}(p) \) is the spatial Fourier transform of the eigenvalue \( \phi(x) \)

\[ \tilde{\phi}(p) = \int e^{-ipx} \phi(x) \, d^3x \]  

(7.171)
and \( c \) is a normalization factor that will cancel in ratios of path integrals.

Apart from \(-2i \ln c\) which we will not keep track of, the wave functions \( \langle \phi_a | 0 \rangle \) and \( \langle 0 | \phi_b \rangle \) add to the action \( S_0[\phi] \) the term

\[ \Delta S_0[\phi] = \frac{i}{2} \int \sqrt{p^2 + m^2} \left( |\tilde{\phi}(p, t)|^2 + |\tilde{\phi}(p, -t)|^2 \right) \frac{d^4p}{(2\pi)^3} \]  

(7.172)
in which we envision taking the limit \( t \to \infty \) with \( \phi(x, t) = \phi_b(x) \) and \( \phi(x, -t) = \phi_a(x) \). The identity [Weinberg 1995, pp. 386–388]

\[ f(\infty) + f(-\infty) = \lim_{\epsilon \to 0+} \epsilon \int_{-\infty}^{\infty} f(t) e^{-\epsilon |t|} \, dt \]  

(7.173)
(exercise ??) allows us to write \( \Delta S_0[\phi] \) as

\[ \Delta S_0[\phi] = \lim_{\epsilon \to 0+} \frac{i\epsilon}{2} \int \sqrt{p^2 + m^2} \int_{-\infty}^{\infty} |\tilde{\phi}(p, t)|^2 e^{-\epsilon |t|} \, dt \, d^4p \frac{(2\pi)^4}{(2\pi)^3}. \]  

(7.174)

To first order in \( \epsilon \), the change in the action is (exercise ??)

\[ \Delta S_0[\phi] = \lim_{\epsilon \to 0+} \frac{i\epsilon}{2} \int \sqrt{p^2 + m^2} \int_{-\infty}^{\infty} |\tilde{\phi}(p, t)|^2 \, dt \, d^4p \frac{(2\pi)^4}{(2\pi)^3} \]

\[ = \lim_{\epsilon \to 0+} \frac{i\epsilon}{2} \int \sqrt{p^2 + m^2} |\tilde{\phi}(p)|^2 \frac{d^4p}{(2\pi)^4}. \]  

(7.175)
Thus the modified action is
\[ S_0[\phi, \epsilon] = S_0[\phi] + \Delta S_0[\phi] = -\frac{1}{2} \int |\tilde{\phi}(p)|^2 \left( p^2 + m^2 - i\epsilon \sqrt{p^2 + m^2} \right) \frac{d^4 p}{(2\pi)^4} \]
\[ = -\frac{1}{2} \int |\tilde{\phi}(p)|^2 \left( p^2 + m^2 - i\epsilon \right) \frac{d^4 p}{(2\pi)^4} \]  
(7.176)
since the square root is positive. In terms of the modified action, our formula (7.166) for the time-ordered product is the ratio
\[ \langle 0 | T \left[ \phi(x_1) \cdots \phi(x_n) \right] | 0 \rangle = \frac{\int \phi(x_1) \cdots \phi(x_n) e^{iS_0[\phi, \epsilon]} D\phi}{\int e^{iS_0[\phi, \epsilon]} D\phi}. \]  
(7.177)

We can use this formula (7.177) to express the mean value in the vacuum \( |0 \rangle \) of the time-ordered exponential of a spacetime integral of \( j(x)\phi(x) \), in which \( j(x) \) is a classical (c-number, external) current, as the ratio
\[ Z_0[j] \equiv \langle 0 | T \left\{ \exp \left[ i \int j(x) \phi(x) d^4x \right] \right\} | 0 \rangle = \frac{\int \exp \left[ i \int j(x) \phi(x) d^4x \right] e^{iS_0[\phi, \epsilon]} D\phi}{\int e^{iS_0[\phi, \epsilon]} D\phi}. \]  
(7.178)

Since the state \( |0 \rangle \) is normalized, the mean value \( Z_0[0] \) is unity,
\[ Z_0[0] = 1. \]  
(7.179)

If we absorb the current into the action
\[ S_0[\phi, \epsilon, j] = S_0[\phi, \epsilon] + \int j(x) \phi(x) d^4x \]  
(7.180)
then in terms of the current’s Fourier transform
\[ \tilde{j}(p) = \int e^{-ipx} j(x) d^4x \]  
(7.181)
the modified action \( S_0[\phi, \epsilon, j] \) is (exercise ??)
\[ S_0[\phi, \epsilon, j] = -\frac{1}{2} \int \left[ |\tilde{\phi}(p)|^2 \left( p^2 + m^2 - i\epsilon \right) - \tilde{\phi}^*(p)\tilde{\phi}(p) - \tilde{\phi}^*(p)\tilde{j}(p) \right] \frac{d^4 p}{(2\pi)^4}. \]  
(7.182)
Changing variables to
\[ \tilde{\psi}(p) = \tilde{\phi}(p) - \tilde{j}(p)/(p^2 + m^2 - i\epsilon) \]  
(7.183)
we write the action $S_0[\phi, \epsilon, j]$ as (exercise ??)

$$S_0[\phi, \epsilon, j] = -\frac{1}{2} \int \left[ \tilde{\psi}(p)^2 \left( p^2 + m^2 - i\epsilon \right) - \frac{\tilde{j}^*(p)\tilde{j}(p)}{(p^2 + m^2 - i\epsilon)} \right] \frac{d^4p}{(2\pi)^4}$$

$$= S_0[\psi, \epsilon] + \frac{1}{2} \int \left[ \frac{\tilde{j}^*(p)\tilde{j}(p)}{(p^2 + m^2 - i\epsilon)} \right] \frac{d^4p}{(2\pi)^4}.$$  

(7.184)

And since $D\phi = D\psi$, our formula (7.178) gives simply (exercise ??)

$$Z_0[j] = \exp \left( i \int \frac{|\tilde{j}(p)|^2}{p^2 + m^2 - i\epsilon} \right).$$  

(7.185)

Going back to position space, one finds (exercise ??)

$$Z_0[j] = \exp \left( i \int j(x) \Delta(x - x') j(x') d^4x d^4x' \right)$$

(7.186)

in which $\Delta(x - x')$ is Feynman’s propagator \(2.24\)

$$\Delta(x - x') = \int \frac{e^{ip(x-x')}}{p^2 + m^2 - i\epsilon} \frac{d^4p}{(2\pi)^4}.$$  

(7.187)

The functional derivative (chapter ???) of $Z_0[j]$, defined by (7.178), is

$$\frac{1}{i} \frac{\delta Z_0[j]}{\delta j(x)} = \langle 0 | T \left[ \phi(x) \exp \left( i \int j(x')\phi(x')d^4x' \right) \right] | 0 \rangle$$

(7.188)

while that of equation (7.186) is

$$\frac{1}{i} \frac{\delta Z_0[j]}{\delta j(x)} = Z_0[j] \int \Delta(x - x') j(x') d^4x'.$$  

(7.189)

Thus the second functional derivative of $Z_0[j]$ evaluated at $j = 0$ gives

$$\langle 0 | T \left[ \phi(x)\phi(x') \right] | 0 \rangle = \frac{1}{i^2} \frac{\delta^2 Z_0[j]}{\delta j(x)\delta j(x')} \bigg|_{j=0} = -i \Delta(x - x').$$  

(7.190)

Similarly, one may show (exercise ???) that

$$\langle 0 | T \left[ \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) \right] | 0 \rangle = \frac{1}{i^4} \frac{\delta^4 Z_0[j]}{\delta j(x_1)\delta j(x_2)\delta j(x_3)\delta j(x_4)} \bigg|_{j=0}$$

$$= - \Delta(x_1 - x_2)\Delta(x_3 - x_4) - \Delta(x_1 - x_3)\Delta(x_2 - x_4)$$

$$- \Delta(x_1 - x_4)\Delta(x_2 - x_3).$$  

(7.191)

Suppose now that we add a potential $V = P(\phi)$ to the free hamiltonian \(7.164\). Scattering amplitudes are matrix elements of the time-ordered exponential $T \exp \left[ -i \int P(\phi) d^4x \right]$ (Weinberg 1995 p. 260) Our formula \(7.177\)
for the mean value in the ground state $|0\rangle$ of the free hamiltonian $H_0$ of any time-ordered product of fields leads us to

$$\langle 0| T \left\{ \exp \left[ -i \int P(\phi) \, d^4 x \right] \right\} |0\rangle = \int \frac{\exp \left[ -i \int P(\phi) \, d^4 x \right] e^{iS_0[\phi, \epsilon]} \, D\phi}{e^{iS_0[\phi, \epsilon]} \, D\phi}.$$  

(7.192)

Using (7.190 & 7.191), we can cast this expression into the magical form

$$\langle 0| T \left\{ \exp \left[ -i \int P(\phi) \, d^4 x \right] \right\} |0\rangle = \exp \left[ -i \int P \left( \frac{\delta}{i\delta f(x)} \right) \, d^4 x \right] Z_0[j] \bigg|_{j=0}.$$  

(7.193)

The generalization of the path-integral formula (7.177) to the ground state $|\Omega\rangle$ of an interacting theory with action $S$ is

$$\langle \Omega| T \left\{ \phi(x_1) \ldots \phi(x_n) \right\} |\Omega\rangle = \int \frac{\phi(x_1) \ldots \phi(x_n) \, e^{iS[\phi, \epsilon]} \, D\phi}{e^{iS[\phi, \epsilon]} \, D\phi}$$  

(7.194)

in which a term like $i\epsilon\phi^2$ is added to make the modified action $S[\phi, \epsilon]$.

These are some of the techniques one uses to make states of incoming and outgoing particles and to compute scattering amplitudes [Weinberg 1995, 1996; Srednicki 2007; Zee 2010].

### 7.11 Application to quantum electrodynamics

In the Coulomb gauge $\nabla \cdot A = 0$, the QED hamiltonian is

$$H = H_m + \int \left[ \frac{1}{2} \pi^2 + \frac{1}{2} (\nabla \times A)^2 - A \cdot j \right] \, d^3 x + V_C$$  

(7.195)

in which $H_m$ is the matter hamiltonian, and $V_C$ is the Coulomb term

$$V_C = \frac{1}{2} \int \frac{j^0(x, t) \, j^0(y, t)}{4\pi|x - y|} \, d^3 x \, d^3 y.$$  

(7.196)

The operators $A$ and $\pi$ are canonically conjugate, but they satisfy the Coulomb-gauge conditions

$$\nabla \cdot A = 0 \quad \text{and} \quad \nabla \cdot \pi = 0.$$  

(7.197)

One may show (Weinberg 1995, pp. 413–418) that in this theory, the
analog of equation \(\text{(7.194)}\) is
\[
\langle \Omega | T[O_1 \ldots O_n] | \Omega \rangle = \frac{\int O_1 \ldots O_n e^{iS_C} \delta[\nabla \cdot A] DA D\psi}{\int e^{iS_C} \delta[\nabla \cdot A] DA D\psi} \tag{7.198}
\]
in which the Coulomb-gauge action is
\[
S_C = \int \frac{1}{2} \dot{A}^2 - \frac{1}{2} (\nabla \times A)^2 + A \cdot j + \mathcal{L}_m \, d^4x - \int V_C \, dt \tag{7.199}
\]
and the functional delta function
\[
\delta[\nabla \cdot A] = \prod_x \delta(\nabla \cdot A(x)) \tag{7.200}
\]
enforces the Coulomb-gauge condition. The term \(\mathcal{L}_m\) is the action density of the matter field \(\psi\).

Tricks are available. We introduce a new field \(A^0(x)\) and consider the factor
\[
F = \int \exp \left[ i \int \frac{1}{2} (\nabla A^0 + \nabla \triangle^{-1} j^0)^2 \, d^4x \right] DA^0 \tag{7.201}
\]
which is just a number independent of the charge density \(j^0\) since we can cancel the \(j^0\) term by shifting \(A^0\). By \(\triangle^{-1}\), we mean \(-1/4\pi|x - y|\). By integrating by parts, we can write the number \(F\) as (exercise ??)
\[
F = \int \exp \left[ i \int \frac{1}{2} (\nabla A^0)^2 - A^0 j^0 - \frac{1}{2} j^0 \triangle^{-1} j^0 \, d^4x + i \int V_C \, dt \right] DA^0. \tag{7.202}
\]
So when we multiply the numerator and denominator of the amplitude \(\text{(7.198)}\) by \(F\), the awkward Coulomb term \(V_C\) cancels, and we get
\[
\langle \Omega | T[O_1 \ldots O_n] | \Omega \rangle = \frac{\int O_1 \ldots O_n e^{iS'} \delta[\nabla \cdot A] DA D\psi}{\int e^{iS'} \delta[\nabla \cdot A] DA D\psi} \tag{7.203}
\]
where now \(DA\) includes all four components \(A^\mu\) and
\[
S' = \int \frac{1}{2} \dot{A}^2 - \frac{1}{2} (\nabla \times A)^2 + \frac{1}{2} (\nabla A^0)^2 + A \cdot j - A^0 j^0 + \mathcal{L}_m \, d^4x. \tag{7.204}
\]
Since the delta-functional \(\delta[\nabla \cdot A]\) enforces the Coulomb-gauge condition, we can add to the action \(S'\) the term \((\nabla \cdot \dot{A}) A^0\) which is \(-\dot{A} \cdot \nabla A^0\) after
we integrate by parts and drop the surface term. This extra term makes the action gauge invariant

\[ S = \int \tfrac{1}{2} (\dot{A} - \nabla A^0)^2 - \tfrac{1}{2} (\nabla \times A)^2 + A \cdot j - A^0 j^0 + \mathcal{L}_m \, d^4x \]  

\[ (7.205) \]

Thus at this point we have

\[ \langle \Omega | T [O_1 \ldots O_n] | \Omega \rangle = \frac{\int O_1 \ldots O_n \, e^{iS} \delta[\nabla \cdot A] \, DA \, D\psi}{\int e^{iS} \delta[\nabla \cdot A] \, DA \, D\psi} \]  

\[ (7.206) \]

in which \( S \) is the gauge-invariant action \((7.205)\), and the integral is over all fields. The only relic of the Coulomb gauge is the gauge-fixing delta functional \( \delta[\nabla \cdot A] \).

We now make the gauge transformation

\[ A'_b(x) = A_b(x) + \partial_b \Lambda(x) \quad \text{and} \quad \psi'(x) = e^{i\Lambda(x)} \psi(x) \]  

\[ (7.207) \]

in the numerator and also, using a different gauge transformation \( \Lambda' \), in the denominator of the ratio \((7.206)\) of path integrals. Since we are integrating over all gauge fields, these gauge transformations merely change the order of integration in the numerator and denominator of that ratio. They are like replacing \( \int_{-\infty}^{\infty} f(x) \, dx \) by \( \int_{-\infty}^{\infty} f(y) \, dy \). They change nothing, and so

\[ \langle \Omega | T [O_1 \ldots O_n] | \Omega \rangle = \langle \Omega | T [O_1 \ldots O_n] | \Omega \rangle' \]  

\[ (7.208) \]

in which the prime refers to the gauge transformations \( \Lambda \) and \( \Lambda' \).

We’ve seen that the action \( S \) is gauge invariant. So is the measure \( DA \, D\psi \). We now restrict ourselves to operators \( O_1 \ldots O_n \) that are gauge invariant. So in the right-hand side of equation \((7.208)\), the replacement of the fields by their gauge transforms affects only the term \( \delta[\nabla \cdot A] \) that enforces the Coulomb-gauge condition

\[ \langle \Omega | T [O_1 \ldots O_n] | \Omega \rangle = \frac{\int O_1 \ldots O_n \, e^{iS} \delta[\nabla \cdot A + \triangle \Lambda] \, DA \, D\psi}{\int e^{iS} \delta[\nabla \cdot A + \triangle \Lambda'] \, DA \, D\psi} \]  

\[ (7.209) \]

We now have two choices. If we integrate over all gauge functions \( \Lambda(x) \) and \( \Lambda'(x) \) in both the numerator and the denominator of this ratio \((7.209)\), then apart from over-all constants that cancel, the mean value in the vacuum of
the time-ordered product is the ratio
\[ \langle \Omega | T [ O_1 \ldots O_n ] | \Omega \rangle = \frac{\int O_1 \ldots O_n e^{iS} DA D\psi}{\int e^{iS} DA D\psi} \] (7.210)
in which we integrate over all matter fields, gauge fields, and gauges. That is, we do not fix the gauge.

The analogous formula for the euclidian time-ordered product is
\[ \langle \Omega | T [ O_{e,1} \ldots O_{e,n} ] | \Omega \rangle = \frac{\int O_1 \ldots O_n e^{-S_e} DA D\psi}{\int e^{-S_e} DA D\psi} \] (7.211)
in which the euclidian action \( S_e \) is the spacetime integral of the energy density. This formula is quite general; it holds in nonabelian gauge theories and is important in lattice gauge theory.

Our second choice is to multiply the numerator and the denominator of the ratio \( \frac{\int O_1 \ldots O_n e^{iS} DA D\psi}{\int e^{iS} DA D\psi} \) (7.209) by the exponential \( \exp[-i\frac{1}{2} \alpha \int (\Lambda^\alpha)^2 d^4x] \) and then integrate over \( \Lambda(x) \) in the numerator and over \( \Lambda'(x) \) in the denominator. This operation just multiplies the numerator and denominator by the same constant factor, which cancels. But if before integrating over all gauge transformations, we shift \( \Lambda \) so that \( \Lambda^\alpha \) changes to \( \Lambda^\alpha - \dot{A}_0 \), then the exponential factor is \( \exp[-i\frac{1}{2} \alpha \int (\dot{A}^0 - \Delta \Lambda)^2 d^4x] \). Now when we integrate over \( \Lambda(x) \), the delta function \( \delta(\nabla \cdot A + \Lambda) \) replaces \( \Delta \Lambda \) by \( -\nabla \cdot A \) in the inserted exponential, converting it to \( \exp[-i\frac{1}{2} \alpha \int (\dot{A}^0 + \nabla \cdot A)^2 d^4x] \). This term changes the gauge-invariant action \( S_{invariant} \) to the gauge-fixed action
\[ S_{\alpha} = \int \frac{-1}{4} F_{\mu \nu} F^{\mu \nu} - \frac{\alpha}{2} (\partial_b A^b)^2 + A^b j_b + L_m d^4x. \] (7.212)
This Lorentz-invariant, gauge-fixed action is much easier to use than the Coulomb-gauge action \( S_{coulomb} \) with the Coulomb potential \( U_{coulomb} \). We can use it to compute scattering amplitudes perturbatively. The mean value of a time-ordered product of operators in the ground state \( |0\rangle \) of the free theory is
\[ \langle 0 | T [ O_1 \ldots O_n ] | 0 \rangle = \frac{\int O_1 \ldots O_n e^{iS_{\alpha}} DA D\psi}{\int e^{iS_{\alpha}} DA D\psi}. \] (7.213)
By following steps analogous to those the led to \( \langle 0 | T [ O_1 \ldots O_n ] | 0 \rangle \) (7.187), one may show
That in Feynman’s gauge, $\alpha = 1$, the photon propagator is

$$
\langle 0 | T [ A_\mu (x) A_\nu (y) ] | 0 \rangle = - i \Delta_{\mu\nu} (x - y) = - i \int \frac{\eta_{\mu\nu}}{q^2 - i\epsilon} e^{iq(x-y)} \frac{d^4q}{(2\pi)^4}.
$$

(7.214)

7.12 Fermionic path integrals

In our brief introduction (??–??) and (??–??), to Grassmann variables, we learned that because

$$\theta^2 = 0$$

(7.215)

the most general function $f(\theta)$ of a single Grassmann variable $\theta$ is

$$f(\theta) = a + b \theta.$$  

(7.216)

So a complete integral table consists of the integral of this linear function

$$\int f(\theta) d\theta = \int a + b \theta \, d\theta = a \int d\theta + b \int \theta \, d\theta.$$  

(7.217)

This equation has two unknowns, the integral $\int d\theta$ of unity and the integral $\int \theta \, d\theta$ of $\theta$. We choose them so that the integral of $f(\theta + \zeta)$

$$\int f(\theta + \zeta) \, d\theta = \int a + b (\theta + \zeta) \, d\theta = (a + b \zeta) \int d\theta + b \int \theta \, d\theta$$

(7.218)

is the same as the integral (7.217) of $f(\theta)$. Thus the integral $\int d\theta$ of unity must vanish, while the integral $\int \theta \, d\theta$ of $\theta$ can be any constant, which we choose to be unity. Our complete table of integrals is then

$$\int d\theta = 0 \quad \text{and} \quad \int \theta \, d\theta = 1.$$  

(7.219)

The anticommutation relations for a fermionic degree of freedom $\psi$ are

$$\{ \psi, \psi^\dagger \} = \psi \psi^\dagger + \psi^\dagger \psi = 1 \quad \text{and} \quad \{ \psi, \psi \} = \{ \psi^\dagger, \psi^\dagger \} = 0.$$  

(7.220)

Because $\psi$ has $\psi^\dagger$, it is conventional to introduce a variable $\theta^* = \theta^\dagger$ that anti-commutes with itself and with $\theta$

$$\{ \theta^*, \theta^* \} = \{ \theta^*, \theta \} = \{ \theta, \theta \} = 0.$$  

(7.221)

The logic that led to (7.219) now gives

$$\int d\theta^* = 0 \quad \text{and} \quad \int \theta^* \, d\theta^* = 1.$$  

(7.222)
We define the reference state $|0\rangle$ as $|0\rangle \equiv \psi|s\rangle$ for a state $|s\rangle$ that is not annihilated by $\psi$. Since $\psi^2 = 0$, the operator $\psi$ annihilates the state $|0\rangle$

$$\psi|0\rangle = \psi^2|s\rangle = 0. \quad (7.223)$$

The effect of the operator $\psi$ on the state

$$|\theta\rangle = \exp \left( \psi^\dagger \theta - \frac{1}{2} \theta^* \theta \right) |0\rangle = \left( 1 + \psi^\dagger \theta - \frac{1}{2} \theta^* \theta \right) |0\rangle \quad (7.224)$$

is

$$\psi|\theta\rangle = \psi(1 + \psi^\dagger \theta - \frac{1}{2} \theta^* \theta)|0\rangle = \psi \psi^\dagger \theta |0\rangle = (1 - \psi^\dagger \psi)|\theta\rangle = \theta|0\rangle \quad (7.225)$$

while that of $\theta$ on $|\theta\rangle$ is

$$\theta|\theta\rangle = \theta (1 + \psi^\dagger \theta - \frac{1}{2} \theta^* \theta)|0\rangle = \theta|0\rangle. \quad (7.226)$$

The state $|\theta\rangle$ therefore is an eigenstate of $\psi$ with eigenvalue $\theta$

$$\psi|\theta\rangle = \theta|\theta\rangle. \quad (7.227)$$

The bra corresponding to the ket $|\zeta\rangle$ is

$$\langle \zeta | = \langle 0 | \left( 1 + \zeta^* \psi - \frac{1}{2} \zeta^* \zeta \right) \quad (7.228)$$

and the inner product $\langle \zeta | \theta \rangle$ is (exercise ??)

$$\langle \zeta | \theta \rangle = \langle 0 | \left( 1 + \zeta^* \psi - \frac{1}{2} \zeta^* \zeta \right) \left( 1 + \psi^\dagger \theta - \frac{1}{2} \theta^* \theta \right) |0\rangle$$

$$= \langle 0 | 1 + \zeta^* \psi \psi^\dagger \theta - \frac{1}{2} \zeta^* \zeta - \frac{1}{2} \zeta^* \theta^* \theta + \frac{1}{4} \zeta^* \zeta \theta^* \theta |0\rangle$$

$$= \langle 0 | 1 + \zeta^* \theta - \frac{1}{2} \zeta^* \zeta - \frac{1}{2} \theta^* \theta + \frac{1}{4} \zeta^* \zeta \theta^* \theta |0\rangle$$

$$= \exp \left[ \zeta^* \theta - \frac{1}{2} \left( \zeta^* \zeta + \theta^* \theta \right) \right]. \quad (7.229)$$

**Example 7.9 (A gaussian integral)** For any number $c$, we can compute the integral of $\exp(c \theta^* \theta)$ by expanding the exponential

$$\int e^{c \theta^* \theta} \, d\theta^* d\theta = \int (1 + c \theta^* \theta) \, d\theta^* d\theta = \int (1 - c \theta \theta^*) \, d\theta^* d\theta = -c. \quad (7.230)$$

The identity operator for the space of states

$$c|0\rangle + d|1\rangle \equiv c|0\rangle + d\psi^\dagger |0\rangle \quad (7.231)$$
is (exercise ??) the integral

\[ I = \int \langle \theta | d\theta^* d\theta = |0\rangle \langle 0| + |1\rangle \langle 1| \]  

(7.232)

in which the differentials anti-commute with each other and with other fermionic variables: \( \{d\theta, d\theta^*\} = 0, \{d\theta, \theta\} = 0, \{d\theta, \psi\} = 0 \), and so forth.

The case of several Grassmann variables \( \theta_1, \theta_2, \ldots, \theta_n \) and several Fermi operators \( \psi_1, \psi_2, \ldots, \psi_n \) is similar. The \( \theta_k \) anticommute among themselves

\[ \{\theta_i, \theta_j\} = \{\theta_i, \theta_j^*\} = \{\theta_i^*, \theta_j^*\} = 0 \]  

(7.233)

while the \( \psi_k \)satisfy

\[ \{\psi_k, \psi_l^\dagger\} = \delta_{kl} \quad \text{and} \quad \{\psi_k, \psi_l\} = \{\psi_k^\dagger, \psi_l^\dagger\} = 0. \]  

(7.234)

The reference state \(|0\rangle \) is

\[ |0\rangle = \left( \prod_{k=1}^n \psi_k \right) |s\rangle \]  

(7.235)

in which \(|s\rangle \) is any state not annihilated by any \( \psi_k \) (so the resulting \(|0\rangle \) isn’t zero). The direct-product state

\[ |\theta\rangle \equiv \exp\left( \sum_{k=1}^n \psi_k^\dagger \theta_k - \frac{1}{2} \theta_k^* \theta_k \right) |0\rangle \]  

(7.236)

is (exercise ??) a simultaneous eigenstate of each \( \psi_k \)

\[ \psi_k |\theta\rangle = \theta_k |\theta\rangle. \]  

(7.237)

It follows that

\[ \psi_l \psi_k |\theta\rangle = \psi_l \theta_k |\theta\rangle = -\theta_k \psi_l |\theta\rangle = -\theta_k \theta_l |\theta\rangle = \theta_l \theta_k |\theta\rangle \]  

(7.238)

and so too \( \psi_k \psi_l |\theta\rangle = \theta_k \theta_l |\theta\rangle \). Since the \( \psi \)'s anticommute, their eigenvalues must also

\[ \theta_l \theta_k |\theta\rangle = \psi_l \psi_k |\theta\rangle = -\psi_k \psi_l |\theta\rangle = -\theta_k \theta_l |\theta\rangle \]  

(7.239)

(even if the eigenvalues commuted with the \( \psi \)'s in which case we’d have \( \psi_l \psi_k |\theta\rangle = \theta_k \theta_l |\theta\rangle = -\psi_k \psi_l |\theta\rangle = -\theta_k \theta_l |\theta\rangle \)).
The inner product $\langle \zeta | \theta \rangle$ is

$$\langle \zeta | \theta \rangle = \langle 0 | \prod_{k=1}^{n} (1 + \zeta_k^* \psi_k - \frac{1}{2} \zeta_k^* \zeta_k) \prod_{\ell=1}^{n} (1 + \psi_{\ell}^* \theta_{\ell} - \frac{1}{2} \theta_{\ell}^* \theta_{\ell}) | 0 \rangle$$

$$= \exp \left[ \sum_{k=1}^{n} \zeta_k^* \theta_k - \frac{1}{2} (\zeta_k^* \zeta_k + \theta_k^* \theta_k) \right] = e^{\zeta^\dagger \theta - (\zeta^\dagger \zeta + \theta^\dagger \theta)/2}. \quad (7.240)$$

The identity operator is

$$I = \int |\theta \rangle \langle \theta | \prod_{k=1}^{n} d\theta_k^* d\theta_k. \quad (7.241)$$

**Example 7.10 (Gaussian Grassmann Integral)**  For any $2 \times 2$ matrix $A$, we may compute the gaussian integral

$$g(A) = \int e^{-\theta^\dagger A \theta} d\theta_1^* d\theta_1 d\theta_2^* d\theta_2 \quad (7.242)$$

by expanding the exponential. The only terms that survive are the ones that have exactly one of each of the four variables $\theta_1$, $\theta_2$, $\theta_1^*$, and $\theta_2^*$. Thus, the integral is the determinant of the matrix $A$

$$g(A) = \int \frac{1}{2} (\theta_k^* A_{k\ell} \theta_{\ell})^2 d\theta_1^* d\theta_1 d\theta_2^* d\theta_2$$

$$= \int (\theta_1^* A_{11} \theta_1 \theta_2^* A_{22} \theta_2 + \theta_1^* A_{12} \theta_2 \theta_2^* A_{21} \theta_1) d\theta_1^* d\theta_1 d\theta_2^* d\theta_2$$

$$= A_{11} A_{22} - A_{12} A_{21} = \det A. \quad (7.243)$$

The natural generalization to $n$ dimensions

$$\int e^{-\theta^\dagger A \theta} \prod_{k=1}^{n} d\theta_k^* d\theta_k = \det A \quad (7.244)$$

is true for any $n \times n$ matrix $A$. If $A$ is invertible, then the invariance of
Grassmann integrals under translations implies that

\[
\int e^{-\theta^† A \theta + \theta^† \zeta + \zeta^† \theta} \prod_{k=1}^{n} d\theta^*_k d\theta_k = \int e^{-\theta^† A (\theta + A^{-1} \zeta) + \theta^† \zeta + \zeta^† (\theta + A^{-1} \zeta)} \prod_{k=1}^{n} d\theta^*_k d\theta_k \\
= \int e^{-\theta^† A \theta + \theta^† \zeta + \zeta^† A^{-1} \zeta} \prod_{k=1}^{n} d\theta^*_k d\theta_k \\
= \int e^{-(\theta^† A^{-1}) \theta + \theta^† \zeta + \zeta^† A^{-1} \zeta} \prod_{k=1}^{n} d\theta^*_k d\theta_k \\
= \int e^{-\theta^† A \theta + \theta^† A^{-1} \zeta} \prod_{k=1}^{n} d\theta^*_k d\theta_k \\
= \det A e^{\zeta^† A^{-1} \zeta}.
\]

(7.245)

The values of \( \theta \) and \( \theta^† \) that make the argument \(-\theta^† A \theta + \theta^† \zeta + \zeta^† \theta \) of the exponential stationary are \( \overline{\theta} = A^{-1} \zeta \) and \( \overline{\theta^†} = \zeta^† A^{-1} \). So a gaussian Grassmann integral is equal to its exponential evaluated at its stationary point, apart from a prefactor involving the determinant \( \det A \). This result is a fermionic echo of the bosonic result (7.235).

One may further extend these definitions to a Grassmann field \( \chi_m(x) \) and an associated Dirac field \( \psi_m(x) \). The \( \chi_m(x) \)'s anticommute among themselves and with all fermionic variables at all points of spacetime

\[
\{ \chi_m(x), \chi_n(x') \} = \{ \chi_m^*(x), \chi_n(x') \} = \{ \chi_m(x), \chi_n^*(x') \} = 0
\]

(7.246)

and the Dirac field \( \psi_m(x) \) obeys the equal-time anticommutation relations

\[
\{ \psi_m(x, t), \psi_n^\dagger(x', t) \} = \delta_{mn} \delta(x - x') \quad (n, m = 1, \ldots, 4) \\
\{ \psi_m(x, t), \psi_n(x', t) \} = \{ \psi_m^\dagger(x, t), \psi_n^\dagger(x', t) \} = 0.
\]

(7.247)

As in (7.235), we use eigenstates of the field \( \psi \) at \( t = 0 \). If \( |0⟩ \) is defined in terms of a state \(|s⟩\) that is not annihilated by any \( \psi_m(x, 0) \) as

\[
|0⟩ = \left[ \prod_{m, x} \psi_m(x, 0) \right] |s⟩
\]

(7.248)

then (exercise ??) the state

\[
|\chi⟩ = \exp \left( \int \sum_m \psi_m^\dagger(x, 0) \chi_m(x) - \frac{1}{2} \chi_m^* (x) \chi_m(x) \, d^3x \right) |0⟩ \\
= \exp \left( \int \psi^\dagger \chi - \frac{1}{2} \chi^\dagger \chi \, d^3x \right) |0⟩
\]

(7.249)
Path integrals

is an eigenstate of the operator $\psi_m(x, 0)$ with eigenvalue $\chi_m(x)$

$$\psi_m(x, 0)|\chi\rangle = \chi_m(x)|\chi\rangle. \quad (7.250)$$

The inner product of two such states is (exercise ??)

$$\langle \chi'|\chi \rangle = \exp \left[ \int \chi'^{\dagger} \chi - \frac{1}{2} \chi'^{\dagger} \chi' - \frac{1}{2} \chi \chi \right]. \quad (7.251)$$

The identity operator is the integral

$$I = \int |\chi\rangle\langle \chi| d\chi^{\dagger}d\chi \quad (7.252)$$

in which

$$d\chi^{\dagger}d\chi \equiv \prod_{m,x} d\chi^{\dagger}_m(x)d\chi_m(x). \quad (7.253)$$

The hamiltonian for a free Dirac field $\psi$ of mass $m$ is the spatial integral

$$H_0 = \int \bar{\psi} (\gamma \cdot \nabla + m) \psi \, d^3x \quad (7.254)$$

in which $\bar{\psi} \equiv i\psi^{\dagger}\gamma^0$ and the gamma matrices (?) satisfy

$$\{\gamma^a, \gamma^b\} = 2\eta^{ab} \quad (7.255)$$

where $\eta$ is the $4 \times 4$ diagonal matrix with entries $(-1, 1, 1, 1)$. Since $\psi|\chi\rangle = \chi|\chi\rangle$ and $\langle \chi'|\psi^{\dagger} = \langle \chi'|\chi^{\dagger}$, the quantity $\langle \chi'|\exp(-i\epsilon H_0)|\chi\rangle$ is by (7.251)

$$\langle \chi'|e^{-i\epsilon H_0}|\chi \rangle = \langle \chi'|\chi \rangle \exp \left[ -i \epsilon \int \bar{\chi}' (\gamma \cdot \nabla + m) \chi \, d^3x \right] \quad (7.256)$$

$$= \exp \left[ \frac{1}{2}(\chi'^{\dagger} - \chi^{\dagger})(\chi' - \chi) - \frac{i}{2} \chi'^{\dagger} \chi' \right] \exp \left( \epsilon \int \left[ \frac{1}{2} \chi'^{\dagger} \chi - \frac{1}{2} \chi^{\dagger} \chi' - i\bar{\chi}' (\gamma \cdot \nabla + m) \chi \right] \, d^3x \right)$$

in which $\chi'^{\dagger} - \chi^{\dagger} = \epsilon \bar{\chi}'$ and $\chi' - \chi = \epsilon \bar{\chi}$. Everything within the square brackets is multiplied by $\epsilon$, so we may replace $\chi'^{\dagger}$ by $\chi^{\dagger}$ and $\bar{\chi}'$ by $\bar{\chi}$ so as to write to first order in $\epsilon$

$$\langle \chi'|e^{-i\epsilon H_0}|\chi \rangle = \exp \left[ \epsilon \int \bar{\chi}' \chi - \frac{1}{2} \chi^{\dagger} \chi' - i\bar{\chi} (\gamma \cdot \nabla + m) \chi \, d^3x \right] \quad (7.257)$$

in which the dependence upon $\chi'$ is through the time derivatives.
Putting together \( n = 2t/\epsilon \) such matrix elements, integrating over all intermediate-state dyadics \( |\chi\rangle \langle \chi| \), and using our formula (7.252), we find

\[
\langle \chi_t | e^{-2itH_0} | \chi_{-t} \rangle = \int \exp \left[ \int \frac{1}{2} \dot{\chi}^\dagger \chi - \frac{1}{2} \chi^\dagger \chi - i \overline{\chi} (\gamma \cdot \nabla + m) \chi \right] d^4x \, D\chi^* D\chi.
\]  

(7.258)

Integrating \( \dot{\chi}^\dagger \chi \) by parts and dropping the surface term, we get

\[
\langle \chi_t | e^{-2itH_0} | \chi_{-t} \rangle = \int \exp \left( i \int \overline{\chi} (\gamma \cdot \nabla + m) \chi \right) d^4x \, D\chi^* D\chi.
\]  

(7.259)

Since \( -\overline{\chi} \dot{\chi} = -i\overline{\chi} \gamma^0 \dot{\chi} \), the argument of the exponential is

\[
i \int -\overline{\chi} \gamma^0 \dot{\chi} - \overline{\chi} (\gamma \cdot \nabla + m) \chi \, d^4x = i \int -\overline{\chi} (\gamma^\mu \partial_\mu + m) \chi \, d^4x.
\]  

(7.260)

We then have

\[
\langle \chi_t | e^{-2itH_0} | \chi_{-t} \rangle = \int e^{i \int L_0(\chi) \, d^4x} \, D\chi^* D\chi
\]  

(7.261)

in which \( L_0(\chi) = -\overline{\chi} (\gamma^\mu \partial_\mu + m) \chi \) is the action density (??) for a free Dirac field. Thus the amplitude is a path integral with phases given by the classical action \( S_0[\chi] \)

\[
\langle \chi_t | e^{-2itH_0} | \chi_{-t} \rangle = \int e^{i \int S_0[\chi] \, d^4x} \, D\chi^* D\chi
\]  

(7.262)

and the integral is over all fields that go from \( \chi(x, -t) = \chi_{-t}(x) \) to \( \chi(x, t) = \chi_t(x) \). Any normalization factor will cancel in ratios of such integrals.

Since Fermi fields anticommute, their time-ordered product has an extra minus sign

\[
T \left[ \overline{\psi}(x_1) \psi(x_2) \right] = \theta(x_1^0 - x_2^0) \overline{\psi}(x_1) \psi(x_2) - \theta(x_2^0 - x_1^0) \psi(x_2) \overline{\psi}(x_1).
\]  

(7.263)

The logic behind our formulas (7.148) and (7.166) for the time-ordered product of bosonic fields now leads to an expression for the time-ordered product of \( 2n \) Dirac fields (with \( D\chi'' \) and \( D\chi' \) and so forth suppressed)

\[
\langle 0 | T \left[ \overline{\psi}(x_1) \ldots \overline{\psi}(x_{2n}) \right] | 0 \rangle = \frac{\int \langle 0 | \chi''(x_1) \ldots \chi(x_{2n}) e^{iS_0[\chi]} \rangle \, D\chi'' D\chi}{\int \langle 0 | \chi''(x_1) \ldots \chi''(x_{2n}) e^{iS_0[\chi']} \rangle \, D\chi'' D\chi'}
\]  

(7.264)

As in (7.177), the effect of the inner products \( \langle 0 | \chi'' \rangle \) and \( \langle \chi' | 0 \rangle \) is to insert
\[ \langle 0 | T \left[ \overline{\psi}(x_1) \cdots \psi(x_{2n}) \right] | 0 \rangle = \frac{\int \overline{\chi}(x_1) \cdots \chi(x_{2n}) e^{i S_0[x|\epsilon]} \, D\chi^* D\chi}{\int e^{i S_0[x|\epsilon]} \, D\chi^* D\chi}. \tag{7.265} \]

Imitating (7.178), we introduce a Grassmann external current \( \zeta(x) \) and define a fermionic analog of \( Z_0[j] \)

\[ Z_0[\zeta] \equiv \langle 0 | T \left[ e^{i \int \overline{\zeta} \psi + \overline{\psi} \zeta \, d^4 x} \right] | 0 \rangle = \frac{\int e^{i \int \overline{\zeta} \chi + \chi \zeta \, d^4 x} e^{i S_0[x|\epsilon]} \, D\chi^* D\chi}{\int e^{i S_0[x|\epsilon]} \, D\chi^* D\chi}. \tag{7.266} \]

**Example 7.11** (Feynman’s fermion propagator) Since

\[ i (\gamma^\mu \partial_\mu + m) \Delta(x - y) \equiv i (\gamma^\mu \partial_\mu + m) \int \frac{d^4 p}{(2\pi)^4} e^{ip(x-y)} \left\{ \frac{-i \gamma^\nu p_\nu + m}{p^2 + m^2 - i\epsilon} \right\} \]

\[ = \int \frac{d^4 p}{(2\pi)^4} e^{ip(x-y)} \frac{(-i \gamma^\nu p_\nu + m)}{p^2 + m^2 - i\epsilon} \]

\[ = \int \frac{d^4 p}{(2\pi)^4} e^{ip(x-y)} \frac{p^2 + m^2}{p^2 + m^2 - i\epsilon} = \delta^4(x - y), \tag{7.267} \]

the function \( \Delta(x - y) \) is the inverse of the differential operator \( i (\gamma^\mu \partial_\mu + m) \). Thus the Grassmann identity (7.245) implies that \( Z_0[\zeta] \) is

\[ \langle 0 | T \left[ e^{i \int \overline{\zeta} \psi + \overline{\psi} \zeta \, d^4 x} \right] | 0 \rangle = \frac{\int e^{i \int \overline{\zeta} \chi + i \int \overline{\zeta} (\gamma^\mu \partial_\mu + m) \chi} \, d^4 x \, D\chi^* D\chi}{\int e^{i S_0[x|\epsilon]} \, D\chi^* D\chi} \]

\[ = \exp \left[ \int \overline{\zeta}(x) \Delta(x - y) \zeta(y) \, d^4 x d^4 y \right]. \tag{7.268} \]

Differentiating we get

\[ \langle 0 | T \left[ \psi(x) \overline{\psi}(y) \right] | 0 \rangle = \Delta(x - y) = -i \int \frac{d^4 p}{(2\pi)^4} e^{ip(x-y)} \frac{-i \gamma^\nu p_\nu + m}{p^2 + m^2 - i\epsilon}. \tag{7.269} \]
7.13 Application to nonabelian gauge theories

The action of a generic non-abelian gauge theory is

$$S = \int -\frac{1}{4} F_{a\mu\nu} F^{a\mu\nu} - \overline{\psi} (\gamma^\mu D_\mu + m) \psi \, d^4x \tag{7.270}$$

in which the Maxwell field is

$$F_{a\mu\nu} \equiv \partial_\mu A_{a\nu} - \partial_\nu A_{a\mu} + g f_{abc} A_{b\mu} A_{c\nu} \tag{7.271}$$

and the covariant derivative is

$$D_\mu \psi \equiv \partial_\mu \psi - ig t_a A_{a\mu} \psi. \tag{7.272}$$

Here $g$ is a coupling constant, $f_{abc}$ is a structure constant (?), and $t_a$ is a generator (?) of the Lie algebra (section ??) of the gauge group.

One may show (Weinberg, 1996, pp. 14–18) that the analog of equation (7.206) for quantum electrodynamics is

$$\langle \Omega | T [O_1 \ldots O_n] | \Omega \rangle = \int O_1 \ldots O_n e^{iS} \delta[A_{a3}] DAD\psi \int e^{iS} \delta[A_{a3}] DAD\psi \tag{7.273}$$

in which the functional delta function

$$\delta[A_{a3}] \equiv \prod_{x,b} \delta(A_{a3}(x)) \tag{7.274}$$

enforces the axial-gauge condition, and $D\psi$ stands for $D\psi^* D\psi$.

Initially, physicists had trouble computing nonabelian amplitudes beyond the lowest order of perturbation theory. Then DeWitt showed how to compute to second order (DeWitt, 1967), and Faddeev and Popov, using path integrals, showed how to compute to all orders (Faddeev and Popov, 1967).

7.14 Faddeev-Popov trick

The path-integral tricks of Faddeev and Popov are described in (Weinberg, 1996, pp. 19–27). We will use gauge-fixing functions $G_a(x)$ to impose a gauge condition on our non-abelian gauge fields $A_{a\mu}^\lambda(x)$. For instance, we can use $G_a(x) = A_{a\mu}^a(x)$ to impose an axial gauge or $G_a(x) = i\partial_\mu A_{a\mu}^a(x)$ to impose a Lorentz-invariant gauge.

Under an infinitesimal gauge transformation (??)

$$A_{a\mu}^\lambda = A_{a\mu} - \partial_\mu \lambda_a - g f_{abc} A_{b\mu} \lambda_c \tag{7.275}$$
the gauge fields change, and so the gauge-fixing functions \( G_b(x) \), which depend upon them, also change. The Jacobian \( J \) of that change at \( \lambda = 0 \) is

\[
J = \det \left( \frac{\delta G_a^\lambda(x)}{\delta \lambda_b(y)} \right) \bigg|_{\lambda=0} \equiv \frac{DG^\lambda}{D\lambda} \bigg|_{\lambda=0}
\]

(7.276)

and it typically involves the delta function \( \delta^4(x - y) \).

Let \( B[G] \) be any functional of the gauge-fixing functions \( G_b(x) \) such as

\[
B[G] = \prod_{x,a} \delta(G_a(x)) = \prod_{x,a} \delta(A^\beta_3(x))
\]

(7.277)
in an axial gauge or

\[
B[G] = \exp \left[ \frac{i}{2} \int (G_a(x))^2 \, d^4x \right] = \exp \left[ -\frac{i}{2} \int (\partial_\mu A^\mu_3(x))^2 \, d^4x \right]
\]

(7.278)
in a Lorentz-invariant gauge.

We want to understand functional integrals like (7.273)

\[
\langle \Omega | T [O_1 \ldots O_n] | \Omega \rangle = \int O_1 \ldots O_n e^{iS} B[G] J \, D A \, D \psi \int e^{iS} B[G] J \, D A \, D \psi
\]

(7.279)
in which the operators \( O_k \), the action functional \( S[A] \), and the differentials \( D A \, D \psi \) (but not the gauge-fixing functional \( B[G] \) or the Jacobian \( J \)) are gauge invariant. The axial-gauge formula (7.273) is a simple example in which \( B[G] = \delta[A^3_3] \) enforces the axial-gauge condition \( A^\alpha_3(x) = 0 \) and the determinant \( J = \det (\delta_{ab} \delta(x - y)) \) is a constant that cancels.

If we translate the gauge fields by gauge transformations \( \Lambda \) and \( \Lambda' \), then the ratio (7.279) does not change

\[
\langle \Omega | T [O_1 \ldots O_n] | \Omega \rangle = \int O_1 \ldots O_n e^{iS} B[G^\Lambda] J^\Lambda \, D A^\Lambda \, D \psi^\Lambda \int e^{iS} B[G^{\Lambda'}] J^{\Lambda'} \, D A^{\Lambda'} \, D \psi^{\Lambda'}
\]

(7.280)

any more than \( \int f(y) \, dy \) is different from \( \int f(x) \, dx \). Since the operators \( O_k \), the action functional \( S[A] \), and the differentials \( D A \, D \psi \) are gauge invariant, most of the \( \Lambda \)-dependence goes away

\[
\langle \Omega | T [O_1 \ldots O_n] | \Omega \rangle = \int O_1 \ldots O_n e^{iS} B[G^\Lambda] J^\Lambda \, D A \, D \psi \int e^{iS} B[G^{\Lambda'}] J^{\Lambda'} \, D A \, D \psi
\]

(7.281)

Let \( \Lambda \lambda \) be a gauge transformation \( \Lambda \) followed by an infinitesimal gauge
transformation $\lambda$. The Jacobian $J^\Lambda$ is a determinant of a product of matrices which is a product of their determinants

$$J^\Lambda = \det \left( \frac{\delta G^a_\Lambda(x)}{\delta \lambda_b(y)} \right)_{\lambda=0} = \det \left( \int \frac{\delta G^a_\Lambda(x)}{\delta \lambda_c(z)} \frac{\delta \lambda_c(z)}{\delta \lambda_b(y)} d^4 z \right)_{\lambda=0}$$

$$= \det \left( \frac{\delta G^a_\Lambda(x)}{\delta \lambda_c(z)} \right)_{\lambda=0} \det \left( \frac{\delta \lambda_c(z)}{\delta \lambda_b(y)} \right)_{\lambda=0} = \det \left( \frac{\delta G^a_\Lambda(x)}{\delta \lambda_c(z)} \right)_{\lambda=0} \equiv \frac{\partial G^a_\Lambda}{\partial \lambda} \frac{\partial G^a_\Lambda}{\partial \lambda} \bigg|_{\lambda=0}.$$ (7.282)

Now we integrate over the gauge transformations $\Lambda$ (and $\Lambda'$) with weight function $\rho(\Lambda) = \left( \frac{\partial G^a_\Lambda}{\partial \lambda} \bigg|_{\lambda=0} \right)$ and find, since the ratio (7.281) is $\Lambda$-independent

$$\langle \Omega | T [O_1 \ldots O_n] | \Omega \rangle = \frac{\int O_1 \ldots O_n e^{iS} B[G^\Lambda] \frac{D G^\Lambda}{D \Lambda} DA DA D\psi}{\int e^{iS} B[G^\Lambda'] \frac{D G^\Lambda'}{D \Lambda'} DA' DA D\psi}$$

$$= \frac{\int O_1 \ldots O_n e^{iS} B[G^\Lambda] D G^\Lambda DA D\psi}{\int e^{iS} B[G^\Lambda] DA D\psi}$$

$$= \frac{\int O_1 \ldots O_n e^{iS} DA D\psi}{\int e^{iS} DA D\psi}.$$ (7.283)

Thus the mean-value in the vacuum of a time-ordered product of gauge-invariant operators is a ratio of path integrals over all gauge fields without any gauge fixing. No matter what gauge condition $G$ or gauge-fixing functional $B[G]$ we use, the resulting gauge-fixed ratio (7.279) is equal to the ratio (7.283) of path integrals over all gauge fields without any gauge fixing. All gauge-fixed ratios (7.279) give the same time-ordered products, and so we can use whatever gauge condition $G$ or gauge-fixing functional $B[G]$ is most convenient.

The analogous formula for the euclidean time-ordered product is

$$\langle \Omega | T_e [O_1 \ldots O_n] | \Omega \rangle = \frac{\int O_1 \ldots O_n e^{-S_e} DA D\psi}{\int e^{-S_e} DA D\psi}.$$ (7.284)
where the euclidian action $S_e$ is the spacetime integral of the energy density. This formula is the basis for lattice gauge theory.

The path-integral formulas (7.210 & 7.211) derived for quantum electrodynamics therefore also apply to nonabelian gauge theories.

7.15 Ghosts

Faddeev and Popov showed how to do perturbative calculations in which one does fix the gauge. To continue our description of their tricks, we return to the gauge-fixed expression (7.279) for the time-ordered product

$$\langle \Omega | T [O_1 \ldots O_n] | \Omega \rangle = \int O_1 \ldots O_n e^{iS} B[G] J \det D\psi$$

(7.285)

set $G_b(x) = -i\partial_\mu A^\mu_b(x)$ and use (7.278) as the gauge-fixing functional

$$B[G] = \exp \left[ \frac{i}{2} \int (G_a(x))^2 \right] = \exp \left[ -\frac{i}{2} \int (\partial_\mu A^\mu_a(x))^2 \right].$$

(7.286)

This functional adds to the action density the term $- (\partial_\mu A^\mu_a(x))^2/2$ which leads to a gauge-field propagator like the photon’s (7.214)

$$\langle 0 | T [A^\mu_a(x) A^\nu_b(y)] | 0 \rangle = -i \delta_{ab} \Delta_{\mu\nu}(x-y) = - i \int \frac{\eta_{\mu\nu} \delta_{ab}}{q^2 - i\epsilon} e^{i(q(x-y))} \frac{d^4 q}{(2\pi)^4}.$$  

(7.287)

What about the determinant $J$? Under an infinitesimal gauge transformation (7.275), the gauge field becomes

$$A^\lambda_{a\mu} = A_{a\mu} - \partial_\mu \lambda_a - g f_{abc} A_{b\mu} \lambda_c$$

(7.288)

and so $G^\lambda_{a\mu}(x) = i\partial^\mu A^\lambda_{a\mu}(x)$ is

$$G^\lambda_{a\mu}(x) = i\partial^\mu A_{a\mu}(x) + i\partial^\mu \int [-\delta_{ac} \partial_\mu - g f_{abc} A_{b\mu}(x)] \delta^4(x-y) \lambda_c(y) d^4 y.$$  

(7.289)

The jacobian $J$ then is the determinant (7.276) of the matrix

$$\left( \frac{\delta G^\lambda_{a\mu}(x)}{\delta \lambda_c(y)} \right)_{\lambda=0} = -i \delta_{ac} \square \delta^4(x-y) - i g f_{abc} \frac{\partial}{\partial x^\mu} [A^\mu_b(x) \delta^4(x-y)]$$

(7.290)

that is

$$J = \det \left( -i \delta_{ac} \square \delta^4(x-y) - i g f_{abc} \frac{\partial}{\partial x^\mu} [A^\mu_b(x) \delta^4(x-y)] \right).$$  

(7.291)
But we’ve seen (7.244) that a determinant can be written as a fermionic path integral
\[
\det A = \int e^{-\theta^\dagger A \theta} \prod_{k=1}^{n} d\theta^*_k d\theta_k. \tag{7.292}
\]
So we can write the Jacobian \( J \) as
\[
J = \int \exp \left[ \int i\omega^*_a \Box \omega_a + ig f_{abc} \omega^*_a \partial_\mu (A^\mu_b \omega_c) \, d^4 x \right] D\omega^* D\omega \tag{7.293}
\]
which contributes the terms \( -\partial^\mu \omega^*_a \partial^\mu \omega_a \) and
\[
-\partial^\mu \omega^*_a g f_{abc} A^\mu_b \omega_c = \partial^\mu \omega^*_a g f_{abc} A^\mu_c \omega_b \tag{7.294}
\]
to the action density.

Thus we can do perturbation theory by using the modified action density
\[
\mathcal{L}' = -\frac{1}{4} F_{a\mu\nu} F^{a\mu\nu} - \frac{1}{2} (\partial_\mu A^\mu_a)^2 - \partial_\mu \omega^*_a \partial^\mu \omega_a + \partial^\mu \omega^*_a g f_{abc} A^\mu_c \omega_b - \bar{\psi} (\bar{D} + m) \psi \tag{7.295}
\]
in which \( \bar{D} \equiv \gamma^\mu D_\mu = \gamma^\mu (\partial_\mu - ig A_\mu) \). The ghost field \( \omega \) is a mathematical device, not a physical field describing real particles, which would be spinless fermions violating the spin-statistics theorem (example ??).

### 7.16 Integrating over the momenta

When a Hamiltonian is quadratic in the momenta like (7.128 & 7.149), one easily integrates over its momentum and converts it into its Lagrangian. If, however, the Hamiltonian is a more complicated function of the momenta, one usually cannot path-integrate over the momenta analytically. The partition function is then a path integral over both \( \phi \) and \( \pi \)
\[
Z(\beta) = \int \exp \left\{ \int_0^\beta \int [i\dot{\phi}(x)\pi(x) - H(\phi, \pi)] \, dt \, d^3 x \right\} D\phi D\pi. \tag{7.296}
\]
This exponential is not positive, and so is not a probability distribution for \( \phi \) and \( \pi \). The Monte Carlo methods of chapter ?? are designed for probability distributions, not for distributions that assume negative or complex values. This is one aspect of the **sign problem**.

The integral over the momentum
\[
P[\phi] = \int \exp \left\{ \int_0^\beta \int [i\dot{\phi}(x)\pi(x) - H(\phi, \pi)] \, dt \, d^3 x \right\} D\pi \tag{7.297}
\]
is positive, however, and is a probability distribution. So one can numerically integrate over the momentum, make a look-up table for \( P[\phi] \), and then apply
the usual Monte Carlo method to the probability functional $P[\phi]$ (Amdahl and Cahill 2016). Programs that do this are in the repository Path_integrals at github.com/kevinecahill.
8

Landau theory of phase transitions

8.1 First- and second-order phase transitions

In a first-order phase transition, a physical quantity, called an order parameter, changes discontinuously. Often there is a point where the discontinuous phase transition becomes continuous. Such a point is called a critical point. The continuous phase transition is called a second-order phase transition.

Binary alloys, fluids, and helium-4 exhibit phase transitions.

The archetypical example is the ferromagnet. In an idealized ferromagnet with spins parallel or antiparallel to a given axis, the order parameter is the total magnetization $M$ along this axis. At low temperatures, an external magnetic field $H$ will favor one spin direction. At $H = 0$ the states will be in equilibrium, but small changes in $H$ will make $M$ change discontinuously. There is a first-order phase transition at $H = 0$ at low temperatures.

At slightly higher temperatures, the spins fluctuate and the mean value of $|M|$ decreases. At the critical temperature $T_c$, tiny changes in $H$ no longer make $M$ change discontinuously. At $T > T_c$, the phase transition is continuous and second order.

Landau used a Gibbs free-energy potential $G$ which obeys

$$\frac{\partial G}{\partial M} \bigg|_T = -H. \quad (8.1)$$

He worked at $T \approx T_c$ and $M \approx 0$ and expanded $G(M)$ at $H = 0$ as

$$G(M) = A(T) + B(T)M^2 + C(T)M^4 + \ldots \quad (8.2)$$

keeping only even terms because of the symmetry under $M \rightarrow -M$. So at
Landau theory of phase transitions

$H = 0$ equation (8.1) gives

$$0 = - H = \frac{\partial G}{\partial M} \bigg|_{T} = 2B(T)M + 4C(T)M^3. \quad (8.3)$$

If $B$ and $C$ are positive, the only solution is $M = 0$. But if $C > 0$ and if for $T < T_c$ it happens that $B < 0$, then we get

$$2CM^2 + B = 0 \quad \text{or} \quad M = \sqrt{-B/2C}. \quad (8.4)$$

The approximations

$$B(T) \approx b(T - T_c) \quad \text{and} \quad C(T) = c \quad (8.5)$$

give

$$M = \pm \sqrt{(b/2c)(T_c - T)} \quad \text{for} \quad T < T_c \quad (8.6)$$

while $M = 0$ for $T > T_c$.

The minimum of the potential

$$G(M, H) = A(T) + B(T)M^2 + C(T)M^4 - HM \quad (8.7)$$

as a function of $M$ at constant $T$ gives equation (8.3).

One may represent the magnetization in terms of a spin density $s(\mathbf{x})$ as

$$M = \int s(\mathbf{x}) \, d^3x. \quad (8.8)$$

Gibb’s free energy then becomes

$$G = \int \frac{1}{2}(\nabla s)^2 + b(T - T_c)s^2 + cs^4 - Hs \, d^3x. \quad (8.9)$$

The spin density that minimizes the free energy must make $G$ stationary

$$0 = \frac{\delta G}{\delta s(\mathbf{x})} = -\nabla^2 s(\mathbf{x}) + 2b(T - T_c)s(\mathbf{x}) + 4cs^3(\mathbf{x}) - H(\mathbf{x}). \quad (8.10)$$

For $T > T_c$, the macroscopic magnetization vanishes and so the spin density $s$ is small. So we drop the nonlinear term and get

$$(-\nabla^2 + 2b(T - T_c)) s(\mathbf{x}) = H(\mathbf{x}). \quad (8.11)$$

The spin density $D(\mathbf{x})$ generated by $H(\mathbf{x}) = H_0\delta^3(\mathbf{x})$ is the Green’s function of the differential equation

$$(-\nabla^2 + 2b(T - T_c)) D(\mathbf{x}) = H_0 \delta(\mathbf{x}). \quad (8.12)$$
8.1 First- and second-order phase transitions

Expressing both sides as Fourier transforms, we get for $T > T_c$

$$D(x) = \int \frac{d^3k}{(2\pi)^3} \frac{H_0 e^{ik \cdot x}}{k^2 + 2b(T - T_c)}$$

$$= \frac{H_0}{4\pi r} e^{-r/\xi}$$  \hspace{1cm} (8.13)

where the correlation length

$$\xi = [2b(T - T_c)]^{-1/2}$$  \hspace{1cm} (8.14)

which diverges as $T \to T_c$ indicating the onset of large-scale fluctuations.

Different classes of systems have different critical exponents. For example, when $T < T_c$, the magnetization can go as

$$M \approx (T_c - T)^\beta$$  \hspace{1cm} (8.15)

in which instead of $\beta = 1/2$ one has $\beta = 0.313$ for some fluids and some 3D magnets.

The susceptibility $\chi = \frac{\partial M}{\partial H}|_H = 0$ diverges as $|T - T_c|^{-\gamma}$. We differentiate the equation (8.3) with respect to $H$

$$0 = -H = \frac{\partial G}{\partial M}|_T = 2B(T)M + 4C(T)M^3$$  \hspace{1cm} (8.16)

getting

$$-1 = 2B\chi + 4C3M^2\chi.$$  \hspace{1cm} (8.17)

So

$$\chi = \frac{-1}{2B + 12CM^2}.$$  \hspace{1cm} (8.18)

Setting $B = b(T - T_c)$, $C = c$, and $M = \pm \sqrt{(b/2c)(T_c - T)}$, we get

$$\chi = \frac{-1}{2b(T - T_c) + 6b(T_c - T)} = \frac{-1}{4b(T_c - T)} = \frac{1}{4b(T - T_c)}$$  \hspace{1cm} (8.19)

which says $\gamma = 1$. 

9.1 Christoffel symbols as nonabelian gauge fields

A contravariant vector \( V^i \) transforms like

\[
V'^i(x') = \frac{\partial x'^i}{\partial x'^k} V^k(x) = x'^i_k V^k(x) = E^i_k(x) V^k(x).
\] (9.1)

The 4 × 4 matrix \( E^i_k(x) = x'^i_k \) depends upon the spacetime point \( x \) and is a member of the huge noncompact group \( GL(4, \mathbb{R}) \).

The insight of Yang and Mills (section 5.1) lets us define a covariant derivative \( D_\ell = \partial_\ell + A_\ell \) of a contravariant vector \( V^i \)

\[
(D_\ell V)^k = (\partial_\ell \delta^k_j + A^k_\ell j) V^j
\] (9.2)

that transforms as

\[
[(D_\ell V)^k]' = \frac{\partial x'^i}{\partial x'^m} (D_\ell V)^m.
\] (9.3)

We need

\[
[(\partial_\ell \delta^k_j + A^k_\ell j)V^j]' = (\partial'_\ell \delta^k_j + A'^k_\ell j)V'^j = V'^k + A'^k_\ell V'^j
\]
\[
x_i x'_j (\partial_k \delta^m_n + A^m_n) V^m = x_i x'_j (V^m + A^m_n V^n)
\] (9.4)

or

\[
V'^k + A'^k_\ell V'^j = x_i x'_j (V^m + A^m_n V^n).
\] (9.5)

Decoding the left-hand side, we get

\[
\partial_\ell (x'^m_i V^m) + A'^k_\ell V'^j = x'_i x'^m_j (V^m + A^m_n V^n)
\] (9.6)
or
\[ x^i_{,\ell} \partial_{\ell}(x^k_{,m} V^m) + A^k_{ij} V^{ij} = x^i_{,\ell} x^k_{,m} (V^m + A^m_{in} V^n) \]  
(9.7)

or
\[ x^i_{,\ell} x^k_{,m} V^m_i + x^i_{,\ell} x^k_{,m} V^m + A^k_{ij} V^{ij} = x^i_{,\ell} x^k_{,m} (V^m_i + A^m_{in} V^n). \]  
(9.8)

After a cancelation, we get
\[ x^i_{,\ell} x^k_{,m} V^m + A^k_{ij} V^{ij} = x^i_{,\ell} x^k_{,m} A^m_{in} V^n \]  
(9.9)

or
\[ A^k_{ij} V^{ij} = x^p_{,\ell} x^k_{,m} A^m_{pn} V^n - x^i_{,\ell} x^{kn}_{,mn} V^m \]  
(9.10)

which is
\[ A^k_{ij} x^{ij}_{,m} V^m = x^p_{,\ell} x^k_{,m} A^m_{pn} V^n - x^i_{,\ell} x^{kn}_{,mn} V^m. \]  
(9.11)

or after the interchange of \( m \) and \( n \) in the second term
\[ A^k_{ij} x^{ij}_{,m} V^m = x^p_{,\ell} x^k_{,n} A^m_{pn} V^n - x^i_{,\ell} x^{kn}_{,nm} V^m. \]  
(9.12)

Since the vector \( V^m \) is arbitrary, we can promote this to an equation with three free indexes
\[ A^k_{ij} x^{ij}_{,m} = x^p_{,\ell} x^k_{,n} A^m_{pn} - x^i_{,\ell} x^{kn}_{,mn}. \]  
(9.13)

and then multiply by the inverse of \( x^{ij}_{,m} \)
\[ A^k_{ij} x^{ij}_{,m} A^m_{np} x^n_{,n,\ell} = x^p_{,\ell} x^k_{,n} A^m_{pn} x^n_{,n,\ell} - x^i_{,\ell} x^{kn}_{,nm} x^n_{,n,\ell}. \]  
(9.14)

so as to get
\[ A^k_{i} = x^m_{,i} x^k_{,n} A^m_{np} x^n_{,\ell} - x^p_{,\ell} x^{nk}_{,np} x^n_{,\ell}. \]  
(9.15)

Partial derivatives of the same kind commute, so \( x^{ik}_{,jn} = x^{jk}_{,im} \). It follows that the inhomogeneous term in the transformation law (9.15) for \( A \) is symmetric in \( \ell \) and \( i \)
\[ - x^p_{,\ell} x^{nk}_{,n,\ell} = - x^n_{,i} x^{np}_{,\ell} = - x^p_{,\ell} x^{jk}_{,pm} x^n_{,\ell} = - x^p_{,\ell} x^{jk}_{,np} x^n_{,\ell}. \]  
(9.16)

The homogeneous term \( x^m_{,i} x^k_{,n} A^m_{np} x^n_{,\ell} \) in the transformation law (9.15) for \( A \) is symmetric in \( \ell \) and \( i \) if \( A^m_{np} \) is symmetric in \( n \) and \( m \), \( A^m_{nm} = A^m_{mn} \). Thus we can use gauge fields that are symmetric in their lower indexes, \( A^m_{nm} = A^m_{nm} \).

The transformations are
\[ V^k = x^k_{,j} V^j \equiv E^k_{j} V^j \quad \text{and} \quad V^i_\ell = x^i_{,\ell} V_i \equiv E^{-1} T^i_{\ell} V_i = E^{-1} i_{\ell} V_i \]  
(9.17)
and
\[ A^k_{\ell i} = x^m x^{\ell k} A^p_{nm} x^n_{i'} - x^p x^{\ell k} A^m_{np} x^n_{i'}, \]
\[ = E^{-1m}_\ell E^k_p A^p_{nm} E^{-1n}_{i'} - x^p x^{\ell k} x^n_{i'}. \]

We decode the inhomogeneous term
\[ -x^p x^{\ell k} x^n_{i'} = -(\partial_k x^p) x^n_{i'} = -(\partial_k E^k_n) x^n_{i'} = -(\partial_k E^k_n) E^{-1n}_{i'} \]
\[ = -E^{-1}_{i'}(\partial_k E^k_n) = E^k_n(\partial_k E^{-1n}_{i'}) = x^{\ell k}(\partial_k x^n_{i'}) \]
(9.18)

So on a contravariant vector \( V' = E^{-1}V, A' \) is
\[ A'_\ell = E^{-1m}_\ell E A_m E^{-1} + E \partial_k E^{-1} \]
(9.19)
or
\[ A^k_{\ell i} = E^{-1m}_\ell E^p_m A^p_{mn} E^{-1n}_{i'} + E^k_{ni} \partial_k E^{-1n}_{i'}. \]
(9.20)

This gauge field is the affine connection, aka, Christoffel symbol (of the second kind) \( A^k_{\ell i} = \Gamma^k_{\ell i} \). The usual formula for the covariant derivative of a contravariant vector is
\[ (D_\ell V)^k = (\partial_\ell \delta^k_j + A^k_{\ell j})V^j = (\partial_\ell \delta^k_j + \Gamma^k_{\ell j})V^j = \partial_\ell V^k + \Gamma^k_{\ell j} V^j. \]
(9.21)

We can write the chain rule identities
\[ \frac{\partial x^i}{\partial x^k} \frac{\partial x^k}{\partial x^\ell} = \delta^i_\ell = \frac{\partial x^i}{\partial x^k} \frac{\partial x^k}{\partial x^\ell} \]
(9.22)
in terms of the matrix \( E^k_\ell = x^k_{\ell} \) as
\[ \frac{\partial x^i}{\partial x^k} E^k_\ell = \delta^i_\ell = E^i_k \frac{\partial x^k}{\partial x^\ell} \]
(9.23)
which show that the inverse of the matrices \( E^k_\ell \) and \( E^i_k \) are
\[ E^{-1i}_{k} = \frac{\partial x^i}{\partial x^k} = x^i_{,k} \text{ and } E^{-1k}_\ell = \frac{\partial x^k}{\partial x^\ell} = x^k_{,\ell}. \]
(9.24)
The very general identity
\[ 0 = \partial_\ell 1 = \partial_\ell (C C^{-1}) = C_\ell i C^{-1} + C^{-1} C_{,i} \]
(9.25)
will soon be useful. So we can write
\[ E^m_n E^{-1m}_{k,\ell} = E^m_n \partial_\ell E^{-1m}_{k,} = E^m_n \partial_{k'} \frac{\partial x^n}{\partial x^\ell} = E^m_n E^{-1n}_{\ell k'} \]
\[ = E^m_n E_{\ell, k'} = -E^m_{n,k'} E^{-1n}_{\ell}. \]
(9.26)
A covariant vector $V_j$ transforms like a derivative $\partial_j$

$$V'_j(x') = \frac{\partial x^k}{\partial x'^j} V_k(x) = x^k_j V_k(x) \equiv E^{-1} k_j V_k. \quad (9.27)$$

The Yang-Mills trick now requires that we find a gauge field $B_\ell$ so that the covariant derivative of a covariant vector

$$(D_\ell V)_k = (\partial_\ell \delta^k_\ell + B^j_{\ell k}) V_j \quad (9.28)$$

transforms as

$$[ (D_\ell V)_k ]' = \frac{\partial x^i}{\partial x'^k} \frac{\partial x^n}{\partial x'^i} (D_\ell V)_n. \quad (9.29)$$

We need

$$[ (\partial_\ell \delta^j_\ell + B^j_{\ell k}) V_j ]' = (\partial_\ell \delta^j_\ell + B^j_{\ell k}) V_j' = V_{k', \ell} + B^j_{\ell k} V_j' = x^i_{\ell'} x'^n_{k', \ell} (\partial_\delta^p_n + B^p_n)_\ell = x^i_{\ell'} x'^n_{k', \ell} (V_{n, i} + B^p_n V_p)$$

or

$$V_{k', \ell} + B^j_{\ell k} V_j' = x^i_{\ell'} x'^n_{k', \ell} (V_{n, i} + B^p_n V_p). \quad (9.30)$$

Decoding the left-hand side, we get

$$(x^n_{k', \ell} V_n + B^j_{\ell k} x^n_{j', \ell} V_n = x^n_{k', \ell} V_n + x^n_{j', \ell} V_n, V_{n, \ell} + x^i_{\ell'} x'^n_{k', \ell} V_{n, i} + x^i_{\ell'} x'^n_{k', \ell} B^p_n V_p) \quad (9.32)$$

So combining the last two equations, we find

$$x^n_{k', \ell} V_n + x^n_{j', \ell} V_n, i x^i_{\ell'} V_{n, i} + x^i_{\ell'} x'^n_{k', \ell} B^p_n V_p \quad (9.33)$$

or

$$x^n_{k', \ell} V_n + B^j_{\ell k} x^n_{j', \ell} V_n = x^i_{\ell'} x'^n_{k', \ell} B^p_n V_p. \quad (9.34)$$

Interchanging $p$ and $n$ in the third term

$$x^n_{k', \ell} V_n + B^j_{\ell k} x^n_{j', \ell} V_n = x^i_{\ell'} x'^n_{k', \ell} B^p_{n, \ell} V_{n, \ell} \quad (9.35)$$

and then using the arbitrariness of $V_n$, we get

$$x^n_{k', \ell} + B^j_{\ell k} x^n_{j', \ell} = x^i_{\ell'} x'^n_{k', \ell} B^p_{n, \ell} \quad (9.36)$$

Multiplying by $x^m_{\ell}$ gives

$$x^n_{k', \ell} x^m_{\ell} + B^j_{\ell k} x^n_{j', \ell} x^m_{\ell} = x^n_{k', \ell} x^m_{\ell} + B^m_{\ell k} = x^i_{\ell'} x'^p_{k', \ell} B^m_{n, \ell} x^m_{\ell} \quad (9.37)$$

or

$$B^m_{\ell k} = x^i_{\ell'} x'^p_{k', \ell} B^m_{n, \ell} x^m_{\ell} - x^m_{k', \ell} x^m_{\ell} \quad (9.38)$$
The inhomogeneous term $-x_{,kl'}^n x_{,l'}^m$ is symmetric in $k$ and $\ell$. The homogeneous term $x_{,p}^p x_{,p}^m B_{ip}^n x_{,n}^p$ also is symmetric in $k$ and $\ell$. Thus we can use gauge fields that are symmetric in their lower indexes, $B_{ip}^n = B_{pi}^n$.

In terms of the matrices

$$E^k_\ell = x^k_{,\ell} \quad \text{and} \quad E^{-1k}_\ell = \frac{\partial x^k}{\partial x^{\ell'}} = x^k_{,\ell'}$$

and in view of the identities (9.25) and (9.26), the transformation rule (9.38) for $B$ has the equivalent form

$$B_{mk}^p = (E^{-1T}_i)^m_p E_{n}^m B_{np}^k E^{-1p}_k + E_{n,\ell}^m E^{-1n}_k.$$  \hspace{1cm} (9.40)

The transformation law for $A_\ell$ is

$$A_{k\ell}^l = E^{-1m}_i E_{n}^m A_{npm}^k E^{-1p}_p - E^{-1n}_{i,\ell} \partial_{\ell'} E_{n}^k.$$  \hspace{1cm} (9.41)

We write the formula (9.40) for $B'$ as

$$-B_{mk}^p = - (E^{-1T}_i)^m_p E_{n}^m B_{np}^k E^{-1p}_k - E_{n,\ell}^m E^{-1n}_k.$$  \hspace{1cm} (9.42)

and do the substitutions $k \rightarrow i$, $i \rightarrow m$, and $m \rightarrow k$

$$-B_{ki}^p = - E^{-1m}_i E_{n}^m B_{mp}^n E^{-1p}_i - E^{-1n}_{i,\ell} \partial_{\ell'} E_{n}^k.$$  \hspace{1cm} (9.43)

The inhomogeneous terms in the equations for $A'$ and $B'$ are the same. We interchange $p$ and $n$ in the formula for $B'$

$$-B_{ki}^p = - E^{-1m}_i E_{n}^m B_{mp}^n E^{-1n}_i - E^{-1n}_{i,\ell} \partial_{\ell'} E_{n}^k.$$  \hspace{1cm} (9.44)

Since the gauge fields are symmetric in their lower indexes, we can write this as

$$-B_{ki}^p = - E^{-1m}_i E^{-1n}_i B_{mn}^k E_{p}^k - E^{-1n}_{i,\ell} \partial_{\ell'} E_{n}^k.$$  \hspace{1cm} (9.45)

Since the transformation laws are the same, we can identify the two gauge fields apart from a minus sign

$$A_{ki}^p = - B_{ki}^p = \Gamma_{ki}^p.$$  \hspace{1cm} (9.46)

The gauge fields $A$ and $B$ correspond to the Christoffel symbols, the $\Gamma$’s of the usual notation,

$$A_{ik}^j = D_k V^i = V_{i,\ell}^i = V_{i,\ell}^i + \Gamma_{ik}^i V^k = V_{i,\ell}^i + e^i \cdot e_{k,\ell} V^k,$$

$$B_{ki}^p = D_k V_i = V_{i,\ell}^i = V_{i,\ell}^i - \Gamma_{ik}^i V^k = V_{i,\ell}^i - e^k \cdot e_{i,\ell} V_k.$$  \hspace{1cm} (9.47)
9.2 Spin connection

The Lorentz index $a$ on a tetrad $c^a_i$ is subject to local Lorentz transformations. A spin-one-half field $\psi_b$ also has an index $b$ that is subject to local Lorentz transformations. We can use the insight of Yang and Mills to introduce a gauge field, called the \textbf{spin connection} $\omega^{ab}_i$ which multiplies the generators $J_{ab}$ of the Lorentz group

$$\omega_i = \omega^{ab}_i J_{ab}. \quad (9.48)$$

Since the generators $J_{ab}$ are antisymmetric $J_{ab} = -J_{ba}$, the spin connection is also antisymmetric

$$\omega^{ab}_i = -\omega_{ba}^i. \quad (9.49)$$

So for each spacetime index $i$, there are six independent $\omega^{ab}_i$'s.

The spin connection is

$$\omega^{a}_{b\ell} = -c^k_b (c^a_{k,\ell} - \Gamma^j_{k\ell} c^a_j) = c^a_j c^k_b \Gamma^j_{k\ell} - c^a_k c^k_b. \quad (9.50)$$

Under a general coordinate transformation and a local Lorentz transformation, the spin connection transforms as

$$\omega'^{a}_{b\ell} = \frac{\partial x^i}{\partial x'^\ell} \left[ L^a_d \omega_d^i - (\partial_i L^a_c) \right] L^{-1}b^c. \quad (9.51)$$

This example is a work in progress....

\textbf{Example 9.1 (Relativistic particle)} The lagrangian of a relativistic particle of mass $m$ is $L = -m\sqrt{1 - \dot{q}^2}$ in units with $c = 1$. The hamiltonian is $H = \sqrt{p^2 + m^2}$. The path integral for the first step (7.63) toward $Z(\beta)$ is

$$\langle q_1 | e^{-\epsilon H} | q_0 \rangle = \int_{-\infty}^{\infty} e^{-\epsilon \sqrt{p^2 + m^2}} e^{i p' (q_1 - q_0)/\hbar} \frac{dp'}{2\pi \hbar} \quad (9.52)$$
which is a nontrivial integral. In the \( m \to 0 \) limit, it is
\[
\langle q_1 | e^{-iH} | q_a \rangle = \int_{-\infty}^{\infty} e^{-i\epsilon p'/(\hbar)} e^{i\epsilon(q_1 - q_a)/\hbar} \frac{dp'}{2\pi\hbar} \\
= 2 \int_0^{\infty} e^{-\epsilon p'} \cos \left( \frac{p'(q_1 - q_a)}{\hbar} \right) \frac{dp'}{2\pi\hbar} \\
= \int_0^{\infty} e^{-\epsilon p' + i\epsilon(q_1 - q_a)/\hbar} + e^{-\epsilon p' - i\epsilon(q_1 - q_a)/\hbar} \frac{dp'}{2\pi\hbar} \\
= \left( \frac{1}{\epsilon - i(q_1 - q_a)/\hbar} + \frac{1}{\epsilon + i(q_1 - q_a)/\hbar} \right) \frac{1}{2\pi\hbar} \\
= \left( \frac{\epsilon^2 + (q_1 - q_a)^2/(\hbar^2)}{2\epsilon} \right) \frac{1}{2\pi\hbar} \\
= \frac{1}{\pi\hbar} \left( 1 + \frac{1}{(q_1 - q_a)^2/(\hbar^2)^2} \right). \tag{9.53}
\]

So the partition function is \( n \to \infty \) limit of the product of the \( n \) integrals
\[
Z(\beta) = \left( \frac{1}{\pi\hbar} \right)^n \int_{-\infty}^{\infty} dq_n \frac{dq_n}{1 + (q_n - q_{n-1})^2/(\hbar^2)^2} \cdots \int_{-\infty}^{\infty} dq_1 \frac{dq_1}{1 + (q_1 - q_n)^2/(\hbar^2)^2}
\]
in which \( q_n = q_a \).
\[
Z(\beta) = \left( \frac{1}{\pi\hbar} \right)^n \left( \frac{\pi\hbar}{\pi} \right)^n. \tag{9.54}
\]

In one space dimension, quantum mechanics gives the result
\[
Z(\beta) = \frac{L}{\pi\hbar\beta c} \tag{9.55}
\]
in which \( c \) has reappeared, and we must take the \( L \to \infty \) limit. In two space dimensions, it gives
\[
Z(\beta) = \frac{L^2}{(2\pi\hbar)^2} \int_0^{\infty} e^{-\beta c \sqrt{m^2c^2 + p^2}} 2\pi pdp = \frac{L^2}{4\pi\hbar^2} \int_0^{\infty} e^{-\beta c \sqrt{m^2c^2 + u}} du \\
= \left( \frac{Lmc}{4\pi\hbar^2} \right)^2 \int_0^{\infty} e^{-\beta mc^2 \sqrt{1 + v^2}} dv = \left( \frac{Lmc}{4\pi\hbar^2} \right)^2 \frac{2(1 + \beta mc^2)}{(\beta mc^2)^2} e^{-\beta mc^2} \tag{9.56}
\]
\[
= \frac{L^2}{2\pi\hbar^2} \frac{1 + \beta mc^2}{(\beta c)^2} e^{-\beta mc^2}. \]