Physics 523 & 524: Quantum Field Theory I

Kevin Cahill
Department of Physics and Astronomy
University of New Mexico, Albuquerque, NM 87131

Contents

1

Quantum fields and special relativity

1.1 States

A Lorentz transformation Λ is implemented by a unitary operator $U(\Lambda)$ which replaces the state $|p,\sigma\rangle$ of a massive particle of momentum p and spin σ along the z-axis by the state

$$U(\Lambda)|p,\sigma\rangle = \sqrt{\frac{(\Lambda p)^0}{p^0}} \sum_{s'} D^{(j)}_{s'\sigma}(W(\Lambda,p))\,|\Lambda p, s'\rangle \tag{1.1}$$

where $W(\Lambda,p)$ is a Wigner rotation

$$W(\Lambda,p) = L^{-1}(\Lambda p)\Lambda L(p) \tag{1.2}$$

and $L(p)$ is a standard Lorentz transformation that takes $(m,\vec{0})$ to p.

1.2 Creation operators

The vacuum is invariant under Lorentz transformations and translations

$$U(\Lambda,a)|0\rangle = |0\rangle. \tag{1.3}$$

A creation operator $a^\dagger(p,\sigma)$ makes the state $|p,\sigma\rangle$ from the vacuum state $|0\rangle$

$$|p\sigma\rangle = a^\dagger(p,\sigma)|0\rangle. \tag{1.4}$$

The creation and annihilation operators obey either the commutation relation

$$[a(p,s), a^\dagger(p',s')]_- = a(p,s)\,a^\dagger(p',s') - a^\dagger(p',s')\,a(p,s) = \delta_{ss'}\,\delta^{(3)}(\boldsymbol{p}-\boldsymbol{p}') \tag{1.5}$$

or the anticommutation relation

$$[a(p,s), a^\dagger(p',s')]_+ = a(p,s)\, a^\dagger(p',s') + a^\dagger(p',s')\, a(p,s) = \delta_{ss'}\, \delta^{(3)}(\boldsymbol{p}-\boldsymbol{p}'). \tag{1.6}$$

The two kinds of relations are written together as

$$[a(p,s), a^\dagger(p',s')]_\mp = a(p,s)\, a^\dagger(p',s') \mp a^\dagger(p',s')\, a(p,s) = \delta_{ss'}\, \delta^{(3)}(\boldsymbol{p}-\boldsymbol{p}'). \tag{1.7}$$

A bracket $[A, B]$ with no signed subscript is interpreted as a commutator.

Equations (1.1 & 1.4) give

$$U(\Lambda)a^\dagger(p,\sigma)|0\rangle = \sqrt{\frac{(\Lambda p)^0}{p^0}} \sum_{s'} D^{(j)}_{s'\sigma}(W(\Lambda,p))\, a^\dagger(\Lambda p, s')|0\rangle. \tag{1.8}$$

And (1.3) gives

$$U(\Lambda)a^\dagger(p,\sigma)U^{-1}(\Lambda)|0\rangle = \sqrt{\frac{(\Lambda p)^0}{p^0}} \sum_{s'} D^{(j)}_{s'\sigma}(W(\Lambda,p))\, a^\dagger(\Lambda p, s')|0\rangle. \tag{1.9}$$

SW in chapter 4 concludes that

$$U(\Lambda)a^\dagger(p,\sigma)U^{-1}(\Lambda) = \sqrt{\frac{(\Lambda p)^0}{p^0}} \sum_{s'} D^{(j)}_{s'\sigma}(W(\Lambda,p))\, a^\dagger(\Lambda p, s'). \tag{1.10}$$

If $U(\Lambda, b)$ follows Λ by a translation by b, then

$$\begin{aligned} U(\Lambda,b)a^\dagger(p,\sigma)U^{-1}(\Lambda,b) &= e^{-i(\Lambda p)\cdot a}\sqrt{\frac{(\Lambda p)^0}{p^0}} \sum_{s'} D^{(j)}_{s'\sigma}(W(\Lambda,p))\, a^\dagger(\Lambda p, s') \\ &= e^{-i(\Lambda p)\cdot a}\sqrt{\frac{(\Lambda p)^0}{p^0}} \sum_{s'} D^{\dagger(j)}_{s'\sigma}(W^{-1}(\Lambda,p))\, a^\dagger(\Lambda p, s') \\ &= e^{-i(\Lambda p)\cdot a}\sqrt{\frac{(\Lambda p)^0}{p^0}} \sum_{s'} D^{*(j)}_{\sigma s'}(W^{-1}(\Lambda,p))\, a^\dagger(\Lambda p, s') \end{aligned} \tag{1.11}$$

The adjoint of this equation is

$$\begin{aligned} U(\Lambda,b)a(p,\sigma)U^{-1}(\Lambda,b) &= e^{i(\Lambda p)\cdot a}\sqrt{\frac{(\Lambda p)^0}{p^0}}\sum_{s'} D^{*(j)}_{s'\sigma}(W(\Lambda,p))\,a(\Lambda p,s') \\ &= e^{i(\Lambda p)\cdot a}\sqrt{\frac{(\Lambda p)^0}{p^0}}\sum_{s'} D^{\dagger(j)}_{\sigma s'}(W(\Lambda,p))\,a(\Lambda p,s') \\ &= e^{i(\Lambda p)\cdot a}\sqrt{\frac{(\Lambda p)^0}{p^0}}\sum_{s'} D^{(j)}_{\sigma s'}(W^{-1}(\Lambda,p))\,a(\Lambda p,s'). \end{aligned} \tag{1.12}$$

These equations (1.11 & 1.12) are (5.1.11 & 5.1.12) of SW.

1.3 How fields transform

The "positive frequency" part of a field is a linear combination of annihilation operators

$$\psi^+_\ell(x) = \sum_\sigma \int d^3p\, u_\ell(x;p,\sigma)\,a(p,\sigma). \tag{1.13}$$

The "negative frequency" part of a field is a linear combination of creation operators of the antiparticles

$$\psi^-_\ell(x) = \sum_\sigma \int d^3p\, v_\ell(x;p,\sigma)\,b^\dagger(p,\sigma). \tag{1.14}$$

To have the fields (1.13 & 1.14) transform properly under Poincaré transformations

$$\begin{aligned} U(\Lambda,a)\psi^+_\ell(x)U^{-1}(\Lambda,a) &= \sum_{\bar\ell} D_{\ell\bar\ell}(\Lambda^{-1})\psi^+_{\bar\ell}(\Lambda x+a) \\ &= \sum_{\bar\ell} D_{\ell\bar\ell}(\Lambda^{-1})\sum_\sigma\int d^3p\, u_{\bar\ell}(\Lambda x+a;p,\sigma)\,a(p,\sigma) \\ U(\Lambda,a)\psi^-_\ell(x)U^{-1}(\Lambda,a) &= \sum_{\bar\ell} D_{\ell\bar\ell}(\Lambda^{-1})\psi^-_{\bar\ell}(\Lambda x+a) \\ &= \sum_{\bar\ell} D_{\ell\bar\ell}(\Lambda^{-1})\sum_\sigma\int d^3p\, v_{\bar\ell}(\Lambda x+a;p,\sigma)\,b^\dagger(p,\sigma) \end{aligned} \tag{1.15}$$

the spinors $u_\ell(x;p,\sigma)$ and $v_\ell(x;p,\sigma)$ must obey certain rules which we'll now determine.

First (1.12 & 1.13) give

$$\begin{aligned}
U(\Lambda,a)\psi_\ell^+(x)U^{-1}(\Lambda,a) &= U(\Lambda,a)\sum_\sigma\int d^3p\, u_\ell(x;p,\sigma)\, a(p,\sigma)U^{-1}(\Lambda,a)\\
&= \sum_\sigma\int d^3p\, u_\ell(x;p,\sigma)\, U(\Lambda,a)a(p,\sigma)U^{-1}(\Lambda,a)\\
&= \sum_\sigma\int d^3p\, u_\ell(x;p,\sigma)\, e^{i(\Lambda p)\cdot a}\sqrt{\frac{(\Lambda p)^0}{p^0}}\sum_{s'} D^{(j)}_{\sigma s'}(W^{-1}(\Lambda,p))\, a(\Lambda p,s').
\end{aligned} \tag{1.16}$$

Now we use the identity

$$\frac{d^3p}{p^0} = \frac{d^3(\Lambda p)}{(\Lambda p)^0} \tag{1.17}$$

to turn (1.16) into

$$\begin{aligned}
U(\Lambda,a)\psi_\ell^+(x)U^{-1}(\Lambda,a) = \sum_\sigma\int d^3(\Lambda p)\, u_\ell(x;p,\sigma)\, e^{i(\Lambda p)\cdot a}\\
\times\sqrt{\frac{p^0}{(\Lambda p)^0}}\sum_{s'} D^{(j)}_{\sigma s'}(W^{-1}(\Lambda,p))\, a(\Lambda p,s').
\end{aligned} \tag{1.18}$$

Similarly (1.11, 1.14, & 1.17) give

$$\begin{aligned}
U(\Lambda,a)\psi_\ell^-(x)U^{-1}(\Lambda,a) &= U(\Lambda,a)\sum_\sigma\int d^3p\, v_\ell(x;p,\sigma)\, b^\dagger(p,\sigma)U^{-1}(\Lambda,a)\\
&= \sum_\sigma\int d^3p\, v_\ell(x;p,\sigma)\, U(\Lambda,a)b^\dagger(p,\sigma)U^{-1}(\Lambda,a) \qquad (1.19)\\
&= \sum_\sigma\int d^3p\, v_\ell(x;p,\sigma)\, e^{-i(\Lambda p)\cdot a}\sqrt{\frac{(\Lambda p)^0}{p^0}}\sum_{s'} D^{*(j)}_{\sigma s'}(W^{-1}(\Lambda,p))\, b^\dagger(\Lambda p,s')\\
&= \sum_\sigma\int d^3(\Lambda p)\, v_\ell(x;p,\sigma)\, e^{-i(\Lambda p)\cdot a}\sqrt{\frac{p^0}{(\Lambda p)^0}}\sum_{s'} D^{*(j)}_{\sigma s'}(W^{-1}(\Lambda,p))\, b^\dagger(\Lambda p,s').
\end{aligned}$$

So to get the fields to transform as in (1.15), equations (1.18 & 1.19) say

that we need

$$
\begin{aligned}
\sum_{\bar\ell} D_{\ell\bar\ell}(\Lambda^{-1})\psi^+_{\bar\ell}(\Lambda x+a) &= \sum_{\bar\ell} D_{\ell\bar\ell}(\Lambda^{-1}) \sum_\sigma \int d^3p\, u_{\bar\ell}(\Lambda x+a;p,\sigma)\, a(p,\sigma) \\
&= \sum_{\bar\ell} D_{\ell\bar\ell}(\Lambda^{-1}) \sum_\sigma \int d^3(\Lambda p)\, u_{\bar\ell}(\Lambda x+a;\Lambda p,\sigma)\, a(\Lambda p,\sigma) \\
&= \sum_\sigma \int d^3(\Lambda p)\, u_\ell(x;p,\sigma)\, e^{i(\Lambda p)\cdot a} \qquad (1.20) \\
&\quad \times \sqrt{\frac{p^0}{(\Lambda p)^0}} \sum_{s'} D^{(j)}_{\sigma s'}(W^{-1}(\Lambda,p))\, a(\Lambda p,s') \\
&= \sum_{s'} \int d^3(\Lambda p)\, u_\ell(x;p,s')\, e^{i(\Lambda p)\cdot a} \\
&\quad \times \sqrt{\frac{p^0}{(\Lambda p)^0}} \sum_\sigma D^{(j)}_{s'\sigma}(W^{-1}(\Lambda,p))\, a(\Lambda p,\sigma)
\end{aligned}
$$

and

$$
\begin{aligned}
\sum_{\bar\ell} D_{\ell\bar\ell}(\Lambda^{-1})\psi^-_{\bar\ell}(\Lambda x+a) &= \sum_{\bar\ell} D_{\ell\bar\ell}(\Lambda^{-1}) \sum_\sigma \int d^3p\, v_{\bar\ell}(\Lambda x+a;p,\sigma)\, b^\dagger(p,\sigma) \\
&= \sum_{\bar\ell} D_{\ell\bar\ell}(\Lambda^{-1}) \sum_\sigma \int d^3(\Lambda p)\, v_{\bar\ell}(\Lambda x+a;\Lambda p,\sigma)\, b^\dagger(\Lambda p,\sigma) \\
&= \sum_\sigma \int d^3(\Lambda p)\, v_\ell(x;p,\sigma)\, e^{-i(\Lambda p)\cdot a} \qquad (1.21) \\
&\quad \times \sqrt{\frac{p^0}{(\Lambda p)^0}} \sum_{s'} D^{*(j)}_{\sigma s'}(W^{-1}(\Lambda,p))\, b^\dagger(\Lambda p,s') \\
&= \sum_{s'} \int d^3(\Lambda p)\, v_\ell(x;p,s')\, e^{-i(\Lambda p)\cdot a} \\
&\quad \times \sqrt{\frac{p^0}{(\Lambda p)^0}} \sum_\sigma D^{*(j)}_{s'\sigma}(W^{-1}(\Lambda,p))\, b^\dagger(\Lambda p,\sigma).
\end{aligned}
$$

Equating coefficients of the red annihilation and blue creation operators, we find that the fields will transform properly if the spinors u and v satisfy the

rules

$$\sum_{\bar{\ell}} D_{\ell\bar{\ell}}(\Lambda^{-1})\, u_{\bar{\ell}}(\Lambda x + a; \Lambda p, \sigma) = \sqrt{\frac{p^0}{(\Lambda p)^0}} \sum_{s'} D^{(j)}_{s'\sigma}(W^{-1}(\Lambda, p)) u_\ell(x; p, s')\, e^{i(\Lambda p)\cdot a} \tag{1.22}$$

$$\sum_{\bar{\ell}} D_{\ell\bar{\ell}}(\Lambda^{-1}) v_{\bar{\ell}}(\Lambda x + a; \Lambda p, \sigma) = \sqrt{\frac{p^0}{(\Lambda p)^0}} \sum_{s'} D^{*(j)}_{s'\sigma}(W^{-1}(\Lambda, p)) v_\ell(x; p, s') e^{-i(\Lambda p)\cdot a} \tag{1.23}$$

which differ from SW's by an interchange of the subscripts σ, s' on the rotation matrices $D^{(j)}$. (I think SW has a typo there.) If we multiply both sides of these equations (1.22 & 1.23) by the two kinds of D matrices, then we get first

$$\begin{aligned}\sum_{\bar{\ell},\ell} D_{\ell'\ell}(\Lambda) D_{\ell\bar{\ell}}(\Lambda^{-1})\, u_{\bar{\ell}}(\Lambda x + a; \Lambda p, \sigma) &= u_{\ell'}(\Lambda x + a; \Lambda p, \sigma) \\ &= \sqrt{\frac{p^0}{(\Lambda p)^0}} \sum_{s',\ell} D^{(j)}_{s'\sigma}(W^{-1}(\Lambda, p)) D_{\ell'\ell}(\Lambda) u_\ell(x; p, s')\, e^{i(\Lambda p)\cdot a} \end{aligned} \tag{1.24}$$

$$\begin{aligned}\sum_{\bar{\ell},\ell} D_{\ell'\ell}(\Lambda) D_{\ell\bar{\ell}}(\Lambda^{-1}) v_{\bar{\ell}}(\Lambda x + a; \Lambda p, \sigma) &= v_{\ell'}(\Lambda x + a; \Lambda p, \sigma) \\ &= \sqrt{\frac{p^0}{(\Lambda p)^0}} \sum_{s',\ell} D^{*(j)}_{s'\sigma}(W^{-1}(\Lambda, p)) D_{\ell'\ell}(\Lambda) v_\ell(x; p, s') e^{-i(\Lambda p)\cdot a} \end{aligned} \tag{1.25}$$

and then with $W \equiv W(\Lambda, p)$

$$\sum_\sigma D^{(j)}_{\sigma\bar{s}}(W) u_{\ell'}(\Lambda x + a; \Lambda p, \sigma)$$

$$= \sqrt{\frac{p^0}{(\Lambda p)^0}} \sum_{s',\sigma,\ell} D^{(j)}_{s'\sigma}(W^{-1}) D^{(j)}_{\sigma\bar{s}}(W) D_{\ell'\ell}(\Lambda) u_\ell(x; p, s')\, e^{i(\Lambda p)\cdot a}$$

$$= \sqrt{\frac{p^0}{(\Lambda p)^0}} \sum_\ell D_{\ell'\ell}(\Lambda) u_\ell(x; p, \bar{s})\, e^{i(\Lambda p)\cdot a} \tag{1.26}$$

$$\sum_\sigma D^{*(j)}_{\sigma\bar{s}}(W) v_{\ell'}(\Lambda x + a; \Lambda p, \sigma)$$

$$= \sqrt{\frac{p^0}{(\Lambda p)^0}} \sum_{\sigma,s',\ell} D^{*(j)}_{s'\sigma}(W^{-1}) D^{*(j)}_{\sigma\bar{s}}(W) D_{\ell'\ell}(\Lambda) v_\ell(x; p, s') e^{-i(\Lambda p)\cdot a}$$

$$= \sqrt{\frac{p^0}{(\Lambda p)^0}} \sum_\ell D_{\ell'\ell}(\Lambda) v_\ell(x; p, \bar{s}) e^{-i(\Lambda p)\cdot a} \tag{1.27}$$

which are equations (5.1.13 & 5.1.14) of SW:

$$\sum_{\bar{s}} u_{\bar{\ell}}(\Lambda x + a; \Lambda p, \bar{s}) D^{(j)}_{\bar{s}\sigma}(W(\Lambda, p)) = \sqrt{\frac{p^0}{(\Lambda p)^0}} \sum_\ell D_{\bar{\ell}\ell}(\Lambda) u_\ell(x; p, \sigma)\, e^{i(\Lambda p)\cdot a}$$

$$\sum_{\bar{s}} v_{\bar{\ell}}(\Lambda x + a; \Lambda p, \bar{s}) D^{*(j)}_{\bar{s}\sigma}(W(\Lambda, p)) = \sqrt{\frac{p^0}{(\Lambda p)^0}} \sum_\ell D_{\bar{\ell}\ell}(\Lambda) v_\ell(x; p, \sigma) e^{-i(\Lambda p)\cdot a}. \tag{1.28}$$

These are the equations that determine the spinors u and v up to a few arbitrary phases.

1.4 Translations

When $\Lambda = I$, the D matrices are equal to unity, and these last equations (1.28) say that for $x = 0$

$$\begin{aligned} u_\ell(a; p, \sigma) &= u_\ell(0; p, \sigma)\, e^{ip\cdot a} \\ v_\ell(a; p, \sigma) &= v_\ell(0; p, \sigma) e^{-ip\cdot a}. \end{aligned} \tag{1.29}$$

Thus the spinors u and v depend upon spacetime by the usual phase $e^{\pm ip\cdot x}$

$$\begin{aligned} u_\ell(x; p, \sigma) &= (2\pi)^{-3/2} u_\ell(p, \sigma)\, e^{ip\cdot x} \\ v_\ell(x; p, \sigma) &= (2\pi)^{-3/2} v_\ell(p, \sigma) e^{-ip\cdot x} \end{aligned} \tag{1.30}$$

in which the 2π's are conventional. The fields therefore are Fourier transforms:

$$\begin{aligned}\psi_\ell^+(x) &= (2\pi)^{-3/2}\sum_\sigma \int d^3p\, e^{ip\cdot x} u_\ell(p,\sigma)\, a(p,\sigma)\\ \psi_\ell^-(x) &= (2\pi)^{-3/2}\sum_\sigma \int d^3p\, e^{-ip\cdot x} v_\ell(p,\sigma)\, b^\dagger(p,\sigma)\end{aligned} \tag{1.31}$$

and every field of mass m obeys the Klein-Gordon equation

$$(\nabla^2 - \partial_0^2 - m^2)\,\psi_\ell(x) = (\Box - m^2)\,\psi_\ell(x) = 0. \tag{1.32}$$

Since $\exp[i(\Lambda p\cdot(\Lambda x + a))] = \exp(ip\cdot x + i\Lambda p\cdot a)$, the conditions (1.28) simplify to

$$\begin{aligned}\sum_{\bar s} u_{\bar\ell}(\Lambda p,\bar s) D^{(j)}_{\bar s\sigma}(W(\Lambda,p)) &= \sqrt{\frac{p^0}{(\Lambda p)^0}}\sum_\ell D_{\bar\ell\ell}(\Lambda) u_\ell(p,\sigma)\\ \sum_{\bar s} v_{\bar\ell}(\Lambda p,\bar s) D^{*(j)}_{\bar s\sigma}(W(\Lambda,p)) &= \sqrt{\frac{p^0}{(\Lambda p)^0}}\sum_\ell D_{\bar\ell\ell}(\Lambda) v_\ell(p,\sigma)\end{aligned} \tag{1.33}$$

for all Lorentz transformations Λ.

1.5 Boosts

Set $p = k = (m,\vec 0)$ and $\Lambda = L(q)$ where $L(q)k = q$. So $L(p) = 1$ and

$$W(\Lambda,p) \equiv L^{-1}(\Lambda p)\Lambda L(p) = L^{-1}(q)L(q) = 1. \tag{1.34}$$

Then the equations (1.33) are

$$\begin{aligned}u_{\bar\ell}(q,\sigma) &= \sqrt{\frac{m}{q^0}}\sum_\ell D_{\bar\ell\ell}(L(q)) u_\ell(\vec 0,\sigma)\\ v_{\bar\ell}(q,\sigma) &= \sqrt{\frac{m}{q^0}}\sum_\ell D_{\bar\ell\ell}(L(q)) v_\ell(\vec 0,\sigma).\end{aligned} \tag{1.35}$$

Thus a spinor at finite momentum is given by a representation $D(\Lambda)$ of the Lorentz group (see the online notes of chapter 10 of my book for its finite-dimensional nonunitary representations) acting on the spinor at zero 3-momentum $p = k = (m,\vec 0)$. We need to find what these spinors are.

1.6 Rotations

Now set $p = k = (m, \vec{0})$ and $\Lambda = R$ a rotation so that $W = R$. For rotations, the spinor conditions (1.33) are

$$\begin{aligned} \sum_{\bar{s}} u_{\bar{\ell}}(\vec{0}, \bar{s}) D^{(j)}_{\bar{s}\sigma}(R) &= \sum_{\ell} D_{\bar{\ell}\ell}(R) u_{\ell}(\vec{0}, \sigma) \\ \sum_{\bar{s}} v_{\bar{\ell}}(\vec{0}, \bar{s}) D^{*(j)}_{\bar{s}\sigma}(R) &= \sum_{\ell} D_{\bar{\ell}\ell}(R) v_{\ell}(\vec{0}, \sigma). \end{aligned} \tag{1.36}$$

The representations $D^{(j)}_{\bar{s}\sigma}(R)$ of the rotation group are $(2j+1) \times (2j+1)$-dimensional unitary matrices. For a rotation of angle θ about the $\vec{\theta} = \boldsymbol{\theta}$ axis, they are the ones taught in courses on quantum mechanics (and discussed in the notes of chapter 10)

$$D^{(j)}_{\bar{s}\sigma}(\boldsymbol{\theta}) = \left[e^{-i\boldsymbol{\theta}\cdot\boldsymbol{J}^{(j)}} \right]_{\bar{s}\sigma} \tag{1.37}$$

where $[J_a, J_b] = i\epsilon_{abc} J_c$. The representations $D_{\bar{\ell}\ell}(R)$ of the rotation group are finite-dimensional unitary matrices. For a rotation of angle θ about the $\vec{\theta} = \boldsymbol{\theta}$ axis, they are

$$D_{\bar{\ell}\ell}(\boldsymbol{\theta}) = \left[e^{-i\boldsymbol{\theta}\cdot\boldsymbol{\mathcal{J}}} \right]_{\bar{\ell}\ell} \tag{1.38}$$

in which $[\mathcal{J}_a, \mathcal{J}_b] = i\epsilon_{abc}\mathcal{J}$. For tiny rotations, the conditions (1.36) require (because of the complex conjugation of the antiparticle condition) that the spinors obey the rules

$$\begin{aligned} \sum_{\bar{s}} u_{\bar{\ell}}(\vec{0}, \bar{s}) (J^{(j)}_a)_{\bar{s}\sigma} &= \sum_{\ell} (\mathcal{J}_a)_{\bar{\ell}\ell} u_{\ell}(\vec{0}, \sigma) \\ \sum_{\bar{s}} v_{\bar{\ell}}(\vec{0}, \bar{s}) (-J_a)^{*(j)}_{\bar{s}\sigma}) &= \sum_{\ell} (\mathcal{J}_a)_{\bar{\ell}\ell} v_{\ell}(\vec{0}, \sigma) \end{aligned} \tag{1.39}$$

for $a = 1, 2, 3$.

1.7 Spin-zero fields

Spin-zero fields have no spin or Lorentz indexes. So the boost conditions (1.210) merely require that $u(q) = \sqrt{m/q^0}\, u(0)$ and $v(q) = \sqrt{m/q^0}\, v(0)$. The conventional normalization is $u(0) = 1/\sqrt{2m}$ and $v(0) = 1/\sqrt{2m}$. The spin-zero spinors then are

$$u(p) = (2p^0)^{-1/2} \quad \text{and} \quad v(p) = (2p^0)^{-1/2}. \tag{1.40}$$

For simpicity, let's first consider a neutral scalar field so that $b(p, s) =$

$a(p,s)$. The definitions (1.13) and (1.14) of the positive-frequency and negative-frequency fields and their behavior (1.30) under translations then give us

$$\begin{aligned}\phi^+(x) &= \int \frac{d^3p}{\sqrt{(2\pi)^3 2p^0}}\, a(p)\, e^{ip\cdot x}\\ \phi^-(x) &= \int \frac{d^3p}{\sqrt{(2\pi)^3 2p^0}}\, a^\dagger(p)\, e^{-ip\cdot x}.\end{aligned} \tag{1.41}$$

Note that

$$\left[\phi^\pm(x)\right]^\dagger = \phi^\mp(x). \tag{1.42}$$

Since $[a(p), a(p')]_\pm = 0$, it follows that

$$[\phi^+(x), \phi^+(y)]_\mp = 0 \quad \text{and} \quad [\phi^-(x), \phi^-(y)]_\mp = 0 \tag{1.43}$$

whatever the values of x and y as long as we use commutators for bosons and anticommutators for fermions.

But the commutation relation

$$[a(p,s), a^\dagger(q,t)]_\mp = \delta_{st}\, \delta^{(3)}(\boldsymbol{p}-\boldsymbol{q}) \tag{1.44}$$

makes the commutator

$$\begin{aligned}[\phi^+(x), \phi^-(y)]_\mp &= \int \frac{d^3p d^3p'}{(2\pi)^3\sqrt{2p^0 2p'^0}}\, e^{ip\cdot x} e^{-ip'\cdot y}\, \delta^3(\boldsymbol{p}-\boldsymbol{p}')\\ &= \int \frac{d^3p}{(2\pi)^3 2p^0}\, e^{ip\cdot(x-y)} = \Delta_+(x-y)\end{aligned} \tag{1.45}$$

nonzero even for $(x-y)^2 > 0$ as we'll now verify.

For space-like x, the Lorentz-invariant function $\Delta_+(x)$ can only depend upon $x^2 > 0$ since the time x^0 and its sign are not Lorentz invariant. So we choose a Lorentz frame with $x^0 = 0$ and $|\boldsymbol{x}| = \sqrt{x^2}$. In this frame,

$$\begin{aligned}\Delta_+(x) &= \int \frac{d^3p}{(2\pi)^3 2\sqrt{\boldsymbol{p}^2+m^2}}\, e^{i\boldsymbol{p}\cdot\boldsymbol{x}}\\ &= \int \frac{p^2 dp\; d\cos\theta}{(2\pi)^2 2\sqrt{\boldsymbol{p}^2+m^2}}\, e^{ipx\cos\theta}\end{aligned} \tag{1.46}$$

where $p = |\boldsymbol{p}|$ and $x = |\boldsymbol{x}|$. Now

$$\int d\cos\theta\, e^{ipx\cos\theta} = \left(e^{ipx} - e^{-ipx}\right)/(ipx) = 2\sin(px)/(px), \tag{1.47}$$

so the integral (1.46) is

$$\Delta_+(x) = \frac{1}{4\pi^2 x}\int_0^\infty \frac{\sin(px)\, p dp}{\sqrt{p^2+m^2}} \tag{1.48}$$

with $u \equiv p/m$

$$\Delta_+(x) = \frac{m}{4\pi^2 x}\int_0^\infty \frac{\sin(mxu)\,udu}{\sqrt{\boldsymbol{u}^2+1}} = \frac{m}{4\pi^2 x}\,K_1(mx^2) \tag{1.49}$$

a Hankel function.

To get a Lorentz-invariant, causal theory, we use the arbitrary parameters κ and λ setting

$$\phi(x) = \kappa\phi^+(x) + \lambda\phi^-(x) \tag{1.50}$$

Now the adjoint rule (1.42) and the commutation relations (1.45 and 1.45) give

$$\begin{aligned}
[\phi(x),\phi^\dagger(y)]_\mp &= [\kappa\phi^+(x)+\lambda\phi^-(x),\kappa^*\phi^-(y)+\lambda\phi^+(y)]_\mp \\
&= |\kappa|^2[\phi^+(x),\phi^-(y)]_\mp + |\lambda|^2[\phi^-(x),\phi^+(y)]_\mp \\
&= |\kappa|^2\,\Delta_+(x-y) \mp |\lambda|^2\,\Delta_+(y-x) \\
[\phi(x),\phi(y)]_\mp &= [\kappa\phi^+(x)+\lambda\phi^-(x),\kappa\phi^+(y)+\lambda\phi^-(y)]_\mp \\
&= \kappa\lambda\left([\phi^+(x),\phi^-(y)]_\mp + [\phi^-(x),\phi^+(y)]_\mp\right) \\
&= \kappa\lambda\left(\Delta_+(x-y) \mp \Delta_+(y-x)\right).
\end{aligned} \tag{1.51}$$

But when $(x-y)^2 > 0$, $\Delta_+(x-y) = \Delta_+(y-x)$. Thus these conditions are

$$\begin{aligned}
[\phi(x),\phi^\dagger(y)]_\mp &= \left(|\kappa|^2 \mp |\lambda|^2\right)\,\Delta_+(x-y) \\
[\phi(x),\phi(y)]_\mp &= \kappa\lambda\Delta_+(x-y)(1\mp 1).
\end{aligned} \tag{1.52}$$

The first of these equations implies that we choose the minus sign and so that we use commutation relations and not anticommutation relations for spin-zero fields. This is the **spin-statistics theorem** for spin-zero fields. SW proves the theorem for arbitrary massive fields in section 5.7.

We also must set

$$|\kappa| = |\lambda|. \tag{1.53}$$

The second equation then is automatically satisfied. The common magnitude and the phases of κ and λ are arbitrary, so we choose $\kappa = \lambda = 1$. We then have

$$\phi(x) = \phi^+(x) + \phi^-(x) = \phi^+(x) + \phi^{+\dagger}(x) = \phi^\dagger(x). \tag{1.54}$$

Now the interaction density $\mathcal{H}(x)$ will commute with $\mathcal{H}(y)$ for $(x-y)^2 > 0$, and we have a chance of having a Lorentz-invariant, causal theory.

The field (1.54)

$$\phi(x) = \int \frac{d^3p}{\sqrt{(2\pi)^3 2p^0}}\left[a(p)\,e^{ip\cdot x} + a^\dagger(p)\,e^{-ip\cdot x}\right] \tag{1.55}$$

obeys the **Klein-Gordon equation**

$$(\nabla^2 - \partial_0^2 - m^2)\,\phi(x) \equiv (\Box - m^2)\,\phi(x) = 0. \tag{1.56}$$

1.8 Conserved charges

If the field ϕ adds and deletes charged particles, an interaction $\mathcal{H}(x)$ that is a polynomial in ϕ will not commute with the charge operator Q because ϕ^+ will lower the charge and ϕ^- will raise it. The standard way to solve this problem is to start with two hermitian fields ϕ_1 and ϕ_2 of the same mass. One defines a complex scalar field as a complex linear combination of the two fields

$$\begin{aligned}\phi(x) &= \frac{1}{\sqrt{2}}\,(\phi_1(x) + i\phi_2(x)) \\ &= \int \frac{d^3p}{\sqrt{(2\pi)^3 2p^0}} \left[\frac{1}{\sqrt{2}}\,(a_1(p) + ia_2(p))\; e^{ip\cdot x} + \frac{1}{\sqrt{2}}\left(a_1^\dagger(p) + ia_2^\dagger(p)\right)\; e^{-ip\cdot x}\right].\end{aligned} \tag{1.57}$$

Setting

$$a(p) = \frac{1}{\sqrt{2}}\,(a_1(p) + ia_2(p)) \quad \text{and} \quad b^\dagger(p) = \frac{1}{\sqrt{2}}\left(a_1^\dagger(p) + ia_2^\dagger(p)\right) \tag{1.58}$$

so that

$$b(p) = \frac{1}{\sqrt{2}}\,(a_1(p) - ia_2(p)) \quad \text{and} \quad a^\dagger(p) = \frac{1}{\sqrt{2}}\left(a_1^\dagger(p) - ia_2^\dagger(p)\right) \tag{1.59}$$

we have

$$\phi(x) = \int \frac{d^3p}{\sqrt{(2\pi)^3 2p^0}} \left[a(p)\, e^{ip\cdot x} + b^\dagger(p)\, e^{-ip\cdot x}\right] \tag{1.60}$$

and

$$\phi^\dagger(x) = \int \frac{d^3p}{\sqrt{(2\pi)^3 2p^0}} \left[b(p)\, e^{ip\cdot x} + a^\dagger(p)\, e^{-ip\cdot x}\right]. \tag{1.61}$$

Since the commutation relations of the real creation and annihilation operators are for $i, j = 1, 2$

$$[a_i(p), a_j^\dagger(p')] = \delta_{ij}\,\delta^3(\boldsymbol{p} - \boldsymbol{p}') \quad \text{and} \quad [a_i(p), a_j(p')] = 0 = [a_i^\dagger(p), a_j^\dagger(p')] \tag{1.62}$$

the commutation relations of the complex creation and annihilation operators are

$$[a(p), a^\dagger(p')] = \delta^3(\boldsymbol{p} - \boldsymbol{p}') \quad \text{and} \quad [b(p), b^\dagger(p')] = \delta^3(\boldsymbol{p} - \boldsymbol{p}') \tag{1.63}$$

with all other commutators vanishing.

Now $\phi(x)$ lowers the charge of a state by q if $a^\dagger$ adds a particle of charge q and if $b^\dagger$ adds a particle of charge $-q$. Similarly, $\phi^\dagger(x)$ raises the charge of a state by q

$$[Q, \phi(x)] = -q\phi(x) \quad \text{and} \quad [Q, \phi^\dagger(x)] = q\phi^\dagger(x). \tag{1.64}$$

So an interaction with as many $\phi(x)$'s as $\phi^\dagger(x)$'s conserves charge.

1.9 Parity, charge conjugation, and time reversal

If the unitary operator P represents parity on the creation operators

$$\mathsf{P}a_1^\dagger(\boldsymbol{p})\mathsf{P}^{-1} = \eta\, a_1^\dagger(-\boldsymbol{p}) \quad \text{and} \quad \mathsf{P}a_2^\dagger(\boldsymbol{p})\mathsf{P}^{-1} = \eta\, a_2^\dagger(-\boldsymbol{p}) \tag{1.65}$$

with the same phase η. Then

$$\mathsf{P}a_1(\boldsymbol{p})\mathsf{P}^{-1} = \eta^* a_1(-\boldsymbol{p}) \quad \text{and} \quad \mathsf{P}a_2(\boldsymbol{p})\mathsf{P}^{-1} = \eta^* a_2(-\boldsymbol{p}) \tag{1.66}$$

and so both

$$\mathsf{P}a^\dagger(\boldsymbol{p})\mathsf{P}^{-1} = \eta a^\dagger(-\boldsymbol{p}) \quad \text{and} \quad \mathsf{P}a(\boldsymbol{p})\mathsf{P}^{-1} = \eta^* a(-\boldsymbol{p}) \tag{1.67}$$

and

$$\mathsf{P}b^\dagger(\boldsymbol{p})\mathsf{P}^{-1} = \eta b^\dagger(-\boldsymbol{p}) \quad \text{and} \quad \mathsf{P}b(\boldsymbol{p})\mathsf{P}^{-1} = \eta^* b(-\boldsymbol{p}). \tag{1.68}$$

Thus if the field

$$\phi_1(x) = \int \frac{d^3p}{\sqrt{(2\pi)^3 2p^0}} \left[a_1(p)\, e^{ip\cdot x} + a_1^\dagger(p)\, e^{-ip\cdot x} \right] \tag{1.69}$$

or $\phi_2(x)$, or the complex field (1.60) is to go into a multiple of itself under parity, then we need $\eta = \eta^*$ so that η is real. Then the fields transform under parity as

$$\begin{aligned} \mathsf{P}\phi_1(x)\mathsf{P}^{-1} &= \eta^*\phi_1(x^0, -\boldsymbol{x}) = \eta\phi_1(x^0, -\boldsymbol{x}) \\ \mathsf{P}\phi_2(x)\mathsf{P}^{-1} &= \eta^*\phi(x^0, -\boldsymbol{x}) = \eta\phi_2(x^0, -\boldsymbol{x}) \\ \mathsf{P}\phi(x)\mathsf{P}^{-1} &= \eta^*\phi(x^0, -\boldsymbol{x}) = \eta\phi(x^0, -\boldsymbol{x}). \end{aligned} \tag{1.70}$$

Since $\mathsf{P}^2 = I$, we must have $\eta = \pm 1$. SW allows for a more general phase by having parity act with the same phase on a and $b^\dagger$. Both schemes imply that the parity of a hermitian field is ± 1 and that the state

$$|ab\rangle = \int d^3p\, f(\boldsymbol{p}^2)\, a^\dagger(\boldsymbol{p})\, b^\dagger(-\boldsymbol{p})\, |0\rangle \tag{1.71}$$

has even or positive parity, $\mathsf{P}|ab\rangle = |ab\rangle$.

Charge conjugation works similarly. If the unitary operator C represents charge conjugation on the creation operators

$$\mathsf{C}a_1^\dagger(\boldsymbol{p})\mathsf{C}^{-1} = \xi a_1^\dagger(\boldsymbol{p}) \quad \text{and} \quad \mathsf{C}a_2^\dagger(\boldsymbol{p})\mathsf{C}^{-1} = -\xi a_2^\dagger(\boldsymbol{p}) \tag{1.72}$$

with the same phase ξ. Then

$$\mathsf{C}a_1(\boldsymbol{p})\mathsf{C}^{-1} = \xi^* a_1(\boldsymbol{p}) \quad \text{and} \quad \mathsf{C}a_2(\boldsymbol{p})\mathsf{C}^{-1} = -\xi^* a_2(\boldsymbol{p}) \tag{1.73}$$

and so since $a = (a_1 + ia_2)/\sqrt{2}$ and $b = (a_1 - ia_2)/\sqrt{2}$

$$\mathsf{C}a(\boldsymbol{p})\mathsf{C}^{-1} = \xi^*\, b(\boldsymbol{p}) \quad \text{and} \quad \mathsf{C}b(\boldsymbol{p})\mathsf{C}^{-1} = \xi^* a(\boldsymbol{p}) \tag{1.74}$$

and since $a^\dagger = (a_1^\dagger - ia_2^\dagger)/\sqrt{2}$ and $b^\dagger = (a_1^\dagger + ia_2^\dagger)/\sqrt{2}$

$$\mathsf{C}a^\dagger(\boldsymbol{p})\mathsf{C}^{-1} = \xi\, b^\dagger(\boldsymbol{p}) \quad \text{and} \quad \mathsf{C}b^\dagger(\boldsymbol{p})\mathsf{C}^{-1} = \xi a^\dagger(\boldsymbol{p}). \tag{1.75}$$

Thus under charge conjugation, the field (1.60) becomes

$$\mathsf{C}\phi(x)\mathsf{C}^{-1} = \int \frac{d^3p}{\sqrt{(2\pi)^3 2p^0}} \Big[\, \xi^*\, b(p)\, e^{ip\cdot x} + \xi\, a^\dagger(p)\, e^{-ip\cdot x} \Big] \tag{1.76}$$

and so if it is to go into a multiple of itself or of its adjoint under charge conjugation then we need $\xi = \xi^*$ so that ξ is real. We then get

$$\mathsf{C}\phi(x)\mathsf{C}^{-1} = \xi^*\, \phi^\dagger(x) = \xi\, \phi^\dagger(x). \tag{1.77}$$

Since $\mathsf{C}^2 = I$, we must have $\xi = \pm 1$. SW allows for a more general phase by having charge conjugation act with the same phase on a and $b^\dagger$. Both schemes imply that the charge-conjugation parity of a hermitian field is ± 1 and that the state

$$|ab\rangle = \int d^3p\, f(\boldsymbol{p}^2)\, a^\dagger(\boldsymbol{p})\, b^\dagger(\boldsymbol{p})\, |0\rangle \tag{1.78}$$

has even or positive charge-conjugation parity, $\mathsf{C}|ab\rangle = |ab\rangle$.

The time-reversal operator T is antilinear and antiunitary. So if

$$\begin{aligned} \mathsf{T}a_1(\boldsymbol{p})\mathsf{T}^{-1} &= \zeta^* a_1(-\boldsymbol{p}) \quad \text{and} \quad \mathsf{T}a_2(\boldsymbol{p})\mathsf{T}^{-1} = -\zeta^* a_2(-\boldsymbol{p}) \\ \mathsf{T}a_1^\dagger(\boldsymbol{p})\mathsf{T}^{-1} &= \zeta a_1^\dagger(-\boldsymbol{p}) \quad \text{and} \quad \mathsf{T}a_2^\dagger(\boldsymbol{p})\mathsf{T}^{-1} = -\zeta a_2^\dagger(-\boldsymbol{p}) \end{aligned} \tag{1.79}$$

then

$$\begin{aligned} \mathsf{T}a(\boldsymbol{p})\mathsf{T}^{-1} &= \mathsf{T}\frac{1}{\sqrt{2}}\big(a_1(\boldsymbol{p}) + ia_2(\boldsymbol{p})\big)\mathsf{T}^{-1} = \frac{1}{\sqrt{2}}\big(\mathsf{T}a_1(\boldsymbol{p})\mathsf{T}^{-1} - i\mathsf{T}a_2(\boldsymbol{p})\mathsf{T}^{-1}\big) \\ &= \zeta^* \frac{1}{\sqrt{2}}\big(a_1(-\boldsymbol{p}) + ia_2(-\boldsymbol{p})\big) = \zeta^*\, a(-\boldsymbol{p}) \end{aligned} \tag{1.80}$$

and

$$\begin{aligned}\mathsf{T}b^\dagger(\boldsymbol{p})\mathsf{T}^{-1} &= \mathsf{T}\frac{1}{\sqrt{2}}\big(a_1^\dagger(\boldsymbol{p}) + ia_2^\dagger(\boldsymbol{p})\big)\mathsf{T}^{-1} = \frac{1}{\sqrt{2}}\big(\mathsf{T}a_1^\dagger(\boldsymbol{p})\mathsf{T}^{-1} - i\mathsf{T}a_2^\dagger(\boldsymbol{p})\mathsf{T}^{-1}\big)\\ &= \zeta\frac{1}{\sqrt{2}}\big(a_1^\dagger(-\boldsymbol{p}) + ia_2^\dagger(-\boldsymbol{p})\big) = \zeta\, b^\dagger(-\boldsymbol{p})\end{aligned} \tag{1.81}$$

then one has

$$\begin{aligned}\mathsf{T}\phi(x)\mathsf{T}^{-1} &= \mathsf{T}\int\frac{d^3p}{\sqrt{(2\pi)^3 2p^0}}\Big[a(p)\,e^{ip\cdot x} + b^\dagger(p)\,e^{-ip\cdot x}\Big]\,\mathsf{T}^{-1}\\ &= \int\frac{d^3p}{\sqrt{(2\pi)^3 2p^0}}\Big[\mathsf{T}a(p)\mathsf{T}^{-1}\,e^{-ip\cdot x} + \mathsf{T}b^\dagger(p)\mathsf{T}^{-1}\,e^{ip\cdot x}\Big]\\ &= \int\frac{d^3p}{\sqrt{(2\pi)^3 2p^0}}\Big[\zeta^*\,a(-\boldsymbol{p})\,e^{-ip\cdot x} + \zeta\,b^\dagger(-\boldsymbol{p})\,e^{ip\cdot x}\Big].\end{aligned} \tag{1.82}$$

So if ζ is real, then after replacing $-\boldsymbol{p}$ by $\boldsymbol{p}$, we get

$$\begin{aligned}\mathsf{T}\phi(x)\mathsf{T}^{-1} &= \zeta^*\int\frac{d^3p}{\sqrt{(2\pi)^3 2p^0}}\Big[a(\boldsymbol{p})\,e^{ip^0x^0+i\boldsymbol{p}\cdot\boldsymbol{x}} + b^\dagger(\boldsymbol{p})\,e^{-ip^0x^0+i\boldsymbol{p}\cdot\boldsymbol{x}}\Big]\\ &= \zeta^*\phi(-x^0,\boldsymbol{x}) = \zeta\phi(-x^0,\boldsymbol{x}).\end{aligned} \tag{1.83}$$

Since $\mathsf{T}^2 = I$, the phase $\zeta = \pm 1$. SW lets ζ be complex but defined only for complex scalar fields and not for their real and imaginary parts.

1.10 Vector fields

Vector fields transform like the 4-vector x^i of spacetime. So

$$D_{\bar{\ell}\ell}(\Lambda) = \Lambda^{\bar{\ell}}{}_{\ell} \tag{1.84}$$

for $\bar{\ell}, \ell = 0, 1, 2, 3$. Again we start with a hermitian field labelled by $i = 0, 1, 2, 3$

$$\begin{aligned}\phi^{+i}(x) &= (2\pi)^{-3/2}\sum_s\int d^3p\,e^{ip\cdot x}u^i(p,s)\,a(p,s)\\ \phi^{-i}(x) &= (2\pi)^{-3/2}\sum_s\int d^3p\,e^{-ip\cdot x}v^i(p,s)\,a^\dagger(p,s).\end{aligned} \tag{1.85}$$

The boost conditions (1.210) say that

$$\begin{aligned} u^i(p,s) &= \sqrt{\frac{m}{p^0}} \sum_k L(p)^i{}_k u^k(\vec{0},s) \\ v^i(p,s) &= \sqrt{\frac{m}{p^0}} \sum_k L(p)^i{}_k v^k(\vec{0},s). \end{aligned} \tag{1.86}$$

The rotation conditions (1.39) give

$$\begin{aligned} \sum_{\bar{s}} u^i(\vec{0},\bar{s})(J_a^{(j)})_{\bar{s}s} &= \sum_k (\mathcal{J}_a)^i{}_k u^k(\vec{0},s) \\ -\sum_{\bar{s}} v^i(\vec{0},\bar{s})(J_a^{*(j)})_{\bar{s}s} &= \sum_k (\mathcal{J}_a)^i{}_k v^k(\vec{0},s). \end{aligned} \tag{1.87}$$

The $(2j+1)\times(2j+1)$ matrices $(J_a^{(j)})_{\bar{s}s}$ are the generators of the $(2j+1)\times(2j+1)$ representation of the rotation group. (See my online notes on group theory.) You learned that

$$\sum_{a=1}^{3} \left[(J_a^{(j)})^2\right]_{\bar{s}s'} = \sum_{a=1}^{3}\sum_{s=-j}^{j} (J_a^{(j)})_{\bar{s}s}\,(J_a^{(j)})_{ss'} = j(j+1)\delta_{\bar{s}s'} \tag{1.88}$$

and that

$$J_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad J_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{1.89}$$

in courses on quantum mechanics.

For $k = 1,2,3$, the three 4×4 matrices $(\mathcal{J}_k)^i{}_j$ are the generators of rotations in the vector representation of the Lorentz group. Their nonzero components are

$$(\mathcal{J}_k)^i{}_j = -i\epsilon_{ijk} \tag{1.90}$$

for $i,j,k = 1,2,3$, while $(\mathcal{J}_k)^0{}_0 = 0$, $(\mathcal{J}_k)^0{}_j = 0$, and $(\mathcal{J}_k)^i{}_0 = 0$ for $i,j,k = 1,2,3$. So

$$(\mathcal{J}^2)^i{}_j = 2\delta^i{}_j \tag{1.91}$$

with $(\mathcal{J}^2)^0{}_0 = 0$, $(\mathcal{J}^2)^0{}_j = 0$, and $(\mathcal{J}^2)^i{}_0 = 0$ for $i,j = 1,2,3$. Apart from a factor of i, the $\mathcal{J}_k$'s are the 4×4 matrices $J_a = iR_a$ of my online notes on

the Lorentz group

$$\mathcal{J}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} \quad \mathcal{J}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix} \quad \mathcal{J}_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{1.92}$$

Since $(\mathcal{J}_a)^0{}_k = 0$ for $a, k = 1, 2, 3$, the spin conditions (1.87) give for $i = 0$

$$\sum_{\bar{s}} u^0(\vec{0}, \bar{s})(J_a^{(j)})_{\bar{s}s} = 0 \quad \text{and} \quad -\sum_{\bar{s}} v^0(\vec{0}, \bar{s})(J_a^{*(j)})_{\bar{s}s} = 0. \tag{1.93}$$

Multiplying these equations from the right by $(J_a^{(j)})_{ss'}$ while summing over $a = 1, 2, 3$ and using the formula (1.88) $[(\boldsymbol{J}^{(j)})^2]_{ss'} = j(j+1)\,\delta_{ss'}$, we find

$$j(j+1)\,u^0(\vec{0}, s) = 0 \quad \text{and} \quad j(j+1)\,v^0(\vec{0}, s) = 0. \tag{1.94}$$

Thus $u^0(\vec{0}, \sigma)$ and $v^0(\vec{0}, \sigma)$ can be anything if the field represents particles of spin $j = 0$, but $u^0(\vec{0}, \sigma)$ and $v^0(\vec{0}, \sigma)$ must both vanish if the field represents particles of spin $j > 0$.

Now we set $i = 1, 2, 3$ in the spin conditions (1.87) and again multiply from the right by $(J_a^{(j)})_{ss'}$ while summing over $a = 1, 2, 3$ and using the formula (1.88) $(\boldsymbol{J}^{(j)})^2 = j(j+1)$. The Lorentz rotation matrices generate a $j = 1$ representation of the group of rotations.

$$\sum_{ka=1}^{3} (\mathcal{J}_a)^i{}_k\,(\mathcal{J}_a)^k{}_\ell = j(j+1)\,\delta^i_\ell = 2\delta^i_\ell. \tag{1.95}$$

So the remaining conditions on the fields are

$$\begin{aligned} j(j+1)\,u^i(\vec{0}, s') &= \sum_{\bar{s}sa} u^i(\vec{0}, \bar{s})(J_a^{(j)})_{\bar{s}s}\,(J_a^{(j)})_{ss'} = \sum_{ksa} (\mathcal{J}_a)^i{}_k u^k(\vec{0}, s)\,(J_a^{(j)})_{ss'} \\ &= \sum_{k\ell a} (\mathcal{J}_a)^i{}_k\,(\mathcal{J}_a)^k{}_\ell\,u^\ell(\vec{0}, s') = \sum_k 2\,\delta^i_\ell\,u^\ell(\vec{0}, s') = 2\,u^i(\vec{0}, s') \\ j(j+1)\,v^i(\vec{0}, s') &= \sum_{\bar{s}sa} v^i(\vec{0}, \bar{s})(J_a^{*(j)})_{\bar{s}s}\,(J_a^{*(j)})_{ss'} = \sum_{ksa} (\mathcal{J}_a)^i{}_k v^k(\vec{0}, s)\,(J_a^{(j)})_{ss'} \\ &= \sum_{k\ell a} (\mathcal{J}_a)^i{}_k\,(\mathcal{J}_a)^k{}_\ell\,v^\ell(\vec{0}, s') = \sum_k 2\,\delta^i_\ell\,v^\ell(\vec{0}, s') = 2\,v^i(\vec{0}, s'). \end{aligned} \tag{1.96}$$

Thus if $j = 0$, then for $i = 1, 2, 3$ both $u^i(\vec{0}, s)$ and $v^i(\vec{0}, s)$ must vanish, while if $j > 0$, then since $j(j+1) = 2$, the spin j must be unity, $j = 1$.

1.11 Vector field for spin-zero particles

The only nonvanishing components are constants taken conventionally as

$$u^0(\vec{0}) = i\sqrt{m/2} \quad \text{and} \quad v^0(\vec{0}) = -i\sqrt{m/2}. \tag{1.97}$$

At finite momentum the boost conditions (1.210) give them as

$$u^\mu(\vec{p}) = ip^\mu/\sqrt{2p^0} \quad \text{and} \quad v^\mu(\vec{p}) = -ip^\mu/\sqrt{2p^0}. \tag{1.98}$$

The vector field $\phi^\mu(x)$ of a spin-zero particle is then the derivative of a scalar field $\phi(x)$

$$\phi^\mu(x) = \partial^\mu\phi(x) = \int \frac{d^3p}{\sqrt{(2\pi)^3 2p^0}} \left[ip^\mu\, a(p)\, e^{ip\cdot x} - ip^\mu\, b^\dagger(p)\, e^{-ip\cdot x}\right] \tag{1.99}$$

1.12 Vector field for spin-one particles

We start with the $s = 0$ spinors $u^i(\vec{0}, 0)$ and $v^i(\vec{0}, 0)$ and note that since $(J_3^{(j)})_{\bar{s}0} = 0$, the $a = 3$ rotation conditions (1.87) imply that

$$(\mathcal{J}_3)^i{}_k\, u^k(\vec{0}, 0) = iR_3\, u^i(\vec{0}, 0) = 0 \quad \text{and} \quad (\mathcal{J}_3)^i{}_k\, v^k(\vec{0}, 0) = iR_3\, v^i(\vec{0}, 0) = 0. \tag{1.100}$$

Referring back to the explicit formulas for the generators of rotations and setting $u, v = (0, x, y, z)$ we see that

$$\mathcal{J}_3\, u = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ -iy \\ x \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{1.101}$$

and

$$\mathcal{J}_3\, v = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ -iy \\ x \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{1.102}$$

Thus only the 3-component z can be nonzero. The conventional choice is

$$u^\mu(\vec{0}, 0) = v^\mu(\vec{0}, 0) = \frac{1}{\sqrt{2m}} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{1.103}$$

We now form the linear combinations of the rotation conditions (1.87)

that correspond to the raising and lowering matrices $J_\pm^{(1)} = J_1^{(1)} \pm iJ_2^{(1)}$

$$J_+^{(1)} = \sqrt{2}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad J_-^{(1)} = \sqrt{2}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \tag{1.104}$$

Their Lorentz counterparts are

$$\mathcal{J}_\pm^{(1)} = \mathcal{J}_1^{(1)} \pm i\mathcal{J}_2^{(1)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mp 1 \\ 0 & 0 & 0 & -i \\ 0 & \pm 1 & i & 0 \end{pmatrix}. \tag{1.105}$$

In these terms, the rotation conditions (1.87) for the $j = 1$ spinors $u^i(\vec{0}, s)$ are

$$\sum_{\bar{s}} u^i(\vec{0}, \bar{s})(J_\pm^{(1)})_{\bar{s}s} = \sum_k (\mathcal{J}_\pm)^i{}_k u^k(\vec{0}, s). \tag{1.106}$$

But

$$J_1^{*(1)} \pm iJ_2^{*(1)} = J_1^{(1)} \mp iJ_2^{(1)} = J_\mp. \tag{1.107}$$

So the rotation conditions (1.87) for the $j = 1$ spinors $v^i(\vec{0}, s)$ are

$$-\sum_{\bar{s}} v^i(\vec{0}, \bar{s})(J_\mp^{(1)})_{\bar{s}s} = \sum_k (\mathcal{J}_\pm)^i{}_k v^k(\vec{0}, s). \tag{1.108}$$

So for the plus sign and the choice $s = 0$, the condition (1.106) gives $u^i(\vec{0}, 1)$ as

$$\sum_{\bar{s}} u^i(\vec{0}, \bar{s})\, J_{+\bar{s}0}^{(1)} = \sqrt{2}\, u^i(\vec{0}, 1) = (\mathcal{J}_+)^i{}_k u^k(\vec{0}, 0) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -i \\ 0 & 1 & i & 0 \end{pmatrix} \frac{1}{\sqrt{2m}} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{1.109}$$

or

$$u^i(\vec{0}, 1) = \frac{1}{2\sqrt{m}} \begin{pmatrix} 0 \\ -1 \\ -i \\ 0 \end{pmatrix}. \tag{1.110}$$

Similarly, the minus sign and the choice $s = 0$ give for $u^i(\vec{0}, -1)$

$$\sum_{\bar{s}} u^i(\vec{0}, \bar{s})\, J^{(1)}_{-\bar{s}0} = \sqrt{2}\, u^i(\vec{0}, -1) = (\mathcal{J}_-)^i{}_k u^k(\vec{0}, 0) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -i \\ 0 & -1 & i & 0 \end{pmatrix} \frac{1}{\sqrt{2m}} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{1.111}$$

or

$$u^i(\vec{0}, -1) = \frac{1}{2\sqrt{m}} \begin{pmatrix} 0 \\ 1 \\ -i \\ 0 \end{pmatrix}. \tag{1.112}$$

The rotation condition (1.108) for the $j = 1$ spinors $v^i(\vec{0}, s)$ with the minus sign and the choice $s = 0$ gives

$$-\sum_{\bar{s}} v^i(\vec{0}, \bar{s}) J^{(1)}_{-\bar{s}0} = -\sqrt{2}\, v^i(\vec{0}, -1) = (\mathcal{J}_+)^i{}_k v^k(\vec{0}, 0) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -i \\ 0 & 1 & i & 0 \end{pmatrix} \frac{1}{\sqrt{2m}} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{1.113}$$

or

$$v^i(\vec{0}, -1) = \frac{1}{2\sqrt{m}} \begin{pmatrix} 0 \\ 1 \\ i \\ 0 \end{pmatrix}. \tag{1.114}$$

Similarly, the plus sign and the choice $s = 0$ give

$$-\sum_{\bar{s}} v^i(\vec{0}, \bar{s}) J^{(1)}_{+\bar{s}0} = -\sqrt{2}\, v^i(\vec{0}, 1) = (\mathcal{J}_-)^i{}_k v^k(\vec{0}, 0) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -i \\ 0 & -1 & i & 0 \end{pmatrix} \frac{1}{\sqrt{2m}} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{1.115}$$

or

$$v^i(\vec{0}, 1) = \frac{1}{2\sqrt{m}} \begin{pmatrix} 0 \\ -1 \\ i \\ 0 \end{pmatrix}. \tag{1.116}$$

The boost conditions (1.210) now give for $i, k = 0, 1, 2, 3$

$$u^i(\vec{p}, s) = v^{*i}(\vec{p}, s) = \sqrt{m/p^0}\, L^i{}_k(\vec{p})\, u^k(\vec{0}, s) = e^i(\vec{p}, s)/\sqrt{2p^0} \tag{1.117}$$

where

$$e^i(\vec{p},s) = L^i{}_k(\vec{p})\, e^k(\vec{0},s) \tag{1.118}$$

and

$$e(\vec{0},0) = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}, \quad e(\vec{0},1) = -\frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\i\\0\end{pmatrix}, \quad \text{and} \quad e(\vec{0},-1) = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-i\\0\end{pmatrix}. \tag{1.119}$$

A single massive vector field is then

$$\phi^i(x) = \phi^{+i}(x) + \phi^{-i}(x) = \sum_{s=-1}^{1}\int \frac{d^3p}{\sqrt{(2\pi)^3 2p^0}} e^i(\vec{p},s)\, a(\vec{p},s) e^{ip\cdot x} + e^{*i}(\vec{p},s)\, a^\dagger(\vec{p},s) e^{-ip\cdot x}. \tag{1.120}$$

The commutator/anticommutator of the positive and negative frequency parts of the field is

$$[\phi^{+i}(x), \phi^{-k}(y)]_\mp = \int \frac{d^3p}{\sqrt{(2\pi)^3 2p^0}} e^{ip\cdot(x-y)}\, \Pi^{ik}(\vec{p}) \tag{1.121}$$

where Π is a sum of outer products of 4-vectors

$$\Pi^{ik}(\vec{p}) = \sum_{s=-1}^{1} e^i(\vec{p},s) e^{*k}(\vec{p},s). \tag{1.122}$$

At $\vec{p} = 0$, the matrix Π is the unit matrix on the spatial coordinates

$$\Pi(\vec{0}) = \sum_{s=-1}^{1} e^i(\vec{0},s) e^{*k}(\vec{0},s) = \begin{pmatrix}0&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}. \tag{1.123}$$

So $\Pi(\vec{p})$ is

$$\Pi(\vec{p}) = L\Pi(0)L^\mathsf{T} = L\eta L^\mathsf{T} + L\begin{pmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix} L^\mathsf{T} \tag{1.124}$$

or

$$\Pi(\vec{p})^{ik} = \eta^{ik} + p^i p^k/m^2. \tag{1.125}$$

This equation lets us write the commutator (1.126) in terms of the Lorentz-invariant function $\Delta_+(x-y)$ (1.45) as

$$\begin{aligned}[\phi^{+i}(x), \phi^{-k}(y)]_\mp &= (\eta^{ik} - \partial^i\partial^k/m^2) \int \frac{d^3p}{\sqrt{(2\pi)^3 2p^0}} e^{ip\cdot(x-y)} \\ &= (\eta^{ik} - \partial^i\partial^k/m^2)\, \Delta_+(x-y).\end{aligned} \tag{1.126}$$

As for a scalar field, we set

$$v^i(x) = \kappa\, \phi^{+i}(x) + \lambda\, \phi^{-i}(x) \tag{1.127}$$

and find for $(x-y)^2 > 0$ since $\Delta_+(x-y) = \Delta_+(y-x)$ for x, y spacelike

$$\begin{aligned}[v(x), v^\dagger(y)]_\mp &= \left(|\kappa|^2 \mp |\lambda|^2\right)\, (\eta^{ik} - \partial^i\partial^k/m^2)\, \Delta_+(x-y) \\ [v(x), v(y)]_\mp &= (1 \mp 1)\, \kappa\lambda(\eta^{ik} - \partial^i\partial^k/m^2)\, \Delta_+(x-y).\end{aligned} \tag{1.128}$$

So we must choose the minus sign and set $|\kappa| = |\lambda|$. So then

$$v^i(x) = v^{+i}(x) + v^{-i}(x) = v^{+i}(x) + v^{+i\dagger}(x) \tag{1.129}$$

is real. This is a second example of the spin-statistics theorem.

If two such fields have the same mass, then we can combine them as we combined scalar fields

$$v^i(x) = v_1^{+i}(x) + i v_2^{-i}(x). \tag{1.130}$$

These fields obey the Klein-Gordon equation

$$(\Box - m^2) v^i(x) = 0. \tag{1.131}$$

And since both

$$p^i = L^i{}_j k^j \quad \text{and} \quad e^k(\vec{p}) = L^k{}_\ell e^\ell(0) \tag{1.132}$$

it follows that

$$p \cdot e(\vec{p}) = k \cdot e(0) = 0. \tag{1.133}$$

So the field v^i also obeys the rule

$$\partial_i\, v^i(x) = 0. \tag{1.134}$$

These equations (1.133) and 1.134 are like those of the electromagnetic field in Lorentz gauge. But one can't get quantum electrodynamics as the $m \to 0$ limit of just any such theory. For the interaction $\mathcal{H} = J_i\, v^i$ would lead to a rate for v-boson production like

$$J_i J_k \Pi^{ik}(\vec{p}) \tag{1.135}$$

which diverges as $m \to 0$ because of the $p^i p^k/m^2$ term in $\Pi^{ik}(\vec{p})$. One can

avoid this divergence by requiring that $\partial_i J^i = 0$ which is current conservation.

Under parity, charge conjugation, and time reversal, a vector field transforms as

$$\begin{aligned} \mathsf{P} v^a(x) \mathsf{P}^{-1} &= -\eta^* \mathcal{P}^a{}_b v^b(\mathcal{P}x) \\ \mathsf{C} v^a(x) \mathsf{C}^{-1} &= \xi^* v^{a\dagger}(x) \\ \mathsf{T} v^a(x) \mathsf{T}^{-1} &= \zeta^* \mathcal{P}^a{}_b v^b(-\mathcal{P}x). \end{aligned} \tag{1.136}$$

1.13 Lorentz group

The Lorentz group $O(3,1)$ is the set of all linear transformations L that leave invariant the Minkowski inner product

$$xy \equiv \boldsymbol{x} \cdot \boldsymbol{y} - x^0 y^0 = x^{\mathsf{T}} \eta y \tag{1.137}$$

in which η is the diagonal matrix

$$\eta = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{1.138}$$

So L is in $O(3,1)$ if for all 4-vectors x and y

$$(Lx)^{\mathsf{T}} \eta L\, y = x^{\mathsf{T}} L^{\mathsf{T}} \eta\, L y = x^{\mathsf{T}} \eta\, y. \tag{1.139}$$

Since x and y are arbitrary, this condition amounts to

$$L^{\mathsf{T}} \eta\, L = \eta. \tag{1.140}$$

Taking the determinant of both sides and recalling that $\det A^{\mathsf{T}} = \det A$ and that $\det(AB) = \det A \det B$, we have

$$(\det L)^2 = 1. \tag{1.141}$$

So $\det L = \pm 1$, and every Lorentz transformation L has an inverse. Multiplying (1.140) by η, we get

$$\eta L^{\mathsf{T}} \eta L = \eta^2 = I \tag{1.142}$$

which identifies L^{-1} as

$$L^{-1} = \eta L^{\mathsf{T}} \eta. \tag{1.143}$$

The subgroup of $O(3,1)$ with $\det L = 1$ is the proper Lorentz group $SO(3,1)$. The subgroup of $SO(3,1)$ that leaves invariant the sign of the time component of timelike vectors is the **proper orthochronous Lorentz group** $SO^+(3,1)$.

To find the Lie algebra of $SO^+(3,1)$, we take a Lorentz matrix $L = I + \omega$ that differs from the identity matrix I by a tiny matrix ω and require L to obey the condition (1.140) for membership in the Lorentz group

$$\left(I + \omega^{\mathsf{T}}\right) \eta \left(I + \omega\right) = \eta + \omega^{\mathsf{T}}\eta + \eta\,\omega + \omega^{\mathsf{T}}\omega = \eta. \tag{1.144}$$

Neglecting $\omega^{\mathsf{T}}\omega$, we have $\omega^{\mathsf{T}}\eta = -\eta\,\omega$ or since $\eta^2 = I$

$$\omega^{\mathsf{T}} = -\,\eta\,\omega\,\eta. \tag{1.145}$$

This equation implies that the matrix ω_{ab} is antisymmetric when both indexes are down

$$\omega_{ab} = -\omega_{ba}. \tag{1.146}$$

To see why, we write it (1.145) as $\omega^e{}_a = -\,\eta_{ab}\,\omega^b{}_c\,\eta^{ce}$ and the multiply both sides by η_{de} so as to get $\omega_{da} = \eta_{de}\,\omega^e{}_a = -\eta_{ab}\,\omega^b{}_c\,\eta^{ce}\,\eta_{de} = -\omega_{ac}\,\delta^c_d = -\omega_{ad}$.

The key equation (1.145) also tells us that under transposition the time-time and space-space elements of ω change sign, while the time-space and spacetime elements do not. That is, the tiny matrix ω is for infinitesimal $\boldsymbol{\theta}$ and $\boldsymbol{\lambda}$ a linear combination

$$\omega = \boldsymbol{\theta}\cdot\boldsymbol{R} + \boldsymbol{\lambda}\cdot\boldsymbol{B} \tag{1.147}$$

of three antisymmetric space-space matrices

$$R_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad R_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad R_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{1.148}$$

and of three symmetric time-space matrices

$$B_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad B_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad B_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{1.149}$$

all of which satisfy condition (1.145). The three R_ℓ are 4×4 versions of the familiar rotation generators; the three B_ℓ generate Lorentz boosts.

If we write $L = I + \omega$ as

$$L = I - i\theta_\ell\, iR_\ell - i\lambda_\ell\, iB_\ell \equiv I - i\theta_\ell J_\ell - i\lambda_\ell K_\ell \tag{1.150}$$

then the three matrices $J_\ell = iR_\ell$ are imaginary and antisymmetric, and therefore hermitian. But the three matrices $K_\ell = iB_\ell$ are imaginary and symmetric, and so are antihermitian. The 4×4 matrix $L = \exp(i\theta_\ell J_\ell - i\lambda_\ell K_\ell)$ is **not unitary** because the Lorentz group is **not compact**.

1.14 Gamma matrices and Clifford algebras

In component notation, $L = I + \omega$ is

$$L^a{}_b = \delta^a{}_b + \omega^a{}_b, \tag{1.151}$$

the matrix η is $\eta_{cd} = \eta^{cd}$, and $\omega^\mathsf{T} = -\eta\,\omega\,\eta$ is

$$\omega^a{}_b = (\omega^\mathsf{T})_b{}^a = -(\eta\omega\eta)_b{}^a = -\eta_{bc}\,\omega^c{}_d\,\eta^{da} = -\omega_{bd}\,\eta^{da} = -\omega_b{}^a. \tag{1.152}$$

Lowering index a we get

$$\omega_{eb} = \eta_{ea}\,\omega^a{}_b = -\omega_{bd}\,\eta^{da}\,\eta_{ea} = -\omega_{bd}\,\delta^d{}_e = -\omega_{be} \tag{1.153}$$

That is, ω_{ab} is antisymmetric

$$\omega_{ab} = -\omega_{ba}. \tag{1.154}$$

A representation of the Lorentz group is generated by matrices $D(L)$ that represent matrices L close to the identity matrix by sums over $a, b = 0, 1, 2, 3$

$$D(L) = 1 + \frac{i}{2}\omega_{ab}\,\mathcal{J}^{ab}. \tag{1.155}$$

The generators $\mathcal{J}^{ab}$ must obey the commutation relations

$$i[\mathcal{J}^{ab}, \mathcal{J}^{cd}] = \eta^{bc}\,\mathcal{J}^{ad} - \eta^{ac}\,\mathcal{J}^{bd} - \eta^{da}\,\mathcal{J}^{cb} + \eta^{db}\,\mathcal{J}^{ca}. \tag{1.156}$$

A remarkable representation of these commutation relations is provided by matrices γ^a that obey the anticommutation relations

$$\{\gamma^a, \gamma^b\} = 2\,\eta^{ab}. \tag{1.157}$$

One sets

$$\mathcal{J}^{ab} = -\frac{i}{4}\,[\gamma^a, \gamma^b] \tag{1.158}$$

where η is the usual flat-space metric (1.138). Any four 4×4 matrices that

satisfy these anticommutation relations form a set of Dirac gamma matrices. They are not unique. Is S is any nonsingular 4×4 matrix, then the matrices

$$\gamma'^a = S\,\gamma^a\,S^{-1} \tag{1.159}$$

also are a set of Dirac's gamma matrices.

Any set of matrices obeying the anticommutation relations (1.157) for any $n \times n$ diagonal matrix η with entries that are ± 1 is called a **Clifford algebra**.

As a homework problem, show that

$$[\mathcal{J}^{ab}, \gamma^c] = -\,i\,\gamma^a\,\eta^{bc} + i\,\gamma^b\,\eta^{ac}. \tag{1.160}$$

One can use these commutation relations to derive the commutation relations (1.156) of the Lorentz group.

The gamma matrices are a vectors in the sense that for L near the identity

$$\begin{aligned}
D(L)\,\gamma^c\,D^{-1}(L) &\approx (I + \frac{i}{2}\omega_{ab}\,\mathcal{J}^{ab})\,\gamma^c\,(I - \frac{i}{2}\omega_{ab}\,\mathcal{J}^{ab}) \\
&= \gamma^c + \frac{i}{2}\omega_{ab}\,[\mathcal{J}^{ab}, \gamma^c] \\
&= \gamma^c + \frac{i}{2}\omega_{ab}\,(-i\,\gamma^a\,\eta^{bc} + i\,\gamma^b\,\eta^{ac}) \\
&= \gamma^c + \frac{1}{2}\omega_{ab}\,\gamma^a\,\eta^{bc} - \frac{1}{2}\omega_{ab}\,\gamma^b\,\eta^{ac} \\
&= \gamma^c - \frac{1}{2}\eta^{cb}\,\omega_{ba}\,\gamma^a - \frac{1}{2}\,\eta^{ca}\,\omega_{ab}\,\gamma^b \\
&= \gamma^c - \frac{1}{2}\,\omega^c{}_a\,\gamma^a - \frac{1}{2}\,\omega^c{}_b\,\gamma^b \\
&= \gamma^c - \omega^c{}_a\,\gamma^a \\
&= \gamma^c + \omega_a{}^c\,\gamma^a \\
&= (\delta_a{}^c + \omega_a{}^c)\,\gamma^a \\
&= L_a{}^c\,\gamma^a
\end{aligned} \tag{1.161}$$

in which we used (1.152) to write $-\,\omega^c{}_a = \omega_a{}^c$. The finite ω form is

$$D(L)\,\gamma^a\,D^{-1}(L) = L_c{}^a\,\gamma^c. \tag{1.162}$$

The unit matrix is a scalar

$$D(L)\,I\,D^{-1}(L) = I. \tag{1.163}$$

The generators of the Lorenz group form an antisymmetric tensor

$$D(L)\,\mathcal{J}^{ab}\,D^{-1}(L) = L_c{}^a\,L_d{}^b\,\mathcal{J}^{cd}. \tag{1.164}$$

Out of four gamma matrices, one can also make totally antisymmetric tensors of rank-3 and rank-4

$$A^{abc} \equiv \gamma^{[a}\,\gamma^{b}\,\gamma^{c]} \quad \text{and} \quad B^{abcd} \equiv \gamma^{[a}\,\gamma^{b}\,\gamma^{c}\,\gamma^{d]} \tag{1.165}$$

where the brackets mean that one inserts appropriate minus signs so as to achieve total antisymmetry. Since there are only four γ matrices in four spacetime dimensions, any rank-5 totally antisymmetric tensor made from them must vanish, $C^{abcde} = 0$.

Notation: The parity transformation is

$$\beta = i\gamma^0. \tag{1.166}$$

It flips the spatial gamma matrices but not the temporal one

$$\beta\,\gamma^i\,\beta^{-1} = -\,\gamma^i \quad \text{and} \quad \beta\,\gamma^0\,\beta^{-1} = \gamma^0. \tag{1.167}$$

It flips the generators of boosts but not those of rotations

$$\beta\,\mathcal{J}^{i0}\,\beta^{-1} = -\,\mathcal{J}^{i0} \quad \text{and} \quad \beta\,\mathcal{J}^{ik}\,\beta^{-1} = \mathcal{J}^{ik}. \tag{1.168}$$

1.15 Dirac's gamma matrices

Weinberg's chosen set of Dirac matrices is

$$\gamma^0 = -\,i\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = -\gamma^{0\dagger} \quad \text{and} \quad \gamma^i = -i\begin{pmatrix}0 & \sigma^i\\ -\,\sigma^i & 0\end{pmatrix} = \gamma^{i\dagger} \tag{1.169}$$

in which the σ's are Pauli's 2×2 hermitian matrices

$$\sigma_1 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \quad \sigma_2 = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}, \quad \text{and} \quad \sigma_3 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \tag{1.170}$$

which are the gamma matrices of 3-dimensional spacetime. With this choice of γ's, the matrix β is

$$\beta = i\,\gamma^0 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = \beta^\dagger. \tag{1.171}$$

In spacetimes of five dimensions, the fifth gamma matrix γ^4 which traditionally is called $\gamma^5 = \gamma_5$ is

$$\gamma^5 = \gamma_5 = -i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}. \tag{1.172}$$

It anticommutes with all four Dirac gammas and its square is unity, as it must if it is to be the fifth gamma in 5-space:

$$\{\gamma^a, \gamma^b\} = 2\eta^{ab} \tag{1.173}$$

for $a, b = 0, 1, 2, 3, 4$ with $\eta^{44} = 1$ and $\eta^{40} = \eta^{04} = 0$.

With Weinberg's choice of γ's, the Lorentz boosts are

$$\begin{aligned}\mathcal{J}^{i0} &= -\frac{i}{4}[\gamma^i, \gamma^0] = -\frac{i}{4}\left[-i\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, -i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right] \\ &= \frac{i}{4}\left[\begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix} - \begin{pmatrix} -\sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}\right] = \frac{i}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix}.\end{aligned} \tag{1.174}$$

The Lorentz rotation matrices are

$$\begin{aligned}\mathcal{J}^{ik} &= -\frac{i}{4}[\gamma^i, \gamma^k] = -\frac{i}{4}\left[-i\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, -i\begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix}\right] \\ &= \frac{i}{4}\left[\begin{pmatrix} -\sigma^i\sigma^k & 0 \\ 0 & -\sigma^i\sigma^k \end{pmatrix} - \begin{pmatrix} -\sigma^k\sigma^i & 0 \\ 0 & -\sigma^k\sigma^i \end{pmatrix}\right] \\ &= \frac{-i}{4}\begin{pmatrix} [\sigma^i, \sigma^k] & 0 \\ 0 & [\sigma^i, \sigma^k] \end{pmatrix} = \frac{-i}{4}\begin{pmatrix} 2i\epsilon_{ikj}\sigma^j & 0 \\ 0 & 2i\epsilon_{ikj}\sigma^j \end{pmatrix} \\ &= \frac{1}{2}\,\epsilon_{ikj}\begin{pmatrix} \sigma^j & 0 \\ 0 & \sigma^j \end{pmatrix}.\end{aligned} \tag{1.175}$$

The Dirac representation of the Lorentz group is reducible, as SW's choice of gamma matrices makes apparent. The Dirac rotation matrices are

$$\mathcal{J}_i = \frac{1}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}. \tag{1.176}$$

Some useful relations are

$$\beta\,\gamma^{a\dagger}\,\beta = -\gamma^a, \quad \beta\,\mathcal{J}^{ab\dagger}\,\beta = \mathcal{J}^{ab} \quad \text{and} \quad \beta\,D(L)^\dagger\,\beta = D(L)^{-1} \tag{1.177}$$

as well as

$$\beta\,\gamma_5^\dagger\,\beta = -\gamma_5 \quad \text{and} \quad \beta(\gamma_5\gamma^a)^\dagger\,\beta = -\gamma_5\gamma^a. \tag{1.178}$$

1.16 Dirac fields

The positive- and negative-frequency parts of a Dirac field are

$$\begin{aligned}\psi_\ell^+(x) &= (2\pi)^{-3/2}\sum_s \int d^3p\;\; u_\ell(\vec{p}, s)\, e^{ip\cdot x}\, a(\vec{p}, s) \\ \psi_\ell^-(x) &= (2\pi)^{-3/2}\sum_s \int d^3p\;\; v_\ell(\vec{p}, s)\, e^{-ip\cdot x}\, b^\dagger(\vec{p}, s).\end{aligned} \tag{1.179}$$

The rotation conditions (1.39) are

$$\begin{aligned}\sum_{\bar{s}} u_{\bar{\ell}}(\vec{0},\bar{s})(J_i^{(j)})_{\bar{s}s} &= \sum_{\ell}(\mathcal{J}_i)_{\bar{\ell}\ell}u_\ell(\vec{0},s)\\ \sum_{\bar{s}} v_{\bar{\ell}}(\vec{0},\bar{s})(-J_i^{(j)})^*_{\bar{s}s}) &= \sum_{\ell}(\mathcal{J}_i)_{\bar{\ell}\ell}v_\ell(\vec{0},s).\end{aligned} \tag{1.180}$$

The Dirac rotation matrices (1.176) are

$$\mathcal{J}_i = \frac{1}{2}\begin{pmatrix}\sigma^i & 0\\ 0 & \sigma^i\end{pmatrix} \tag{1.181}$$

so we set the four values $\ell,\bar{\ell}=1,2,3,4$ to $\ell=(m,\pm)$ with $m=\pm\frac{1}{2}$. And we consider $u_\ell(s)$ to be $u^+_m(s)$ stacked upon $u^-_m(s)$ and similarly take $v_\ell(s)$ to be $v^+_{m\pm}(s)$ above $v^-_{m\pm}(s)$ where $u^\pm_m(s)$ and $v^\pm_{m\pm}(s)$ are, a priori, $2\times(2j+1)$-dimensional matrices with indexes $m=\pm 1/2$ and $s=-j,\ldots,j$. That is,

$$\begin{pmatrix}u_1(s)\\ u_2(s)\\ u_3(s)\\ u_4 v\end{pmatrix} = \begin{pmatrix}u^+_{+1/2}(s)\\ u^+_{-1/2}(s)\\ u^-_{+1/2}(s)\\ u^-_{-1/2}(s)\end{pmatrix} \quad\text{and}\quad \begin{pmatrix}v_1(s)\\ v_2(s)\\ v_3(s)\\ v_4(s)\end{pmatrix} = \begin{pmatrix}v^+_{+1/2}(s)\\ v^+_{-1/2}(s)\\ v^-_{+1/2}(s)\\ v^-_{-1/2}(s)\end{pmatrix}. \tag{1.182}$$

We then have four equations

$$\begin{aligned}\sum_{\bar{s}} u^+_{\bar{m}}(\vec{0},\bar{s})(J_i^{(j)})_{\bar{s}s} &= \sum_m \tfrac{1}{2}\,\sigma^i_{\bar{m}m}u^+_m(\vec{0},s)\\ \sum_{\bar{s}} u^-_{\bar{m}}(\vec{0},\bar{s})(J_i^{(j)})_{\bar{s}s} &= \sum_m \tfrac{1}{2}\,\sigma^i_{\bar{m}m}u^-_m(\vec{0},s)\\ \sum_{\bar{s}} v^+_{\bar{m}}(\vec{0},\bar{s})(-J_i)^{*(j)}_{\bar{s}s}) &= \sum_m \tfrac{1}{2}\,\sigma^i_{\bar{m}m}v^+_m(\vec{0},s)\\ \sum_{\bar{s}} v^-_{\bar{m}}(\vec{0},\bar{s})(-J_i)^{*(j)}_{\bar{s}s}) &= \sum_m \tfrac{1}{2}\,\sigma^i_{\bar{m}m}v^-_m(\vec{0},s).\end{aligned} \tag{1.183}$$

SW defines the four $2\times(2j+1)$ matrices

$$\begin{aligned}U^+_{ms} &= u^+_m(\vec{0},s) \quad\text{and}\quad U^-_{ms} = u^-_m(\vec{0},s)\\ V^+_{ms} &= v^+_m(\vec{0},s) \quad\text{and}\quad V^-_{ms} = v^-_m(\vec{0},s).\end{aligned} \tag{1.184}$$

in terms of which the four Dirac rotation conditions (1.183) are

$$\begin{aligned}U^+\,J_i^{(j)} &= \tfrac{1}{2}\,\sigma_i\,U^+ \quad\text{and}\quad U^-\,J_i^{(j)} = \tfrac{1}{2}\,\sigma_i\,U^-\\ V^+\,(-J_i^{*(j)}) &= \tfrac{1}{2}\,\sigma_i\,V^+ \quad\text{and}\quad V^-\,(-J_i^{*(j)}) = \tfrac{1}{2}\,\sigma_i\,V^-.\end{aligned} \tag{1.185}$$

Taking the complex conjugate of the second of these equations, we get

$$\begin{aligned} -J_i^{(j)} &= V^{+*-1}(\tfrac{1}{2}\sigma^{i*})V^{+*} = V^{+*-1}(-\tfrac{1}{2}\sigma_2\,\sigma^i\,\sigma_2)V^{+*} \\ -J_i^{(j)} &= V^{-*-1}(\tfrac{1}{2}\sigma^{i*})V^{-*} = V^{-*-1}(-\tfrac{1}{2}\sigma_2\,\sigma^i\,\sigma_2)V^{-*} \end{aligned} \tag{1.186}$$

or more simply

$$\begin{aligned} J_i^{(j)} &= (\sigma_2 V^{+*})^{-1}\,\tfrac{1}{2}\sigma^i\,(\sigma_2 V^{+*}) \\ J_i^{(j)} &= (\sigma_2 V^{-*})^{-1}\,\tfrac{1}{2}\sigma^i\,(\sigma_2 V^{-*}). \end{aligned} \tag{1.187}$$

The 2×2 Pauli matrices $\vec{\sigma}$ and the $(2j+1)\times(2j+1)$ matrices $\vec{J}^{(j)}$ both generate irreducible representations of the rotation group. So by writing

$$U^+\,J_i^{(j)}\,J_k^{(j)} = \tfrac{1}{2}\,\sigma_i\,U^+\,J_k^{(j)} = \;=\tfrac{1}{2}\,\sigma_i\,\tfrac{1}{2}\,\sigma_k\,U^+ \tag{1.188}$$

and similar equations for U^-, V^+, V^-, we see that

$$\begin{aligned} U^+\,D^{(j)}(\vec{\theta}) &= U^+\,e^{-i\theta\cdot\vec{J}^{(j)}} = e^{-i\vec{\theta}\cdot\vec{\sigma}}\,U^+ = D^{(1/2)}(\vec{\theta})\,U^+ \\ U^-\,D^{(j)}(\vec{\theta}) &= U^-\,e^{-i\theta\cdot\vec{J}^{(j)}} = e^{-i\vec{\theta}\cdot\vec{\sigma}}\,U^- = D^{(1/2)}(\vec{\theta})\,U^- \end{aligned} \tag{1.189}$$

and similar equations for $V^\pm$.

$$\begin{aligned} \sigma_2 V^{+*}\,D^{(j)}(\vec{\theta}) &= \sigma_2 V^{+*}\,e^{-i\theta\cdot\vec{J}^{(j)}} = e^{-i\vec{\theta}\cdot\vec{\sigma}}\,\sigma_2 V^{+*} = D^{(1/2)}(\vec{\theta})\,\sigma_2 V^{+*} \\ \sigma_2 V^{-*}\,D^{(j)}(\vec{\theta}) &= \sigma_2 V^{-*}\,e^{-i\theta\cdot\vec{J}^{(j)}} = e^{-i\vec{\theta}\cdot\vec{\sigma}}\,\sigma_2 V^{-*} = D^{(1/2)}(\vec{\theta})\,\sigma_2 V^{-*}. \end{aligned} \tag{1.190}$$

Now recall Schur's lemma (section 10.7 of PM):

Part 1: If D_1 and D_2 are inequivalent, irreducible representations of a group G, and if $D_1(g)A = AD_2(g)$ for some matrix A and for all $g \in G$, then the matrix A must vanish, $A = 0$.

Part 2: If for a finite-dimensional, irreducible representation $D(g)$ of a group G, we have $D(g)A = AD(g)$ for some matrix A and for all $g \in G$, then $A = cI$. That is, any matrix that commutes with every element of a finite-dimensional, irreducible representation must be a multiple of the identity matrix.

Part 1 tells us that $D^{(j)}(\vec{\theta})$ and $D^{(1/2)}(\vec{\theta})$ must be equivalent. So $j = 1/2$ and $2j+1 = 2$. A Dirac field must represent particles of spin 1/2.

Part 2 then says that the matrices $U^\pm$ must be multiples of the 2×2 identity matrix

$$U^+ = c_+\,I \quad\text{and}\quad U^- = c_-\,I \tag{1.191}$$

and that the matrices $\sigma_2 V^{\pm *}$ must be multiples of the 2×2 identity matrix

$$\sigma_2 V^{+*} = d'_+\,I \quad\text{and}\quad \sigma_2 V^{-*} = d'_-\,I \tag{1.192}$$

or more simply

$$V^+ = -id_+\,\sigma_2 \quad \text{and} \quad V^- = -id_-\,\sigma_2. \tag{1.193}$$

That is,

$$v_m^+(\vec{0},s) = d_+\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} \quad \text{and} \quad v_m^-(\vec{0},s) = d_-\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}. \tag{1.194}$$

Going back to $\ell = (m,\pm)$ by using the index code (1.182)

$$\begin{bmatrix}u_1(s)\\ u_2(s)\\ u_3(s)\\ u_4(s)\end{bmatrix} = \begin{bmatrix}u^+_{+1/2}(s)\\ u^+_{-1/2}(s)\\ u^-_{+1/2}(s)\\ u^-_{-1/2}(s)\end{bmatrix} = \begin{bmatrix}c_+\,\delta_{1/2,s}\\ c_+\,\delta_{-1/2,s}\\ c_-\,\delta_{1/2,s}\\ c_-\,\delta_{-1/2,s}\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}v_1(s)\\ v_2(s)\\ v_3(s)\\ v_4(s)\end{bmatrix} = \begin{bmatrix}v^+_{+1/2}(s)\\ v^+_{-1/2}(s)\\ v^-_{1/2}(s)\\ v^-_{-1/2}(s)\end{bmatrix} = \begin{bmatrix}d_+\,\delta_{-1/2,s}\\ d_+\,\delta_{1/2,s}\\ d_-\,\delta_{-1/2,s}\\ d_-\,\delta_{+1/2,s}\end{bmatrix}. \tag{1.195}$$

we have for the u's

$$u^+_{1/2}(1/2) = c_+ \quad \text{and} \quad u^+_{-1/2}(1/2) = 0 \tag{1.196}$$

$$u^-_{1/2}(1/2) = c_- \quad \text{and} \quad u^-_{-1/2}(1/2) = 0 \tag{1.197}$$

$$u^+_{1/2}(-1/2) = 0 \quad \text{and} \quad u^+_{-1/2}(-1/2) = c_+ \tag{1.198}$$

$$u^-_{1/2}(-1/2) = 0 \quad \text{and} \quad u^-_{-1/2}(-1/2) = c_- \tag{1.199}$$

$$v^+_{1/2}(1/2) = 0 \quad \text{and} \quad v^+_{-1/2}(1/2) = d_+ \tag{1.200}$$

$$v^-_{1/2}(1/2) = 0 \quad \text{and} \quad v^-_{-1/2}(1/2) = d_- \tag{1.201}$$

$$v^+_{1/2}(-1/2) = -d_+ \quad \text{and} \quad v^+_{-1/2}(-1/2) = 0 \tag{1.202}$$

$$v^-_{1/2}(-1/2) = -d_- \quad \text{and} \quad v^-_{-1/2}(-1/2) = 0 \tag{1.203}$$

So

$$u(\vec{0},m=\tfrac{1}{2}) = \begin{bmatrix}u^+_{1/2}(1/2)\\ u^+_{-1/2}(1/2)\\ u^-_{1/2}(1/2)\\ u^-_{-1/2}(1/2)\end{bmatrix} = \begin{bmatrix}c_+\\ 0\\ c_-\\ 0\end{bmatrix} \quad \text{and} \quad u(\vec{0},m=-\tfrac{1}{2}) = \begin{bmatrix}u^+_{1/2}(-1/2)\\ u^+_{-1/2}(-1/2)\\ u^-_{1/2}(-1/2)\\ u^-_{-1/2}(-1/2)\end{bmatrix} = \begin{bmatrix}0\\ c_+\\ 0\\ c_-\end{bmatrix},$$

$$v(\vec{0},m=\tfrac{1}{2}) = \begin{bmatrix}v^+_{1/2}(1/2)\\ v^+_{-1/2}(1/2)\\ v^-_{1/2}(1/2)\\ v^-_{-1/2}(1/2)\end{bmatrix} = \begin{bmatrix}0\\ d_+\\ 0\\ d_-\end{bmatrix} \quad \text{and} \quad v(\vec{0},m=-\tfrac{1}{2}) = \begin{bmatrix}v^+_{1/2}(-1/2)\\ v^+_{-1/2}(-1/2)\\ v^-_{1/2}(-1/2)\\ v^-_{-1/2}(-1/2)\end{bmatrix} = -\begin{bmatrix}d_+\\ 0\\ d_-\\ 0\end{bmatrix}. \tag{1.204}$$

To put more constraints on $c_\pm$ and $d_\pm$, we recall that under parity

$$\mathsf{P}a(\vec{p},s)\mathsf{P}^{-1} = \eta_a^*\, a(-\vec{p},s) \quad \text{and} \quad \mathsf{P}b^\dagger(\vec{p},s)\mathsf{P}^{-1} = \eta_b\, \beta^\dagger(-\vec{p},s) \tag{1.205}$$

and so

$$\begin{aligned}
\mathsf{P}\psi_\ell^+(x)\mathsf{P}^{-1} &= (2\pi)^{-3/2}\sum_s \int d^3p\;\; u_\ell(\vec{p},s)\, e^{ip\cdot x}\, \eta_a^*\, a(-\vec{p},s) \\
&= (2\pi)^{-3/2}\sum_s \int d^3p\;\; u_\ell(-\vec{p},s)\, e^{ip\cdot \mathcal{P}x}\, \eta_a^*\, a(\vec{p},s) \\
\mathsf{P}\psi_\ell^-(x)\mathsf{P}^{-1} &= (2\pi)^{-3/2}\sum_s \int d^3p\;\; v_\ell(\vec{p},s)\, e^{-ip\cdot x}\, \eta_b\, b^\dagger(-\vec{p},s) \\
&= (2\pi)^{-3/2}\sum_s \int d^3p\;\; v_\ell(-\vec{p},s)\, e^{-ip\cdot \mathcal{P}x}\, \eta_b\, b^\dagger(\vec{p},s).
\end{aligned} \tag{1.206}$$

We recall the relations (1.177)

$$\beta\,\gamma^{a\dagger}\,\beta = -\,\gamma^a, \quad \beta\,\mathcal{J}^{ab\dagger}\,\beta = \mathcal{J}^{ab}, \quad \text{and} \quad \beta\, D(L)^\dagger\,\beta = D(L)^{-1} \tag{1.207}$$

and in particular, since $\mathcal{J}^{0i\dagger} = -\,\mathcal{J}^{0i}$, the rule

$$\beta\mathcal{J}^{0i}\beta = \mathcal{J}^{0i\dagger} = -\mathcal{J}^{0i}. \tag{1.208}$$

We also have the pseudounitarity relation

$$\beta\, D^\dagger(L)\,\beta = D^{-1}(L). \tag{1.209}$$

In general spinors at finite momentum are related to those at zero momentum by

$$\begin{aligned}
u_{\bar{\ell}}(q,s) &= \sqrt{\frac{m}{q^0}}\sum_\ell D_{\bar{\ell}\ell}(L(q))u_\ell(\vec{0},s) \\
v_{\bar{\ell}}(q,s) &= \sqrt{\frac{m}{q^0}}\sum_\ell D_{\bar{\ell}\ell}(L(q))v_\ell(\vec{0},s)
\end{aligned} \tag{1.210}$$

which for Dirac spinors is

$$\begin{aligned}
u(p,s) &= \sqrt{\frac{m}{p^0}}\, D(L(p))\, u(\vec{0},s) \\
v(p,s) &= \sqrt{\frac{m}{p^0}}\, D(L(p))\, v(\vec{0},s).
\end{aligned} \tag{1.211}$$

So now by using the boost rule (1.208) we have

$$u_\ell(-\vec{p},s) = \sqrt{m/p^0}D(L(-\vec{p}))u(0,s) = \sqrt{m/p^0}D(L(\vec{p}))^{-1}u(0,s) \tag{1.212}$$

$$= \sqrt{m/p^0}\,\beta\, D(L(\vec{p}))\,\beta\, u(0,s) \tag{1.213}$$

and

$$v_\ell(-\vec{p},s) = \sqrt{m/p^0}D(L(-\vec{p}))v(0,s) = \sqrt{m/p^0}D(L(\vec{p}))^{-1}v(0,s) \quad (1.214)$$
$$= \sqrt{m/p^0}\,\beta\,D(L(\vec{p}))\,\beta\,v(0,s). \quad (1.215)$$

So under parity

$$\mathsf{P}\psi^+(x)\mathsf{P}^{-1} = (2\pi)^{-3/2}\sum_s \int d^3p\ \ \sqrt{m/p^0}\,\beta\,D(L(\vec{p}))\,\beta\,u(0,s)\,e^{ip\cdot\mathcal{P}x}\,\eta_a^*\,a(\vec{p},s)$$
$$\mathsf{P}\psi^-(x)\mathsf{P}^{-1} = (2\pi)^{-3/2}\sum_s \int d^3p\ \ \sqrt{m/p^0}\,\beta\,D(L(\vec{p}))\,\beta\,v(0,s)\,e^{-ip\cdot\mathcal{P}x}\,\eta_b\,b^\dagger(\vec{p},s). \quad (1.216)$$

So to have $\mathsf{P}\psi_\ell^\pm(x)\mathsf{P}^{-1} \propto \psi_\ell^\pm(x)$, we need

$$\beta\,u(0,s) = b_u\,u(0,s) \quad \text{and} \quad \beta\,v(0,s) = b_v\,u(0,s). \quad (1.217)$$

We then get

$$\mathsf{P}\psi_\ell^+(t,\vec{x})\mathsf{P}^{-1} = b_u\,\beta\,\eta_a^*\,\psi_\ell^+(t,-\vec{x}) \quad \text{and} \quad \mathsf{P}\psi_\ell^-(t,\vec{x})\mathsf{P}^{-1} = b_v\,\beta\,\eta_b\,\psi_\ell^-(t,-\vec{x}). \quad (1.218)$$

Here since $\mathsf{P}^2 = 1$, these factors are just signs, $b_u^2 = b_v^2 = 1$. The eigenvalue equations (1.217) tell us that $c_- = b_u\,c_+$ and that $d_- = b_v\,d_+$. So rescaling the fields we get

$$u(\vec{0},m=\tfrac{1}{2}) = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\b_u\\0\end{bmatrix} \quad \text{and} \quad u(\vec{0},m=-\tfrac{1}{2}) = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\0\\b_u\end{bmatrix},$$
$$v(\vec{0},m=\tfrac{1}{2}) = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\0\\b_v\end{bmatrix} \quad \text{and} \quad v(\vec{0},m=-\tfrac{1}{2}) = \frac{-1}{\sqrt{2}}\begin{bmatrix}1\\0\\b_v\\0\end{bmatrix}. \quad (1.219)$$

If the annihilation and creation operators $a(p,s)$ and $a^\dagger(p,s)$ obey the rule

$$[a(p,s),a^\dagger(p',s')]_\mp = \delta_{ss'}\delta^3(\vec{p}-\vec{p}') \quad (1.220)$$

and if the field is the sum of the positive- and negative-frequency parts (2.55)

$$\psi_\ell^+(x) = (2\pi)^{-3/2}\sum_s \int d^3p\ \ u_\ell(\vec{p},s)\,e^{ip\cdot x}\,a(\vec{p},s)$$
$$\psi_\ell^-(x) = (2\pi)^{-3/2}\sum_s \int d^3p\ \ v_\ell(\vec{p},s)\,e^{-ip\cdot x}\,b^\dagger(\vec{p},s) \quad (1.221)$$

with arbitrary coefficients κ and λ

$$\psi_\ell(x) = \kappa\psi_\ell^+(x) + \lambda\psi_\ell^-(x) \tag{1.222}$$

then

$$\begin{aligned}
[\psi_\ell(x), \psi_{\ell'}^\dagger(y)]_\mp &= [\kappa\psi_\ell^+(x) + \lambda\psi_\ell^-(x), \kappa^*\psi_{\ell'}^{+\dagger}(y) + \lambda^*\psi_{\ell'}^{-\dagger}(y)] \\
&= \int \frac{d^3p}{(2\pi)^3} \sum_s \Big[|\kappa|^2\, u_\ell(\vec{p}, s)\, u_{\ell'}^*(\vec{p}, s) e^{ip\cdot(x-y)} \\
&\quad \mp |\lambda|^2\, v_\ell(\vec{p}, s)\, v_{\ell'}^*(\vec{p}, s) e^{-ip\cdot(x-y)} \Big] \\
&= \int \frac{d^3p}{(2\pi)^3} \sum_s \Big[|\kappa|^2\, N_{\ell\ell'}(p)\, e^{ip\cdot(x-y)} \mp |\lambda|^2\, N_{\ell\ell'}(p)\, e^{-ip\cdot(x-y)} \Big]
\end{aligned} \tag{1.223}$$

where

$$\begin{aligned}
N_{\ell\ell'}(p) &= \sum_s u_\ell(\vec{p}, s)\, u_{\ell'}^*(\vec{p}, s) \\
M_{\ell\ell'}(p) &= \sum_s v_\ell(\vec{p}, s)\, v_{\ell'}^*(\vec{p}, s).
\end{aligned} \tag{1.224}$$

When $\vec{p} = 0$, these matrices are

$$\begin{aligned}
N_{\ell\ell'}(0) &= \sum_s u_\ell(\vec{0}, s)\, u_{\ell'}^*(\vec{0}, s) \\
N(0) &= \frac{1}{2}\begin{bmatrix} 1 \\ 0 \\ b_u \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & b_u & 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 \\ 1 \\ 0 \\ b_u \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & b_u & \end{bmatrix} \\
&= \frac{1}{2}\begin{bmatrix} 1 & 0 & b_u & 0 \\ 0 & 0 & 0 & 0 \\ b_u & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & b_u \\ 0 & 0 & 0 & 0 \\ 0 & b_u & 0 & 1 \end{bmatrix} = \frac{1 + b_u\,\beta}{2}
\end{aligned} \tag{1.225}$$

and

$$
\begin{aligned}
M_{\ell\ell'}(0) &= \sum_s v_\ell(\vec{0},s)\, v^*_{\ell'}(\vec{0},s) \\
M(0) &= \frac{1}{2}\begin{bmatrix}0\\1\\0\\b_v\end{bmatrix}\begin{bmatrix}0 & 1 & 0 & b_v\end{bmatrix} + \frac{1}{2}\begin{bmatrix}1\\0\\b_v\\0\end{bmatrix}\begin{bmatrix}1 & 0 & b_v & 0\end{bmatrix} \\
&= \frac{1}{2}\begin{bmatrix}0&0&0&0\\0&1&0&b_v\\0&0&0&0\\0&b_v&0&1\end{bmatrix} + \frac{1}{2}\begin{bmatrix}1&0&b_v&0\\0&0&0&0\\b_v&0&1&0\\0&0&0&0\end{bmatrix} = \frac{1+b_v\,\beta}{2}.
\end{aligned}
\tag{1.226}
$$

So then using the boost relations (1.211) we find

$$
\begin{aligned}
N(\vec{p}) &= \sum_s u_\ell(\vec{p},s)\, u^*_{\ell'}(\vec{p},s) = \frac{m}{p^0} D(L(p)) \sum_s u_\ell(\vec{0},s)\, u^*_{\ell'}(\vec{0},s) D^\dagger(L(p)) \\
&= \frac{m}{2p^0}\, D(L(p))\,(1+b_u\,\beta)\, D^\dagger(L(p)) \\
M(\vec{p}) &= \sum_s v_\ell(\vec{p},s)\, v^*_{\ell'}(\vec{p},s) = \frac{m}{p^0} D(L(p)) \sum_s v_\ell(\vec{0},s)\, v^*_{\ell'}(\vec{0},s) D^\dagger(L(p)) \\
&= \frac{m}{2p^0}\, D(L(p))\,(1+b_v\,\beta)\, D^\dagger(L(p)).
\end{aligned}
\tag{1.227}
$$

The pseudounitarity relation (1.209)

$$\beta\, D^\dagger(L)\,\beta = D^{-1}(L). \tag{1.228}$$

gives

$$\beta\, D^\dagger(L) = D^{-1}(L)\,\beta \tag{1.229}$$

which implies that

$$D(L)\,\beta\, D^\dagger(L) = \beta. \tag{1.230}$$

The pseudounitarity relation also says that

$$D^\dagger(L) = \beta\, D^{-1}(L)\,\beta \tag{1.231}$$

so that

$$D(L)\, D^\dagger(L) = D(L)\,\beta\, D^{-1}(L)\,\beta. \tag{1.232}$$

Also since the gammas form a 4-vector (1.162)

$$D(L)\,\gamma^a\, D^{-1}(L) = L_c{}^a\,\gamma^c \tag{1.233}$$

and since $\beta = i\gamma^0$, we have

$$D(L(p))\,\beta\,D^{-1}(L(p)) = D(L(p))\,i\gamma^0\,D^{-1}(L(p)) = iL_c{}^0(p)\,\gamma^c = -iL^c{}_0\gamma_c. \tag{1.234}$$

Now

$$p^a = L^a{}_b(p)k^b = L^a{}_0(p)m \tag{1.235}$$

so

$$D(L(p))\,\beta\,D^{-1}(L(p)) = -i\,p^c\gamma_c/m \tag{1.236}$$

which implies that

$$D(L)\,D^\dagger(L) = -\,i\,(p^c\gamma_c/m)\,\beta. \tag{1.237}$$

Thus

$$\begin{aligned} N(\vec{p}) &= \frac{m}{2p^0}\,D(L(p))\,(1+b_u\,\beta)\,D^\dagger(L(p)) = \frac{m}{2p^0}\,[(-i\,p^c\gamma_c/m)\,\beta + b_u\,\beta] \\ &= \frac{1}{2p^0}\,[-i\,p^c\gamma_c + b_u\,m]\;\beta \end{aligned} \tag{1.238}$$

and

$$\begin{aligned} M(\vec{p}) &= \frac{m}{2p^0}\,D(L(p))\,(1+b_v\,\beta)\,D^\dagger(L(p)) = \frac{m}{2p^0}\,[(-i\,p^c\gamma_c/m)\,\beta + b_v\,\beta] \\ &= \frac{1}{2p^0}\,[-i\,p^c\gamma_c + b_v\,m]\;\beta. \end{aligned} \tag{1.239}$$

We now put the spin sums (1.238) and (1.239) in the (anti)commutator (1.223) and get

$$\begin{aligned} [\psi_\ell(x), \psi^\dagger_{\ell'}(y)]_\mp = &\int \frac{d^3p}{(2\pi)^3 2p^0}\Big[|\kappa|^2\,[(-\,i\,p^c\gamma_c + b_u\,m)\beta]_{\ell\ell'}\;e^{ip\cdot(x-y)} \\ &\mp |\lambda|^2\,[(-\,i\,p^c\gamma_c + b_v\,m)\beta]_{\ell\ell'}\;e^{-ip\cdot(x-y)}\Big] \\ = &\,|\kappa|^2\,[(-\,\partial_c\gamma^c + b_u\,m)\beta]_{\ell\ell'}\int \frac{d^3p}{(2\pi)^3 2p^0} e^{ip\cdot(x-y)} \\ &\mp |\lambda|^2\,[(-\,\partial_c\gamma^c + b_v\,m)\beta]_{\ell\ell'}\int \frac{d^3p}{(2\pi)^3 2p^0} e^{-ip\cdot(x-y)} \\ = &\,|\kappa|^2\,[(-\,\partial_c\gamma^c + b_u\,m)\beta]_{\ell\ell'}\,\Delta_+(x-y) \\ &\mp |\lambda|^2\,[(-\,\partial_c\gamma^c + b_v\,m)\beta]_{\ell\ell'}\;\Delta_+(y-x). \end{aligned} \tag{1.240}$$

Recall that for $(x-y)^>0$, i.e. spacelike, $\Delta_+(x-y) = \Delta_+(y-x)$. So its first

derivatives are odd. So for $x - y$ spacelike

$$\begin{aligned}[\psi_\ell(x), \psi^\dagger_{\ell'}(y)]_\mp &= |\kappa|^2\,[(-\partial_c\gamma^c + b_u\,m)\beta]_{\ell\ell'}\,\Delta_+(x-y)\\ &\quad \mp |\lambda|^2\,[(\partial_c\gamma^c + b_v\,m)\beta]_{\ell\ell'}\,\Delta_+(x-y)\\ &= \big(|\kappa|^2 \pm |\lambda|^2\big)[(-\partial_c\gamma^c)\beta]_{\ell\ell'}\Delta_+(x-y)\\ &\quad + \big(|\kappa|^2 b_u \mp |\lambda|^2\, b_v\big) m\,\beta_{\ell\ell'}\,\Delta_+(x-y).\end{aligned} \tag{1.241}$$

To get the first term to vanish, we need to choose the lower sign (that is, use anticommutators) and set $|\kappa| = |\lambda|$. To get the second term to be zero, we must set $b_u = -b_v$. We may adjust κ and and b_u so that

$$\kappa = \lambda \quad \text{and} \quad b_u = -b_v = 1. \tag{1.242}$$

In particular, a spin-one-half field must obey anticommutation relations

$$[\psi_\ell(x), \psi^\dagger_{\ell'}(y)]_+ \equiv \{\psi_\ell(x), \psi^\dagger_{\ell'}(y)\} = 0 \quad \text{for} \quad (x-y)^2 > 0. \tag{1.243}$$

Finally then, the Dirac field is

$$\psi_\ell(x) = \sum_s \int \frac{d^3p}{(2\pi)^{3/2}}\ \Big[u_\ell(\vec{p},s)\,e^{ip\cdot x}\,a(\vec{p},s) + v_\ell(\vec{p},s)\,e^{-ip\cdot x}\,b^\dagger(\vec{p},s)\Big]. \tag{1.244}$$

The zero-momentum spinors are

$$\begin{aligned} u(\vec{0}, m=\tfrac{1}{2}) &= \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\1\\0\end{bmatrix} \quad \text{and} \quad u(\vec{0}, m=-\tfrac{1}{2}) = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\0\\1\end{bmatrix},\\ v(\vec{0}, m=\tfrac{1}{2}) &= \frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\0\\-1\end{bmatrix} \quad \text{and} \quad v(\vec{0}, m=-\tfrac{1}{2}) = \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\0\\1\\0\end{bmatrix}.\end{aligned} \tag{1.245}$$

The spin sums are

$$\begin{aligned}\big[N(\vec{p})\big]_{\ell m} &= \sum_s u_\ell(\vec{p},s)u^*_m(\vec{p},s) = \Big[\frac{1}{2p^0}\,(-i\,p^c\gamma_c + m)\ \beta\Big]_{\ell m}\\ \big[M(\vec{p})\big]_{\ell m} &= \sum_s v_\ell(\vec{p},s)v^*_m(\vec{p},s) = \Big[\frac{1}{2p^0}\,(-i\,p^c\gamma_c - m)\ \beta\Big]_{\ell m}.\end{aligned} \tag{1.246}$$

The Dirac anticommutator is

$$[\psi_\ell(x), \psi^\dagger_{\ell'}(y)]_+ \equiv \{\psi_\ell(x), \psi^\dagger_{\ell'}(y)\} = [(-\partial_c\gamma^c + m)\beta]_{\ell\ell'}\,\Delta_+(x-y). \tag{1.247}$$

Two standard abbreviations are

$$\beta \equiv i\gamma^0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad \overline{\psi} \equiv \psi^\dagger \beta = i\psi^\dagger \gamma^0 = \begin{bmatrix} \psi_\ell^* & \psi_r^* \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \psi_r^* & \psi_\ell^* \end{bmatrix}. \tag{1.248}$$

A Majorana fermion is represented by a field like

$$\psi_\ell(x) = \sum_s \int \frac{d^3p}{(2\pi)^{3/2}} \Big[u_\ell(\vec{p},s)\, e^{ip\cdot x}\, a(\vec{p},s) + v_\ell(\vec{p},s)\, e^{-ip\cdot x}\, a^\dagger(\vec{p},s) \Big]. \tag{1.249}$$

Since $\mathcal{C} = \gamma_2\beta$ it follows that $\mathcal{C}^{-1} = \beta\gamma_2$ and so that $\mathcal{J}^{*ab} = -\beta\mathcal{C}\mathcal{J}^{ab}\mathcal{C}^{-1}\beta = -\beta\gamma_2\beta\mathcal{J}^{ab}\beta\gamma_2\beta$. But $\beta\gamma_2\beta = -\beta\beta\gamma_2 = -i^2\gamma_0^2\gamma_2 = -\gamma_2$. So $\mathcal{J}^{*ab} = -\gamma_2\mathcal{J}^{ab}\gamma_2$. Thus

$$D^*(L) = e^{-i\omega_{ab}\mathcal{J}^{*ab}} = e^{-i\omega_{ab}(-\gamma_2\mathcal{J}^{ab}\gamma_2)} = \gamma_2 e^{i\omega_{ab}\mathcal{J}^{ab}}\gamma_2 = \gamma_2\, D(L)\, \gamma_2. \tag{1.250}$$

Now with SW's γ's,

$$\gamma_2\, u(\vec{0},\pm\tfrac{1}{2}) = v(\vec{0},\pm\tfrac{1}{2}) \quad \text{and} \quad \gamma_2\, v(\vec{0},\pm\tfrac{1}{2}) = u(\vec{0},\pm\tfrac{1}{2}). \tag{1.251}$$

Thus the hermitian conjugate of a Majorana field is

$$\begin{aligned}
\psi^*(x) &= \sum_s \int \frac{d^3p}{(2\pi)^{3/2}} \Big[u^*(\vec{p},s)\, e^{-ip\cdot x}\, a^\dagger(\vec{p},s) + v^*(\vec{p},s)\, e^{ip\cdot x}\, a(\vec{p},s) \Big] \\
&= \sum_s \int \frac{d^3p}{(2\pi)^{3/2}} \Big[v^*(\vec{p},s)\, e^{ip\cdot x}\, a(\vec{p},s) + u^*(\vec{p},s)\, e^{-ip\cdot x}\, a^\dagger(\vec{p},s) \Big] \\
&= \sum_s \int \frac{d^3p}{(2\pi)^{3/2}} \Big[D^*(L(p))\, v^*(\vec{0},s)\, e^{ip\cdot x}\, a(\vec{p},s) + D^*(L(p))\, u^*(\vec{0},s)\, e^{-ip\cdot x}\, a^\dagger(\vec{p},s) \Big] \\
&= \gamma_2 \sum_s \int \frac{d^3p}{(2\pi)^{3/2}} \Big[D(L(p))\gamma_2 v(\vec{0},s)\, e^{ip\cdot x}\, a(\vec{p},s) + D(L(p))\gamma_2 u(\vec{0},s)\, e^{-ip\cdot x}\, a^\dagger(\vec{p},s) \Big] \\
&= \gamma_2 \sum_s \int \frac{d^3p}{(2\pi)^{3/2}} \Big[D(L(p)) u(\vec{0},s)\, e^{ip\cdot x}\, a(\vec{p},s) + D(L(p)) v(\vec{0},s)\, e^{-ip\cdot x}\, a^\dagger(\vec{p},s) \Big] \\
&= \gamma_2 \sum_s \int \frac{d^3p}{(2\pi)^{3/2}} \Big[u(\vec{p},s)\, e^{ip\cdot x}\, a(\vec{p},s) + v(\vec{p},s)\, e^{-ip\cdot x}\, a^\dagger(\vec{p},s) \Big] = \gamma_2\, \psi(x).
\end{aligned} \tag{1.252}$$

This works because the spinors obey the Majorana conditions

$$u(p,s) = \gamma^2\, v^*(p,s) \quad \text{and} \quad v(p,s) = \gamma^2\, u^*(p,s). \tag{1.253}$$

The parity rules (1.254) now are

$$\mathsf{P}\psi_\ell^+(t,\vec{x})\mathsf{P}^{-1} = \beta\,\eta_a^*\,\psi_\ell^+(t,-\vec{x}) \quad \text{and} \quad \mathsf{P}\psi_\ell^-(t,\vec{x})\mathsf{P}^{-1} = -\beta\,\eta_b\,\psi_\ell^-(t,-\vec{x}). \tag{1.254}$$

So to have a Dirac field survive a parity transformation, we need the phase of the particle to be minus the complex conjugate of the phase of the antiparticle

$$\eta_a^* = -\eta_b \quad \text{or} \quad \eta_b = -\eta_a^*. \tag{1.255}$$

So the intrinsic parity of a particle-antiparticle state is odd. So negative-parity bosons like $\pi^0, \rho_0, J/\psi$ can be interpreted as s-wave bound states of quark-antiquark pairs. Under parity a Dirac field goes as

$$\mathsf{P}\psi(t,\vec{x})\mathsf{P}^{-1} = \eta^*\,\beta\,\psi(t,-\vec{x}). \tag{1.256}$$

If a Dirac particle is the same as its antiparticle, then its intrinsic parity must be odd under complex conjugation, $\eta = -\eta^*$. So the intrinsic parity of a Majorana fermion must be imaginary

$$\eta = \pm i. \tag{1.257}$$

But this means that if we express a Dirac field ψ as a complex linear combination

$$\psi = \frac{1}{\sqrt{2}}(\chi_1 + i\chi_2) \tag{1.258}$$

of two Majorana fields with intrinsic parities $\eta_1^* = \pm i$ and $\eta_2^* = \pm i$, then under parity

$$\mathsf{P}\psi(t,\vec{x})\mathsf{P}^{-1} = \frac{1}{\sqrt{2}}\Big(\eta_1^*\,\beta\,\phi_1(t,-\vec{x}) + i\eta_2^*\,\beta\,\phi_2(t,-\vec{x})\Big) \tag{1.259}$$

so we need $\eta_1^* = \eta_2^*$ to have

$$\mathsf{P}\psi(t,\vec{x})\mathsf{P}^{-1} = \frac{1}{\sqrt{2}}\Big(\eta_1^*\,\beta\,\phi_1(t,-\vec{x}) + i\eta_2^*\,\beta\,\phi_2(t,-\vec{x})\Big) = \eta^*\,\beta\,\psi(t,-\vec{x}). \tag{1.260}$$

But in that case the Dirac field has intrinsic parity $\eta = \pm i$.

The equation (1.236) that shows how beta goes under $D(L(p))$

$$D(L(p))\,\beta\,D^{-1}(L(p)) = -i\,p^c\gamma_c/m \tag{1.261}$$

tells us that the spinors (1.211)

$$u(p,s) = \sqrt{\frac{m}{p^0}}\,D(L(p))\,u(\vec{0},s) \quad \text{and} \quad v(p,s) = \sqrt{\frac{m}{p^0}}\,D(L(p))\,v(\vec{0},s) \tag{1.262}$$

are eigenstates of $-i\,p^c\gamma_c/m$ with eigenvalues ± 1

$$
\begin{aligned}
(-i\,p^c\gamma_c/m)\,u(p,s) &= D(L(p))\,\beta\,D^{-1}(L(p))\,\sqrt{\frac{m}{p^0}}\,D(L(p))\,u(\vec{0},s)\\
&= \sqrt{\frac{m}{p^0}}\,D(L(p))\,\beta\;u(\vec{0},s) = \sqrt{\frac{m}{p^0}}\,D(L(p))\;u(\vec{0},s) = u(p,s)\\
(-i\,p^c\gamma_c/m)\,v(p,s) &= D(L(p))\,\beta\,D^{-1}(L(p))\,\sqrt{\frac{m}{p^0}}\,D(L(p))\,v(\vec{0},s)\\
&= \sqrt{\frac{m}{p^0}}\,D(L(p))\,\beta\;v(\vec{0},s) = -\sqrt{\frac{m}{p^0}}\,D(L(p))\;v(\vec{0},s) = -v(p,s).
\end{aligned}
\tag{1.263}
$$

So the spinors u and v obey Dirac's equation in momentum space

$$
(i\,p^c\gamma_c + m)u(p,s) = 0 \quad \text{and} \quad (-i\,p^c\gamma_c + m)v(p,s) = 0 \tag{1.264}
$$

which implies that a Dirac field obeys Dirac's equation

$$
\begin{aligned}
(\gamma^a\,\partial_a + m)\psi(x) &= (\gamma^a\,\partial_a + m)\sum_s \int \frac{d^3p}{(2\pi)^{3/2}}\;\Big[u(\vec{p},s)\,e^{ip\cdot x}\,a(\vec{p},s) + v(\vec{p},s)\,e^{-ip\cdot x}\,b^\dagger(\vec{p},s)\Big]\\
&= \sum_s \int \frac{d^3p}{(2\pi)^{3/2}}\;\Big[(\gamma^a\,\partial_a + m)u(\vec{p},s)\,e^{ip\cdot x}\,a(\vec{p},s)\\
&\quad + (\gamma^a\,\partial_a + m)v(\vec{p},s)\,e^{-ip\cdot x}\,b^\dagger(\vec{p},s)\Big]\\
&= \sum_s \int \frac{d^3p}{(2\pi)^{3/2}}\;\Big[(i\gamma^a\,p_a + m)u(\vec{p},s)\,e^{ip\cdot x}\,a(\vec{p},s)\\
&\quad + (-i\gamma^a\,p_a + m)v(\vec{p},s)\,e^{-ip\cdot x}\,b^\dagger(\vec{p},s)\Big] = 0.
\end{aligned}
\tag{1.265}
$$

The Majorana conditions (1.253) imply that

$$
u^*(p,s) = -\beta\mathcal{C}v(p,s) \quad \text{and} \quad v^*(p,s) = -\beta\mathcal{C}u(p,s). \tag{1.266}
$$

So for a Dirac field to survive charge conjugation, the particle-antiparticle phases must be related

$$
\xi_b = \xi_a^*. \tag{1.267}
$$

Then under charge conjugation a Dirac field goes as

$$
\mathsf{C}\,\psi(x)\,\mathsf{C}^{-1} = -\xi^*\,\beta\mathcal{C}\,\psi^*(x). \tag{1.268}
$$

If a Dirac particle is the same as its antiparticle, then ξ must be real (and

η imaginary), $\xi = \pm 1$, and must satisfy the reality condition

$$\psi(x) = -\beta\,\mathcal{C}\,\psi^*(x). \tag{1.269}$$

Suppose a particle and its antiparticle form a bound state

$$|\Phi\rangle = \sum_{ss'} \int d^3p\, d^3p' \chi(p,s;p',s')\, a^\dagger(p,s)\, b^\dagger(p's')\, |0\rangle. \tag{1.270}$$

Under charge conjugation

$$\begin{aligned}
\mathsf{C}\,|\Phi\rangle &= \xi_a \xi_b \sum_{ss'} \int d^3p\, d^3p' \chi(p,s;p',s')\, b^\dagger(p,s)\, a^\dagger(p's')\, |0\rangle \\
&= -\,\xi_a \xi_b \sum_{ss'} \int d^3p\, d^3p' \chi(p,s;p',s')\, a^\dagger(p's')\; b^\dagger(p,s)\, |0\rangle \\
&= -\,\xi_a \xi_b \sum_{ss'} \int d^3p\, d^3p' \chi(p',s';p,s)\, a^\dagger(p,s)\; b^\dagger(p',s')\, |0\rangle \\
&= -\,\xi_a \xi_b\, |\Phi\rangle = -\xi_a \xi_a^*\, |\Phi\rangle = -|\Phi\rangle.
\end{aligned} \tag{1.271}$$

The intrinsic charge-conjugation parity of a bound state of a particle and its antiparticle is odd.

1.17 Expansion of massive and massless Dirac fields

A Dirac field is a sum over $s = \pm 1/2$ of

$$\psi_a(x) = \sum_s \int \frac{d^3p}{(2\pi)^{3/2}}\; u_a(\vec{p},s)\, e^{ip\cdot x}\, a(\vec{p},s) + v_a(\vec{p},s)\, e^{-ip\cdot x}\, b^\dagger(\vec{p},s) \tag{1.272}$$

in which

$$u(\boldsymbol{p},s) = \frac{m - i\not{p}}{\sqrt{2p^0(p^0+m)}}\, u(\mathbf{0},s) \quad \text{and} \quad v(\boldsymbol{p},s) = \frac{m + i\not{p}}{\sqrt{2p^0(p^0+m)}}\, v(\mathbf{0},s) \tag{1.273}$$

and the zero-momentum spinors (1.245) are

$$\begin{aligned}
u(\mathbf{0}, s = \tfrac{1}{2}) &= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad u(\mathbf{0}, s = -\tfrac{1}{2}) = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \\
v(\mathbf{0}, s = \tfrac{1}{2}) &= \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} \quad \text{and} \quad v(\mathbf{0}, s = -\tfrac{1}{2}) = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}.
\end{aligned} \tag{1.274}$$

These static spinors and the matrix formulas (1.273) give explicit expressions for the spinors at arbitrary momentum $\boldsymbol{p}$:

$$u(\boldsymbol{p}, \tfrac{1}{2}) = \frac{1}{2\sqrt{p^0(p^0+m)}} \begin{pmatrix} m + p^0 - p_3 \\ -p_1 - ip_2 \\ m + p^0 + p_3 \\ p_1 + ip_2 \end{pmatrix} \tag{1.275}$$

$$u(\boldsymbol{p}, -\tfrac{1}{2}) = \frac{1}{2\sqrt{p^0(p^0+m)}} \begin{pmatrix} -p_1 + ip_2 \\ m + p^0 + p_3 \\ p_1 - ip_2 \\ m + p^0 - p_3 \end{pmatrix} \tag{1.276}$$

$$v(\boldsymbol{p}, \tfrac{1}{2}) = \frac{1}{2\sqrt{p^0(p^0+m)}} \begin{pmatrix} -p_1 + ip_2 \\ m + p^0 + p_3 \\ -p_1 + ip_2 \\ -m - p^0 + p_3 \end{pmatrix} \tag{1.277}$$

$$v(\boldsymbol{p}, -\tfrac{1}{2}) = \frac{1}{2\sqrt{p^0(p^0+m)}} \begin{pmatrix} -m - p^0 + p_3 \\ p_1 + ip_2 \\ m + p^0 + p_3 \\ p_1 + ip_2 \end{pmatrix}. \tag{1.278}$$

In the $m \to 0$ limit with $\boldsymbol{p} = p^0 \hat{\boldsymbol{z}}$ in the 3-direction, they are

$$u(p^0\hat{\boldsymbol{z}}, \tfrac{1}{2}) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad u(p^0\hat{\boldsymbol{z}}, -\tfrac{1}{2}) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \tag{1.279}$$

$$v(p^0\hat{\boldsymbol{z}}, \tfrac{1}{2}) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad v(p^0\hat{\boldsymbol{z}}, -\tfrac{1}{2}) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}. \tag{1.280}$$

They tell us that the upper two components which are those of the left-handed Dirac field represent particles of negative helicity $u(p^0\hat{\boldsymbol{z}}, -\frac{1}{2})$ and antiparticles of positive helicity $v(p^0\hat{\boldsymbol{z}}, \frac{1}{2})$. They also say that the lower two components which are those of the right-handed field represent particles of positive helicity $u(p^0\hat{\boldsymbol{z}}, \frac{1}{2})$ and antiparticles of negative helicity $v(p^0\hat{\boldsymbol{z}}, -\frac{1}{2})$.

In the $m \to 0$ limit, the spinors are for the particles

$$u(\boldsymbol{p}, \tfrac{1}{2}) = \frac{1}{2p^0} \begin{pmatrix} 0 \\ 0 \\ p^0 + p_3 \\ p_1 + ip_2 \end{pmatrix} \quad \text{and} \quad u(\boldsymbol{p}, -\tfrac{1}{2}) = \frac{1}{2p^0} \begin{pmatrix} -p_1 + ip_2 \\ p^0 + p_3 \\ 0 \\ 0 \end{pmatrix} \tag{1.281}$$

and for the antiparticles

$$v(\boldsymbol{p}, \tfrac{1}{2}) = \frac{1}{2p^0} \begin{pmatrix} -p_1 + ip_2 \\ p^0 + p_3 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad v(\boldsymbol{p}, -\tfrac{1}{2}) = \frac{1}{2p^0} \begin{pmatrix} 0 \\ 0 \\ p^0 + p_3 \\ p_1 + ip_2 \end{pmatrix}. \tag{1.282}$$

The massless Dirac field is

$$\psi_a(x) = \sum_{s=-\frac{1}{2}}^{\frac{1}{2}} \int \frac{d^3p}{(2\pi)^{3/2}} \Big[u_a(\boldsymbol{p}, s) e^{ip\cdot x}\, a(\boldsymbol{p}, s) + v_a(\boldsymbol{p}, s)\, e^{-ip\cdot x}\, b^\dagger(\boldsymbol{p}, s) \Big]. \tag{1.283}$$

The left-handed field is

$$\begin{aligned} \psi_{\ell a}(x) &= \int \frac{d^3p}{(2\pi)^{3/2}} \Big[u_{\ell a}(\boldsymbol{p}, -\tfrac{1}{2}) e^{ip\cdot x}\, a(\boldsymbol{p}, -\tfrac{1}{2}) + v_{\ell a}(\boldsymbol{p}, \tfrac{1}{2})\, e^{-ip\cdot x}\, b^\dagger(\boldsymbol{p}, \tfrac{1}{2}) \Big] \\ &= \int \frac{d^3p}{(2\pi)^{3/2} 2p^0} \Big[\begin{pmatrix} -p_1 + ip_2 \\ p^0 + p_3 \end{pmatrix} e^{ip\cdot x}\, a(\boldsymbol{p}, -\tfrac{1}{2}) \\ &\qquad + \begin{pmatrix} -p_1 + ip_2 \\ p^0 + p_3 \end{pmatrix} e^{-ip\cdot x}\, b^\dagger(\boldsymbol{p}, \tfrac{1}{2}) \Big]. \end{aligned} \tag{1.284}$$

The right-handed field is

$$\begin{aligned} \psi_{ra}(x) &= \int \frac{d^3p}{(2\pi)^{3/2}} \Big[u_{ra}(\boldsymbol{p}, \tfrac{1}{2}) e^{ip\cdot x}\, a(\boldsymbol{p}, \tfrac{1}{2}) + v_{ra}(\boldsymbol{p}, -\tfrac{1}{2})\, e^{-ip\cdot x}\, b^\dagger(\boldsymbol{p}, -\tfrac{1}{2}) \Big] \\ &= \int \frac{d^3p}{(2\pi)^{3/2} 2p^0} \Big[\begin{pmatrix} p^0 + p_3 \\ p_1 + ip_2 \end{pmatrix} e^{ip\cdot x}\, a(\boldsymbol{p}, +\tfrac{1}{2}) \\ &\qquad + \begin{pmatrix} p^0 + p_3 \\ p_1 + ip_2 \end{pmatrix} e^{-ip\cdot x}\, b^\dagger(\boldsymbol{p}, -\tfrac{1}{2}) \Big]. \end{aligned} \tag{1.285}$$

The massless left-handed field and the massless right-handed field create and absorb different particles. These are different fields. That is why the unbroken gauge group $SU(2)_\ell$ can act on ψ_ℓ without affecting ψ_r.

1.18 Spinors and angular momentum

Under a Lorentz transformation a Dirac field goes as

$$U(L)\psi_a(x)U^{-1}(L) = \sum_b D_{ab}(L^{-1})\psi_b(Lx) \tag{1.286}$$

where $D(L)$ is the $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$ representation of the Lorentz group. If $L = R$ is an active rotation of $|\vec{\theta}|$ radians about the vector $\vec{\theta}$, then the matrix is

$$D(R^{-1}) = \begin{pmatrix} e^{i\vec{\theta}\cdot\vec{\sigma}/2} & 0 \\ 0 & e^{i\vec{\theta}\cdot\vec{\sigma}/2} \end{pmatrix}. \tag{1.287}$$

In particular, if $\vec{\theta} = \theta\hat{z}$, then

$$D(R^{-1}) = \begin{pmatrix} e^{i\theta\sigma_3/2} & 0 \\ 0 & e^{i\theta\sigma_3/2} \end{pmatrix} = \begin{pmatrix} e^{i\theta/2} & 0 & 0 & 0 \\ 0 & e^{-i\theta/2} & 0 & 0 \\ 0 & 0 & e^{i\theta/2} & 0 \\ 0 & 0 & 0 & e^{-i\theta/2} \end{pmatrix}. \tag{1.288}$$

So since the field is

$$\psi_a(x) = \sum_s \int \frac{d^3p}{(2\pi)^{3/2}}\; u_a(\vec{p},s)\, e^{ip\cdot x}\, a(\vec{p},s) + v_a(\vec{p},s)\, e^{-ip\cdot x}\, a_c^\dagger(\vec{p},s), \tag{1.289}$$

we need

$$\begin{aligned} U(R)\psi_a(x)U^{-1}(R) &= D(R^{-1})_{ab}\psi_b(Rx) \\ &= \sum_s \int \frac{d^3p}{(2\pi)^{3/2}} \Big[u_a(\vec{p},s)e^{ip\cdot x}U(R)a(\vec{p},s)U^{-1}(R) \\ &\quad + v_a(\vec{p},s)e^{-ip\cdot x}U(R)a_c^\dagger(\vec{p},s)U^{-1}(R)\Big]. \end{aligned} \tag{1.290}$$

A state of momentum $\vec{p}$, kind n, and spin $s = \pm 1/2$ in the z direction transforms under a right-handed rotation $R(\theta, z)$ of θ radians about the z axis as

$$U(R(\theta,z))|p\hat{z},s,n\rangle = e^{-i\theta J_z}|p\hat{z},s,n\rangle = e^{-i\theta s}|p\hat{z},s,n\rangle. \tag{1.291}$$

The state $|p\hat{z},s,n\rangle$ is made from the vacuum by the creation operator for a particle of momentum $\vec{p}$, kind n, and spin $s = \pm 1/2$ in the z direction

$$|p\hat{z},s,n\rangle = a^\dagger(p\hat{z},s,n)|0\rangle. \tag{1.292}$$

The creation operator transforms as

$$U(R(z,\theta))a^\dagger(p\hat{z},s,n)U(R(z,\theta))^{-1} = e^{-i\theta s}a^\dagger(p\hat{z},s,n). \tag{1.293}$$

The antiparticle creation operator goes as

$$U(R)a_c^\dagger(\vec{p},s)U^{-1}(R) = \sum_{s'} \left(e^{-i\vec{\theta}\cdot\vec{\sigma}/2}\right)_{ss'} a_c^\dagger(R\vec{p},s'), \tag{1.294}$$

and so the particle annihilation operator goes as

$$U(R)a(\vec{p},s)U^{-1}(R) = \sum_{s'} \left(e^{-i\vec{\theta}\cdot\vec{\sigma}/2}\right)^*_{ss'} a(R\vec{p},s') = \sum_{s'} \left(e^{i\vec{\theta}\cdot\vec{\sigma}/2}\right)_{s's} a(R\vec{p},s'). \tag{1.295}$$

Our requirement (1.290) is

$$\begin{aligned} U(R)\psi_a(x)U^{-1}(R) &= D(R^{-1})_{ab}\psi_b(Rx) \\ &= \sum_s \int \frac{d^3p}{(2\pi)^{3/2}} \Big[u_a(\vec{p},s)e^{ip\cdot x} \sum_{s'} \left(e^{i\vec{\theta}\cdot\vec{\sigma}/2}\right)_{s's} a(R\vec{p},s') \\ &\quad + v_a(\vec{p},s)e^{-ip\cdot x} \sum_{s'} \left(e^{-i\vec{\theta}\cdot\vec{\sigma}/2}\right)_{ss'} a_c^\dagger(R\vec{p},s') \Big]. \end{aligned} \tag{1.296}$$

But

$$\begin{aligned} D(R^{-1})_{ab}\psi_b(Rx) &= \begin{pmatrix} e^{i\vec{\theta}\cdot\vec{\sigma}/2} & 0 \\ 0 & e^{i\vec{\theta}\cdot\vec{\sigma}/2} \end{pmatrix}_{ab} \psi_b(Rx) \\ &= \begin{pmatrix} e^{i\vec{\theta}\cdot\vec{\sigma}/2} & 0 \\ 0 & e^{i\vec{\theta}\cdot\vec{\sigma}/2} \end{pmatrix}_{ab} \sum_s \int \frac{d^3p}{(2\pi)^{3/2}} \Big[u_b(\vec{p},s)e^{ip\cdot Rx} a(\vec{p},s) \\ &\quad + v_b(\vec{p},s)e^{-ip\cdot Rx} a_c^\dagger(\vec{p},s) \Big]. \end{aligned} \tag{1.297}$$

Equivalently since $Rp\cdot Rx = p\cdot x$,

$$\begin{aligned} D(R^{-1})_{ab}\psi_b(Rx) &= \begin{pmatrix} e^{i\vec{\theta}\cdot\vec{\sigma}/2} & 0 \\ 0 & e^{i\vec{\theta}\cdot\vec{\sigma}/2} \end{pmatrix}_{ab} \psi_b(Rx) \\ &= \begin{pmatrix} e^{i\vec{\theta}\cdot\vec{\sigma}/2} & 0 \\ 0 & e^{i\vec{\theta}\cdot\vec{\sigma}/2} \end{pmatrix}_{ab} \sum_s \int \frac{d^3p}{(2\pi)^{3/2}} \Big[u_b(\vec{R}p,s)e^{ip\cdot x} a(R\vec{p},s) \\ &\quad + v_b(\vec{R}p,s)e^{-ip\cdot x} a_c^\dagger(R\vec{p},s) \Big]. \end{aligned} \tag{1.298}$$

Let's focus on the zero-momentum terms. We need

$$\sum_{s'} \left(e^{i\vec{\theta}\cdot\vec{\sigma}/2}\right)_{ss'} u_a(\vec{0},s') = \begin{pmatrix} e^{i\vec{\theta}\cdot\vec{\sigma}/2} & 0 \\ 0 & e^{i\vec{\theta}\cdot\vec{\sigma}/2} \end{pmatrix}_{ab} u_b(\vec{0},s) \tag{1.299}$$

and

$$\sum_{s'} \left(e^{-i\vec{\theta}\cdot\vec{\sigma}/2}\right)_{s's} v_a(\vec{0},s') = \begin{pmatrix} e^{i\vec{\theta}\cdot\vec{\sigma}/2} & 0 \\ 0 & e^{i\vec{\theta}\cdot\vec{\sigma}/2} \end{pmatrix}_{ab} v_b(\vec{0},s). \tag{1.300}$$

The spinors

$$u(\vec{0},\tfrac{1}{2}) = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\1\\0\end{bmatrix} \quad \text{and} \quad u(\vec{0},-\tfrac{1}{2}) = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\0\\1\end{bmatrix} \tag{1.301}$$

satisfy the condition (1.299) on the spinor $u_a(\vec{0},s)$. Let's look at the condition (1.300) on $v_a(\vec{0},s)$ for the simpler case of $\vec{\theta}=\theta\hat{z}$. We set $s=\frac{1}{2}$ or equivalently $s=+$ and also $a=1$. Since the rotation is about $\hat{z}$,

$$\left(e^{-i\vec{\theta}\cdot\vec{\sigma}/2}\right)_{s's} = \left(e^{-i\theta\sigma_3/2}\right)_{s's} = e^{-i\theta s}\delta_{s's}. \tag{1.302}$$

So the condition (1.300) says that

$$e^{-i\theta/2}v_1(\vec{0},\tfrac{1}{2}) = e^{i\theta/2}v_1(\vec{0},\tfrac{1}{2}). \tag{1.303}$$

So $v_1(\vec{0},\frac{1}{2})=0$. Now we set $a=2$. Then the condition says that

$$e^{-i\theta/2}v_2(\vec{0},\tfrac{1}{2}) = e^{-i\theta/2}v_2(\vec{0},\tfrac{1}{2}). \tag{1.304}$$

So $v_2(\vec{0},\frac{1}{2})$ can be anything. Now we set $a=3$. Then the same condition (1.300) says that

$$e^{-i\theta/2}v_3(\vec{0},\tfrac{1}{2}) = e^{i\theta/2}v_3(\vec{0},\tfrac{1}{2}) \tag{1.305}$$

which implies that $v_3(\vec{0},\frac{1}{2})=0$. Finally, we set $a=4$. Now the same condition (1.300) says that

$$e^{-i\theta/2}v_4(\vec{0},\tfrac{1}{2}) = e^{-i\theta/2}v_4(\vec{0},\tfrac{1}{2}). \tag{1.306}$$

So $v_4(\vec{0},\frac{1}{2})$ can be anything. That is, $v(\vec{0},\frac{1}{2})$ must be of the form

$$v(\vec{0},\tfrac{1}{2}) = \begin{pmatrix}0\\\alpha\\0\\\beta\end{pmatrix} \tag{1.307}$$

in which α and β are arbitrary.

Now we set $s=-\frac{1}{2}$ or equivalently $s=-$ and also $a=1$. Then the condition (1.300) says that

$$e^{i\theta/2}v_1(\vec{0},-\tfrac{1}{2}) = e^{i\theta/2}v_1(\vec{0},-\tfrac{1}{2}). \tag{1.308}$$

So $v_1(\vec{0},-\frac{1}{2})$ can be anything. But the condition (1.300) for $a=2$ says

$$e^{i\theta/2}v_2(\vec{0},-\tfrac{1}{2}) = e^{-i\theta/2}v_2(\vec{0},-\tfrac{1}{2}) \tag{1.309}$$

which means that $v_2(\vec{0}, -\frac{1}{2}) = 0$. For $a = 3$, we get

$$e^{i\theta/2} v_3(\vec{0}, -\tfrac{1}{2}) = e^{i\theta/2} v_3(\vec{0}, -\tfrac{1}{2}), \tag{1.310}$$

so $v_3(\vec{0}, -\frac{1}{2})$ is unrestricted. Finally, for $a = 4$, we have

$$e^{i\theta/2} v_4(\vec{0}, -\tfrac{1}{2}) = e^{-i\theta/2} v_4(\vec{0}, -\tfrac{1}{2}) \tag{1.311}$$

which makes $v_4(\vec{0}, -\frac{1}{2})$ vanish. So

$$v(\vec{0}, -\tfrac{1}{2}) = \begin{pmatrix} \gamma \\ 0 \\ \delta \\ 0 \end{pmatrix} \tag{1.312}$$

in which γ and δ are arbitrary. To make the v's suitably normalized and orthogonal to the u's, one may set

$$v(\vec{0}, \tfrac{1}{2}) = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} \quad \text{and} \quad v(\vec{0}, -\tfrac{1}{2}) = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}. \tag{1.313}$$

The state $a^\dagger(0, s, n_c)|0\rangle$ is (summed over a and s')

$$\begin{aligned} \frac{1}{(2\pi)^{3/2}} \int d^3x\, \bar{v}_a(0,s)\psi_a(x)|0\rangle &= \bar{v}_a(0,s) \int \frac{d^3x d^3p}{(2\pi)^3}\, v(\vec{p}, s')\, e^{-ip\cdot x}\, a^\dagger(\vec{p}, s', n_c)|0\rangle \\ &= \bar{v}_a(0,s) \int d^3p\, \delta^3(\boldsymbol{p}) v(\vec{p}, s')\, a^\dagger(\vec{p}, s', n_c)|0\rangle \\ &= \bar{v}_a(0,s) v(0,s')\, a^\dagger(0, s', n_c)|0\rangle \\ &= \delta_{ss'}\, a^\dagger(0, s', n_c)|0\rangle = a^\dagger(0, s, n_c)|0\rangle. \end{aligned} \tag{1.314}$$

So

$$|0, s, n_c\rangle = \frac{1}{(2\pi)^{3/2}} \sum_a \int d^3x\, \bar{v}_a(0,s)\psi_a(x)|0\rangle. \tag{1.315}$$

To determine its spin, we act on it with the operator $e^{-i\theta J_3}$ that rotates states about the z axis by angle θ in a right-handed way.

$$\begin{aligned} e^{-i\theta J_3}|0, s, n_c\rangle &= e^{-i\theta J_3} \frac{1}{(2\pi)^{3/2}} \sum_a \int d^3x\, \bar{v}_a(0,s)\psi_a(x)|0\rangle \\ &= \frac{1}{(2\pi)^{3/2}} \sum_a \int d^3x\, \bar{v}_a(0,s) e^{-i\theta J_3} \psi_a(x) e^{i\theta J_3} e^{-i\theta J_3}|0\rangle. \end{aligned} \tag{1.316}$$

Since the vacuum is invariant, this is

$$\begin{aligned} e^{-i\theta J_3}|0,s,n_c\rangle &= \frac{1}{(2\pi)^{3/2}}\sum_a \int d^3x\, \bar{v}_a(0,s) e^{-i\theta J_3}\psi_a(x) e^{i\theta J_3}|0\rangle \\ &= \frac{1}{(2\pi)^{3/2}}\sum_a \int d^3x\, \bar{v}_a(0,s) D(R^{-1})_{ab}\psi_b(Rx)|0\rangle. \end{aligned} \tag{1.317}$$

Since the jacobian of a rotation is unity, we have

$$e^{-i\theta J_3}|0,s,n_c\rangle = \frac{1}{(2\pi)^{3/2}}\sum_a \int d^3x\, \bar{v}_a(0,s) D(R^{-1})_{ab}\psi_b(x)|0\rangle \tag{1.318}$$

in which

$$D(R^{-1})_{ab} = \begin{pmatrix} e^{i\theta/2} & 0 & 0 & 0 \\ 0 & e^{-i\theta/2} & 0 & 0 \\ 0 & 0 & e^{i\theta/2} & 0 \\ 0 & 0 & 0 & e^{-i\theta/2} \end{pmatrix}. \tag{1.319}$$

So this is

$$e^{-i\theta J_3}|0,s,n_c\rangle = \frac{1}{(2\pi)^{3/2}}\sum_a \int d^3x\, \bar{v}_a(0,s) e^{i\sigma(a)\theta/2}\psi_a(x)|0\rangle \tag{1.320}$$

in which $\sigma(a) = 1$ for $a = 1$ & 3, and $\sigma(a) = -1$ for $a = 2$ & 4. For $s = \pm 1/2$, the spinors $v(0,s)$ are

$$v(\vec{0},\tfrac{1}{2}) = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} \quad \text{and} \quad v(\vec{0},-\tfrac{1}{2}) = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}. \tag{1.321}$$

So for $s = 1/2$, slots $a = 2$ & 4 are nonzero, and

$$e^{-i\theta J_3}|0,1/2,n_c\rangle = e^{-i\theta/2}\frac{1}{(2\pi)^{3/2}}\sum_a \int d^3x\, \bar{v}_a(0,s)\psi_a(x)|0\rangle \tag{1.322}$$

which means that the state has spin 1/2 in the z direction. For $s = -1/2$, slots $a = 1$ & 3 are nonzero, and

$$e^{-i\theta J_3}|0,1/2,n_c\rangle = e^{i\theta/2}\frac{1}{(2\pi)^{3/2}}\sum_a \int d^3x\, \bar{v}_a(0,s)\psi_a(x)|0\rangle \tag{1.323}$$

which means that the state has spin $-1/2$ in the z direction.

1.19 Charge conjugation

Under charge conjugation, a massive Dirac field

$$\psi(x) = \sum_s \int \frac{d^3p}{(2\pi)^{3/2}} \; u(\vec{p},s)\, e^{ip\cdot x}\, a(\vec{p},s) + v(\vec{p},s)\, e^{-ip\cdot x}\, a_c^\dagger(\vec{p},s) \tag{1.324}$$

transforms as

$$\mathsf{C}\psi(x)\mathsf{C}^{-1} = -\xi^*\beta\gamma_2\beta\psi^*(x) = \xi^*\gamma_2\psi^*(x) \tag{1.325}$$

in which ξ is the charge-conjugation parity of the particle annihilated by a (which is the complex conjugate of that of the particle annihilated by a_c) and

$$\beta\gamma_2\beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} (-i) \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = i \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix} = -\gamma_2. \tag{1.326}$$

Under charge conjugation, the creation operators for particles and their antiparticles, transform as

$$\mathsf{C}a^\dagger(\vec{p},s)\mathsf{C}^{-1} = \xi a_c^\dagger(\vec{p},s) \quad \text{and} \quad \mathsf{C}a_c^\dagger(\vec{p},s)\mathsf{C}^{-1} = \xi_c a^\dagger(\vec{p},s). \tag{1.327}$$

So

$$\mathsf{C}a(\vec{p},s)\mathsf{C}^{-1} = \xi^* a_c(\vec{p},s) \quad \text{and} \quad \mathsf{C}a_c(\vec{p},s)\mathsf{C}^{-1} = \xi_c^* a(\vec{p},s). \tag{1.328}$$

So, suppressing explicit mention of the sum over spins, we have

$$\begin{aligned} \mathsf{C}\psi(x)\mathsf{C}^{-1} &= \int \frac{d^3p}{(2\pi)^{3/2}} \; u(\vec{p},s)\, e^{ip\cdot x}\, \mathsf{C}a(\vec{p},s)\mathsf{C}^{-1} + v(\vec{p},s)\, e^{-ip\cdot x}\, \mathsf{C}a_c^\dagger(\vec{p},s)\mathsf{C}^{-1} \\ &= \int \frac{d^3p}{(2\pi)^{3/2}} \; u(\vec{p},s)\, e^{ip\cdot x}\, \xi^* a_c(\vec{p},s) + v(\vec{p},s)\, e^{-ip\cdot x}\, \xi_c a^\dagger(\vec{p},s)\mathsf{C}^{-1}. \end{aligned}$$

Under charge conjugation, the field changes to

$$\mathsf{C}\psi(x)\mathsf{C}^{-1} = -i\xi^* \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix} \psi^*(x) = \xi^*\gamma_2\psi^*(x). \tag{1.329}$$

Explicitly and with $\xi_c = \xi^*$, we get

$$\begin{aligned} \mathsf{C}\psi(x)\mathsf{C}^{-1} = -i\xi^* \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix} \sum_s \int \frac{d^3p}{(2\pi)^{3/2}} \Big[& u^*(\vec{p},s)\, e^{-ip\cdot x}\, a^\dagger(\vec{p},s) \\ & + v^*(\vec{p},s)\, e^{ip\cdot x}\, a_c(\vec{p},s) \Big] \end{aligned} \tag{1.330}$$

So leaving aside the factor $-i\xi^*$, the spinors for the a_c particles (aka, antiparticles) are

$$u_b(\vec{p}, s) = \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix} v^*(\vec{p}, s) \tag{1.331}$$

while the antispinors for the a particles are

$$v_a(\vec{p}, s) = \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix} u^*(\vec{p}, s). \tag{1.332}$$

Let's focus on the zero-momentum spinors. The zero-momentum spinors are

$$\begin{aligned} u(\vec{0}, \tfrac{1}{2}) &= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad u(\vec{0}, -\tfrac{1}{2}) = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \\ v(\vec{0}, \tfrac{1}{2}) &= \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} \quad \text{and} \quad v(\vec{0}, -\tfrac{1}{2}) = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}. \end{aligned} \tag{1.333}$$

The spinor $u_b(0, +)$ is

$$u_b(\vec{0}, +) = \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix} v^*(\vec{0}, +) \tag{1.334}$$

or in all its components,

$$u_b(\vec{0}, +) = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} = \frac{i}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = i\, u(\vec{0}, +) \tag{1.335}$$

which is what we expect. Similarly,

$$u_b(\vec{0}, -) = \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix} v^*(\vec{0}, -) \tag{1.336}$$

or in all its components,

$$u_b(\vec{0}, -) = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} = i\, u(\vec{0}, -) \tag{1.337}$$

which is what we expect. Similarly, the spinor $v_a(0,+)$ is

$$v_a(\vec{0},+) = \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix} u^*(\vec{0},+) \tag{1.338}$$

or in all its components,

$$v_a(\vec{0},+) = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} = i\, v(\vec{0},+) \tag{1.339}$$

which is what we expect. Finally, the spinor $v_a(0,-)$ is

$$v_a(\vec{0},-) = \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix} u^*(\vec{0},-) \tag{1.340}$$

or in all its components,

$$v_a(\vec{0},-) = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} = \frac{i}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = i\, v(\vec{0},-) \tag{1.341}$$

which is what we expect.

1.20 Parity

Under parity the field $\psi(x)$ (1.324) changes to

$$\mathsf{P}\psi(x)\mathsf{P}^{-1} = \eta^* \beta \psi(Px) \tag{1.342}$$

where η is the parity of the particle annihilated by a, which is minus the complex conjugate of that of the particle annihilated by b. The annihilation and creation operators transform as

$$\mathsf{P}a^\dagger(\vec{p},s)\mathsf{P}^{-1} = \eta a^\dagger(-\vec{p},s) \quad \text{and} \quad \mathsf{P}a_c^\dagger(\vec{p},s)\mathsf{P}^{-1} = \eta_c a_c^\dagger(-\vec{p},s) \tag{1.343}$$

and

$$\mathsf{P}a(\vec{p},s)\mathsf{P}^{-1} = \eta^* a(-\vec{p},s) \quad \text{and} \quad \mathsf{P}a(\vec{p},s)\mathsf{P}^{-1} = \eta_c^* a(-\vec{p},s). \tag{1.344}$$

So

$$\begin{aligned} \mathsf{P}\psi(x)\mathsf{P}^{-1} &= \int \frac{d^3p}{(2\pi)^{3/2}}\; u(\vec{p},s)\, e^{ip\cdot x}\, \mathsf{P}a(\vec{p},s)\mathsf{P}^{-1} + v(\vec{p},s)\, e^{-ip\cdot x}\, \mathsf{P}a_c^\dagger(\vec{p},s)\mathsf{P}^{-1} \\ &= \int \frac{d^3p}{(2\pi)^{3/2}}\; u(\vec{p},s)\, e^{ip\cdot x}\, \eta^* a(-\vec{p},s) + v(\vec{p},s)\, e^{-ip\cdot x}\, \eta_c a_c^\dagger(-\vec{p},s) \end{aligned}$$

But $\eta_c = -\eta^*$, so

$$\mathsf{P}\psi(x)\mathsf{P}^{-1} = \eta^* \int \frac{d^3p}{(2\pi)^{3/2}} \; u(\vec{p},s)\, e^{ip\cdot x}\, a(-\vec{p},s) - v(\vec{p},s)\, e^{-ip\cdot x}\, a_c^\dagger(-\vec{p},s).$$

Now

$$\mathsf{P}\psi(x)\mathsf{P}^{-1} = \eta^*\beta\psi(Px),$$

that is,

$$\begin{aligned}\mathsf{P}\psi(x)\mathsf{P}^{-1} &= \eta^*\beta \int \frac{d^3p}{(2\pi)^{3/2}} \; u(\vec{p},s)\, e^{iPp\cdot x}\, a(\vec{p},s) + v(\vec{p},s)\, e^{-iPp\cdot x}\, a_c^\dagger(\vec{p},s) \\ &= \eta^*\beta \int \frac{d^3p}{(2\pi)^{3/2}} u(-\vec{p},s) e^{ip\cdot x} a(-\vec{p},s) + v(-\vec{p},s) e^{-ip\cdot x} a_c^\dagger(-\vec{p},s)\end{aligned}$$

So we need

$$\beta u(-\vec{p},s) = u(\vec{p},s) \quad \text{and} \quad \beta v(-\vec{p},s) = -v(\vec{p},s). \tag{1.345}$$

These rules are easy to check for $\vec{p} = 0$

$$\beta u(\vec{0},s) = u(\vec{0},s) \quad \text{and} \quad \beta v(\vec{0},s) = -v(\vec{0},s) \tag{1.346}$$

which the spinors (1.333) satisfy.

2

Feynman diagrams

2.1 Time-dependent perturbation theory

Most physics problems are insoluble, and we must approximate the unknown solution by numerical or analytic methods. The most common analytic method is called perturbation theory and is based on an assumption that something is small compared to something simple that we can analyze.

In time-dependent perturbation theory, we write the hamiltonian H as the sum $H = H_0 + V$ of a simple hamiltonian H_0 and a small, complicated part V which we give the time dependence induced by H_0

$$V(t) = e^{iH_0 t/\hbar}\, V\, e^{-iH_0 t/\hbar}. \tag{2.1}$$

We alter the time dependence of states by the same simple exponential (and set $\hbar = 1$)

$$|\psi, t\rangle = e^{iH_0 t} e^{-iHt} |\psi\rangle \tag{2.2}$$

and find as its time derivative

$$\begin{aligned} i\frac{d}{dt}|\psi, t\rangle &= e^{iH_0 t}(H - H_0)e^{-iHt}|\psi\rangle = e^{iH_0 t}\, V\, e^{-iH_0 t}\, e^{iH_0 t} e^{-iHt}|\psi\rangle \\ &= V(t)\, |\psi, t\rangle. \end{aligned} \tag{2.3}$$

We can formally solve this differential equation by time-ordering factors of $V(t)$ in expansion of the exponential

$$\begin{aligned} |\psi, t\rangle &= T\big[e^{-i\int_0^t V(t')\,dt'}\big]|\psi\rangle \\ &= \Big\{1 - i\int_0^t V(t')\,dt' - \frac{1}{2!}\int T\big[V(t')V(t'')\big]\,dt'dt'' + \dots\Big\}|\psi\rangle \end{aligned} \tag{2.4}$$

so that V's of later times occur to the left of V's of earier times, that is, if $t > t'$ then $T[V(t)V(t')] = V(t)V(t')$ and so forth.

2.2 Dyson's expansion of the S matrix

The time-evolution operator in the interaction picture is the time-ordered exponential of the integral over time of the interaction hamiltonian

$$T\Big[e^{-i\int V(t)\,dt}\Big]. \tag{2.5}$$

Time ordering has $V(t_>)$ to the left of $V(t_<)$ if the time $t_>$ is later than the time $V(t_<)$. The interaction hamiltonian is

$$V(t) = \int H(t,x)\,d^3x. \tag{2.6}$$

The density of the interaction hamiltonian is (or is taken to be) a sum of terms

$$H(t,x) = \sum_i g_i H_i(t,x) \tag{2.7}$$

each of which is a monomial in the fields $\psi_\ell(x)$ and their adjoints $\psi_\ell^\dagger(x)$. A generic field is an integral over momentum

$$\psi_\ell(x) = \sum_s \int \frac{d^3p}{(2\pi)^{3/2}}\Big[u_\ell(p,s,n)\,a(p,s,n)\,e^{ip\cdot x} + v_\ell(p,s,n)\,b^\dagger(p,s,n)\,e^{-ip\cdot x}\Big] \tag{2.8}$$

in which n labels the kind of field.

The elements of the S matrix are amplitudes for an initial state $|p_1,s_1,n_1;\ldots;p_k,s_k,n_k\rangle$ to evolve into a final state $|p'_1,s'_1,n'_1;\ldots;p'_{k'},s'_{k'},n'_{k'}\rangle$. The case considered most often is when $k=2$ incoming particles and $k'=2$ or 3 outgoing particles, but in high-energy collisions, k' can be much larger than 2 or 3. An S-matrix amplitude is

$$\begin{aligned} S_{p'_1,s'_1,n'_1;\ldots;p_1,s_1,n_1;\ldots} &= \langle p'_1,s'_1,n'_1;\ldots;p'_{k'},s'_{k'},n'_{k'}|T\Big[e^{-i\int H(x)\,d^4x}\Big]|p_1,s_1,n_1;\ldots;p_k,s_k,n_k\rangle \\ &= \langle 0|a(p_{k'},s_{k'},n_{k'};\ldots;a(p_1,s_1,n_1) \\ &\quad\times \sum_{N=0}^{\infty}\frac{(-i)^n}{N!}\int d^4x_1\ldots d^4x_N T\Big[H(x_1)\ldots H(x_N)\Big] \\ &\quad\times a^\dagger(p_1,s_1,n_1;\ldots;a^\dagger(p_k,s_k,n_k)|0\rangle \end{aligned} \tag{2.9}$$

in which $|0\rangle$ is the vacuum state. It is the mean value in the vacuum state of an infinite polynomial in creation and annihilation operators.

To make sense of it, the first step is to normally order the time-evolution operator by moving all annihilation operators to the right of all creation operators. We assume for now that $H(x)$ itself is already normally ordered.

To do that we use these commutation relations with plus signs for bosons and minus signs for fermions:

$$\begin{aligned} a(p,s,n)\,a^\dagger(p',s',n') &= \pm\, a^\dagger(p',s',n')\,a(p,s,n) + \delta^3(\vec{p}-\vec{p}')\,\delta_{ss'}\,\delta_{nn'} \\ a(p,s,n)\,a(p',s',n') &= \pm\, a(p',s',n')\,a(p,s,n) \\ a^\dagger(p,s,n)\,a^\dagger(p',s',n') &= \pm\, a^\dagger(p',s',n')\,a^\dagger(p,s,n). \end{aligned} \tag{2.10}$$

Since

$$a(p,s,n)\,|0\rangle = 0 \quad \text{and} \quad \langle 0|\,a^\dagger(p,s,n) = 0, \tag{2.11}$$

the S-matrix amplitude is just the left-over pieces proportional to products of delta functions multiplied by u's and v's and Fourier factors all divided by 2π's.

Each such piece arises from the pairing of one annihilation operator with one creation operator. Such pairs arise in six ways:

1. A particle p', s', n' in the final state can pair with a field adjoint $\psi_\ell^\dagger(x)$ in $H(x)$ and yield the factor

$$[a(p',s',n'), \psi_\ell^\dagger(x)]_\mp = u_\ell^*(p',s',n')e^{-ip'\cdot x}(2\pi)^{-3/2}. \tag{2.12}$$

2. An antiparticle p', s', n' in the final state can pair with a field $\psi_\ell(x)$ in $H(x)$ and yield the factor

$$[b(p',s',n'), \psi_\ell(x)]_\mp = v_\ell(p',s',n')e^{-ip'\cdot x}(2\pi)^{-3/2}. \tag{2.13}$$

3. A particle p, s, n in the initial state can pair with a field $\psi_\ell(x)$ in $H(x)$ and yield the factor

$$[\psi_\ell(x), a^\dagger(p,s,n)]_\mp = u_\ell(p,s,n)e^{ip\cdot x}(2\pi)^{-3/2}. \tag{2.14}$$

4. An antiparticle p, s, n in the initial state can pair with a field adjoint $\psi_\ell^\dagger(x)$ in $H(x)$ and yield the factor

$$[\psi_\ell^\dagger(x), b^\dagger(p,s,n)]_\mp = v_\ell^*(p,s,n)e^{ip\cdot x}(2\pi)^{-3/2}. \tag{2.15}$$

5. A particle (or antiparticle) p', s', n' in the final state can pair with a particle (or antiparticle) p, s, n in the initial state and yield

$$[a(p',s',n'), a^\dagger(p,s,n)]_\mp = \delta^3(\vec{p}-\vec{p}')\,\delta_{ss'}\,\delta_{nn'}. \tag{2.16}$$

6. A field $\psi_\ell(x) = \psi_\ell^+(x) + \psi_\ell^-(x)$ in $H(x)$ can pair with a field adjoint in $H'(y)$, $\psi_m^\dagger(y) = \psi_m^{+\dagger}(y) + \psi_m^{-\dagger}(y)$. But for $\psi_\ell^+(x)$ to cross $\psi_m^{+\dagger}(y)$ the time x^0 must be later than y^0, and for $\psi_m^{-\dagger}(y)$ to cross $\psi_\ell^-(x)$, the time y^0 must

be later than x^0. If for instance $H(x) = \psi^\dagger(x)\psi(x)\phi(x)$, then in Dyson's expansion these terms would appear

$$\theta(x^0-y^0)\,\psi^\dagger(x)\psi(x)\phi(x)\,\psi^\dagger(y)\psi(y)\phi(y)+\theta(y^0-x^0)\,\psi^\dagger(y)\psi(y)\phi(y)\,\psi^\dagger(x)\psi(x)\phi(x). \tag{2.17}$$

The resulting pairings are the **propagator**

$$\theta(x^0-y^0)\,[\psi_\ell^+(x),\psi_m^{+\dagger}(y)]_\mp \pm \theta(y^0-x^0)\,[\psi_m^{-\dagger}(y),\psi_\ell^-(x)]_\mp \equiv -i\Delta_{\ell m}(x,y) \tag{2.18}$$

in which the $\pm$ signs will be explained later.

One then integrates over N spacetimes and the implicit momenta. The result is defined by a set of rules and **Feynman diagrams**. In general, the $1/N!$ in Dyson's expansion is cancelled by the $N!$ ways of labelling the x_i's. For example, in the term

$$\frac{1}{2!}\int d^4x_1 d^4x_2 T\big[H(x_1)H(x_2)\big] \tag{2.19}$$

one can have $H(x_1)$ absorb an incoming electron of momentum p and have $H(x_2)$ absorb an incoming electron of momentum p' or the reverse.

But some processes require special combinatorics. So people often write

$$H(x) = \frac{g}{3!}\phi^3(x) \quad \text{or} \quad H(x) = \frac{g}{4!}\phi^4(x) \tag{2.20}$$

to compensate for multiple possible pairings. But these factorials don't always cancel. Fermions introduce minus signs. The surest way to check the signs and factorials in each process until one has gained sufficient experience.

2.3 The Feynman propagator for scalar fields

Adding $\pm i\epsilon$ to the denominator of a pole term of an integral formula for a function $f(x)$ can slightly shift the pole into the upper or lower half plane, causing the pole to contribute if a ghost contour goes around the upper half-plane or the lower half-plane. Such an $i\epsilon$ can impose a boundary condition on a Green's function.

The Feynman propagator $\Delta_F(x)$ is a Green's function for the Klein-Gordon differential operator (Weinberg, 1995, pp. 274–280)

$$(m^2-\Box)\Delta_F(x) = \delta^4(x) \tag{2.21}$$

in which $x = (x^0, \boldsymbol{x})$ and

$$\Box = \triangle - \frac{\partial^2}{\partial t^2} = \triangle - \frac{\partial^2}{\partial (x^0)^2} \tag{2.22}$$

is the four-dimensional version of the laplacian $\triangle \equiv \nabla \cdot \nabla$. Here $\delta^4(x)$ is the four-dimensional Dirac delta function

$$\delta^4(x) = \int \frac{d^4q}{(2\pi)^4} \exp[i(\boldsymbol{q} \cdot \boldsymbol{x} - q^0 x^0)] = \int \frac{d^4q}{(2\pi)^4} e^{iqx} \tag{2.23}$$

in which $qx = \boldsymbol{q} \cdot \boldsymbol{x} - q^0 x^0$ is the Lorentz-invariant inner product of the 4-vectors q and x. There are many Green's functions that satisfy Eq.(2.21). Feynman's propagator $\Delta_F(x)$

$$\Delta_F(x) = \int \frac{d^4q}{(2\pi)^4} \frac{\exp(iqx)}{q^2 + m^2 - i\epsilon} = \int \frac{d^3q}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{dq^0}{2\pi} \frac{e^{i\boldsymbol{q}\cdot\boldsymbol{x} - iq^0x^0}}{q^2 + m^2 - i\epsilon}. \tag{2.24}$$

is the one that satisfies boundary conditions that will become evident when we analyze the effect of its $i\epsilon$. The quantity $E_{\boldsymbol{q}} = \sqrt{\boldsymbol{q}^2 + m^2}$ is the energy of a particle of mass m and momentum $\boldsymbol{q}$ in natural units with the speed of light $c = 1$. Using this abbreviation and setting $\epsilon' = \epsilon/2E_q$, we may write the denominator as

$$q^2 + m^2 - i\epsilon = \boldsymbol{q} \cdot \boldsymbol{q} - \left(q^0\right)^2 + m^2 - i\epsilon = \left(E_q - i\epsilon' - q^0\right)\left(E_q - i\epsilon' + q^0\right) + \epsilon'^2 \tag{2.25}$$

in which ϵ'^2 is negligible. Dropping the prime on ϵ, we do the q^0 integral

$$I(\boldsymbol{q}) = -\int_{-\infty}^{\infty} \frac{dq^0}{2\pi} e^{-iq^0x^0} \frac{1}{[q^0 - (E_{\boldsymbol{q}} - i\epsilon)]\,[q^0 - (-E_{\boldsymbol{q}} + i\epsilon)]}. \tag{2.26}$$

As shown in Fig. 2.1, the integrand

$$e^{-iq^0x^0} \frac{1}{[q^0 - (E_{\boldsymbol{q}} - i\epsilon)]\,[q^0 - (-E_{\boldsymbol{q}} + i\epsilon)]} \tag{2.27}$$

has poles at $E_{\boldsymbol{q}} - i\epsilon$ and at $-E_{\boldsymbol{q}} + i\epsilon$. When $x^0 > 0$, we can add a ghost contour that goes clockwise around the lower half-plane and get

$$I(\boldsymbol{q}) = ie^{-iE_{\boldsymbol{q}}x^0} \frac{1}{2E_{\boldsymbol{q}}} \quad x^0 > 0. \tag{2.28}$$

When $x^0 < 0$, our ghost contour goes counterclockwise around the upper half-plane, and we get

$$I(\boldsymbol{q}) = ie^{iE_{\boldsymbol{q}}x^0} \frac{1}{2E_{\boldsymbol{q}}} \quad x^0 < 0. \tag{2.29}$$

Using the step function $\theta(x) = (x + |x|)/2$, we combine (2.28) and (2.29)

$$-iI(\boldsymbol{q}) = \frac{1}{2E_{\boldsymbol{q}}} \left[\theta(x^0)\, e^{-iE_{\boldsymbol{q}}x^0} + \theta(-x^0)\, e^{iE_{\boldsymbol{q}}x^0}\right]. \tag{2.30}$$

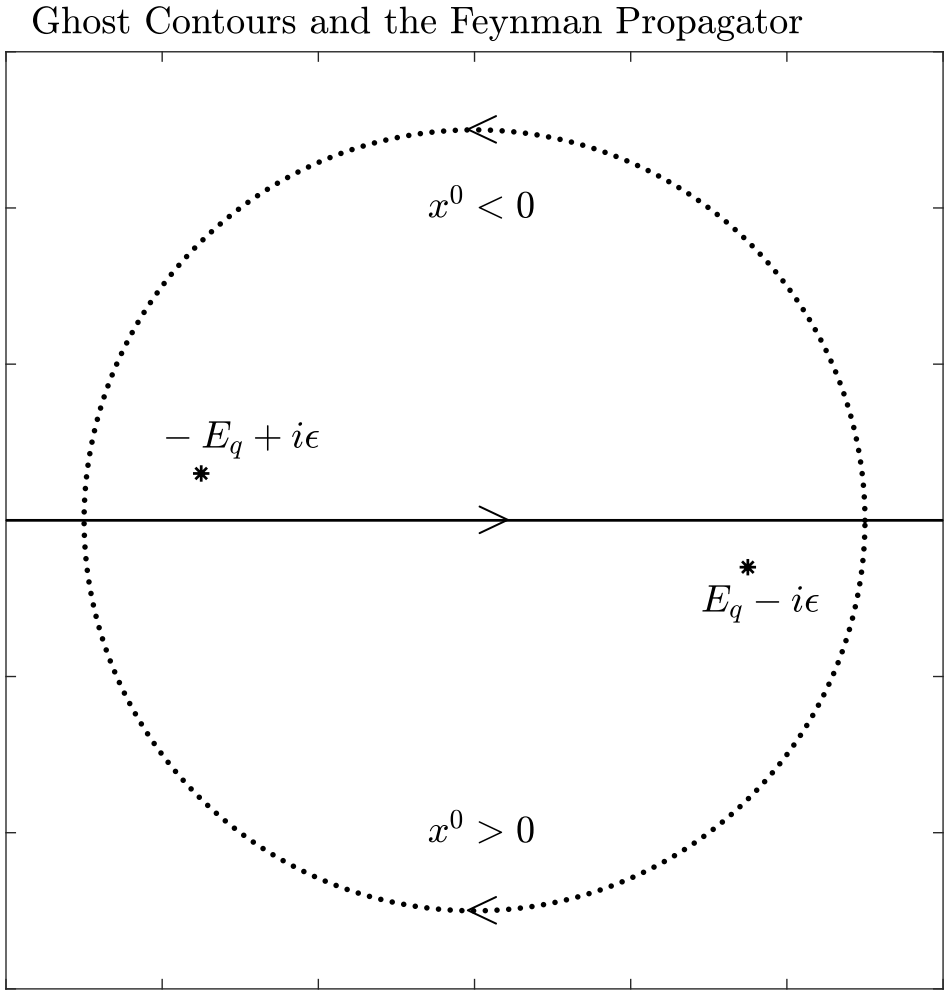

Figure 2.1 In equation (2.27), the function $f(q^0)$ has poles at $\pm(E_{\boldsymbol{q}} - i\epsilon)$, and the function $\exp(-iq^0x^0)$ is exponentially suppressed in the lower half plane if $x^0 > 0$ and in the upper half plane if $x^0 < 0$. So we can add a ghost contour $(\dots)$ in the LHP if $x^0 > 0$ and in the UHP if $x^0 < 0$.

In terms of the Lorentz-invariant function

$$\Delta_+(x) = \frac{1}{(2\pi)^3} \int \frac{d^3q}{2E_{\boldsymbol{q}}} \exp[i(\boldsymbol{q}\cdot\boldsymbol{x} - E_{\boldsymbol{q}}x^0)] \tag{2.31}$$

and with a factor of $-i$, Feynman's propagator (2.24) is

$$-i\Delta_F(x) = \theta(x^0)\,\Delta_+(x) + \theta(-x^0)\,\Delta_+(\boldsymbol{x}, -x^0). \tag{2.32}$$

The integral (2.31) defining $\Delta_+(x)$ is insensitive to the sign of $\boldsymbol{q}$, and so

$$\begin{aligned}\Delta_+(-x) &= \frac{1}{(2\pi)^3}\int \frac{d^3q}{2E_{\boldsymbol{q}}}\,\exp[i(-\boldsymbol{q}\cdot\boldsymbol{x}+E_{\boldsymbol{q}}x^0)] \\ &= \frac{1}{(2\pi)^3}\int \frac{d^3q}{2E_{\boldsymbol{q}}}\,\exp[i(\boldsymbol{q}\cdot\boldsymbol{x}+E_{\boldsymbol{q}}x^0)] = \Delta_+(\boldsymbol{x},-x^0).\end{aligned} \tag{2.33}$$

Thus we arrive at the Standard form of the Feynman propagator

$$-i\Delta_F(x) = \theta(x^0)\,\Delta_+(x) + \theta(-x^0)\,\Delta_+(-x). \tag{2.34}$$

The annihilation operators $a(\boldsymbol{q})$ and the creation operators $a^\dagger(\boldsymbol{p})$ of a scalar field $\phi(x)$ satisfy the commutation relations

$$[a(\boldsymbol{q}),a^\dagger(\boldsymbol{p})] = \delta^3(\boldsymbol{q}-\boldsymbol{p}) \quad\text{and}\quad [a(\boldsymbol{q}),a(\boldsymbol{p})] = [a^\dagger(\boldsymbol{q}),a^\dagger(\boldsymbol{p})] = 0. \tag{2.35}$$

Thus the commutator of the positive-frequency part

$$\phi^+(x) = \int \frac{d^3p}{\sqrt{(2\pi)^3 2p^0}}\,\exp[i(\boldsymbol{p}\cdot\boldsymbol{x}-p^0x^0)]\,a(\boldsymbol{p}) \tag{2.36}$$

of a scalar field $\phi = \phi^+ + \phi^-$ with its negative-frequency part

$$\phi^-(y) = \int \frac{d^3q}{\sqrt{(2\pi)^3 2q^0}}\,\exp[-i(\boldsymbol{q}\cdot\boldsymbol{y}-q^0y^0)]\,a^\dagger(\boldsymbol{q}) \tag{2.37}$$

is the Lorentz-invariant function $\Delta_+(x-y)$

$$\begin{aligned}[\phi^+(x),\phi^-(y)] &= \int \frac{d^3p\,d^3q}{(2\pi)^3 2\sqrt{q^0p^0}}\,e^{ipx-iqy}\,[a(\boldsymbol{p}),a^\dagger(\boldsymbol{q})] \\ &= \int \frac{d^3p}{(2\pi)^3 2p^0}\,e^{ip(x-y)} = \Delta_+(x-y)\end{aligned} \tag{2.38}$$

in which $p(x-y) = \boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y}) - p^0(x^0-y^0)$.

At points x that are space-like, that is, for which $x^2 = \boldsymbol{x}^2-(x^0)^2 \equiv r^2 > 0$, the Lorentz-invariant function $\Delta_+(x)$ depends only upon $r = +\sqrt{x^2}$ and has the value (Weinberg, 1995, p. 202)

$$\Delta_+(x) = \frac{m}{4\pi^2 r}\,K_1(mr) \tag{2.39}$$

in which the Hankel function K_1 is

$$K_1(z) = -\frac{\pi}{2}\,[J_1(iz)+iN_1(iz)] = \frac{1}{z}+\frac{z}{2}\left[\ln\left(\frac{z}{2}\right)+\gamma-\frac{1}{2}\right]+\ldots \tag{2.40}$$

where J_1 is the first Bessel function, N_1 is the first Neumann function, and $\gamma = 0.57721\ldots$ is the Euler-Mascheroni constant.

The Feynman propagator arises most simply as the mean value in the vacuum of the **time-ordered product** of the fields $\phi(x)$ and $\phi(y)$

$$\mathcal{T}\{\phi(x)\phi(y)\} \equiv \theta(x^0 - y^0)\phi(x)\phi(y) + \theta(y^0 - x^0)\phi(y)\phi(x). \tag{2.41}$$

The operators $a(\boldsymbol{p})$ and $a^\dagger(\boldsymbol{p})$ respectively annihilate the vacuum ket $a(\boldsymbol{p})|0\rangle = 0$ and bra $\langle 0|a^\dagger(\boldsymbol{p}) = 0$, and so by (2.36 & 2.37) do the positive- and negative-frequency parts of the field $\phi^+(z)|0\rangle = 0$ and $\langle 0|\phi^-(z) = 0$. Thus the mean value in the vacuum of the time-ordered product is

$$\begin{aligned}\langle 0|\mathcal{T}\{\phi(x)\phi(y)\}|0\rangle &= \langle 0|\theta(x^0 - y^0)\phi(x)\phi(y) + \theta(y^0 - x^0)\phi(y)\phi(x)|0\rangle \\ &= \langle 0|\theta(x^0 - y^0)\phi^+(x)\phi^-(y) + \theta(y^0 - x^0)\phi^+(y)\phi^-(x)|0\rangle \\ &= \langle 0|\theta(x^0 - y^0)[\phi^+(x), \phi^-(y)] \\ &\qquad + \theta(y^0 - x^0)[\phi^+(y), \phi^-(x)]|0\rangle. \end{aligned} \tag{2.42}$$

But by (2.38), these commutators are $\Delta_+(x-y)$ and $\Delta_+(y-x)$. Thus the mean value in the vacuum of the time-ordered product of two real scalar fields

$$\begin{aligned}\langle 0|\mathcal{T}\{\phi(x)\phi(y)\}|0\rangle &= \theta(x^0 - y^0)\Delta_+(x-y) + \theta(y^0 - x^0)\Delta_+(y-x) \\ &= -i\Delta_F(x-y) = \int \frac{d^4q}{(2\pi)^4}\, e^{iq\cdot x}\, \frac{-i}{q^2 + m^2 - i\epsilon} \end{aligned} \tag{2.43}$$

is the Feynman propagator (2.32) multiplied by $-i$.

2.4 Application to a cubic scalar field theory

The action density

$$\mathcal{L} = -\frac{1}{2}\partial_\nu\phi\,\partial^\nu\phi - \frac{1}{2}m^2\phi^2 - \frac{g}{3!}\phi^3, \tag{2.44}$$

describes a scalar field with cubic interactions. We let

$$V(t) = \frac{g}{3!}\int \phi(x)^3\, d^3x \tag{2.45}$$

in which the field ϕ has the free-field time dependence

$$\phi(x) = \int \frac{d^3p}{\sqrt{(2\pi)^3 2p^0}}\Big[a(p)e^{ip\cdot x} + a^\dagger(p)e^{-ip\cdot x}\Big] \tag{2.46}$$

in which $p\cdot x = \vec{p}\cdot\boldsymbol{x} - p^0 t$ and $p^0 = \sqrt{\boldsymbol{p}^2 + m^2}$. The amplitude for the scattering of bosons with momenta p and k to momenta p' and k' is to

lowest order in the coupling constant g

$$\begin{aligned} A &= \langle p', k'| - \frac{1}{2!} \int T\big[V(t')V(t'')\big]\, dt'\, dt''\, |p, k\rangle \\ &= -\frac{g^2}{2!(3!)^2} \langle p', k'| \int T\big[\phi^3(x)\phi^3(y)\big]\, d^4x d^4y\, |p, k\rangle. \end{aligned} \tag{2.47}$$

This process happens in three ways, so $A = A_1 + A_2 + A_3$.

The first way is for the incoming particles to be absorbed at the same vertex x or y and for the outgoing particles to be emitted at the other vertex y or x. We cancel the 2! by choosing the initial particles to be absorbed at y and the final particles to be emitted at x. Then using the expansion (2.46) of the field, we find

$$\begin{aligned} A_1 = &-\frac{g^2}{4} \langle 0|a(k')a(p') \int \frac{d^3p'' d^3k'' d^3p''' d^3k'''}{(2\pi)^6 \sqrt{2p''^0 2k''^0 2p'''^0 2k'''^0}}\, a^\dagger(p''')a^\dagger(k''') \\ &\times e^{-i(p'''+k''')\cdot x} T\big[\phi(x)\phi(y)\big] a(p'')a(k'') e^{i(p''+k'')\cdot y} d^4x d^4y\, a^\dagger(p)a^\dagger(k)|0\rangle \end{aligned} \tag{2.48}$$

after canceling two factors of 3 because each of the 3 fields $\phi^3(x)$ and each of the 3 fields $\phi^3(y)$ could be the one to remain in the time-ordered product $T[\phi(x)\phi(y)]$. The commutation relations $[a(p), a^\dagger(k)] = \delta^3(\boldsymbol{p} - \boldsymbol{k})$ now give

$$A_1 = -g^2 \int \frac{d^4x d^4y}{(2\pi)^6 \sqrt{2p'^0 2k'^0 2p^0 2k^0}} e^{i(p+k)\cdot y - i(p'+k')\cdot x} \langle 0|T\big[\phi(x)\phi(y)\big]|0\rangle \tag{2.49}$$

in which the mean value in the vacuum of the time-ordered product

$$\langle 0|T\big[\phi(x)\phi(y)\big]|0\rangle = -i\Delta_F(x-y) = -i \int \frac{d^4q}{(2\pi)^4} \frac{\exp(iqx)}{q^2 + m^2 - i\epsilon} \tag{2.50}$$

is Feynman's propagator (2.24 & 2.43). Thus the amplitude A_1 is

$$A_1 = ig^2 \int \frac{d^4x d^4y d^4q}{(2\pi)^{10} \sqrt{2p'^0 2k'^0 2p^0 2k^0}} \frac{e^{i(p+k)\cdot y - i(p'+k')\cdot x + iq\cdot(x-y)}}{q^2 + m^2 - i\epsilon}. \tag{2.51}$$

Thanks to Dirac's delta function, the integrals over x, y, and q are easy, and in the smooth $\epsilon \to 0$ limit A_1 is

$$\begin{aligned} A_1 &= ig^2 \int \frac{d^4q}{(2\pi)^2 \sqrt{2p'^0 2k'^0 2p^0 2k^0}} \frac{\delta^4(p+k-q)\delta^4(q-p'-k')}{q^2 + m^2 - i\epsilon} \\ &= \frac{\delta^4(p+k-p'-k')}{(2\pi)^2 \sqrt{2p'^0 2k'^0 2p^0 2k^0}} \frac{ig^2}{(p+k)^2 + m^2}. \end{aligned} \tag{2.52}$$

The sum of the three amplitudes is

$$A = \frac{\delta^4(p+k-p'-k')}{16\pi^2\sqrt{p'^0k'^0p^0k^0}}\Big[\frac{ig^2}{(p+k)^2+m^2} + \frac{ig^2}{(p-k')^2+m^2} + \frac{ig^2}{(p-p')^2+m^2}\Big] \tag{2.53}$$

in which the delta function conserves energy and momentum.

2.5 Feynman's propagator for fields with spin

The time-ordered product for fields with spin is defined so as to compensate for the minus sign that arises when a Fermi field is moved past a Fermi field. Thus the mean value in the vacuum of the time-ordered product $\mathcal{T}\{\psi_\ell(x)\psi_m^\dagger(y)\}$ is

$$\begin{aligned}\langle 0|\mathcal{T}\Big\{\psi_\ell(x)\psi_m^\dagger(y)\Big\}|0\rangle &= \langle 0|\theta(x^0-y^0)\psi_\ell(x)\psi_m^\dagger(y) \pm \theta(y^0-x^0)\psi_m^\dagger(y)\psi_\ell(x)|0\rangle \\ &= \langle 0|\theta(x^0-y^0)\psi_\ell^+(x)\psi_m^{\dagger-}(y) \pm \theta(y^0-x^0)\psi_m^{\dagger+}(y)\psi_\ell^-(x)|0\rangle \\ &= \langle 0|\theta(x^0-y^0)[\psi_\ell^+(x),\psi_m^{\dagger-}(y)]_\mp \\ &\qquad \pm \theta(y^0-x^0)[\psi_m^{\dagger+}(y),\psi_\ell^-(x)]|0\rangle_\mp .\end{aligned} \tag{2.54}$$

in which the upper signs are used for bosons and the lower ones for fermions.

The expansions

$$\begin{aligned}\psi_\ell^+(x) &= (2\pi)^{-3/2}\sum_s\int d^3p\ \ u_\ell(\vec{p},s)\,e^{ip\cdot x}\,a(\vec{p},s) \\ \psi_\ell^-(x) &= (2\pi)^{-3/2}\sum_s\int d^3p\ \ v_\ell(\vec{p},s)\,e^{-ip\cdot x}\,b^\dagger(\vec{p},s)\end{aligned} \tag{2.55}$$

give for the (anti)commutators

$$\begin{aligned}[\psi_\ell^+(x),\psi_m^{\dagger-}(y)]_\mp &= \Big[(2\pi)^{-3/2}\sum_s\int d^3p\ \ u_\ell(\vec{p},s)\,e^{ip\cdot x}\,a(\vec{p},s), \\ &\qquad (2\pi)^{-3/2}\sum_r\int d^3k\ \ u_m^*(\vec{k},r)e^{-ik\cdot y}\,a^\dagger(\vec{k},r)\Big]_\mp \\ &= \sum_s\int\frac{d^3p}{(2\pi)^3}\,u_\ell(\vec{p},s)\,u_m^*(\vec{p},s)\,e^{ip\cdot(x-y)}\end{aligned} \tag{2.56}$$

and

$$
\begin{aligned}
[\psi_m^{\dagger +}(y), \psi_\ell^-(x)]_\mp = \Big[& (2\pi)^{-3/2} \sum_s \int d^3p \;\; v_m^*(\vec{p}, s)\, e^{ip\cdot y}\, b(\vec{p}, s), \\
& (2\pi)^{-3/2} \sum_r \int d^3k \;\; v_\ell(\vec{k}, r) e^{-ik\cdot x}\, b^\dagger(\vec{k}, r) \Big]_\mp \\
= & \sum_s \int \frac{d^3p}{(2\pi)^3}\, v_\ell(\vec{p}, s)\, v_m^*(\vec{p}, s)\, e^{ip\cdot(y-x)}.
\end{aligned}
\tag{2.57}
$$

Putting these expansions into the formula (2.54) for the time-ordered product, we get for its mean value in the vacuum

$$
\begin{aligned}
\langle 0|\mathcal{T}\Big\{\psi_\ell(x)\psi_m^\dagger(y)\Big\}|0\rangle = & \;\theta(x^0 - y^0) \sum_s \int \frac{d^3p}{(2\pi)^3}\, u_\ell(\vec{p}, s)\, u_m^*(\vec{p}, s)\, e^{ip\cdot(x-y)} \\
& \pm \theta(y^0 - x^0) \sum_s \int \frac{d^3p}{(2\pi)^3}\, v_\ell(\vec{p}, s)\, v_m^*(\vec{p}, s)\, e^{ip\cdot(y-x)}.
\end{aligned}
\tag{2.58}
$$

2.6 Feynman's propagator for spin-one-half fields

For fields of spin one half, the spin sums are

$$
\begin{aligned}
\big[N(\vec{p})\big]_{\ell m} &= \sum_s u_\ell(\vec{p}, s) u_m^*(\vec{p}, s) = \Big[\frac{1}{2p^0}\,(-i\,p^c\gamma_c + m)\;\beta\Big]_{\ell m} \\
\big[M(\vec{p})\big]_{\ell m} &= \sum_s v_\ell(\vec{p}, s) v_m^*(\vec{p}, s) = \Big[\frac{1}{2p^0}\,(-i\,p^c\gamma_c - m)\;\beta\Big]_{\ell m},
\end{aligned}
\tag{2.59}
$$

so for spin-one-half fields Feynman's propagator is

$$
\begin{aligned}
\langle 0|\mathcal{T}\Big\{\psi_\ell(x)\psi_m^\dagger(y)\Big\}|0\rangle = & \;\theta(x^0 - y^0) \int \frac{d^3p}{(2\pi)^3} \Big[\frac{1}{2p^0}\,(-i\,p_c\gamma^c + m)\;\beta\Big]_{\ell m} e^{ip\cdot(x-y)} \\
& - \theta(y^0 - x^0) \int \frac{d^3p}{(2\pi)^3} \Big[\frac{1}{2p^0}\,(-i\,p_c\gamma^c - m)\;\beta\Big]_{\ell m} e^{ip\cdot(y-x)}.
\end{aligned}
\tag{2.60}
$$

Using the derivative term $-\partial_c\gamma^c$ to generate $-ip_c\gamma^c$, we get

$$\begin{aligned}\langle 0|\mathcal{T}\Big\{\psi_\ell(x)\psi_m^\dagger(y)\Big\}|0\rangle &= \theta(x^0-y^0)\Big[\,(-\partial_c\gamma^c+m)\;\beta\Big]_{\ell m}\int\frac{d^3p}{(2\pi)^3 2p^0}\;e^{ip\cdot(x-y)}\\ &\quad-\theta(y^0-x^0)\Big[\,(\partial_c\gamma^c-m)\;\beta\Big]_{\ell m}\int\frac{d^3p}{(2\pi)^3 2p^0}\;e^{ip\cdot(y-x)}\\ &= \theta(x^0-y^0)\Big[\,(-\partial_c\gamma^c+m)\;\beta\Big]_{\ell m}\Delta_+(x-y)\\ &\quad+\theta(y^0-x^0)\Big[\,(-\partial_c\gamma^c+m)\,\beta\Big]_{\ell m}\Delta_+(y-x).\end{aligned}\tag{2.61}$$

Since the derivative of the step function is a delta function

$$\partial_0\theta(x^0-y^0)=\delta(x^0-y^0),\tag{2.62}$$

we can write Feynman's propagator as

$$\begin{aligned}\langle 0|\mathcal{T}\Big\{\psi_\ell(x)\psi_m^\dagger(y)\Big\}|0\rangle &= \Big[\,(-\partial_c\gamma^c+m)\,\beta\Big]_{\ell m}\\ &\quad\times\Big(\theta(x^0-y^0)\,\Delta_+(x-y)+\theta(y^0-x^0)\,\Delta_+(y-x)\Big)\\ &\quad+\partial_0\gamma^0\beta\theta(x^0-y^0)\,\Delta_+(x-y)\\ &\quad-\partial_0\gamma^0\beta\theta(y^0-x^0)\,\Delta_+(y-x)\end{aligned}\tag{2.63}$$

and so also as

$$\begin{aligned}\langle 0|\mathcal{T}\Big\{\psi_\ell(x)\psi_m^\dagger(y)\Big\}|0\rangle &= \Big[\,(-\partial_c\gamma^c+m)\,\beta\Big]_{\ell m}\\ &\quad\times\Big(\theta(x^0-y^0)\,\Delta_+(x-y)+\theta(y^0-x^0)\,\Delta_+(y-x)\Big)\\ &\quad+\gamma^0\beta\delta(x^0-y^0)\,\Delta_+(x-y)-\gamma^0\beta\delta(y^0-x^0)\,\Delta_+(y-x)\\ &= \Big[\,(-\partial_c\gamma^c+m)\;\beta\Big]_{\ell m}\\ &\quad\times\Big(\theta(x^0-y^0)\,\Delta_+(x-y)-\theta(y^0-x^0)\,\Delta_+(y-x)\Big)\\ &\quad+\gamma^0\beta\delta(x^0-y^0)\big(\Delta_+(x-y)-\Delta_+(y-x)\big).\end{aligned}\tag{2.64}$$

But at equal times $\Delta_+(x-y)=\Delta_+(y-x)$. So the ugly final term vanishes, and Feynman's propagator for spin-one-half fields is

$$\begin{aligned}\langle 0|\mathcal{T}\Big\{\psi_\ell(x)\psi_m^\dagger(y)\Big\}|0\rangle &= \Big[\,(-\partial_c\gamma^c+m)\;\beta\Big]_{\ell m}\\ &\quad\times\Big(\theta(x^0-y^0)\Delta_+(x-y)+\theta(y^0-x^0)\Delta_+(y-x)\Big).\end{aligned}\tag{2.65}$$

But this last piece is Feynman's propagator for real scalar fields

$$\begin{aligned}\langle 0|\mathcal{T}\{\phi(x)\phi(y)\}|0\rangle &= \theta(x^0-y^0)\Delta_+(x-y)+\theta(y^0-x^0)\Delta_+(y-x)\\ &= -i\Delta_F(x-y) = \int \frac{d^4q}{(2\pi)^4}\,\frac{-i}{q^2+m^2-i\epsilon}\,e^{iq\cdot(x-y)}.\end{aligned}$$

so Feynman's propagator for spin-one-half fields is

$$\begin{aligned}\langle 0|\mathcal{T}\Big\{\psi_\ell(x)\psi^\dagger_m(y)\Big\}|0\rangle &\equiv -i\Delta_{\ell m}(x-y)\\ &= \Big[\,(-\partial_c\gamma^c+m)\;\beta\Big]_{\ell m}\int \frac{d^4q}{(2\pi)^4}\,\frac{-i}{q^2+m^2-i\epsilon}\,e^{iq\cdot(x-y)}.\end{aligned} \tag{2.66}$$

Letting the derivatives act on the exponential, we get

$$\begin{aligned}\langle 0|\mathcal{T}\Big\{\psi_\ell(x)\psi^\dagger_m(y)\Big\}|0\rangle &\equiv -i\Delta_{\ell m}(x-y)\\ &= -i\int \frac{d^4q}{(2\pi)^4}\,\frac{\big[(-iq_a\gamma^a+m)\beta\big]_{\ell m}}{q^2+m^2-i\epsilon}\,e^{iq\cdot(x-y)}.\end{aligned} \tag{2.67}$$

Since

$$(-iq_a\gamma^a+m)(iq_a\gamma^a+m) = q^2+m^2, \tag{2.68}$$

people often write this as

$$\begin{aligned}\langle 0|\mathcal{T}\Big\{\psi_\ell(x)\psi^\dagger_m(y)\Big\}|0\rangle &\equiv -i\Delta_{\ell m}(x-y)\\ &= -i\int \frac{d^4q}{(2\pi)^4}\,\frac{\big[(-iq_a\gamma^a+m)\beta\big]_{\ell m}}{q^2+m^2-i\epsilon}\,e^{iq\cdot(x-y)}\\ &= -i\int \frac{d^4q}{(2\pi)^4}\,\Big[\frac{1}{iq_a\gamma^a+m-i\epsilon}\beta\Big]_{\ell m}e^{iq\cdot(x-y)}.\end{aligned} \tag{2.69}$$

Feynman was a master of notation (and of everything else). He set $\not{p} = p_a\gamma^a$ and wrote

$$\begin{aligned}\langle 0|\mathcal{T}\Big\{\psi_\ell(x)\psi^\dagger_m(y)\Big\}|0\rangle &\equiv -i\Delta_{\ell m}(x-y)\\ &= -i\int \frac{d^4q}{(2\pi)^4}\,\frac{\big[(-i\not{q}+m)\beta\big]_{\ell m}}{q^2+m^2-i\epsilon}\,e^{iq\cdot(x-y)}\\ &= -i\int \frac{d^4q}{(2\pi)^4}\,\Big(\frac{1}{i\not{q}+m-i\epsilon}\beta\Big)_{\ell m}e^{iq\cdot(x-y)}.\end{aligned} \tag{2.70}$$

In the common notation $\overline{\psi} = \psi^\dagger \beta$, his propagator is

$$\begin{aligned}\langle 0|\mathcal{T}\{\psi_\ell(x)\overline{\psi}_m(y)\}|0\rangle &\equiv -i\Delta_{\ell m}(x-y)\,\beta \\ &= -i\int \frac{d^4q}{(2\pi)^4}\,\frac{\left[(-i\not{q}+m)\right]_{\ell m}}{q^2+m^2-i\epsilon}\,e^{iq\cdot(x-y)} \\ &= -i\int \frac{d^4q}{(2\pi)^4}\left(\frac{1}{i\not{q}+m-i\epsilon}\right)_{\ell m} e^{iq\cdot(x-y)}.\end{aligned} \tag{2.71}$$

2.7 Application to a theory of fermions and bosons

Let us consider first the theory

$$H(x) = g_{\ell m}\,\psi_\ell^\dagger(x)\,\psi_m(x)\,\phi(x) \tag{2.72}$$

where $g_{\ell m}$ is a coupling constant, $\psi(x)$ is the field of a fermion f, and $\phi(x) = \phi^\dagger(x)$ is a real boson b. Let's compute the amplitude A for $f + b \to f' + b'$. The lowest-order term is

$$\begin{aligned}A &= -\frac{1}{2!}\int d^4x d^4y\,\langle 0|a(p',s')b(k')\,T\Big[H(x)\,H(y)\Big]\,b^\dagger(k)a^\dagger(p,s)|0\rangle \\ &= -\frac{g_{\ell m}g_{\ell' m'}}{2!}\int d^4x d^4y \langle 0|a(p',s')b(k')T\Big[\psi_\ell^\dagger(x)\,\psi_m(x)\phi(x)\psi_{\ell'}^\dagger(y)\,\psi_{m'}(y)\phi(y)\Big] \\ &\quad \times b^\dagger(k)a^\dagger(p,s)|0\rangle.\end{aligned} \tag{2.73}$$

Here the operators $b(k')$ and $b^\dagger(k)$ are the boson deletion and addition operators. Either the boson field $\phi(y)$ deletes the boson from the initial state and the boson field $\phi(x)$ deletes the boson from the final state or the boson field $\phi(x)$ deletes the boson from the initial state and the boson field $\phi(y)$ deletes the boson from the final state. These give the same result. So we cancel the 2! and let $\phi(y)$ delete the boson from the initial state and have

$\phi(x)$ add the boson of the final state. We then get

$$
\begin{aligned}
A = & -g_{\ell m} g_{\ell' m'} \int d^4x d^4y \langle 0|a(p',s')b(k')T\Big[\psi^\dagger_\ell(x)\,\psi_m(x)\,\phi^-(x)\psi^\dagger_{\ell'}(y)\,\psi_{m'}(y)\,\phi^+(y)\Big] \\
& \times b^\dagger(k)a^\dagger(p,s)|0\rangle \\
= & -g_{\ell m} g_{\ell' m'} \int d^4x d^4y\, \langle 0|a(p',s')b(k')\,T\Big[\psi^\dagger_\ell(x)\,\psi_m(x) \\
& \times \int \frac{d^3k''}{\sqrt{(2\pi)^3 2k''^0}}\, b^\dagger(k'')e^{-ik''\cdot x}\psi^\dagger_{\ell'}(y)\,\psi_{m'}(y)\,\phi^+(y)\Big]\, b^\dagger(k)a^\dagger(p,s)|0\rangle \\
= & -g_{\ell m} g_{\ell' m'} \int d^4x d^4y\, \langle 0|a(p',s')\,T\Big[\psi^\dagger_\ell(x)\,\psi_m(x) \\
& \times \int \frac{d^3k''}{\sqrt{(2\pi)^3 2k''^0}}\, \delta^3(\vec{k}' - \vec{k}'')\, e^{-ik''\cdot x}\psi^\dagger_{\ell'}(y)\,\psi_{m'}(y)\,\phi^+(y)\Big]\, b^\dagger(k)a^\dagger(p,s)|0\rangle \\
= & -g_{\ell m} g_{\ell' m'} \int d^4x d^4y\, \langle 0|a(p',s')\,T\Big[\psi^\dagger_\ell(x)\,\psi_m(x)\,\frac{1}{\sqrt{(2\pi)^3 2k'^0}}\, e^{-ik'\cdot x} \\
& \times \psi^\dagger_{\ell'}(y)\,\psi_{m'}(y)\,\phi^+(y)\Big]\, b^\dagger(k)a^\dagger(p,s)|0\rangle.
\end{aligned}
\tag{2.74}
$$

We now let $\phi^+(y)$ delete the boson from the initial state.

$$
\begin{aligned}
A = & -g^2 \int d^4x d^4y\, \langle 0|a(p',s')\,T\Big[\psi^\dagger_\ell(x)\,\psi_m(x)\,\frac{1}{\sqrt{(2\pi)^3 2k'^0}}\, e^{-ik'\cdot x} \\
& \times \psi^\dagger_{\ell'}(y)\,\psi_{m'}(y)\int \frac{d^3k''}{\sqrt{(2\pi)^3 2k''^0}}\, b(k'')e^{ik''\cdot y}\Big]\, b^\dagger(k)a^\dagger(p,s)|0\rangle \\
= & -g_{\ell m} g_{\ell' m'} \int d^4x d^4y\, \langle 0|a(p',s')\,T\Big[\psi^\dagger_\ell(x)\,\psi_m(x)\,\frac{1}{\sqrt{(2\pi)^3 2k'^0}}\, e^{-ik'\cdot y} \\
& \times \psi^\dagger_{\ell'}(y)\,\psi_{m'}(y)\int \frac{d^3k''}{\sqrt{(2\pi)^3 2k''^0}}\, \delta^3(k''-k)e^{ik''\cdot y}\Big]\, a^\dagger(p,s)|0\rangle \\
= & -g_{\ell m} g_{\ell' m'} \int d^4x d^4y\, \langle 0|a(p',s')\,T\Big[\psi^\dagger_\ell(x)\,\psi_m(x)\,\frac{1}{\sqrt{(2\pi)^3 2k'^0}}\, e^{-ik'\cdot x} \\
& \times \psi^\dagger_{\ell'}(y)\,\psi_{m'}(y)\,\frac{1}{\sqrt{(2\pi)^3 2k^0}}\, e^{ik\cdot y}\Big]\, a^\dagger(p,s)|0\rangle \\
= & -g_{\ell m} g_{\ell' m'} \int \frac{d^4x d^4y}{\sqrt{(2\pi)^6 2k'^0 2k^0}} e^{ik\cdot y - ik'\cdot x}\langle 0|a(p',s')T\Big[\psi^\dagger_\ell(x)\psi_m(x)\psi^\dagger_{\ell'}(y)\psi_{m'}(y)\Big]a^\dagger(p,s)|0\rangle.
\end{aligned}
\tag{2.75}
$$

Now there are two terms. In one the initial fermion is deleted at y and the final fermion is added at x

$$
\begin{aligned}
A_1 = & -g_{\ell m} g_{\ell' m'} \int \frac{d^4x d^4y}{\sqrt{(2\pi)^6 2k'^0 2k^0}} e^{ik\cdot y - ik'\cdot x} \langle 0|a(p',s') T\Big[\psi_\ell^{+\dagger}(x)\psi_m(x)\psi_{\ell'}^\dagger(y)\psi_{m'}^+(y)\Big] a^\dagger(p,s)|0\rangle \\
= & -g_{\ell m} g_{\ell' m'} \int \frac{d^4x d^4y}{\sqrt{(2\pi)^6 2k'^0 2k^0}} e^{ik\cdot y - ik'\cdot x} \langle 0|a(p',s') T\Big[\int \frac{d^3p''}{\sqrt{(2\pi)^3}} e^{-ip''\cdot x} a^\dagger(p'',s'') u_\ell^*(p'',s'') \\
& \times \psi_m(x)\psi_{\ell'}^\dagger(y) \int \frac{d^3p'''}{\sqrt{(2\pi)^3}} e^{ip'''\cdot y} a(p''',s''')\, u_{m'}(p''',s''')\Big] a^\dagger(p,s)|0\rangle
\end{aligned}
\tag{2.76}
$$

$$
\begin{aligned}
= & -g_{\ell m} g_{\ell' m'} \int \frac{d^4x d^4y}{\sqrt{(2\pi)^6 2k'^0 2k^0}} e^{ik\cdot y - ik'\cdot x} \langle 0|T\Big[\int \frac{d^3p''}{\sqrt{(2\pi)^3}} e^{-ip''\cdot x} u_\ell^*(p'',s'') \delta^3(\vec{p}' - \vec{p}'')\delta_{s's''} \\
& \times \psi_m(x)\psi_{\ell'}^\dagger(y) \int \frac{d^3p'''}{\sqrt{(2\pi)^3}} e^{ip'''\cdot y} u_{m'}(p''',s''')\delta^3(\vec{p} - \vec{p}''')\delta_{ss'''}\Big]|0\rangle \\
= & -g_{\ell m} g_{\ell' m'} u_\ell^*(p',s') u_{m'}(p,s) \int \frac{d^4x d^4y}{\sqrt{(2\pi)^{12} 2k'^0 2k^0}} e^{ik\cdot y - ik'\cdot x} e^{-ip'\cdot x} e^{ip\cdot y} \langle 0|T\Big[\psi_m(x)\psi_{\ell'}^\dagger(y)\Big]|0\rangle \\
= & -g_{\ell m} g_{\ell' m'} u_\ell^*(p',s') u_{m'}(p,s) \int \frac{d^4x d^4y}{\sqrt{(2\pi)^{12} 2k'^0 2k^0}} e^{i(k+p)\cdot y - i(k'+p')\cdot x} \langle 0|\, T\Big[\,\psi_m(x)\psi_{\ell'}^\dagger(y)\Big]|0\rangle.
\end{aligned}
$$

In terms of SW's definition (6.2.31) of the fermion propagator, A_1 is

$$
A_1 = -g_{\ell m} g_{\ell' m'} u_\ell^*(p',s') u_{m'}(p,s) \int \frac{d^4x d^4y}{\sqrt{(2\pi)^{12} 2k'^0 2k^0}} e^{i(k+p)\cdot y - i(k'+p')\cdot x} \big(-i\Delta_{m\ell'}(x,y)\big).
\tag{2.77}
$$

In the other term, the initial fermion is deleted at x and the final fermion is added at y

$$
\begin{aligned}
A_2 = & -g_{\ell m} g_{\ell' m'} \int \frac{d^4x d^4y}{\sqrt{(2\pi)^6 2k'^0 2k^0}} e^{ik\cdot y - ik'\cdot x} \langle 0|a(p',s') T\Big[\psi_\ell^\dagger(x)\psi_m^+(x)\psi_{\ell'}^{+\dagger}(y)\psi_{m'}(y)\Big] a^\dagger(p,s)|0\rangle \\
= & -g_{\ell m} g_{\ell' m'} \int \frac{d^4x d^4y}{\sqrt{(2\pi)^6 2k'^0 2k^0}} e^{ik\cdot y - ik'\cdot x} \langle 0|a(p',s') T\Big[\,\psi_\ell^\dagger(x)\psi_{\ell'}^{+\dagger}(y)\psi_{m'}(y)\psi_m^+(x)\Big] a^\dagger(p,s)|0\rangle
\end{aligned}
\tag{2.78}
$$

$$
= g_{\ell m} g_{\ell' m'} \int \frac{d^4x d^4y}{\sqrt{(2\pi)^6 2k'^0 2k^0}} e^{ik\cdot y - ik'\cdot x} \langle 0|a(p',s') T\Big[\psi_{\ell'}^{+\dagger}(y)\psi_\ell^\dagger(x)\psi_{m'}(y)\psi_m^+(x)\Big] a^\dagger(p,s)|0\rangle
$$

in which the minus sign arises from the transposition $\psi_\ell^\dagger(x)\,\psi^{+\dagger}(y) \to \psi^{+\dagger}(y)\,\psi_\ell^\dagger(x)$. The earlier transposition $\psi^+(x)\,\psi^{+\dagger}(y)\,\psi_{m'}(y) \to \psi^{+\dagger}(y)\,\psi_{m'}(y)\,\psi^+(x)$ produced two minus signs or one plus sign. Inserting the expansions of $\psi^{+\dagger}(y)$

and $\psi^+(x)$, we have

$$A_2 = g_{\ell m} g_{\ell' m'} \int \frac{d^4x d^4y}{\sqrt{(2\pi)^6 2k'^0 2k^0}} e^{ik\cdot y - ik'\cdot x} \langle 0|a(p',s') T\Big[\int \frac{d^3p''}{\sqrt{(2\pi)^3}} u^*_{\ell'}(p'',s'') a^\dagger(p'',s'') e^{-ip''\cdot y}$$
$$\times\ \psi^\dagger_\ell(x)\psi_{m'}(y) \int \frac{d^3p'''}{\sqrt{(2\pi)^3}} u_m(p''',s''') a(p''',s''') e^{ip'''\cdot x}\Big] a^\dagger(p,s)|0\rangle$$
$$= g_{\ell m} g_{\ell' m'} u^*_{\ell'}(p',s')\, u_m(p,s) \int \frac{d^4x d^4y}{\sqrt{(2\pi)^{12} 2k'^0 2k^0}}\, e^{i(k-p')\cdot y - i(k'-p)\cdot x} \langle 0|T\Big[\, \psi^\dagger_\ell(x)\, \psi_{m'}(y)\,\Big]\, |0\rangle. \tag{2.79}$$

In terms of SW's definition (6.2.31) of the fermion propagator, A_2 is

$$A_2 = -\, g_{\ell m} g_{\ell' m'} u^*_{\ell'}(p'',s'') u_m(p,s) \int \frac{d^4x d^4y}{\sqrt{(2\pi)^{12} 2k'^0 2k^0}} e^{i(k-p')\cdot y - i(k'-p)\cdot x} \langle 0|T\Big[\psi_{m'}(y)\psi^\dagger_\ell(x)\Big]|0\rangle$$
$$= -\, g_{\ell m} g_{\ell' m'} u^*_{\ell'}(p',s')\, u_m(p,s) \int \frac{d^4x d^4y}{\sqrt{(2\pi)^{12} 2k'^0 2k^0}}\, e^{i(k-p')\cdot y - i(k'-p)\cdot x} \big(-\, i\Delta_{m'\ell}(y,x)\big). \tag{2.80}$$

The full amplitude for $f + b \to f' + b'$ is the sum $A = A_1 + A_2$ of the two amplitudes

$$A = -\, g_{\ell m} g_{\ell' m'} u^*_\ell(p',s') u_{m'}(p,s) \int \frac{d^4x d^4y}{\sqrt{(2\pi)^{12} 2k'^0 2k^0}} e^{i(k+p)\cdot y - i(k'+p')\cdot x} \big(-\, i\Delta_{m\ell'}(x,y)\big)$$
$$-\, g_{\ell m} g_{\ell' m'} u^*_{\ell'}(p',s')\, u_m(p,s) \int \frac{d^4x d^4y}{\sqrt{(2\pi)^{12} 2k'^0 2k^0}}\, e^{i(k-p')\cdot y - i(k'-p)\cdot x} \big(-\, i\Delta_{m'\ell}(y,x)\big). \tag{2.81}$$

Interchanging x and y, m and m', and ℓ and ℓ' in A_1, we get

$$A = -\, g_{\ell' m'} g_{\ell m} u^*_{\ell'}(p',s') u_m(p,s) \int \frac{d^4x d^4y}{\sqrt{(2\pi)^{12} 2k'^0 2k^0}} e^{i(k+p)\cdot x - i(k'+p')\cdot y} \big(-\, i\Delta_{m'\ell}(y,x)\big)$$
$$-\, g_{\ell m} g_{\ell' m'} u^*_{\ell'}(p',s')\, u_m(p,s) \int \frac{d^4x d^4y}{\sqrt{(2\pi)^{12} 2k'^0 2k^0}}\, e^{i(k-p')\cdot y - i(k'-p)\cdot x} \big(-\, i\Delta_{m'\ell}(y,x)\big). \tag{2.82}$$

Combining terms, we get

$$A = -\, g_{\ell' m'} g_{\ell m} u^*_{\ell'}(p',s') u_m(p,s) \int \frac{d^4x d^4y}{\sqrt{(2\pi)^{12} 2k'^0 2k^0}}\, e^{ip\cdot x - ip'\cdot y} \big(-\, i\Delta_{m'\ell}(y,x)\big)$$
$$\times \Big(e^{ik\cdot x - ik'\cdot y} + e^{ik\cdot y - ik'\cdot x} \Big) \tag{2.83}$$

which agrees with SW's (6.1.27) when his boson is restricted to a single scalar field.

We now replace mean value in the vacuum of the time-ordered product by its value (2.70)

$$\langle 0|\mathcal{T}\Big\{\psi_\ell(x)\psi^\dagger_m(y)\Big\}|0\rangle \equiv -i\Delta_{\ell m}(x-y) = -i\int\frac{d^4q}{(2\pi)^4}\,\frac{\big[(-iq_a\gamma^a+m)\beta\big]_{\ell m}}{q^2+m^2-i\epsilon}\,e^{iq\cdot x}. \tag{2.84}$$

Replacing ℓ, m, x, y by m', ℓ, y, x, we get

$$\langle 0|\mathcal{T}\Big\{\psi_{m'}(y)\psi^\dagger_\ell(x)\Big\}|0\rangle \equiv -i\Delta_{m'\ell}(y-x) = -i\int\frac{d^4q}{(2\pi)^4}\,\frac{\big[(-iq_a\gamma^a+m)\beta\big]_{m'\ell}}{q^2+m^2-i\epsilon}\,e^{iq\cdot(x-y)}. \tag{2.85}$$

We then find

$$\begin{aligned} A = &-g_{\ell' m'}g_{\ell m}u^*_{\ell'}(p',s')u_m(p,s)\int\frac{d^4xd^4y}{(2\pi)^6\sqrt{2k'^0 2k^0}}\,e^{ip\cdot x-ip'\cdot y}\\ &\times\Big(e^{ik\cdot x-ik'\cdot y}+e^{ik\cdot y-ik'\cdot x}\Big)(-i)\int\frac{d^4q}{(2\pi)^4}\,\frac{\big[(-i\not{q}+m)\beta\big]_{m'\ell}}{q^2+m^2-i\epsilon}\,e^{iq\cdot(y-x)}. \end{aligned} \tag{2.86}$$

In matrix notation, this is

$$\begin{aligned} A = &-\int\frac{d^4xd^4y}{(2\pi)^6\sqrt{2k'^0 2k^0}}\Big(e^{i(p+k)\cdot x-i(p'+k')\cdot y}+e^{i(k-p')\cdot y+i(p-k')\cdot x}\Big)\\ &\times(-i)\int\frac{d^4q}{(2\pi)^4}\,\frac{u^\dagger(p',s')\,g\,\big[(-i\not{q}+m)\beta\big]\,g\,u(p,s)}{q^2+m^2-i\epsilon}\,e^{iq\cdot(y-x)}. \end{aligned} \tag{2.87}$$

The d^4x and d^4y integrations give

$$\begin{aligned} A = i&\int\frac{d^4q}{(2\pi)^2\sqrt{2k'^0 2k^0}}\Big(\delta(q-k'-p')\delta(p+k-q)+\delta(q+k-p')\delta(p-k'-q)\Big)\\ &\times\frac{u^\dagger(p',s')\,g\,\big[(-i\not{q}+m)\beta\big]\,g\,u(p,s)}{q^2+m^2-i\epsilon} \end{aligned} \tag{2.88}$$

which is

$$A = \frac{i\delta(p'+k'-p-k)}{8\pi^2\sqrt{k'^0k^0}}\,u^\dagger(p',s')g\left[\frac{-i(\not{p}+\not{k})+m}{(p+k)^2+m^2}+\frac{-i(\not{p}+\not{k}')+m}{(p-k')^2+m^2}\right]\beta g u(p,s). \tag{2.89}$$

2.8 Feynman propagator for spin-one fields

The general form of the propagator is

$$\langle 0|\mathcal{T}\big[\psi_a(x)\psi_b^\dagger(y)\big]|0\rangle = \theta(x^0-y^0)\sum_s\int\frac{d^3p}{(2\pi)^3}\,u_a(\vec{p},s)\,u_b^\dagger(\vec{p},s)\,e^{ip\cdot(x-y)} \\ \pm\,\theta(y^0-x^0)\sum_s\int\frac{d^3p}{(2\pi)^3}\,v_a(\vec{p},s)\,v_b^*(\vec{p},s)\,e^{ip\cdot(y-x)}. \tag{2.90}$$

We use the upper (+) sign for spin-one fields because they are bosons. For single real massive vector field

$$\psi^a(x) = \sum_{s=-1}^{1}\int\frac{d^3p}{\sqrt{(2\pi)^3 2p^0}}\,\Big[e^a(\vec{p},s)a(\vec{p},s)e^{ip\cdot x} + e^{*a}(\vec{p},s)a^\dagger(\vec{p},s)e^{-ip\cdot x}\Big], \tag{2.91}$$

the mean value in the vacuum of its time-ordered product is

$$\langle 0|\mathcal{T}\big[\psi_a(x)\psi_b^\dagger(y)\big]|0\rangle = \theta(x^0-y^0)\sum_s\int\frac{d^3p}{(2\pi)^3 2p^0}\,e_a(\vec{p},s)\,e_b^*(\vec{p},s)\,e^{ip\cdot(x-y)} \\ +\,\theta(y^0-x^0)\sum_s\int\frac{d^3p}{(2\pi)^3 2p^0}\,e_a(\vec{p},s)\,e_b^*(\vec{p},s)\,e^{ip\cdot(y-x)}. \tag{2.92}$$

The spin-one spin sum is

$$\sum_s e_a(\vec{p},s)\,e_b^*(\vec{p},s) = \eta_{ab} + p_a\,p_b/m^2. \tag{2.93}$$

So the mean value (2.58) is

$$\langle 0|\mathcal{T}\big[\psi_a(x)\psi_b^\dagger(y)\big]|0\rangle = \theta(x^0-y^0)\int\frac{d^3p}{(2\pi)^3 2p^0}\,\Big(\eta_{ab}+\frac{p_a\,p_b}{m^2}\Big)\,e^{ip\cdot(x-y)} \\ +\,\theta(y^0-x^0)\int\frac{d^3p}{(2\pi)^3 2p^0}\,\Big(\eta_{ab}+\frac{p_a\,p_b}{m^2}\Big)\,e^{ip\cdot(y-x)} \tag{2.94}$$

in which the momentum p_a is physical (aka, on the mass shell) in that $p^0 = \sqrt{\vec{p}^2 + m^2}$ and $p_0^2 = \vec{p}^2 + m^2$. In terms of derivatives, we have

$$\begin{aligned}\langle 0|\mathcal{T}\big[\psi_a(x)\psi_b^\dagger(y)\big]|0\rangle = \theta(x^0 - y^0)\left(\eta_{ab} - \frac{\partial_a\,\partial_b}{m^2}\right)\int \frac{d^3p}{(2\pi)^3 2p^0}\; e^{ip\cdot(x-y)} \\ + \theta(y^0 - x^0)\left(\eta_{ab} - \frac{\partial_a\,\partial_b}{m^2}\right)\int \frac{d^3p}{(2\pi)^3 2p^0}\; e^{ip\cdot(y-x)}.\end{aligned} \tag{2.95}$$

In this formula, we may interpret the double time derivative as $\partial_0^2 = \nabla^2 - m^2$. For spatial values of a and b, we can move the derivatives to the left of the step functions. And we can move the product $\partial_0\,\partial_i$ of one spatial and one time derivative by the argument we used for spin one-half. We find

$$\begin{aligned}\langle 0|\mathcal{T}\big[\psi_a(x)\psi_b^\dagger(y)\big]|0\rangle &= \left(\eta_{ab} - \frac{\partial_a\,\partial_b}{m^2}\right)\Big[\theta(x^0 - y^0)\int \frac{d^3p}{(2\pi)^3 2p^0}\; e^{ip\cdot(x-y)} \\ &\quad + \theta(y^0 - x^0)\int \frac{d^3p}{(2\pi)^3 2p^0}\; e^{ip\cdot(y-x)}\Big] \\ &= \left(\eta_{ab} - \frac{\partial_a\,\partial_b}{m^2}\right)\Big[\theta(x^0 - y^0)\,\Delta_+(x-y) + \theta(y^0 - x^0)\,\Delta_+(y-x)\Big] \\ &= \left(\eta_{ab} - \frac{\partial_a\,\partial_b}{m^2}\right)[-i\Delta_F(x-y)] \\ &= -\,i\left(\eta_{ab} - \frac{\partial_a\,\partial_b}{m^2}\right)\int \frac{d^4q}{(2\pi)^4}\frac{e^{iq\cdot(x-y)}}{q^2 + m^2 - i\epsilon}\end{aligned} \tag{2.96}$$

in which $\partial_0^2 = \nabla^2 - m^2$.

We can relax the rule $\partial_0^2 = \nabla^2 - m^2$ if we add an extra term to the propagator. The extra term is

$$\begin{aligned}\frac{i}{m^2}\left(\nabla^2 - m^2 - \partial_0^2\right)\int \frac{d^4q}{(2\pi)^4}\frac{e^{iq\cdot(x-y)}}{q^2 + m^2 - i\epsilon} &= \frac{i}{m^2}\int \frac{d^4q}{(2\pi)^4}\frac{-\vec{q}^2 - m^2 + q_0^2}{q^2 + m^2 - i\epsilon}\, e^{iq\cdot(x-y)} \\ &= -\,\frac{i}{m^2}\,\delta^4(x-y).\end{aligned} \tag{2.97}$$

So the Feynman propagator for spin-one fields is

$$\langle 0|\mathcal{T}\big[\psi_a(x)\psi_b^\dagger(y)\big]|0\rangle = -\,i\int \frac{d^4q}{(2\pi)^4}\frac{(\eta_{ab} + p_a\,p_b/m^2)}{q^2 + m^2 - i\epsilon}\, e^{iq\cdot(x-y)} - \frac{i}{m^2}\,\delta_a^0\,\delta_b^0\,\delta^4(x-y). \tag{2.98}$$

Suppose the vector field $\psi_a(x)$ interacts with a current $j^a(x)$ thru a term $\psi_a(x)j^a(x)$ in $\mathcal{H}(x)$. Then in Dyson's expansion, the awkward second term

in the Feynman propagator (2.98) contributes the term

$$-i\mathcal{H}_2(x) = \frac{1}{2}\int [-ij^a(x)][-ij^b(y)]\left[-\frac{i}{m^2}\,\delta^0_a\,\delta^0_b\,\delta^4(x-y)\right]d^4y = \frac{i[j^0(x)]^2}{2m^2}. \tag{2.99}$$

So if $\mathcal{H}(x)$ also contains the term

$$\mathcal{H}_C(x) = \frac{[j^0(x)]^2}{2m^2}, \tag{2.100}$$

then the effect of the awkward second term in the Feynman propagator is cancelled. This actually happens in a natural way as SW explains in chapter 7.

2.9 The Feynman rules

1) Draw all diagrams of the order you are working at. Label each internal line with its (unphysical) 4-momentum considered to flow in the direction of the arrow or in either direction if the particle is uncharged.

2) For each vertex of type i, include the factor

$$-i(2\pi)^4 g_i \delta^4\left(\sum p + q - p' - q'\right) \tag{2.101}$$

which makes the sum of the 4-momenta $p+q$ entering each vertex add up to sum of the 4-momenta $p'+q'$ leaving each vertex.

For each outgoing line include the factor

$$\frac{u^*_\ell(\vec{p}', s', n')}{(2\pi)^{3/2}} \quad \text{or} \quad \frac{v_\ell(\vec{p}', s', n')}{(2\pi)^{3/2}} \tag{2.102}$$

for arrows pointing out ($u^*_\ell(\vec{p}', s', n')$) or pointing in ($v_\ell(\vec{p}', s', n')$).

For each incoming line include the factor

$$\frac{u_\ell(\vec{p}, s, n)}{(2\pi)^{3/2}} \quad \text{or} \quad \frac{v^*_\ell(\vec{p}, s, n)}{(2\pi)^{3/2}} \tag{2.103}$$

for arrows pointing in ($u^*_\ell(\vec{p}', s', n')$) or pointing out ($v_\ell(\vec{p}', s', n')$).

For each internal line of a spin-zero particle carrying momentum q^a include the factor

$$\frac{-i}{(2\pi)^4}\,\frac{1}{q^2 + m_\ell^2 - i\epsilon}. \tag{2.104}$$

For each internal line of a spin-one-half particle with ends labelled by ℓ and m and carrying momentum q^a include the factor

$$\frac{-i}{(2\pi)^4}\,\frac{\left[(-i\not{q} + m)\beta\right]_{\ell m}}{q^2 + m_\ell^2 - i\epsilon}. \tag{2.105}$$

For each internal line of a spin-one particle with ends labelled by ℓ and m and carrying momentum q^a include the factor

$$\frac{-i}{(2\pi)^4}\,\frac{\eta_{\ell m}+q_\ell q_m/m^2}{q^2+m_\ell^2-i\epsilon} \tag{2.106}$$

and keep in mind the delta-function term in (2.98).

3). Integrate the product of all these factors over all the internal momenta and sum over ℓ and m, etc.

4) Add the results of all the Feynman diagrams.

2.10 Fermion-antifermion scattering

We can watch Feynman's rules emerge in fermion-antifermion scattering. We consider a fermion interacting with a neutral scalar boson (2.72)

$$H(x) = g_{\ell m}\,\psi_\ell^\dagger(x)\,\psi_m(x)\,\phi(x). \tag{2.107}$$

The initial state is $|p,s;q,t\rangle = a^\dagger(p,s)b^\dagger(q,t)|0\rangle$ and the final state is $|p',s';q',t'\rangle = a^\dagger(p',s')b^\dagger(q',t')|0\rangle$. The lowest-order term is

$$\begin{aligned} A =& -\frac{1}{2!}\int d^4x d^4y\,\langle 0|b(q',t')\,a(p',s')T\Big[H(x)\,H(y)\Big]\,a^\dagger(p,s)\,b^\dagger(q,t)|0\rangle \\ =& -\frac{g_{\ell m}g_{\ell' m'}}{2!}\int d^4x d^4y\langle 0|b(q',t')a(p',s')T\Big[\psi_\ell^\dagger(x)\,\psi_m(x)\phi(x)\psi_{\ell'}^\dagger(y)\,\psi_{m'}(y)\phi(y)\Big] \\ &\times a^\dagger(p,s)b^\dagger(q,t)|0\rangle \end{aligned} \tag{2.108}$$

in which the operators a and b delete fermions and antifermions. There is only one Feynman diagram (Fig. 2.2).

We cancel the 2! by choosing to absorb the incoming fermion-antifermion pair at vertex y and to add the outgoing fermion-antifermion pair at vertex x. We are left with

$$\begin{aligned} A =& -\frac{g_{\ell m}g_{\ell' m'}}{(2\pi)^6}\int d^4x d^4y\langle 0|u_\ell^*(p',s')\,v_m(q',t')\,e^{-ix\cdot(p'+q')}\,T\Big[\phi(x)\,\phi(y)\Big] \\ &\times u_{m'}(p,s)\,v_{\ell'}^*(q,t)\,e^{iy\cdot(p+q)}|0\rangle. \end{aligned} \tag{2.109}$$

Adding in the scalar propagator (2.50), we get

$$\begin{aligned} A =& -\frac{g_{\ell m}g_{\ell' m'}}{(2\pi)^6}\int d^4x\,d^4y\,u_\ell^*(p',s')\,v_m^*(q',t')\,e^{-ix\cdot(p'+q')} \\ &\times u_{m'}(p,s)\,v_{\ell'}^*(q,t)\,e^{iy\cdot(p+q)}\int\frac{d^4k}{(2\pi)^4}\,\frac{-ie^{ik\cdot(x-y)}}{k^2+m^2-i\epsilon}. \end{aligned} \tag{2.110}$$

Fermion-antifermion scattering

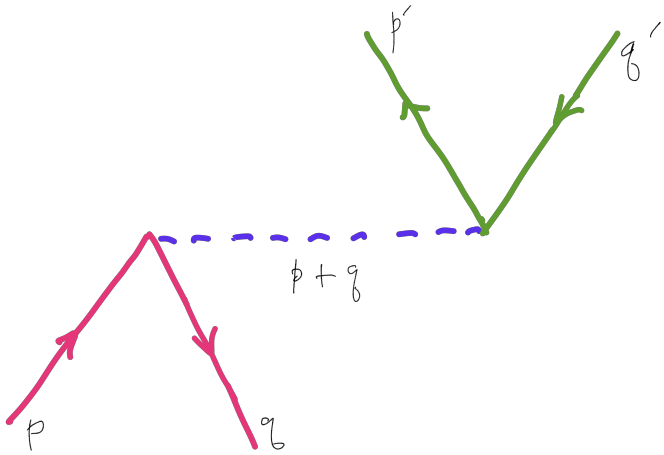

Figure 2.2 Fermion-antifermion scattering $p+q \to p'+q'$ (arbitrary colors).

The integration over y conserves 4-momentum at vertex y, and the integration over x conserves 4-momentum at vertex x. We then have

$$A = i\,\frac{g_{\ell m} g_{\ell' m'}}{(2\pi)^2} \int d^4k\, u^*_\ell(p',s')\, v_m(q',t')\, \delta(k-p'-q')\, \delta(p+q-k) \\ \times u_{m'}(p,s)\, v^*_{\ell'}(q,t) \frac{1}{k^2+m^2-i\epsilon} \tag{2.111}$$

or

$$A = i\,\delta(p+q-k'-q')\, \frac{g_{\ell m} g_{\ell' m'}}{(2\pi)^2}\, \frac{u^*_\ell(p',s')\, v_m(q',t')\, u_{m'}(p,s)\, v^*_{\ell'}(q,t)}{(p+k)^2+m^2}. \tag{2.112}$$

3
Action

3.1 Lagrangians and hamiltonians

A transformation is a symmetry of a theory if the **action** is invariant or changes by a surface term. So we choose to work with actions that are symmetrical. The action is normally an integral over spacetime of an **action density** often called a **lagrangian**. Often the action density itself is invariant under the transformation of the symmetry.

There are procedures, sometimes clumsy procedures, for computing the hamiltonian from the lagrangian. The hamiltonian often is not invariant under the transformation of the symmetry. So it's very hard to find a suitably symmetrical theory by starting with a hamiltonian. But once one has a hamiltonian, one can compute scattering amplitudes energies, and states with these energies.

3.2 Canonical variables

In quantum mechanics, we use the equal-time commutation relations

$$[q_i, p_k] = i\delta_{ik}, \quad [q_i, q_k] = 0, \quad \text{and} \quad [p_i, p_k] = 0. \tag{3.1}$$

In general, the operators q_i and $q_i(t) = e^{iHt} q_i e^{-iHt}$ do not commute. Quantum field theory promotes these equal-time commutation relations to ones in which the indexes i and k denote different points of space

$$\begin{aligned} &[q^n(\vec{x}, t), p_m(\vec{y}, t)]_\mp = i\delta(\vec{x} - \vec{y})\delta^n_m, \\ &[q^n(\vec{x}, t), q^m(\vec{y}, t)]_\mp = 0, \quad \text{and} \quad [p_n(\vec{x}, t), p_m(\vec{y}, t)]_\mp = 0. \end{aligned} \tag{3.2}$$

The commutator of a real scalar field

$$\phi(x) = \int \frac{d^3p}{\sqrt{(2\pi)^3 2p^0}} \left[a(\vec{p})\, e^{ip\cdot x} + a^\dagger(\vec{p})\, e^{-ip\cdot x} \right] \tag{3.3}$$

is

$$[\phi(x), \phi(y)] = \Delta(x-y) = \int \frac{d^3p}{(2\pi)^3 2p^0} \left(e^{ip\cdot(x-y)} - e^{-ip\cdot(x-y)}\right). \tag{3.4}$$

At equal times, one has

$$\begin{aligned} \Delta(\vec{x}-\vec{y}, 0) = 0, \quad \frac{\partial}{\partial x^0}\, \Delta(x-y)|_{x^0=y^0} = -\, i\, \delta^3(\vec{x}-\vec{y}), \quad \text{and} \\ \frac{\partial^2}{\partial x^0 \partial y^0}\, \Delta(x-y)|_{x^0=y^0} = 0. \end{aligned} \tag{3.5}$$

So a real field ϕ and its time derivative $\dot{\phi}$ satisfy the equal-time commutation relations (3.60)

$$\begin{aligned} &[\phi(\vec{x},t), \dot{\phi}(\vec{y},t)]_- = i\, \delta^3(\vec{x}-\vec{y}), \\ &[\phi(\vec{x},t), \phi(\vec{y},t)]_- = 0, \quad \text{and} \quad [\dot{\phi}(\vec{x},t), \dot{\phi}(\vec{y},t)]_- = 0. \end{aligned} \tag{3.6}$$

A complex scalar field

$$\phi(x) = \frac{1}{\sqrt{2}}\big[\phi_1(x) + i\phi_2(x)\big] \tag{3.7}$$

obeys the commutation relations

$$\begin{aligned} [\phi(x), \phi^\dagger(y)]_- &= \frac{1}{2}\, [\phi_1(x) + i\phi_2(x), \phi_1(x) - i\phi_2(x)] \\ &= \frac{1}{2}\, ([\phi_1(x), \phi_1(x)] + [\phi_2(x), \phi_2(x)]) = \Delta(x-y) \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} [\phi(x), \phi(y)]_- &= \frac{1}{2}\, [\phi_1(x) + i\phi_2(x), \phi_1(x) + i\phi_2(x)] \\ &= \frac{1}{2}\, ([\phi_1(x), \phi_1(x)] - [\phi_2(x), \phi_2(x)]) = 0. \end{aligned} \tag{3.9}$$

So the complex scalar fields $\phi(x)$ and $\dot{\phi}^\dagger(x)$ obey the equal-time commutation relations (3.6)

$$[\phi(\vec{x},t), \dot{\phi}^\dagger(\vec{y},t)]_- = i\, \delta^3(\vec{x}-\vec{y}) \quad \text{and} \quad [\phi(\vec{x},t), \phi^\dagger(\vec{y},t)]_- = 0 \tag{3.10}$$

$$[\dot{\phi}(\vec{x},t), \dot{\phi}(\vec{y},t)] = 0 \quad \text{and} \quad [\dot{\phi}^\dagger(\vec{x},t), \dot{\phi}^\dagger(\vec{y},t)] = 0 \tag{3.11}$$

$$[\phi(\vec{x},t), \phi(\vec{y},t)] = 0 \quad \text{and} \quad [\phi^\dagger(\vec{x},t), \phi^\dagger(\vec{y},t)] = 0. \tag{3.12}$$

3.3 Principle of stationary action in field theory

If $\phi(x)$ is a scalar field, and $L(\phi)$ is its action density, then its action $S[\phi]$ is the integral over all of spacetime

$$S[\phi] = \int L(\phi(x))\, d^4x. \tag{3.13}$$

The principle of least (or stationary) action says that the field $\phi(x)$ that satisfies the classical equation of motion is the one for which the first-order change in the action due to any tiny variation $\delta\phi(x)$ in the field vanishes, $\delta S[\phi] = 0$. And to keep things simple, we'll assume that the action (or Lagrange) density $L(\phi)$ is a function only of the field ϕ and its first derivatives $\partial_a\phi = \partial\phi/\partial x^a$. The first-order change in the action then is

$$\delta S[\phi] = \int \left[\frac{\partial L}{\partial\phi}\delta\phi + \frac{\partial L}{\partial(\partial_a\phi)}\delta(\partial_a\phi)\right] d^4x \tag{3.14}$$

in which we sum over the repeated index a from 0 to 3. Now $\delta(\partial_a\phi) = \partial_a(\phi+\delta\phi) - \partial_a\phi = \partial_a\delta\phi$. So we may integrate by parts and drop the surface terms because we set $\delta\phi = 0$ on the surface at infinity

$$\begin{aligned}\delta S[\phi] &= \int \left[\frac{\partial L}{\partial\phi}\delta\phi + \frac{\partial L}{\partial(\partial_a\phi)}\partial_a(\delta\phi)\right] d^4x \\ &= \int \left[\frac{\partial L}{\partial\phi} - \partial_a\frac{\partial L}{\partial(\partial_a\phi)}\right] \delta\phi\, d^4x.\end{aligned} \tag{3.15}$$

This first-order variation is zero, for arbitrary $\delta\phi$ only if the field $\phi(x)$ satisfies Lagrange's equation

$$\partial_a\left(\frac{\partial L}{\partial(\partial_a\phi)}\right) \equiv \frac{\partial}{\partial x^a}\left[\frac{\partial L}{\partial(\partial\phi/\partial x^a)}\right] = \frac{\partial L}{\partial\phi} \tag{3.16}$$

which is the classical equation of motion.

Example 3.1 (Theory of a scalar field) The action density of a scalar field ϕ of mass m is

$$L = \tfrac{1}{2}\,(\dot\phi)^2 - \tfrac{1}{2}\,(\nabla\phi)^2 - \tfrac{1}{2}\,m^2\phi^2 = -\tfrac{1}{2}\,\partial_a\phi\,\partial^a\phi - \tfrac{1}{2}\,m^2\phi^2. \tag{3.17}$$

Lagrange's equation (3.16) for this action density is

$$\nabla^2\phi - \ddot\phi = \partial_a\,\partial^a\phi = m^2\phi \tag{3.18}$$

which is the Klein-Gordon equation (1.56). □

In a theory of several fields $\phi_1, \ldots, \phi_n$ with action density $L(\phi_k, \partial_a \phi_k)$, the fields obey n copies of Lagrange's equation

$$\frac{\partial}{\partial x^a}\left(\frac{\partial L}{\partial(\partial_a \phi_k)}\right) = \frac{\partial L}{\partial \phi_k} \tag{3.19}$$

one for each k.

3.4 Symmetries and conserved quantities in field theory

An action density $L(\phi_i, \partial_a \phi_i)$ that is invariant under a transformation of the coordinates x^a or of the fields ϕ_i and their derivatives $\partial_a \phi_i$ is a **symmetry** of the action density. Such a symmetry implies that something is conserved or time independent.

Suppose that an action density $L(\phi_i, \partial_a \phi_i)$ is unchanged when the fields ϕ_i and their derivatives $\partial_a \phi_i$ change by $\delta\phi_i$ and by $\delta(\partial_a \phi_i)$

$$0 = \delta L = \sum_i \frac{\partial L}{\partial \phi_i}\delta\phi_i + \frac{\partial L}{\partial \partial_a \phi_i}\,\delta\left(\partial_a \phi_i\right). \tag{3.20}$$

For many transformations (but not for Lorentz transformations), we can set $\delta(\partial_a \phi_i) = \partial_a(\delta\phi_i)$. In such cases by using Lagrange's equations (3.19) to rewrite $\partial L/\partial\phi_i$, we find

$$0 = \sum_i \left(\partial_a \frac{\partial L}{\partial \partial_a \phi_i}\right)\delta\phi_i + \frac{\partial L}{\partial \partial_a \phi_i}\partial_a \delta\phi_i = \partial_a \sum_i \frac{\partial L}{\partial \partial_a \phi_i}\delta\phi_i \tag{3.21}$$

which says that the **current**

$$J^a = \sum_i \frac{\partial L}{\partial \partial_a \phi_i}\delta\phi_i \tag{3.22}$$

has zero divergence

$$\partial_a J^a = 0. \tag{3.23}$$

Thus the time derivative of the volume integral of the charge density J^0

$$Q_V = \int_V J^0\, d^3x \tag{3.24}$$

is the flux of current $\vec{J}$ entering through the boundary S of the volume V

$$\dot{Q}_V = \int_V \partial_0 J^0\, d^3x = -\int_V \partial_k J^k\, d^3x = -\int_S J^k d^2 S_k. \tag{3.25}$$

If no current enters V, then the charge Q inside V is conserved. When the volume V is the whole universe, the charge is the integral over all of space

$$Q = \int J^0 \, d^3x = \int \sum_i \frac{\partial L}{\partial \dot{\phi}_i} \, \delta\phi_i \, d^3x = \int \sum_i \pi_i \, \delta\phi_i \, d^3x \tag{3.26}$$

in which π_i is the momentum conjugate to the field ϕ_i

$$\pi_i = \frac{\partial L}{\partial \dot{\phi}_i}. \tag{3.27}$$

Example 3.2 ($O(n)$ symmetry and its charge) Suppose the action density L is the sum of n copies of the quadratic action density (3.17)

$$L = \sum_{i=1}^{n} \tfrac{1}{2} \, (\dot{\phi}_i)^2 - \tfrac{1}{2} \, (\nabla\phi_i)^2 - \tfrac{1}{2} \, m^2 \phi_i^2 = -\tfrac{1}{2} \, \partial_a\phi \, \partial^a\phi - \tfrac{1}{2} \, m^2 \phi^2, \tag{3.28}$$

and A_{ij} is any constant antisymmetric matrix, $A_{ij} = -A_{ji}$. Then if the fields change by $\delta\phi_i = \epsilon \sum_j A_{ij} \, \phi_j$, the change (3.20) in the action density

$$\delta L = -\epsilon \sum_{i,j=1}^{n} \left[m^2 \phi_i A_{ij} \phi_j + \partial^a \phi_i A_{ij} \partial_a \phi_j \right] = 0 \tag{3.29}$$

vanishes. Thus the charge (3.26) associated with the matrix A

$$Q_A = \int \sum_i \pi_i \, \delta\phi_i \, d^3x = \epsilon \int \sum_i \pi_i \, A_{ij} \, \phi_j \, d^3x \tag{3.30}$$

is conserved. There are $n(n-1)/2$ antisymmetric $n \times n$ imaginary matrices; they generate the group $O(n)$ of $n \times n$ orthogonal matrices. □

An action density $L(\phi_i, \partial_a\phi_i)$ that is invariant under a spacetime translation, $x'^a = x^a + \delta x^a$, depends upon x^a only through the fields ϕ_i and their derivatives $\partial_a\phi_i$

$$\frac{\partial L}{\partial x^a} = \sum_i \left(\frac{\partial L}{\partial \phi_i} \frac{\partial \phi_i}{\partial x^a} + \frac{\partial L}{\partial \partial_b \phi_i} \frac{\partial^2 \phi_i}{\partial x^b \partial x^a} \right). \tag{3.31}$$

Using Lagrange's equations (3.19) to rewrite $\partial L / \partial \phi_i$, we find

$$\begin{aligned} 0 &= \left(\sum_i \partial_b \left(\frac{\partial L}{\partial \partial_b \phi_i} \right) \partial_a \phi_i + \frac{\partial L}{\partial \partial_b \phi_i} \frac{\partial^2 \phi_i}{\partial x^b \partial x^a} \right) - \frac{\partial L}{\partial x^a} \\ 0 &= \partial_b \left[\sum_i \left(\frac{\partial L}{\partial \partial_b \phi_i} \frac{\partial \phi_i}{\partial x^a} \right) - \delta_a^b L \right] \end{aligned} \tag{3.32}$$

that the **energy-momentum tensor**

$$T^b{}_a = \sum_i \frac{\partial L}{\partial \partial_b \phi_i} \frac{\partial \phi_i}{\partial x^a} - \delta^b_a L \tag{3.33}$$

has zero divergence, $\partial_b T^b{}_a = 0$.

Thus the time derivative of the 4-momentum P_{aV} inside a volume V

$$P_{aV} = \int_V \left(\sum_i \frac{\partial L}{\partial \partial_0 \phi_i} \frac{\partial \phi_i}{\partial x^a} - \delta^0_a L \right) d^3x = \int_V T^0{}_a \, d^3x \tag{3.34}$$

is equal to the flux entering through V's boundary S

$$\partial_0 P_{aV} = \int_V \partial_0 T^0{}_a \, d^3x = -\int_V \partial_k T^k{}_a \, d^3x = -\int_S T^k{}_a \, d^2S_k. \tag{3.35}$$

The invariance of the action density L under spacetime translations implies the **conservation of energy** P_0 and **momentum** $\vec{P}$.

The momentum $\pi_i(x)$ that is canonically conjugate to the field $\phi_i(x)$ is the derivative of the action density L with respect to the time derivative of the field

$$\pi_i = \frac{\partial L}{\partial \dot{\phi}_i}. \tag{3.36}$$

If one can express the time derivatives $\dot{\phi}_i$ of the fields in terms of the fields ϕ_i and their momenta π_i, then hamiltonian of the theory is the spatial integral of

$$H = P_0 = T^0{}_0 = \left(\sum_{i=1}^n \pi_i \, \dot{\phi}_i \right) - L \tag{3.37}$$

in which $\dot{\phi}_i = \dot{\phi}_i(\phi, \pi)$.

Example 3.3 (Hamiltonian of a scalar field) The hamiltonian density (3.37) of the theory (3.17) is

$$H = \tfrac{1}{2} \pi^2 + \tfrac{1}{2} (\nabla \phi)^2 + \tfrac{1}{2} m^2 \phi^2. \tag{3.38}$$

□

A Lorentz transformation (1.15) of a field is like this

$$U(\Lambda, a) \psi_\ell(x) U^{-1}(\Lambda, a) = \sum_{\bar{\ell}} D_{\ell\bar{\ell}}(\Lambda^{-1}) \psi_{\bar{\ell}}(\Lambda x + a). \tag{3.39}$$

The action is invariant under Lorentz transformations. The action density

is constructed so as to be invariant when the fields transform this way:

$$\psi'_\ell(x) = \sum_{\bar\ell} D_{\ell\bar\ell}(\Lambda^{-1})\psi_{\bar\ell}(x) \tag{3.40}$$

The action density is constructed so as to be invariant when the fields and their derivatives of the fields transform under infinitesimal Lorentz transformations as

$$\begin{aligned} \delta\psi^\ell(x) &= \frac{i}{2}\,\omega^{ab}(J_{ab})^\ell{}_m\,\psi^m(x) \\ \delta(\partial_k\,\psi^\ell)(x) &= \frac{i}{2}\,\omega^{ab}(J_{ab})^\ell{}_m\,\partial_k\,\psi^m(x) + \omega_k{}^j\partial_j\psi^\ell(x). \end{aligned} \tag{3.41}$$

The invariance of the action density says that

$$\begin{aligned} 0 &= \frac{\partial L}{\partial\psi^\ell}\,\delta\psi^\ell + \frac{\partial L}{\partial\partial_k\psi^\ell}\,\delta(\partial_k\,\psi^\ell) \\ &= \frac{\partial L}{\partial\psi^\ell}\,\frac{i}{2}\,\omega^{ab}(J_{ab})^\ell{}_m\,\psi^m + \frac{\partial L}{\partial\partial_k\psi^\ell}\Big[\frac{i}{2}\,\omega^{ab}(J_{ab})^\ell{}_m\,\partial_k\,\psi_m + \omega_k{}^j\partial_j\psi^\ell\Big]. \end{aligned} \tag{3.42}$$

Remembering that $\omega_{ab} = -\omega_{ba}$ and $\omega^{ab} = -\omega^{ba}$, we rewrite the last bit as

$$\omega_k{}^b\partial_b\psi_\ell = \eta_{ka}\,\omega^{ab}\partial_b = \frac{1}{2}\big(\eta_{ka}\,\omega^{ab}\partial_b\psi_\ell - \eta_{kb}\,\omega^{ab}\partial_a\big)\psi_\ell. \tag{3.43}$$

We then have

$$\begin{aligned} 0 = {}& \frac{\partial L}{\partial\psi^\ell}\,\frac{i}{2}\,\omega^{ab}(J_{ab})^\ell{}_m\,\psi^m \\ &+ \frac{\partial L}{\partial\partial_k\psi^\ell}\Big[\frac{i}{2}\,\omega^{ab}(J_{ab})^\ell{}_m\,\partial_k\,\psi_m + \frac{1}{2}\big(\eta_{ka}\,\omega^{ab}\partial_b - \eta_{kb}\,\omega^{ab}\partial_a\big)\psi_\ell\Big] \end{aligned} \tag{3.44}$$

or

$$\begin{aligned} 0 = {}& \frac{\partial L}{\partial\psi^\ell}\,\frac{i}{2}\,(J_{ab})^\ell{}_m\,\psi^m \\ &+ \frac{\partial L}{\partial\partial_k\psi^\ell}\Big[\frac{i}{2}\,(J_{ab})^\ell{}_m\,\partial_k\,\psi_m + \frac{1}{2}\big(\eta_{ka}\,\partial_b - \eta_{kb}\,\partial_a\big)\psi_\ell\Big]. \end{aligned} \tag{3.45}$$

The equations of motion now give

$$\begin{aligned} 0 = {}& \left(\partial_k\frac{\partial L}{\partial\partial_k\psi^\ell}\right)\frac{i}{2}\,(J_{ab})^\ell{}_m\,\psi^m \\ &+ \frac{\partial L}{\partial\partial_k\psi^\ell}\Big[\frac{i}{2}\,(J_{ab})^\ell{}_m\,\partial_k\,\psi_m + \frac{1}{2}\big(\eta_{ka}\,\partial_b - \eta_{kb}\,\partial_a\big)\psi_\ell\Big] \end{aligned} \tag{3.46}$$

or

$$0 = \partial_k\left(\frac{\partial L}{\partial\partial_k\psi^\ell}\,\frac{i}{2}\,(J_{ab})^\ell{}_m\,\psi^m\right) + \frac{\partial L}{\partial\partial_k\psi^\ell}\,\frac{1}{2}\big(\eta_{ka}\,\partial_b - \eta_{kb}\,\partial_a\big)\psi_\ell\Big]$$

or

$$0 = \partial_k \left(\frac{\partial L}{\partial \partial_k \psi^\ell} \frac{i}{2} \left(J_{ab}\right)^\ell{}_m \psi^m \right) + \frac{1}{2} \left(\frac{\partial L}{\partial \partial^a \psi^\ell} \partial_b - \frac{\partial L}{\partial \partial^b \psi^\ell} \partial_a \psi_\ell \right).$$

We recall (3.33) the energy-momentum tensor

$$T^b{}_a = \sum_i \frac{\partial L}{\partial \partial_b \phi_i} \frac{\partial \phi_i}{\partial x^a} - \delta^b_a L \tag{3.47}$$

or

$$T_{ba} = \frac{\partial L}{\partial \partial^b \psi_\ell} \frac{\partial \psi_\ell}{\partial x^a} - \eta_{ba} L \tag{3.48}$$

which has zero divergence, $\partial_b T^b{}_a = 0$. In terms of it, we have

$$0 = \partial_k \left(\frac{\partial L}{\partial \partial_k \psi^\ell} \frac{i}{2} \left(J_{ab}\right)^\ell{}_m \psi^m \right) - \frac{1}{2} \left(T_{ab} - T_{ba}\right).$$

So Belinfante defined the symmetric energy-momentum tensor as

$$\begin{aligned} \Theta^{ab} = T^{ab} - \frac{i}{2} \partial_k \Big[& \frac{\partial L}{\partial \partial_k \psi^\ell} (J^{ab})^\ell{}_m \psi^m \\ & - \frac{\partial L}{\partial \partial_a \psi^\ell} (J^{kb})^\ell{}_m \psi^m - \frac{\partial L}{\partial \partial_b \psi^\ell} (J^{ka})^\ell{}_m \psi^m \Big]. \end{aligned} \tag{3.49}$$

The quantity in the square brackets is antisymmetric in a, k. So

$$\begin{aligned} \partial_a \Theta^{ab} &= \partial_a T^{ab} - \frac{i}{2} \partial_a \partial_k \Big[\frac{\partial L}{\partial \partial_k \psi^\ell} (J^{ab})^\ell{}_m \psi^m \\ &\quad - \frac{\partial L}{\partial \partial_a \psi^\ell} (J^{kb})^\ell{}_m \psi^m - \frac{\partial L}{\partial \partial_b \psi^\ell} (J^{ka})^\ell{}_m \psi^m \Big] \\ &= \partial_a T^{ab} = 0. \end{aligned} \tag{3.50}$$

The Belinfante energy-momentum tensor is symmetric

$$\Theta^{ab} = \Theta^{ba} \tag{3.51}$$

and so it is the one to use when gravity is involved.

The quantity

$$M^{abc} = x^b \Theta^{ac} - x^c \Theta^{ab} \tag{3.52}$$

is a conserved current

$$\partial_a M^{abc} = \partial_a \left(x^b \Theta^{ac} - x^c \Theta^{ab} \right) = \Theta^{bc} - \Theta^{cb} = 0. \tag{3.53}$$

So the angular-momentum operators

$$J^{bc} = \int M^{0bc} \, d^3x = \int \left(x^b \Theta^{0c} - x^c \Theta^{0b} \right) d^3x \tag{3.54}$$

are conserved.

3.5 Poisson Brackets

In a classical theory with coordinates q_n and conjugate momenta

$$p_n = \frac{\partial L}{\partial \dot{q}_n}, \tag{3.55}$$

the Poisson bracket is

$$[A, B]_P = \sum_n \frac{\partial A}{\partial q_n}\frac{\partial B}{\partial p_n} - \frac{\partial B}{\partial q_n}\frac{\partial A}{\partial p_n}. \tag{3.56}$$

One of Dirac's many insights was to say that in the quantum version of such a theory the equal-time commutation relations are

$$[A, B] = i\hbar\,[A, B]_P. \tag{3.57}$$

In field theories, with coordinates $\phi_a = \phi_n(\boldsymbol{x})$ and conjugate momenta

$$\pi_a = \pi_n(x) = \frac{\partial L(x)}{\partial \dot{\phi}_n(x)}, \tag{3.58}$$

the Poisson bracket is

$$[A, B]_P = \sum_n \int d^3x \frac{\partial A}{\partial \phi_n(\boldsymbol{x})}\frac{\partial B}{\partial \pi_n(\boldsymbol{x})} - \frac{\partial B}{\partial \phi_n(\boldsymbol{x})}\frac{\partial A}{\partial \pi_n(\boldsymbol{x})}. \tag{3.59}$$

In the quantum theory, the equal-time commutator is

$$[A, B] = i\hbar\,[A, B]_P. \tag{3.60}$$

When these equations and definitions (3.55–3.60) make sense, one sometimes can express the velocities $\dot{\phi}_n(\boldsymbol{x})$ in terms of the coordinates $\phi_n(\boldsymbol{x})$ and their conjugate momenta $\pi_n(\boldsymbol{x})$. In such cases, one can make a hamiltonian density

$$H = \dot{\phi}_n(\boldsymbol{x})\pi_n(\boldsymbol{x}) - L(x) \tag{3.61}$$

in which $L(x)$ is the action density. And the action principle gives us the equations of motion

$$\partial_i \frac{\partial L}{\partial \partial_i \phi_n} = \frac{\partial L}{\partial \phi_n}. \tag{3.62}$$

But things can go wrong. For instance, the Lagrange's equations for the action densities $L = q$ and $L = \phi_n$ lead to the inconsistent equation

$$0 = 1. \tag{3.63}$$

Similarly,

So Nature doesn't care about our ideas. The theory of a massive vector field V

$$L = -\tfrac{1}{4}F_{ik}F^{ik} - \tfrac{1}{2}V_iV^i - J_iV^i \tag{3.64}$$

provides another example in which the momentum π_0 is constrained to vanish

$$\pi_0 = \frac{\partial L(x)}{\partial \dot{V}_0(x)} = 0 \tag{3.65}$$

and the equation of motion for the field V^0 (3.62)

$$\partial_i \frac{\partial L}{\partial \partial_i V^0} = \frac{\partial L}{\partial V^0} \tag{3.66}$$

is

$$-\partial_i F^i_{\ 0} = \partial_i F^{i0} = -m^2 V_0 - J_0 \tag{3.67}$$

or with $\pi_i = -F^{0i} = F^{i0}$

$$\partial_i F^{i0} = \nabla \cdot \boldsymbol{\pi} = m^2 V^0 + J^0. \tag{3.68}$$

This last equation constrains the $\nabla \cdot \boldsymbol{\pi}$ to be $m^2V^0 + J^0$ at each point $\boldsymbol{x}, t$. To cope with such problems and thereby to extend the variety of quantum field theories, Dirac invented a kind of bracket that is different from his $\langle a|b\rangle$ bracket.

3.6 Dirac Brackets

First, one makes a list of all the constraints

$$\chi = 0 \tag{3.69}$$

that occur due to the definitions of the conjugate momenta and to the equations of motion.

A constraint $\chi_f = 0$ is **first class** if its Poisson bracket with all the other constraints vanishes

$$[\chi_f, \chi_a]_P = 0. \tag{3.70}$$

One satisfies these constraints by choosing a **gauge**.

One then makes a matrix of the Poisson brackets of the **second-class** constraints

$$C_{ab} = [\chi_a, \chi_b]_P. \tag{3.71}$$

In computing this matrix C, one does not apply the constraints (3.69) until after one has computed their Poisson brackets. Otherwise C would vanish.

Dirac defined his new bracket in terms of the second-class constraints as

$$[A, B]_D = [A, B]_P - [A, \chi_a]_P \, (C^{-1})^{ab} \, [\chi_b, B]_P. \tag{3.72}$$

This new Dirac bracket inherits the algebraic properties of Poisson's bracket

$$\begin{aligned} [A, B]_D &= -[B, A]_D, \\ [A, BC]_D &= [A, B]_D \, C + B \, [A, C]_D, \\ [A, [B, C]_D]_D + [B, [C, A]_D]_D &+ [C, [A, B]_D]_D = 0. \end{aligned} \tag{3.73}$$

Dirac then extended his rule that the commutator of two operators should be $i\hbar$ times their Poisson bracket to his new rule that it should be $i\hbar$ times their Dirac bracket

$$[A, B] = i\hbar \, [A, B]_D. \tag{3.74}$$

3.7 Dirac Brackets and QED

The action density is

$$L = -\tfrac{1}{4} F_{ik} F^{ik} - J_i A^i. \tag{3.75}$$

The usual rules give the momenta as

$$\pi^i = \frac{\partial L}{\partial(\partial_0 A_i)} \tag{3.76}$$

and the commutation relations as

$$[A_i(\boldsymbol{x}, t), \pi^k(\boldsymbol{y}, t)] = i\delta_i^k \delta^3(\boldsymbol{x} - \boldsymbol{y}). \tag{3.77}$$

These equations contain the first-class constraints

$$\begin{aligned} \pi^0 &= 0 \\ \boldsymbol{\nabla} \cdot \boldsymbol{\pi} - J_0 &= 0. \end{aligned} \tag{3.78}$$

We first choose a gauge that satisfies these first-class constraints. One such gauge is the radiation or Coulomb gauge

$$\nabla \cdot \boldsymbol{A} = 0. \tag{3.79}$$

In this gauge, the first-class constraints say that

$$-\partial_i F^{i0} = -\nabla^2 A^0 + \nabla \cdot \dot{\nabla} \boldsymbol{A} = -\nabla^2 A^0 = J^0. \tag{3.80}$$

So we can solve for A^0

$$A^0(\boldsymbol{x},t) = \int d^3y \, \frac{J^0(\boldsymbol{y},t)}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}. \tag{3.81}$$

Once we have eliminated A^0 and π^0 from the theory, we list the second-class constraints

$$\begin{aligned} \chi_1(\boldsymbol{x},t) &= \nabla\cdot\boldsymbol{A}(\boldsymbol{x},t) = 0 \\ \chi_2(\boldsymbol{x},t) &= \nabla\cdot\boldsymbol{\pi}(\boldsymbol{x},t) + J^0(\boldsymbol{x},t) = 0. \end{aligned} \tag{3.82}$$

Their Poisson bracket is

$$\begin{aligned} [\chi_1(\boldsymbol{x},t),\chi_2(\boldsymbol{y},t)]_P = \sum_i \int d^3z \, & \frac{\partial\nabla\cdot\boldsymbol{A}(\boldsymbol{x},t)}{\partial A_i(\boldsymbol{z})} \frac{\partial\nabla\cdot\boldsymbol{\pi}(\boldsymbol{y},t)+J^0(\boldsymbol{y},t)}{\partial\pi_i(\boldsymbol{z})} \\ & - \frac{\partial\nabla\cdot\boldsymbol{\pi}(\boldsymbol{y},t)+J^0(\boldsymbol{y},t)}{\partial A_i(\boldsymbol{z})} \frac{\partial\nabla\cdot\boldsymbol{A}(\boldsymbol{x},t)}{\partial\pi_i(\boldsymbol{z})}. \end{aligned} \tag{3.83}$$

The second terms vanish as do terms with J^0. Since

$$\frac{\partial\nabla\cdot\boldsymbol{A}(\boldsymbol{x},t)}{\partial A_i(\boldsymbol{z})} = \partial_i\delta^3(\boldsymbol{x}-\boldsymbol{z}) \quad \text{and} \quad \frac{\partial\nabla\cdot\boldsymbol{\pi}(\boldsymbol{y},t)}{\partial\pi_i(\boldsymbol{z})} = \partial_i\delta^3(\boldsymbol{y}-\boldsymbol{z}), \tag{3.84}$$

the Poisson bracket is

$$[\chi_1(\boldsymbol{x},t),\chi_2(\boldsymbol{y},t)]_P = -\nabla^2\delta^3(\boldsymbol{x}-\boldsymbol{y}). \tag{3.85}$$

The other Poisson brackets vanish, so

$$\begin{aligned} C_{1x,2y} &= -\nabla^2\delta^3(\boldsymbol{x}-\boldsymbol{y}) = -C_{2y,1x} \\ C_{1x,1y} &= 0 \\ C_{2x,2y} &= 0. \end{aligned} \tag{3.86}$$

The matrix C is nonsingular, $\det C \neq 0$. Its inverse is

$$(C^{-1})^{1x,2y} = -\int \frac{d^3k}{(2\pi)^3} \frac{e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})}}{\boldsymbol{k}^2} = -\frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{y}|} = -(C^{-1})^{2y,1x} \tag{3.87}$$

in which the last minus sign is the one I missed in class. (Incidentally, there are typos in the b-ok.org versions of Weinberg's QTFI which I put on the class website. Does anyone have a pdf of the 2005 paperback edition?)

To check the formula (3.87) for the inverse C^{-1}, we do the integral

$$\begin{aligned}
\int d^3z\, C_{1x,2z}(C^{-1})^{2z,1y} &= -\int d^3z\, C_{1x,2z}(C^{-1})^{1y,2z} \\
&= -\int d^3z\, [-\nabla^2\delta^3(\boldsymbol{x}-\boldsymbol{z})][-\int \frac{d^3k}{(2\pi)^3}\,\frac{e^{i\boldsymbol{k}\cdot(\boldsymbol{y}-\boldsymbol{z})}}{\boldsymbol{k}^2}] \\
&= -\int d^3z\, \delta^3(\boldsymbol{x}-\boldsymbol{z})\nabla^2 \int \frac{d^3k}{(2\pi)^3}\,\frac{e^{i\boldsymbol{k}\cdot(\boldsymbol{y}-\boldsymbol{z})}}{\boldsymbol{k}^2} \\
&= \int d^3z\, \delta^3(\boldsymbol{x}-\boldsymbol{z}) \int \frac{d^3k}{(2\pi)^3}\, e^{i\boldsymbol{k}\cdot(\boldsymbol{y}-\boldsymbol{z})} \\
&= \int d^3z\, \delta^3(\boldsymbol{x}-\boldsymbol{z})\delta(\boldsymbol{y}-\boldsymbol{z}) = \delta(\boldsymbol{x}-\boldsymbol{y}). \qquad (3.88)
\end{aligned}$$

To evaluate the Dirac bracket $[A^i(\boldsymbol{x},t), \pi_k(\boldsymbol{y},t)]_D$ we need to compute the Poisson brackets

$$\begin{aligned}
[A^i(\boldsymbol{x},t), \chi_{2y}]_P &= \sum_k \int d^3z \left[\frac{\partial A^i(\boldsymbol{x},t)}{\partial A_k(\boldsymbol{z},t)}\frac{\partial\nabla\cdot\boldsymbol{\pi}(\boldsymbol{y},t)+J^0(\boldsymbol{y},t)}{\partial\pi_k(\boldsymbol{z},t)}\right. \\
&\qquad \left. -\frac{\partial\nabla\cdot\boldsymbol{\pi}(\boldsymbol{y},t)+J^0(\boldsymbol{y},t)}{\partial A_k(\boldsymbol{z},t)}\frac{\partial A^i(\boldsymbol{x},t)}{\partial\pi_k(\boldsymbol{z},t)}\right] \\
&= \sum_k \int d^3z\, \frac{\partial A^i(\boldsymbol{x},t)}{\partial A_k(\boldsymbol{z},t)}\frac{\partial\nabla\cdot\boldsymbol{\pi}(\boldsymbol{y},t)}{\partial\pi_k(\boldsymbol{z},t)} \qquad (3.89) \\
&= \sum_k \int d^3z\, \delta^i_k\delta^3(\boldsymbol{x}-\boldsymbol{z})\,\partial_k\delta^3(\boldsymbol{y}-\boldsymbol{z}) \\
&= \frac{\partial}{\partial y^i}\,\delta^3(\boldsymbol{x}-\boldsymbol{y}) = -\frac{\partial}{\partial x^i}\,\delta^3(\boldsymbol{x}-\boldsymbol{y})
\end{aligned}$$

and

$$
\begin{aligned}
[\pi_i(\boldsymbol{x},t),\chi_{1y}]_P &= \sum_k \int d^3z \left[\frac{\partial \pi_i(\boldsymbol{x},t)}{A_k(\boldsymbol{z},t)}\frac{\partial \nabla\cdot\boldsymbol{A}(\boldsymbol{y},t)}{\partial \pi_k(\boldsymbol{z},t)}\right.\\
&\qquad \left. - \frac{\partial \nabla\cdot\boldsymbol{A}(\boldsymbol{y},t)}{A_k(\boldsymbol{z},t)}\frac{\partial \pi_i(\boldsymbol{x},t)}{\partial \pi_k(\boldsymbol{z},t)}\right]\\
&= -\int d^3z\, \frac{\partial \nabla\cdot\boldsymbol{A}(\boldsymbol{y},t)}{A_k(\boldsymbol{z},t)}\frac{\partial \pi_i(\boldsymbol{x},t)}{\partial \pi_k(\boldsymbol{z},t)}\\
&= -\int d^3z\, \frac{\partial}{\partial y_k}\,\delta(\boldsymbol{y}-\boldsymbol{z})\,\delta_{ki}\,\delta(\boldsymbol{x}-\boldsymbol{z})\\
&= -\int d^3z\, \frac{\partial}{\partial y_i}\,\delta(\boldsymbol{y}-\boldsymbol{z})\,\delta(\boldsymbol{x}-\boldsymbol{z})\\
&= -\frac{\partial}{\partial y_i}\,\delta(\boldsymbol{y}-\boldsymbol{x})\\
&= \frac{\partial}{\partial x_i}\,\delta(\boldsymbol{y}-\boldsymbol{x}) = \frac{\partial}{\partial x_i}\,\delta(\boldsymbol{x}-\boldsymbol{y}).
\end{aligned}
\tag{3.90}
$$

So finally we compute the Dirac bracket (3.72) of $A^i(x)$ with $\pi_k(y)$ at equal times

$$
\begin{aligned}
[A^i(\boldsymbol{x},t),&\pi_k(\boldsymbol{y},t)]_D = [A^i(\boldsymbol{x},t),\pi_k(\boldsymbol{y},t)]_P \\
&- \int d^3z\, d^3w\, [A^i(\boldsymbol{x},t),\chi_2(\boldsymbol{z},t)]_P (C^{-1})^{2z,1w}[\chi_1(\boldsymbol{w},t),\pi_k(\boldsymbol{y},t)]_P.
\end{aligned}
\tag{3.91}
$$

The first Poisson bracket is

$$
\begin{aligned}
[A^i(\boldsymbol{x},t),\pi_k(\boldsymbol{y},t)]_P &= \sum_j \int d^3z \left[\frac{\partial A^i(\boldsymbol{x},t)}{\partial A_j(\boldsymbol{z})}\frac{\partial \pi_k(\boldsymbol{y},t)}{\partial \pi_j(\boldsymbol{z})} - \frac{\partial \pi_k(\boldsymbol{y},t)}{\partial A_j(\boldsymbol{z})}\frac{\partial A^i(\boldsymbol{x},t)}{\partial \pi_j(\boldsymbol{z})}\right]\\
&= \sum_j \int d^3z\, \frac{\partial A^i(\boldsymbol{x},t)}{\partial A_j(\boldsymbol{z})}\frac{\partial \pi_k(\boldsymbol{y},t)}{\partial \pi_j(\boldsymbol{z})}\\
&= \sum_j \int d^3z\, \delta_{ij}\delta^3(\boldsymbol{x}-\boldsymbol{z})\delta_{jk}\delta^3(\boldsymbol{y}-\boldsymbol{z}) = \delta_{ik}\,\delta^3(\boldsymbol{x}-\boldsymbol{y}).
\end{aligned}
\tag{3.92}
$$

The matrix element (3.87) of C^{-1} is

$$
(C^{-1})^{2z,1w} = \frac{1}{4\pi|\boldsymbol{z}-\boldsymbol{w}|}. \tag{3.93}
$$

The Poisson brackets (3.89) and (3.90) are

$$
\begin{aligned}
[A^i(\boldsymbol{x},t),\chi_{2y}]_P &= -\frac{\partial}{\partial x^i}\,\delta^3(\boldsymbol{x}-\boldsymbol{y})\\
[\pi_i(\boldsymbol{x},t),\chi_{1y}]_P &= \frac{\partial}{\partial x_i}\,\delta(\boldsymbol{x}-\boldsymbol{y}).
\end{aligned}
\tag{3.94}
$$

So the second term T_2 in the Dirac bracket (3.91) is

$$\begin{aligned}
T_2 &= -\int d^3z\, d^3w\, [A^i(\boldsymbol{x},t), \chi_2(\boldsymbol{z},t)]_P \frac{1}{4\pi|\boldsymbol{z}-\boldsymbol{w}|}[\chi_1(\boldsymbol{w},t), \pi_k(\boldsymbol{y},t)]_P \\
&= \int d^3z\, d^3w\, \frac{\partial}{\partial x^i}\, \delta^3(\boldsymbol{x}-\boldsymbol{z}) \frac{1}{4\pi|\boldsymbol{z}-\boldsymbol{w}|} \frac{\partial}{\partial w_k}\, \delta(\boldsymbol{w}-\boldsymbol{y}) \\
&= \int d^3z\, d^3w\, \delta^3(\boldsymbol{x}-\boldsymbol{z}) \frac{\partial}{\partial z^i}\, \frac{1}{4\pi|\boldsymbol{z}-\boldsymbol{w}|} \frac{\partial}{\partial w_k}\, \delta(\boldsymbol{w}-\boldsymbol{y}) \\
&= \int d^3z\, d^3w\, \delta^3(\boldsymbol{x}-\boldsymbol{z}) \frac{\partial^2}{\partial z^i \partial z^k}\, \frac{1}{4\pi|\boldsymbol{z}-\boldsymbol{w}|}\, \delta(\boldsymbol{w}-\boldsymbol{y}) \\
&= \frac{\partial^2}{\partial x^i \partial x^k}\, \frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}.
\end{aligned} \tag{3.95}$$

So Dirac's rule of quantization says that the commutator of A^i with π_k is the **transverse delta function**

$$\begin{aligned}
[A^i(\boldsymbol{x},t), \pi_k(\boldsymbol{y},t)] &= i[A^i(\boldsymbol{x},t), \pi_k(\boldsymbol{y},t)]_D \\
&= i\delta_{ik}\, \delta^3(\boldsymbol{x}-\boldsymbol{y}) + i\frac{\partial^2}{\partial x^i \partial x^k}\, \frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}
\end{aligned} \tag{3.96}$$

and that

$$[A^i(\boldsymbol{x},t), A^k(\boldsymbol{y},t)] = 0 \quad \text{and} \quad [\pi_i(\boldsymbol{x},t), \pi_k(\boldsymbol{y},t)] = 0. \tag{3.97}$$

Apart from matter terms, the hamiltonian for QED is

$$H = \int d^3x \big[\boldsymbol{\pi}_\perp \cdot \dot{\boldsymbol{A}} - L\big] \tag{3.98}$$

in which $\boldsymbol{\pi}_\perp$ is the transverse part of $\boldsymbol{\pi}$, that is, the part that has no divergence

$$\nabla \cdot \boldsymbol{\pi}_\perp = 0. \tag{3.99}$$

The longitudinal part $\boldsymbol{\pi}_\parallel$ of $\boldsymbol{\pi}$ does not contribute because it is the gradient of a scalar S which when dotted into a transverse vector vanishes as an integration by parts shows

$$\int d^3x\, \nabla S \cdot \dot{\boldsymbol{A}} = -\int d^3x\, S\, \nabla \cdot \dot{\boldsymbol{A}} = 0. \tag{3.100}$$

So we can take the conjugate momentum $\boldsymbol{\pi}$ to be transverse

$$\nabla \cdot \boldsymbol{\pi}_\perp = 0. \tag{3.101}$$

in the term $\boldsymbol{\pi} \cdot \dot{\boldsymbol{A}}$ The transverse part $\boldsymbol{\pi}_\perp$ is

$$\boldsymbol{\pi}_\perp = \frac{\partial L}{\partial \dot{\boldsymbol{A}}} - \nabla A^0 = \dot{\boldsymbol{A}}. \tag{3.102}$$

The hamiltonian for QED is then

$$H = \int d^3x \big[\boldsymbol{\pi}_\perp \cdot \dot{\boldsymbol{A}} - L\big] = \int d^3x \big[{\boldsymbol{\pi}_\perp}^2 - L\big] = \int d^3x \big[\dot{\boldsymbol{A}}^2 - L\big] \tag{3.103}$$

in which L is the action density

$$L = -\tfrac{1}{4}F_{ik}F^{ik} + J_iA^i = \tfrac{1}{2}(\dot{\boldsymbol{A}} + \nabla A^0)^2 - \tfrac{1}{2}(\nabla \times \boldsymbol{A})^2 + J_iA^i. \tag{3.104}$$

So H is

$$H = \int d^3x \left[{\boldsymbol{\pi}_\perp}^2 - \tfrac{1}{2}(\boldsymbol{\pi}_\perp + \nabla A^0)^2 + \tfrac{1}{2}(\nabla \times \boldsymbol{A})^2 - \boldsymbol{J} \cdot \boldsymbol{A} + J^0A^0\right] \tag{3.105}$$

apart from terms due to the matter fields. The integral of $\boldsymbol{\pi}_\perp \cdot \nabla A^0$ vanishes so H is

$$H = \int d^3x \left[\tfrac{1}{2}{\boldsymbol{\pi}_\perp}^2 + \tfrac{1}{2}(\nabla \times \boldsymbol{A})^2 - \boldsymbol{J} \cdot \boldsymbol{A} + J^0A^0 - \tfrac{1}{2}(\nabla A^0)^2\right]. \tag{3.106}$$

By Gauss's law and the Coulomb gauge condition $\nabla \cdot \boldsymbol{A} = 0$

$$-\partial_i F^{i0} = -\nabla^2 A^0 = J^0, \tag{3.107}$$

so the last term in H is

$$\int d^3x\, -\tfrac{1}{2}(\nabla A^0)^2 = \int d^3x\, \tfrac{1}{2}A^0\nabla^2 A^0 = \int d^3x\, -\tfrac{1}{2}A^0J^0. \tag{3.108}$$

The hamiltonian then is

$$H = \int d^3x \left[\tfrac{1}{2}{\boldsymbol{\pi}_\perp}^2 + \tfrac{1}{2}(\nabla \times \boldsymbol{A})^2 - \boldsymbol{J} \cdot \boldsymbol{A} + \tfrac{1}{2}J^0A^0\right] \tag{3.109}$$

in which the last term is the Coulomb energy

$$V_C = \tfrac{1}{2}\int d^3x\, J^0A^0 = \frac{1}{2}\int d^3x\, d^3y\, \frac{J^0(\boldsymbol{x},t)J^0(\boldsymbol{y},t)}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}. \tag{3.110}$$

One may satisfy the commutation relations (3.96 & 3.97) by setting

$$\boldsymbol{A}(x) = \int \frac{d^3p}{\sqrt{(2\pi)^3 2p^0}} \sum_s \left[e^{ip\cdot x}\, \boldsymbol{e}(p,s)\, a(p,s) + e^{-ip\cdot x}\, \boldsymbol{e}^*(p,s)\, a^\dagger(p,s)\right] \tag{3.111}$$

and using $\boldsymbol{\pi}_\perp(x) = \dot{\boldsymbol{A}}$

$$\boldsymbol{\pi}_\perp(x) = -i\int \frac{\sqrt{p^0}\, d^3p}{\sqrt{(2\pi)^3 2}} \sum_s \left[e^{ip\cdot x}\, \boldsymbol{e}(p,s)\, a(p,s) - e^{-ip\cdot x}\, \boldsymbol{e}^*(p,s)\, a^\dagger(p,s)\right] \tag{3.112}$$

in both of which $x = (\boldsymbol{x}, t)$ and

$$[a(p,s), a^\dagger(q,t)] = \delta_{st}\, \delta^3(\boldsymbol{p}-\boldsymbol{q}) \quad \text{and} \quad [a(p,s), a(q,t)] = 0. \tag{3.113}$$

One gets the transverse delta function because the two polarization 3-vectors $\boldsymbol{e}(p,s)$ are perpendicular to $\boldsymbol{p}$ and obey the sum rule

$$\sum_s \boldsymbol{e}^i(p,s)\,\boldsymbol{e}^{k*}(p,s) = \delta_{ik} - \frac{p_i p_k}{\boldsymbol{p}^2}. \tag{3.114}$$

The 3-vectors $\boldsymbol{e}(p,s)$ can be thought of as the vector parts of 4-vectors with vanishing time components.

4

Quantum Electrodynamics

4.1 Global $U(1)$ Symmetry

In most theories of charged fields, the action density is invariant when the charged fields change by a phase transformation

$$\psi'_\ell(x) = e^{i\theta_\ell}\psi_\ell(x). \tag{4.1}$$

If the phase θ is independent of x, then the symmetry is called a **global $U(1)$ symmetry**.

Section 3.4 describes Noether's theorem according to which a current

$$J^a = \sum_\ell \frac{\partial L}{\partial \partial_a \psi_\ell} \delta\psi_\ell \tag{4.2}$$

has zero divergence

$$\partial_a J^a = 0. \tag{4.3}$$

The charge

$$Q = \int J^0 \, d^3x \tag{4.4}$$

is conserved due to the global $U(1)$ symmetry. Here for tiny θ_ℓ, $\delta\psi_\ell = i\theta_\ell\psi_\ell$, and the charge density J^0 is

$$J^0 = \sum_\ell \frac{\partial L}{\partial \partial_0 \psi_\ell} \delta\psi_\ell = \sum_\ell \frac{\partial L}{\partial \partial_0 \psi_\ell} i\theta_\ell\psi_\ell = \sum_\ell \pi_\ell i\theta_\ell\psi_\ell. \tag{4.5}$$

One imagines that the angle is proportional to the charge of the field ψ_ℓ, $\theta_\ell = q_\ell\theta$.

4.2 Abelian gauge invariance

Quantum electrodynamics is a theory in which the action density is invariant when the charged fields change by a phase transformation that varies with the spacetime point x

$$\psi'_\ell(x) = e^{i\theta_\ell(x)}\psi_\ell(x). \tag{4.6}$$

Such a symmetry is called a **local $U(1)$ symmetry**. A theory with a local $U(1)$ symmetry also has a global $U(1)$ symmetry and so it conserves charge.

Quantities like $\psi^\dagger_\ell(x)\,\psi_\ell(x)$ are intrinsically invariant under local $U(1)$ symmetries because

$$(\psi^\dagger_\ell(x))'\,\psi'_\ell(x) = \psi^\dagger_\ell(x)e^{-i\theta_\ell(x)}\,e^{i\theta_\ell(x)}\psi(x) = \psi^\dagger_\ell(x)\,\psi(x). \tag{4.7}$$

Derivatives of fields present problems, however, because

$$\partial_a\big(e^{i\theta_\ell(x)}\psi_\ell(x)\big) = e^{i\theta_\ell(x)}\left[\partial_a\big(\psi_\ell(x)\big) + i(\partial_a\theta(x))\,\psi_\ell(x)\right] \neq e^{i\theta_\ell(x)}\partial_a\big(\psi_\ell(x)\big) \tag{4.8}$$

so things like $\psi^\dagger_\ell(x)\,\partial_a\psi_\ell(x)$ and like $(\partial^a\psi^\dagger_\ell(x))\,\partial_a\psi_\ell(x)$ are not invariant under local phase transformations.

The trick is to introduce a field $A_a(x)$ that transforms so as to cancel the awkward term $e^{i\theta_\ell}i(\partial_a\theta)\,\psi_\ell$ in the derivative (4.8). We want a new derivative $D_a\psi_\ell$ that transforms like ψ_ℓ .

$$(D_a\psi_\ell)' = e^{i\theta_\ell}D_a\psi_\ell. \tag{4.9}$$

So we set

$$D_a = \partial_a + iA_a \tag{4.10}$$

and require that

$$(D_a\psi_\ell)' = (\partial_a + iA'_a)\psi'_a = e^{i\theta_\ell}D_a\psi_\ell = e^{i\theta_\ell}(\partial_a + iA_a)\psi_\ell. \tag{4.11}$$

That is, we insist that

$$(\partial_a + iA'_a)\psi'_a = (\partial_a + iA'_a)e^{i\theta_\ell}\psi_a = e^{i\theta_\ell}(\partial_a + iA_a)\psi_\ell. \tag{4.12}$$

So we need

$$i\partial_a\theta_\ell + iA'_a = iA_a \tag{4.13}$$

or

$$A'_a(x) = A_a(x) - \partial_a\theta_\ell. \tag{4.14}$$

This is what we need except that the field A_a does not carry the index ℓ.

The solution to this problem is to define the symmetry transformation (4.6) so that the angle θ_ℓ is proportional to the charge of the field ψ_ℓ

$$\psi'_\ell(x) = e^{iq_\ell\theta(x)}\psi_\ell(x). \tag{4.15}$$

This definition has the advantage that the charge density (4.5) becomes

$$J^0 = i\theta \sum_\ell q_\ell\, \pi_\ell \psi_\ell \tag{4.16}$$

which makes more sense than the old formula (4.5). More importantly, the definition (4.15) means that equations (4.9–4.22) change to

$$(D_a\psi_\ell)' = e^{iq_\ell\theta} D_a\psi_\ell. \tag{4.17}$$

So we make the **covariant derivative**

$$D_a\psi_\ell = (\partial_a + iq_\ell A_a)\psi_\ell \tag{4.18}$$

depend upon the field ψ_ℓ and require that

$$(D_a\psi_\ell)' = (\partial_a + iq_\ell A'_a)\psi'_a = e^{iq_\ell\theta} D_a\psi_\ell = e^{iq_\ell\theta}(\partial_a + iq_\ell A_a)\psi_\ell. \tag{4.19}$$

That is, we insist that

$$(\partial_a + iq_\ell A'_a)\psi'_a = (\partial_a + iq_\ell A'_a)e^{iq_\ell\theta}\psi_a = e^{iq_\ell\theta}(\partial_a + iq_\ell A_a)\psi_\ell. \tag{4.20}$$

So we need

$$iq_\ell\partial_a\theta + iq_\ell A'_a = iq_\ell A_a \tag{4.21}$$

or

$$A'_a(x) = A_a(x) - \partial_a\theta(x) \tag{4.22}$$

which is much better than the old rule (4.14). The twin rules

$$\begin{aligned} \psi'_\ell(x) &= e^{iq_\ell\theta(x)}\psi_\ell(x) \\ A'_a(x) &= A_a(x) - i\partial_a\theta(x) \end{aligned} \tag{4.23}$$

constitute an **abelian gauge transformation**.

Note that a field ψ_ℓ couples to the electromagnetic field A_a in a way $q_\ell A_a\psi_\ell$ that is proportional to its charge q_ℓ.

To insure that the right charge q_ℓ appears in the right place, we can introduce a charge operator q such that $q\,\psi_\ell = q_\ell\psi_\ell$ and redefine the abelian gauge transformation (4.23) as

$$\begin{aligned} \psi'_\ell(x) &= e^{iq\theta(x)}\psi_\ell(x) = e^{iq_\ell\theta(x)}\psi_\ell(x) \\ A'_a(x) &= A_a(x) - \partial_a\theta(x). \end{aligned} \tag{4.24}$$

So

$$\partial_b A'_a = \partial_b A_a - \partial_b \partial_a \theta \tag{4.25}$$

is not invariant, but the antisymmetric combination

$$\partial_b A'_a - \partial_a A'_b = \partial_b A_a - \partial_b \partial_a \theta - \partial_a A_b + \partial_a \partial_b \theta = \partial_b A_a - \partial_a A_b \tag{4.26}$$

is invariant. Maxwell introduced this combination

$$F_{ba} = \partial_b A_a - \partial_a A_b. \tag{4.27}$$

Thus

$$L = -\frac{1}{4} F_{ba} F^{ba} \tag{4.28}$$

is a Lorentz-invariant, gauge-invariant action density for the electromagnetic field.

4.3 Coulomb-gauge quantization

The first step in the canonical quantization of a gauge theory is to pick a gauge. The most physical gauge for electrodynamics is the Coulomb gauge defined by the gauge condition

$$0 = \nabla \cdot \vec{A}. \tag{4.29}$$

If the action density (4.28) is modified by an interaction with a current J^a

$$L = -\frac{1}{4} F_{ba} F^{ba} + A_a J^a \tag{4.30}$$

then the equation of motion is

$$\partial_b F^{ba} = -J^a \tag{4.31}$$

while the homogeneous equations

$$0 = \partial_a F_{bc} + \partial_b F_{ca} + \partial_c F_{ab} \tag{4.32}$$

follow from the antisymmetry of F_{ab}.

Because of the antisymmetry of F_{ab}, the derivative $\dot{A}_0$ does not appear in the action density (4.30). So for $a = 0$, the equation of motion (4.31) actually is a **constraint**

$$\partial_i F^{i0} = -\partial_i E^i = -J^0 \tag{4.33}$$

known as Gauss's law

$$\nabla \cdot \vec{E} = \rho = J^0. \tag{4.34}$$

This constraint together with the Coulomb gauge condition (4.29) lets us express A^0 in terms of the charge density $\rho = J^0$. We find

$$\nabla^2 A_0 - \partial_0 \nabla \cdot \vec{A} = \nabla \cdot E = J^0 \tag{4.35}$$

or

$$\nabla^2 A_0 = J^0. \tag{4.36}$$

The solution is

$$A^0(\vec{x}, t) = \int \frac{J^0(\vec{y}, t)}{4\pi|\vec{x} - \vec{y}|}\, d^3y. \tag{4.37}$$

The quantum fields in this gauge are the transverse parts of $\vec{A}$ and their conjugate momentum $\vec{E}$.

4.4 QED in the interaction picture

The Coulomb-gauge hamiltonian in the interaction picture is

$$\begin{aligned} H &= H_0 + V \\ H_0 &= \frac{1}{2}\int \vec{E}^2 + \vec{B}^2\, d^3x + H_{\psi,0} \\ V &= -\int \vec{J} \cdot \vec{A}\, d^3x + V_C + V_\psi \end{aligned} \tag{4.38}$$

in which $B = \nabla \times \vec{A}$ and both $\vec{E}$ and $\vec{A}$ are travsverse, that is

$$\nabla \cdot \vec{E} = 0 \quad \text{and} \quad \nabla \cdot \vec{A} = 0 \tag{4.39}$$

and the Coulomb potential energy is

$$V_C = \frac{1}{2}\int \frac{J^0(\vec{x}, 0) J^0(\vec{y}, 0)}{4\pi|\vec{x} - \vec{y}|}\, d^3x d^3y. \tag{4.40}$$

The electromagnetic field is

$$A^b(x) = \int \frac{d^3p}{\sqrt{(2\pi)^3 2p^0}} \sum_s \left[e^{ip\cdot x} e^b(p, s) a(p, s) + e^{-ip\cdot x} e^{b*}(p, s) a^\dagger(p, s) \right]. \tag{4.41}$$

The polarization vectors may be chosen to be

$$e^b(p, \pm 1) = R(\hat{p}) \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ \pm i \\ 0 \end{pmatrix} \tag{4.42}$$

where $R(\hat{p})$ rotates $\hat{z}$ into $\hat{p}$. Thus

$$\vec{p}\cdot\vec{e}(p,s)=0 \tag{4.43}$$

and

$$e^0(p,s)=0 \tag{4.44}$$

because A^0 is a dependent variable (4.37). The commutation relations are

$$[a(p,s),a^\dagger(p',s')]=\delta_{ss'}\,\delta^3(\vec{p}-\vec{p}') \quad \text{and} \quad [a(p,s),a(p',s')]=0. \tag{4.45}$$

The term H_0 in the hamiltonian (4.38) is

$$\begin{aligned} H_0 &= \int\sum_s \frac{1}{2}p^0[a(p,s),a^\dagger(p,s)]_+\,d^3p \\ &= \int\sum_s p^0\left(a^\dagger(p,s)a(p,s)+\frac{1}{2}\delta^3(\vec{p}-\vec{p})\right)d^3p. \end{aligned} \tag{4.46}$$

The interaction is

$$V(t)=e^{iH_0t}\Big[-\int\vec{J}(\vec{x},0)\cdot\vec{A}(\vec{x},0)d^3x+\int\frac{J^0(\vec{x},0)J^0(\vec{y},0)}{8\pi|\vec{x}-\vec{y}|}d^3xd^3y+V_m(0)\Big]e^{-iH_0t} \tag{4.47}$$

in which $V_m(0)$ is the non-electromagnetic part of the matter interaction. Since $A^0=0$, $\vec{J}\cdot\vec{A}=J\cdot A$.

4.5 Photon propagator

The photon propagator is

$$-i\Delta_{ab}(x-y)=\langle 0|T[A_a(x),A_b(y)]|0\rangle. \tag{4.48}$$

Inserting the formula for the electromagnetic field (4.41), we get

$$\begin{aligned} -i\Delta_{ij}(x-y) &= \int\frac{d^3p}{(2\pi)^3 2p^0}\left(\delta_{ij}-\frac{p_ip_j}{\vec{p}^2}\right) \\ &\quad\times\left[e^{ip\cdot(x-y)}\theta(x^0-y^0)+e^{-ip\cdot(x-y)}\theta(y^0-x^0)\right] \\ &= \int\frac{d^4q}{(2\pi)^4}\frac{-i}{q^2-i\epsilon}\left(\delta_{ij}-\frac{q_iq_j}{\vec{q}^2}\right)e^{iq\cdot(x-y)} \end{aligned} \tag{4.49}$$

with the understanding that this propagator vanishes when a or b is zero. If $n^c=(1,0,0,0)$ has only a time component, one can write the $\delta_{ij}-q_iq_j/\vec{q}^2$ as

$$\left(\delta_{ab}-\frac{q_aq_b}{\vec{q}^2}\right)\theta(ab)=\eta_{ab}+\frac{q^0(q_an_b+q_bn_a)-q_aq_b+q^2n_an_b}{\vec{q}^2} \tag{4.50}$$

in which q^0 is arbitrary and may be chosen to be determined by conservation of energy. The terms $q_a n_b$, $q_b n_a$, and $q_a q_b$ act like $\partial_a n_b$, $\partial_b n_a$, and $\partial_a \partial_b$ so they appear in the S-matrix as $\partial_a J^a n_b$, and $\partial_b J^b n_a$, etc., which vanish because of current conservation. The remaining term

$$\frac{q^2 n_a n_b}{\vec{q}^2} \frac{-i}{q^2 - i\epsilon} = \frac{-i n_a n_b}{\vec{q}^2} \tag{4.51}$$

gives in the S-matrix a term

$$\begin{aligned} T &= \frac{1}{2} \int d^4x d^4y [-iJ^0(x)][-iJ^0(y)] \frac{-i}{(2\pi)^4} \int \frac{d^4q}{\vec{q}^2} e^{iq\cdot(x-y)} \\ &= i\frac{1}{2} \int d^4x d^4y J^0(x) J^0(y) \frac{d^3q}{(2\pi)^3} \frac{\delta(x^0 - y^0)}{\vec{q}^2} e^{i\vec{q}\cdot(\vec{x}-\vec{y})} \qquad (4.52) \\ &= i\frac{1}{2} \int d^3x\, d^3y\, dt\, \frac{J^0(\vec{x},t) J^0(\vec{y},t)}{4\pi|\vec{x}-\vec{y}|} \qquad (4.53) \end{aligned}$$

which cancels the Coulomb term

$$T_C = -i\frac{1}{2} \int d^3x\, d^3y\, dt\, \frac{J^0(\vec{x},t) J^0(\vec{y},t)}{4\pi|\vec{x}-\vec{y}|}. \tag{4.54}$$

The bottom line is that the effective photon propagator is

$$-i\Delta_{ab}(x-y) = \int \frac{d^4q}{(2\pi)^4} \frac{-i\eta_{ab}}{q^2 - i\epsilon} e^{iq\cdot(x-y)} = -i\Delta_{ab}(y-x). \tag{4.55}$$

Dirac's action density is

$$L_\psi = -\bar{\psi}(\gamma^a \partial_a + m)\psi \tag{4.56}$$

with $\bar{\psi} = i\psi^\dagger \gamma^0$. The conjugate momentum is

$$\pi = \frac{\partial L}{\partial \dot{\psi}} = -\bar{\psi}\gamma^0 = -i\psi^\dagger \gamma^0 \gamma^0 = i\psi^\dagger. \tag{4.57}$$

The hamiltonian is

$$\begin{aligned} H &= \int (\pi\dot{\psi} - L) d^3x = \int [i\psi^\dagger \dot{\psi} + \bar{\psi}(\gamma^a \partial_a + m)\psi]\, d^3x \\ &= \int \bar{\psi}(\vec{\gamma}\cdot\vec{\nabla} + m)\psi\, d^3x. \end{aligned} \tag{4.58}$$

The free Dirac field aka the Dirac field in the interaction picture is

$$\psi(x) = \int \frac{d^3p}{(2\pi)^{3/2}} \sum_s \left[u(p,s) e^{ipx} a(p,s) + v(p,s) e^{-ipx} b^\dagger(p,s) \right] \tag{4.59}$$

The addition and deletion operators obey the anticommutation relations

$$[a(p,s), a^\dagger(p',s')]_+ = [b(p,s), b^\dagger(p',s')]_+ = \sigma_{ss'}\,\delta^3(\vec{p}-\vec{p}') \tag{4.60}$$

with the other anticommutators equal to zero. Putting $\psi(x)$ into H gives

$$H = \int \sum_s p^0 \left[a^\dagger(p,s)a(p,s) + b^\dagger(p,s)b(p,s) - \delta^3(\vec{p}-\vec{p})\right] d^3p. \tag{4.61}$$

4.6 Feynman's rules for QED

The action density is

$$L = -\frac{1}{4}F_{ab}F^{ab} - \bar{\psi}[\gamma^a(\partial_a + ieA_a) + m]\psi. \tag{4.62}$$

The electric current is

$$J^a = \frac{\partial L}{\partial A_a} = -ie\bar{\psi}\gamma^a\psi. \tag{4.63}$$

The interaction is

$$V(t) = ie\int \bar{\psi}(\vec{x},t)\gamma^a\psi(\vec{x},t)A_a(\vec{x},t)\,d^3x + V_C(t) \tag{4.64}$$

Draw all appropriate diagrams.

Label each vertex with $\alpha = 1, 2, 3,$ or 4 on an electron line of momentum p entering the vertex and $\beta = 1, .., 4$ on an electron line of momentum p' leaving the vertex, and an index a on the photon line of momentum q. The vertex carries the factor

$$(2\pi)^4 e\gamma^a_{\beta\alpha}\delta^4(p - p' + q). \tag{4.65}$$

An outgoing electron line gives

$$\frac{\bar{u}_\beta(p,s)}{(2\pi)^{3/2}}. \tag{4.66}$$

An outgoing positron line gives

$$\frac{v_\alpha(p,s)}{(2\pi)^{3/2}}. \tag{4.67}$$

An incoming electron line gives

$$\frac{u_\alpha(p,s)}{(2\pi)^{3/2}}. \tag{4.68}$$

An incoming positron line gives

$$\frac{\bar{v}_\beta(p,s)}{(2\pi)^{3/2}}. \tag{4.69}$$

An outgoing photon gives

$$\frac{e_a^*(p,s)}{\sqrt{(2\pi)^3 2p^0}}. \tag{4.70}$$

An incoming photon gives

$$\frac{e_a(p,s)}{\sqrt{(2\pi)^3 2p^0}}. \tag{4.71}$$

An internal electron line of momentum p from vertex β to vertex α gives

$$\frac{-i}{(2\pi)^4}\,\frac{(-i\not{p}+m)_{\alpha\beta}}{p^2+m^2-i\epsilon}. \tag{4.72}$$

An internal photon line of momentum q linking vertexes a and b gives

$$\frac{-i}{(2\pi)^4}\,\frac{\eta_{ab}}{q^2-i\epsilon}. \tag{4.73}$$

Integrate over all momenta, sum over all indexes.
Add up all terms and get the combinatorics and minus signs right.

4.7 Electron-positron scattering

There are two Feynman diagrams for electron-positron scattering at order e^2 where $\alpha = e^2/(4\pi\epsilon_0\hbar c) = 0.00729735256 \approx 1/137$ is the fine-structure constant. The direct or s-channel diagram is represented in Fig. 4.1.

S-channel electron-positron scattering

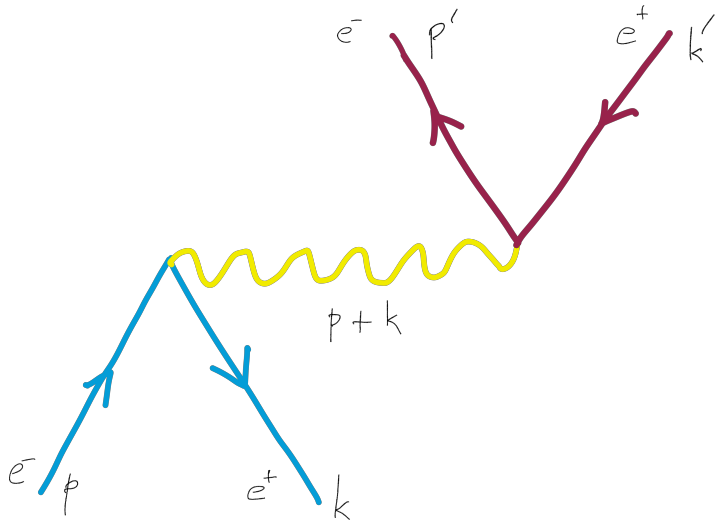

Figure 4.1 Direct or s-channel diagram for electron-positron scattering $e^- + e^+ \to e^- + e^+$ via electron-positron annihilation into an electron-positron pair (arbitrary colors).

Feynman's rules give for the annihilation diagram

$$
\begin{aligned}
A_s = & \int d^4q\,(2\pi)^4 e\gamma^a_{\beta\alpha}\delta^4(p+k-q)(2\pi)^4 e\gamma^b_{\gamma\delta}\delta^4(-p'-k'+q) \\
& \times \frac{\bar{u}_\gamma(p',s')}{(2\pi)^{3/2}}\frac{v_\delta(k',t')}{(2\pi)^{3/2}}\frac{u_\alpha(p,s)}{(2\pi)^{3/2}}\frac{\bar{v}_\beta(k,t)}{(2\pi)^{3/2}}\frac{-i}{(2\pi)^4}\frac{\eta_{ab}}{q^2-i\epsilon} \\
= & (2\pi)^4 e\gamma^a_{\beta\alpha}(2\pi)^4 e\gamma^b_{\gamma\delta}\delta^4(p+k-p'-k') \\
& \times \frac{\bar{u}_\gamma(p',s')}{(2\pi)^{3/2}}\frac{v_\delta(k',t')}{(2\pi)^{3/2}}\frac{u_\alpha(p,s)}{(2\pi)^{3/2}}\frac{\bar{v}_\beta(k,t)}{(2\pi)^{3/2}}\frac{-i}{(2\pi)^4}\frac{\eta_{ab}}{(p+k)^2} \qquad (4.74) \\
= & e\gamma^a_{\beta\alpha}e\gamma^b_{\gamma\delta}\delta^4(p+k-p'-k') \\
& \times \bar{u}_\gamma(p',s')v_\delta(k',t')u_\alpha(p,s)\bar{v}_\beta(k,t)\frac{-i}{(2\pi)^2}\frac{\eta_{ab}}{(p+k)^2} \\
= & e^2\bar{v}_\beta(k,t)\gamma^a_{\beta\alpha}u_\alpha(p,s)\bar{u}_\gamma(p',s')\gamma^b_{\gamma\delta}v_\delta(k',t')\delta^4(p+k-p'-k') \\
& \times \frac{-i}{(2\pi)^2}\frac{\eta_{ab}}{(p+k)^2} \\
= & -i\frac{e^2}{(2\pi)^2}\frac{\delta^4(p+k-p'-k')}{(p+k)^2}\bar{v}_\beta(k,t)\gamma^a_{\beta\alpha}u_\alpha(p,s)\bar{u}_\gamma(p',s')\gamma_{a\gamma\delta}v_\delta(k',t').
\end{aligned}
$$

T-channel electron-positron scattering

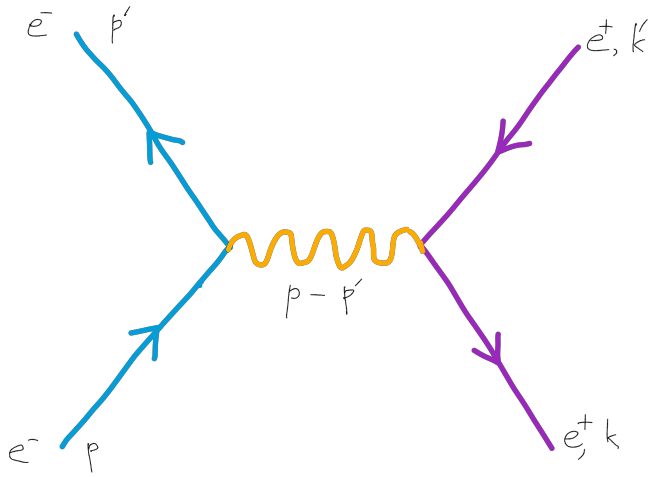

Figure 4.2 Exchange or t-channel diagram for electron-positron scattering (arbitrary colors).

Suppressing indexes, we get

$$A_s = -i\frac{e^2}{(2\pi)^2}\delta^4(p+k-p'-k')\ \frac{\bar{v}\gamma^a u\ \bar{u}'\gamma_a v'}{(p+k)^2}. \tag{4.75}$$

The exchange diagram is

$$A_t = \pm i\frac{e^2}{(2\pi)^2}\delta^4(p+k-p'-k')\ \frac{\bar{u}'\gamma^a u\ \bar{v}\gamma_a v'}{(p-p')^2}, \tag{4.76}$$

and is illustrated in Fig. 4.2 and we must check our minus signs.

The initial state is $|p,s;k,t\rangle = a^\dagger(p,s)b^\dagger(k,t)|0\rangle$ and the final state is $|p',s';k',t'\rangle = a^\dagger(p',s')b^\dagger(k',t')|0\rangle$. The direct diagram comes from

$$\begin{aligned} &T[\bar{\psi}(x)\gamma^a\psi(x)A_a(x)\bar{\psi}(y)\gamma^b\psi(y)A_b(y)] \\ &\qquad = (\bar{\psi})^-(x)\gamma^a\psi^-(x)T[A_a(x)A_b(y)](\bar{\psi})^+(y)\gamma^b\psi^+(y) \\ &\qquad \sim a^\dagger(p's')\gamma^a b^\dagger(k',t')T[A_a(x)A_b(y)]b(k,s)\gamma^b a(p,s), \end{aligned} \tag{4.77}$$

while the exchange diagram illustrated in Fig. 4.2comes from

$$\begin{aligned}
&T[\bar\psi(x)\gamma^a\psi(x)A_a(x)\bar\psi(y)\gamma^b\psi(y)A_b(y)]\\
&\qquad = (\bar\psi)^+(x)\gamma^a\psi^-(x)T[A_a(x)A_b(y)](\bar\psi)^-(y)\gamma^b\psi^+(y)\\
&\qquad \sim b(k,t)\gamma^a b^\dagger(k',t')T[A_a(x)A_b(y)]a^\dagger(p',s')\gamma^b a(p,s) \quad (4.78)\\
&\qquad \sim a^\dagger(p',s')b(k,t)\gamma^a b^\dagger(k',t')T[A_a(x)A_b(y)]\gamma^b a(p,s)\\
&\qquad \sim -a^\dagger(p',s')\gamma^a b^\dagger(k',t')T[A_a(x)A_b(y)]b(k,t)\gamma^b a(p,s)
\end{aligned}$$

which differs by a minus sign. So the total amplitude is

$$\begin{aligned}
A &= A_s + A_t\\
&= -i\frac{e^2}{(2\pi)^2}\delta^4(p+k-p'-k')\ \frac{\bar v\gamma^a u\ \bar u'\gamma_a v'}{(p+k)^2}\\
&\quad -\Big[-i\frac{e^2}{(2\pi)^2}\,\delta^4(p+k-p'-k')\ \frac{\bar u'\gamma^b u\ \bar v\gamma_b v'}{(p-p')^2}\Big] \qquad (4.79)\\
&= -i\frac{e^2}{(2\pi)^2}\,\delta^4(p+k-p'-k')\left[\frac{\bar v\gamma^a u\ \bar u'\gamma_a v'}{(p+k)^2}-\frac{\bar u'\gamma^b u\ \bar v\gamma_b v'}{(p-p')^2}\right]\Big].
\end{aligned}$$

The probability is the square $|A|^2$. To compute such things, we need to know some trace identities.

4.8 Trace identities

Since γ^5 is a fourth spatial gamma matrix (1.173), its square is unity, $\gamma^5\gamma^5 = 1$, and it anticommutes with the ordinary gammas. So since the trace is cyclic,

$$\mathrm{Tr}(\gamma^a) = \mathrm{Tr}(\gamma^5\gamma^5\gamma^a) = -\mathrm{Tr}(\gamma^5\gamma^a\gamma^5) = -\mathrm{Tr}(\gamma^5\gamma^5\gamma^a) = -\mathrm{Tr}(\gamma^a) \qquad (4.80)$$

the trace of a single gamma matrix is zero.

The trace of two gammas

$$\mathrm{Tr}(\gamma^a\gamma^b) = \mathrm{Tr}(2\eta^{ab} - \gamma^b\gamma^a) \qquad (4.81)$$

is

$$\mathrm{Tr}(\gamma^a\gamma^b) = \eta^{ab}\mathrm{Tr}(1) = 4\eta^{ab}. \qquad (4.82)$$

The trace of an odd number of gammas vanishes because

$$\begin{aligned}
\mathrm{Tr}(\gamma^{a_1}\cdots\gamma^{a_{2n+1}}) &= \mathrm{Tr}(\gamma^5\gamma^5\gamma^{a_1}\cdots\gamma^{a_{2n+1}}) = -\mathrm{Tr}(\gamma^5\gamma^{a_1}\cdots\gamma^{a_{2n+1}}\gamma^5)\\
&= -\mathrm{Tr}(\gamma^5\gamma^5\gamma^{a_1}\cdots\gamma^{a_{2n+1}}) = -\mathrm{Tr}(\gamma^{a_1}\cdots\gamma^{a_{2n+1}}) \quad (4.83)
\end{aligned}$$

which implies that

$$\mathrm{Tr}(\gamma^{a_1}\cdots\gamma^{a_{2n+1}}) = 0. \tag{4.84}$$

To find the trace of four gammas, we use repeatedly the fundamental rule

$$\gamma^a\gamma^b = 2\eta^{ab} - \gamma^b\gamma^a. \tag{4.85}$$

We thus find

$$\begin{aligned}\mathrm{Tr}(\gamma^a\gamma^b\gamma^c\gamma^d) &= \mathrm{Tr}[(2\eta^{ab} - \gamma^b\gamma^a)\gamma^c\gamma^d] = \mathrm{Tr}[2\eta^{ab}\gamma^c\gamma^d - \gamma^b\gamma^a\gamma^c\gamma^d] \\ &= \mathrm{Tr}[2\eta^{ab}\gamma^c\gamma^d - \gamma^b(2\eta^{ac} - \gamma^c\gamma^a)\gamma^d] \\ &= \mathrm{Tr}[2\eta^{ab}\gamma^c\gamma^d - 2\eta^{ac}\gamma^b\gamma^d + \gamma^b\gamma^c\gamma^a\gamma^d] \\ &= \mathrm{Tr}[2\eta^{ab}\gamma^c\gamma^d - 2\eta^{ac}\gamma^b\gamma^d + 2\eta^{ad}\gamma^b\gamma^c - \gamma^b\gamma^c\gamma^d\gamma^a]\end{aligned} \tag{4.86}$$

or

$$\begin{aligned}\mathrm{Tr}(\gamma^a\gamma^b\gamma^c\gamma^d) &= \mathrm{Tr}[\eta^{ab}\gamma^c\gamma^d - \eta^{ac}\gamma^b\gamma^d + \eta^{ad}\gamma^b\gamma^c] \\ &= 4\big(\eta^{ab}\eta^{cd} - \eta^{ac}\eta^{bd} + \eta^{ad}\eta^{bc}\big).\end{aligned} \tag{4.87}$$

Reversing completely the order of the gammas changes nothing:

$$\mathrm{Tr}\big(\gamma^a\gamma^b\cdots\gamma^y\gamma^z\big) = \mathrm{Tr}\big(\gamma^z\gamma^y\cdots\gamma^b\gamma^a\big). \tag{4.88}$$

4.9 Electron-positron to muon-antimuon

To avoid the extra complications that the sum of two amplitudes $A_s + A_t$ entail, let's consider the process $e^-(p) + e^+(p') \to \mu^-(k) + \mu^+(k')$ which requires only the direct amplitude

$$\begin{aligned}A &= -\,i\frac{e^2}{(2\pi)^2}\delta^4(p+p'-k-k')\;\frac{\bar v(p',s')\gamma^a u(p,s)\;\bar u(k,t)\gamma_a v(k',t')}{(p+p')^2} \\ &= -\,2\pi i\,\delta^4(p+p'-k-k')\,M\end{aligned} \tag{4.89}$$

where

$$M = \frac{e^2}{(2\pi)^3}\frac{\bar v(p',s')\gamma^a u(p,s)\;\bar u(k,t)\gamma_a v(k',t')}{(p+p')^2} \tag{4.90}$$

in which I have relabeled the momenta so that $k\,\&\,k'$ belong to the muons.

The probability $|A|^2$ includes the term

$$\bar v\gamma^a u\,\bar u_\mu\gamma_a v_\mu\;(\bar v\gamma^b u\,\bar u_\mu\gamma_b v_\mu)^* \tag{4.91}$$

in which μ's label muons. Since $\bar v = iv^\dagger\gamma^0 = v^\dagger\beta$ as well as $\beta^2 = (i\gamma^0)^2 = 1$, and since (1.177) $\beta\gamma^{a\dagger}\beta = -\gamma^a$, we get

$$(\bar v\gamma^a u)^* = (v^\dagger\beta\gamma^a u)^* = u^\dagger\gamma^{a\dagger}\beta v = u^\dagger\beta\beta\gamma^{a\dagger}\beta v = u^\dagger\beta\gamma^a v = \bar u\gamma^a v. \tag{4.92}$$

So the electron part of the term (4.91) is

$$\bar{v}\gamma^a u\,(\bar{v}\gamma^b u)^* = \bar{v}\gamma^a u\,\bar{u}\gamma^b v. \tag{4.93}$$

The spin sums are

$$\begin{aligned}\sum_s u_\ell(\vec{p},s)u^*_m(\vec{p},s) &= \Big[\frac{1}{2p^0}\,(-i\,p^c\gamma_c + m)\;\beta\Big]_{\ell m}\\ \sum_s v_\ell(\vec{p},s)v^*_m(\vec{p},s) &= \Big[\frac{1}{2p^0}\,(-i\,p^c\gamma_c - m)\;\beta\Big]_{\ell m}.\end{aligned} \tag{4.94}$$

Equivalently and more simply,

$$\begin{aligned}\sum_s u_\ell(\vec{p},s)\bar{u}_m(\vec{p},s) &= \Big[\frac{1}{2p^0}\,(-i\,p^c\gamma_c + m)\Big]_{\ell m}\\ \sum_s v_\ell(\vec{p},s)\bar{v}_m(\vec{p},s) &= \Big[\frac{1}{2p^0}\,(-i\,p^c\gamma_c - m)\Big]_{\ell m}.\end{aligned} \tag{4.95}$$

So

$$\begin{aligned}\sum_{s,s'}\bar{v}\gamma^a u\,\bar{u}\gamma^b v &= \sum_{s',j,\ell,m,n}\bar{v}_j(p',s')\gamma^a_{j\ell}u_\ell(p,s)\,\bar{u}_m(p,s)\gamma^b_{mn}v_n(p',s')\\ &= \frac{1}{2p^0}\sum_{s',j,\ell,m,n}\bar{v}_j(p',s')\gamma^a_{j\ell}(-i\not{p}+m)_{\ell m}\gamma^b_{mn}v_n(p',s')\\ &= \frac{1}{2p^0}\sum_{s',j,\ell,m,n}v_n(p',s')\bar{v}_j(p',s')\gamma^a_{j\ell}(-i\not{p}+m)_{\ell m}\gamma^b_{mn}\\ &= \frac{1}{2p^0 2p'^0}\sum_{sj,\ell,m,n}(-i\not{p}'-m)_{nj}\gamma^a_{j\ell}(-i\not{p}+m)_{\ell m}\gamma^b_{mn}\\ &= \frac{1}{2p^0 2p'^0}\mathrm{Tr}[(-i\not{p}'-m_e)\gamma^a(-i\not{p}+m_e)\gamma^b].\end{aligned} \tag{4.96}$$

The muon part of the term (4.91) is

$$\bar{u}\gamma_a v\,(\bar{u}\gamma_b v)^* = \bar{u}\gamma_a v\,\bar{v}\gamma_b u = \bar{v}\gamma_b u\,\bar{u}\gamma_a v. \tag{4.97}$$

So the sum over all spins gives for the muons

$$\sum_{t,t'}\bar{u}\gamma_a v\,\bar{v}\gamma_b u = \frac{1}{2k^0 2k'^0}\mathrm{Tr}[(-i\not{k}'-m_\mu)\gamma_b(-i\not{k}+m_\mu)\gamma_a]. \tag{4.98}$$

Using the formula (4.86) for the trace of 4 gammas, we evaluate the trace for the electrons in stages:

$$\mathrm{Tr}[(-i\not{p}'-m_e)\gamma^a(-i\not{p}+m_e)\gamma^b] = -\,\mathrm{Tr}\big(\not{p}'\gamma^a\not{p}\gamma^b\big) - m_e^2\mathrm{Tr}\big(\gamma^a\gamma^b\big). \tag{4.99}$$

The 4-gamma term is

$$\begin{aligned}\mathrm{Tr}[(-i\not{p}')\gamma^a(-i\not{p})\gamma^b] &= -\mathrm{Tr}[p'_c\gamma^c\gamma^a p_d\gamma^d\gamma^b]\\ &= -p'_c p_d \mathrm{Tr}[\gamma^c\gamma^a\gamma^d\gamma^b]\\ &= -4p'_c p_d\big(\eta^{ca}\eta^{db}-\eta^{cd}\eta^{ab}+\eta^{cb}\eta^{ad}\big)\\ &= -4\big(p'^a p^b - p'p\,\eta^{ab}+p'^b p^c\big). \end{aligned} \tag{4.100}$$

The 2-gamma term is

$$-m_e^2\mathrm{Tr}\big(\gamma^a\gamma^b\big) = -4m_e^2\eta^{ab}. \tag{4.101}$$

So their sum is

$$\mathrm{Tr}[(-i\not{p}'-m_e)\gamma^a(-i\not{p}+m_e)\gamma^b] = -4\big[p'^a p^b+p'^b p^a+\eta^{ab}(m_e^2-p'p)\big]. \tag{4.102}$$

The similar term for muons is

$$\mathrm{Tr}[(-i\not{k}'-m_\mu)\gamma_a(-i\not{k}+m_\mu)\gamma_b] = -4\big[k'_a k_b+k'_b k_a+\eta_{ab}(m_\mu^2-k'k)\big]. \tag{4.103}$$

Their product is

$$\begin{aligned}&\mathrm{Tr}[(-i\not{p}'-m_e)\gamma^a(-i\not{p}+m_e)\gamma^b]\,\mathrm{Tr}[(-i\not{k}'-m_\mu)\gamma_a(-i\not{k}+m_\mu)\gamma_b] \\ &\quad= 16\big[p'^a p^b+p'^b p^a+\eta^{ab}(m_e^2-p'p)\big]\big[k'_a k_b+k'_b k_a+\eta_{ab}(m_\mu^2-k'k)\big]\\ &\quad= 32\Big[(p'k')(pk)+(p'k)(pk')+m_\mu^2(pp')+m_e^2(k'k)\Big].\end{aligned} \tag{4.104}$$

So apart from the delta function and the (2π)'s, the squared amplitude summed over final spins and averaged over initial spins is

$$\frac{1}{4}\sum_{s,s'}|M|^2 = \frac{e^4}{2(2\pi)^6 p^0 p'^0 k^0 k'^0}\frac{(p'k')(pk)+(p'k)(pk')+m_\mu^2(pp')+m_e^2(k'k)}{(p+p')^4} \tag{4.105}$$

where $(p+p')^4 = [(p+p')^2]^2$.

In the rest frame of a collider, $q^2 = (p+p')^2 = 4E^2 = 4p^{02}$. The cosine of the scattering angle θ is $\cos\theta = \vec{p}\cdot\vec{k}/(|\vec{p}||\vec{k}|)$.

Since $m_e \approx m_\mu/200$, I will neglect m_e in what follows. One then gets $p\cdot p' = -2E^2$, $p\cdot k = E(|\vec{k}|\cos\theta - E)$, and $p\cdot k' = p'\cdot k = -E(|\vec{k}|\cos\theta + E)$.

Thus

$$\begin{aligned}\frac{1}{4}\sum_{s,s'}|M|^2 &= \frac{e^4\left[E^2(E-k\cos\theta)^2+E^2(E+k\cos\theta)^2+2m_\mu^2E^2\right]}{2(2\pi)^6E^4 16E^4}\\ &= \frac{e^4\left[(E-k\cos\theta)^2+(E+k\cos\theta)^2+2m_\mu^2\right]}{32(2\pi)^6E^6}\\ &= \frac{e^4\left[2E^2+2k^2\cos^2\theta+2m_\mu^2\right]}{32(2\pi)^6E^6} = \frac{e^4\left[E^2+k^2\cos^2\theta+m_\mu^2\right]}{16(2\pi)^6E^6}\end{aligned} \tag{4.106}$$

The squared amplitude is

$$|A|^2 = (2\pi)^2|\delta^4(p+p'-k-k')|^2\,\frac{1}{4}\sum_{ss'}|M|^2 \tag{4.107}$$

$$= \frac{VT}{(2\pi)^2}\delta^4(p+p'-k-k')\,\frac{1}{4}\sum_{ss'}|M|^2. \tag{4.108}$$

since

$$\delta^4(0) = \int\frac{d^4x}{(2\pi)^4} = \frac{VT}{(2\pi)^4} \tag{4.109}$$

in which V is the volume of the universe and T is its infinite time. We also need to switch from delta-function normalization of states to unit normalization. The relation is that between a continuum delta function and a Kronecker delta

$$\delta^3(\vec{p'}-\vec{p}) = \frac{V}{(2\pi)^3}\delta_{pp'}. \tag{4.110}$$

Since there are two particles in the initial and final states, the probability is

$$P = \left[\frac{(2\pi)^3}{V}\right]^4|A|^2N_f \tag{4.111}$$

where N_f is the number of final states. For a range of momenta d^3kd^3k', the number of final states is

$$N_f = \left[\frac{V}{(2\pi)^3}\right]^2 d^3kd^3k'. \tag{4.112}$$

So the rate P/T is

$$
\begin{aligned}
R &= [\frac{(2\pi)^3}{V}]^4 \frac{|A|^2}{T} N_f = [\frac{(2\pi)^3}{V}]^4 \frac{|A|^2}{T} [\frac{V}{(2\pi)^3}]^2 d^3k d^3k' \\
&= [\frac{(2\pi)^3}{V}]^4 \frac{|A|^2}{T} N_f = [\frac{(2\pi)^3}{V}]^2 \frac{|A|^2}{T} d^3k d^3k' \\
&= \frac{(2\pi)^4}{V} \delta^4(p + p' - k - k') \, \frac{1}{4} \sum_{ss'} |M|^2 d^3k d^3k'
\end{aligned}
\tag{4.113}
$$

The flux of incoming particles is u/V where $u = |\vec{v}_1 - \vec{v}_2$ is the relative velocity, which with $c = 1$ for massless electrons is $u = 2$. So the differential cross-section is

$$
d\sigma = \frac{1}{2}(2\pi)^4 \delta^4(p + p' - k - k') \, \frac{1}{4} \sum_{ss'} |M|^2 d^3k d^3k'. \tag{4.114}
$$

Integrating over d^3k' sets $\vec{k}' = -\vec{k}$, and so

$$
\begin{aligned}
d\sigma &= \frac{1}{2}(2\pi)^4 \delta(p^0 + p'^0 - 2k^0) \, \frac{1}{4} \sum_{ss'} |M|^2 d^3k \\
&= \frac{1}{2}(2\pi)^4 \delta\left(p^0 + p'^0 - 2\sqrt{k^2 + m_\mu^2}\right) \, \frac{1}{4} \sum_{ss'} |M|^2 k^2 dk d\Omega
\end{aligned}
\tag{4.115}
$$

where $k = |\vec{k}|$. The derivative of the delta function is $k2p^0/k^{02}$, so

$$
\begin{aligned}
d\sigma &= \frac{1}{2}(2\pi)^4 \, \frac{1}{4} \sum_{ss'} |M|^2 k^2 d\Omega \frac{k^{02}}{2kp^0} = 2\pi^4 \frac{1}{4} \sum_{ss'} |M|^2 \frac{kk^{02}}{p^0} \, d\Omega \\
&= 4\pi^4 \frac{1}{4} \sum_{ss'} |M|^2 \frac{kk^{02}}{p^0} \, d\Omega.
\end{aligned}
\tag{4.116}
$$

So then

$$
\begin{aligned}
d\sigma &= 4\pi^4 \frac{e^4\left[E^2 + k^2\cos^2\theta + m_\mu^2\right]}{4(2\pi)^6 E^6}\frac{kk^{02}}{p^0}\,d\Omega \\
&= \frac{e^4\left[E^2 + k^2\cos^2\theta + m_\mu^2\right]}{4(2\pi)^2 E^5}\,k\,d\Omega \\
&= \frac{e^4}{4(2\pi)^2 E^3}\left[1 + \frac{k^2}{E^2}\cos^2\theta + \frac{m_\mu^2}{E^2}\right]k\,d\Omega \\
&= \frac{e^4}{4(2\pi)^2 E^3}\left[1 + \frac{E^2 - m_\mu^2}{E^2}\cos^2\theta + \frac{m_\mu^2}{E^2}\right]k\,d\Omega \qquad (4.117) \\
&= \frac{e^4}{4(2\pi)^2 E^2}\sqrt{1 - \frac{m_\mu^2}{E^2}}\left[1 + \frac{m_\mu^2}{E^2} + \left(1 - \frac{m_\mu^2}{E^2}\right)\cos^2\theta\right]d\Omega \\
&= \frac{\alpha^2}{16E^2}\sqrt{1 - \frac{m_\mu^2}{E^2}}\left[1 + \frac{m_\mu^2}{E^2} + \left(1 - \frac{m_\mu^2}{E^2}\right)\cos^2\theta\right]d\Omega.
\end{aligned}
$$

The total cross-section is the inetgral over solid angle

$$
\begin{aligned}
\sigma &= \frac{\alpha^2}{16E^2}\sqrt{1 - \frac{m_\mu^2}{E^2}}\left[\left(1 + \frac{m_\mu^2}{E^2}\right)4\pi + \left(1 - \frac{m_\mu^2}{E^2}\right)\int \cos^2\theta\,d\Omega\right] \\
&= \frac{\alpha^2}{16E^2}\sqrt{1 - \frac{m_\mu^2}{E^2}}\left[\left(1 + \frac{m_\mu^2}{E^2}\right)4\pi + \left(1 - \frac{m_\mu^2}{E^2}\right)\int \cos^2\theta\,2\pi d\cos\theta\right] \\
&= \frac{\pi\alpha^2}{4E^2}\sqrt{1 - \frac{m_\mu^2}{E^2}}\left[\left(1 + \frac{m_\mu^2}{E^2}\right) + \frac{1}{3}\left(1 - \frac{m_\mu^2}{E^2}\right)\right] \\
&= \frac{\pi\alpha^2}{3E^2}\sqrt{1 - \frac{m_\mu^2}{E^2}}\left(1 + \frac{1}{2}\frac{m_\mu^2}{E^2}\right) \qquad (4.118)
\end{aligned}
$$

where E is the energy of each electron in the lab frame of the collider. The energy of the collider is $E_{\text{cm}} = 2E$. At very high energies, the x-section is $\sigma = \pi\alpha^2/(3E^2)$.

4.10 Electron-muon scattering

For the process $e^-(p_1)+\mu^-(p_2)\to e^-(p_1')+\mu^-(p_2')$, the Feynman rules give

$$\begin{aligned} A = & -i\frac{e^2}{(2\pi)^2}\delta^4(p_1+p_2-p_1'-p_2')\ \frac{\bar{u}(p_1',s')\gamma^a u(p_1,s)\ \bar{u}(p_2',t')\gamma_a u(p_2,t)}{(p_1-p_1')^2} \\ = & -2\pi i\,\delta^4(p_1+p_2-p_1'-p_2')\,M. \end{aligned} \tag{4.119}$$

The product of the traces over $q^2=(p_1-p_1')^2$ is

$$\frac{\mathrm{Tr}[(-i\not{p}_1'+m_e)\gamma^a(-i\not{p}_1+m_e)\gamma^b]\,\mathrm{Tr}[(-i\not{p}_2'+m_\mu)\gamma_a(-i\not{p}_2+m_\mu)\gamma_b]}{(p_1-p_1')^2}. \tag{4.120}$$

For $e^+-e^-\to\mu^++\mu^-$ we got

$$\frac{\mathrm{Tr}[(-i\not{p}'-m_e)\gamma^a(-i\not{p}+m_e)\gamma^b]\,\mathrm{Tr}[(-i\not{k}'-m_\mu)\gamma_a(-i\not{k}+m_\mu)\gamma_b]}{(p+p')^2}. \tag{4.121}$$

If in this amplitude for $e^+-e^-\to\mu^++\mu^-$ we make these replacements

$$p\to p_1,\quad p'\to -p_1',\quad k\to p_2',\quad \text{and}\quad k'\to -p_2, \tag{4.122}$$

it becomes

$$\begin{aligned} &\mathrm{Tr}[(i\not{p}_1'-m_e)\gamma^a(-i\not{p}_1'+m_e)\gamma^b]\mathrm{Tr}[(i\not{p}_2-m_\mu)\gamma_a(-i\not{p}_2'+m_\mu)\gamma_b]/(p_1+p_2')^2 \\ &\quad=\frac{\mathrm{Tr}[(-i\not{p}_1'+m_e)\gamma^a(-i\not{p}_1'+m_e)\gamma^b]\,\mathrm{Tr}[(-i\not{p}_2+m_\mu)\gamma_a(-i\not{p}_2'+m_\mu)\gamma_b]}{(p_1-p_1')^2} \end{aligned} \tag{4.123}$$

which is exactly the same as the amplitude (4.120) for $e^-(p_1)+\mu^-(p_2)\to e^-(p_1')+\mu^-(p_2')$ scattering. So apart from the delta function and the (2π)'s, the squared amplitude summed over final spins and averaged over initial spins is the one for $e^+-e^-\to\mu^++\mu^-$ but with the substitutions (4.122). That is, for $e^-(p_1)+\mu^-(p_2)\to e^-(p_1')+\mu^-(p_2')$ scattering, the summed amplitudes are

$$\frac{1}{4}\sum_{s,s',t,t'}|M|^2=\frac{e^4\left[(p_1'p_2)(p_1p_2')+(p_1'p_2')(p_1p_2)-m_\mu^2(p_1p_1')-m_e^2(p_2p_2')\right]}{2(2\pi)^6p_1^0p_1'^0p_2^0p_2'^0\,(p_1-p_1')^4}. \tag{4.124}$$

The equality of the two amplitudes (4.120) and (4.121) under the correspondence (4.122) is an example of **crossing symmetry**. More generally, the scattering amplitude for a process in which an incoming particle of 4-momentum p that interacts with other particles is equal to the scattering

amplitude for the process in which the antiparticle of the incoming particle leaves with 4-momentum $-p$ after interacting with the same other particles

$$S(-p, n_c; \cdots | \cdots) = S(\cdots | p, n; \cdots) \tag{4.125}$$

in which type n_c is labels the antiparticle of type n. Since $-p^0 + p^0 = 0$, this is not the amplitude for a physical process in which each particle has its appropriate energy, $p_i^0 = \sqrt{m_i^2 + \vec{p}_i^2}$. This symmetry caused huge excitement on the West Coast during the 1960s.

Let $k = |\vec{p}_1'| = p_1'^0$ be energy or equivalently the modulus (magnitude) of the 3-momentum of the (massless) final electron. Let $E = p_2^0$ be the energy of the initial muon. Then the differential x-section in the lab frame of the collider is

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{2k^2(E+k)^2(1-\cos\theta)^2}\Big[(E+k)^2 + (E + k\cos\theta)^2 - m_\mu^2(1-\cos\theta)\Big] \tag{4.126}$$

in which the cosine of the scattering angle of the electron is

$$\cos\theta = \frac{\vec{p}_1 \cdot \vec{p}_1}{|\vec{p}_1||\vec{p}_1|}. \tag{4.127}$$

The total x-section σ is infinite.

4.11 One-loop QED

The action density is

$$L = -\frac{1}{4}F_B^{ab}F_{Bab} - \bar{\psi}_B\left[\gamma_a\left(\partial^a + ie_B A_B^a\right) + m_B\right]\psi_B \tag{4.128}$$

in which the subscript B means that the values of these "bare" terms include infinite quantities that may be used to cancel the infinities that arise in perturbation theory. The actual physical terms are defined as

$$\psi \equiv Z_2^{-1/2}\,\psi_B \tag{4.129}$$

$$A^a \equiv Z_3^{-1/2}\,A_B^a \tag{4.130}$$

$$e \equiv Z_3^{1/2}\,e_B \tag{4.131}$$

$$m \equiv m_B + \delta m \tag{4.132}$$

so that in particular $eA^a = e_B A_B^a$. In these terms, the action density is

$$L = L_0 + L_1 + L_2 \tag{4.133}$$

Vacuum polarization

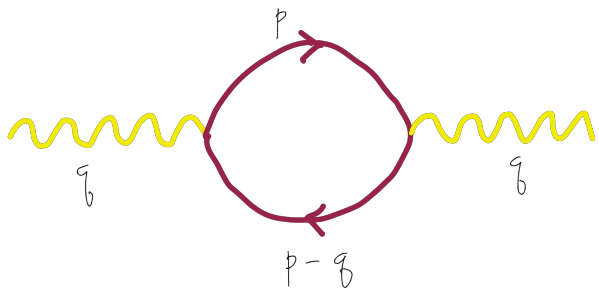

Figure 4.3 One-loop Feynman diagram for vacuum polarization (arbitrary colors).

where L_0 is the physical or naive free action density

$$L_0 = -\frac{1}{4}F^{ab}F_{ab} - \bar{\psi}\left(\gamma_a\partial^a + m\right)\psi, \tag{4.134}$$

L_1 is the interaction

$$L_1 = -ieA_a\bar{\psi}\gamma^a\psi, \tag{4.135}$$

and L_2 is a sum of infinite "counterterms"

$$\begin{aligned} L_2 = & -\frac{1}{4}(Z_3-1)F^{ab}F_{ab} - (Z_2-1)\bar{\psi}\left(\gamma_a\partial^a + m\right)\psi \\ & + Z_2\,\delta m\,\bar{\psi}\psi - ie(Z_2-1)A_a\bar{\psi}\gamma^a\psi. \end{aligned} \tag{4.136}$$

Vacuum polarization is a one-loop correction to the photon propagator. Its Feynman diagram is in figure 4.3. Let's use L_1 to compute it. The amplitude S is

$$S = \langle q', s'|\mathcal{T}\left[e^{i\int L_1\,d^4x}\right]|q,s\rangle = \langle q,s|\mathcal{T}\left[e^{e\int A_a(x)\bar{\psi}(x)\gamma^a\psi(x)\,d^4x}\right]|q,s\rangle. \tag{4.137}$$

The one-loop correction is due to the order-e^2 term

$$S_2 = \frac{1}{2}\langle q', s'|\mathcal{T}\left[e^2 \int A_a(x)\bar{\psi}(x)\gamma^a\psi(x)\,d^4x \int A_b(y)\bar{\psi}(y)\gamma^b\psi(y)\,d^4y\right]|q, s\rangle. \tag{4.138}$$

We choose to absorb the photon at y and cancel the factor of one-half. The electromagnetic field is

$$A_b(y) = \int \frac{d^3p}{\sqrt{(2\pi)^3 2p^0}} \sum_{s'} \left[e^{ip\cdot y} e_b(p, s')a(p, s') + e^{-ip\cdot y} e_b^*(p, s')a^\dagger(p, s')\right]. \tag{4.139}$$

So S_2 is

$$\begin{aligned} S_2 = \frac{e^2}{(2\pi)^3 2q^0}\langle 0|\mathcal{T}\Big[\int e^{-iq'\cdot x} e_a^*(q', s')\bar{\psi}(x)\gamma^a\psi(x)\,d^4x \\ \times \int e^{iq\cdot y} e_b(q, s)\bar{\psi}(y)\gamma^b\psi(y)\,d^4y\Big]|0\rangle. \end{aligned} \tag{4.140}$$

The term L_1 is normally ordered (it should be surrounded by colons), so fermions emitted by $\bar{\psi}(y)$ and $\psi(y)$ must be absorbed by $\psi(x)$ and $\bar{\psi}(x)$. Thus S_2 is

$$\begin{aligned} S_2 = \frac{e^2 e_a^*(q', s')e_b(q, s)}{(2\pi)^3 2q^0} \int e^{-iq'\cdot x} e^{iq\cdot y} \langle 0|\mathcal{T}[\bar{\psi}_\alpha(x)\gamma^a_{\alpha\beta}\psi_\delta(y)]|0\rangle \\ \times \langle 0|\mathcal{T}[\psi_\beta(x)\bar{\psi}_\gamma(y)\gamma^b_{\gamma\delta}]|0\rangle\,d^4x\,d^4y. \end{aligned} \tag{4.141}$$

Moving $\psi_\delta(y)$ to the left of $\bar{\psi}_\alpha(x)$ adds a minus sign, so

$$\begin{aligned} S_2 = -\frac{e^2 e_a^*(q', s')e_b(q, s)}{(2\pi)^3 2q^0} \int e^{-iq'\cdot x} e^{iq\cdot y} \langle 0|\mathcal{T}[\psi_\delta(y)\bar{\psi}_\alpha(x)\gamma^a_{\alpha\beta}]|0\rangle \\ \times \langle 0|\mathcal{T}[\psi_\beta(x)\bar{\psi}_\gamma(y)\gamma^b_{\gamma\delta}]|0\rangle\,d^4x\,d^4y. \end{aligned} \tag{4.142}$$

This minus sign is an instance of Feynman's rule that closed fermion loops make minus signs. In the common notation $\overline{\psi} = \psi^\dagger\beta$, his propagator is

$$\begin{aligned} \langle 0|\mathcal{T}\{\psi_\ell(x)\overline{\psi}_m(y)\}|0\rangle &\equiv -i\Delta_{\ell m}(x-y)\,\beta \\ &= -i\int \frac{d^4q}{(2\pi)^4}\,\frac{(-i\not{q}+m)_{\ell m}}{q^2+m^2-i\epsilon}\,e^{iq\cdot(x-y)} \\ &= -i\int \frac{d^4q}{(2\pi)^4}\left(\frac{1}{i\not{q}+m-i\epsilon}\right)_{\ell m} e^{iq\cdot(x-y)}. \end{aligned} \tag{4.143}$$

Putting all this together, we get

$$S_2 = \frac{e^2 e_a^*(q',s') e_b(q,s)}{(2\pi)^3 2q^0} \int e^{-iq'\cdot x} e^{iq\cdot y} \int \frac{d^4k}{(2\pi)^4} \frac{(-i\not{k}+m)_{\delta\alpha}}{k^2+m^2-i\epsilon} e^{ik\cdot(y-x)} \gamma^a_{\alpha\beta}$$
$$\times \int \frac{d^4p}{(2\pi)^4} \frac{(-i\not{p}+m)_{\beta\gamma}}{p^2+m^2-i\epsilon} e^{ip\cdot(x-y)} \gamma^b_{\gamma\delta}\, d^4x\, d^4y. \tag{4.144}$$

Dirac's delta functions simplify this 16-dimensional integral to

$$S_2 = \frac{e^2 e_a^*(q',s') e_b(q,s) \delta^4(q'-q)}{(2\pi)^3 2q^0} \int d^4p \frac{\big[-i(\not{p}-\not{q})+m\big]_{\delta\alpha}}{(p-q)^2+m^2-i\epsilon}$$
$$\times \gamma^a_{\alpha\beta} \frac{(-i\not{p}+m)_{\beta\gamma}}{p^2+m^2-i\epsilon} \gamma^b_{\gamma\delta}. \tag{4.145}$$

The product of matrices is a trace

$$S_2 = \frac{e^2 e_a^*(q',s') e_b(q,s) \delta^4(q'-q)}{(2\pi)^3 2q^0} \int d^4p \frac{\mathrm{Tr}\big\{\big[-i(\not{p}-\not{q})+m\big]\gamma^a(-i\not{p}+m)\gamma^b\big\}}{[(p-q)^2+m^2-i\epsilon](p^2+m^2-i\epsilon)}. \tag{4.146}$$

Equivalently, since $\mathrm{Tr}AB = \mathrm{Tr}BA$, we have

$$S_2 = \frac{e^2 e_a^*(q',s') e_b(q,s) \delta^4(q'-q)}{(2\pi)^3 2q^0} \int d^4p \frac{\mathrm{Tr}\big\{(-i\not{p}+m)\gamma^b\big[-i(\not{p}-\not{q})+m\big]\gamma^a\big\}}{(p^2+m^2-i\epsilon)[(p-q)^2+m^2-i\epsilon]}. \tag{4.147}$$

or interchanging a and b

$$S_2 = \frac{e^2 e_a(q,s) e_b^*(q',s') \delta^4(q'-q)}{(2\pi)^3 2q^0} \int d^4p \frac{\mathrm{Tr}\big\{(-i\not{p}+m)\gamma^a\big[-i(\not{p}-\not{q})+m\big]\gamma^b\big\}}{(p^2+m^2-i\epsilon)[(p-q)^2+m^2-i\epsilon]} \tag{4.148}$$

SW extracts from this the definition

$$\Pi^{*ab}(q) = \frac{-ie^2}{(2\pi)^4} \int d^4p \frac{\mathrm{Tr}\big\{(-i\not{p}+m)\gamma^a\big[-i(\not{p}-\not{q})+m\big]\gamma^b\big\}}{(p^2+m^2-i\epsilon)[(p-q)^2+m^2-i\epsilon]}. \tag{4.149}$$

We use the Feynman trick

$$\frac{1}{AB} = \int_0^1 \frac{dx}{[(1-x)A+xB]^2} \tag{4.150}$$

to write

$$\begin{aligned}\frac{1}{(p^2+m^2-i\epsilon)[(p-q)^2+m^2-i\epsilon]} &= \int_0^1 \frac{dx}{\big\{(p^2+m^2-i\epsilon)(1-x)] + [(p-q)^2+m^2-i\epsilon]x\big\}^2} \\ &= \int_0^1 \frac{dx}{\big[p^2+m^2-i\epsilon-2p\cdot q\,x+q^2\,x\big]^2} \qquad (4.151)\\ &= \int_0^1 \frac{dx}{\big[(p-xq)^2+m^2-i\epsilon+x(1-x)\,q^2\big]^2}.\end{aligned}$$

The next step is to make the integral finite somehow. One can use the counterterms, one can regard quantum field theory as merely a way to interpolate between values taken from experiment, or one can impose dimensional regularization as we'll see momentarily. If we assume that we have made the integral finite one way or another, then we can shift p to $p+x\,q$. After this shift, the quantity (4.152) becomes

$$\Pi^{*ab}(q) = \frac{-ie^2}{(2\pi)^4}\int_0^1 dx \int d^4p\, \frac{\mathrm{Tr}\big\{(-i(\not p+x\not q)+m)\gamma^a\big[-i(\not p-\not q(1-x))+m\big]\gamma^b\big\}}{\big[p^2+m^2-i\epsilon+x(1-x)\,q^2\big]^2}. \qquad (4.152)$$

The trace identities of section 4.8 give

$$\begin{aligned}&\mathrm{Tr}\big\{(-i(\not p+x\not q)+m)\gamma^a\big[-i(\not p-\not q(1-x))+m\big]\gamma^b\big\}\\ &\qquad = 4\Big[-(p+xq)^a(p-(1-x)q)^b + (p+xq)\cdot(p-(1-x)q)\,\eta^{ab}\\ &\qquad\quad -(p+xq)^b(p-(1-x)q)^a + m^2\eta^{ab}\Big]. \qquad (4.153)\end{aligned}$$

The variable p^0 is being integrated along the real axis of the complex p^0 plane from $-\infty$ to $+\infty$. We now imagine that we have made the p-integral sufficiently finite that we can rotate the p^0 contour by 90 degrees so that it runs up the imaginary axis in the complex p^0 plane from $p^0=-i\infty$ to $p^0=i\infty$. This is a Wick rotation. We set $p^0=ip^4$ and integrate over $p^4=-ip^0$, which is real, from $-\infty$ to ∞. We also set $q^0=iq^4$ and $q^4=-iq^0$. So now

$$p\cdot q = \vec p\cdot\vec q - p^0q^0 = \vec p\cdot\vec q + p^4q^4. \qquad (4.154)$$

All the Lorentz signs are now positive and

$$p^0p^0\eta^{00} = ip^4ip^4(-1) = p^4p^4 = p^4p^4\delta^{44} \qquad (4.155)$$

as well as

$$q^0q^0\eta^{00} = iq^4iq^4(-1) = q^4q^4 = q^4q^4\delta^{44}. \qquad (4.156)$$

The quantity q^2 is real whether q^0 is real or imaginary; it is positive when $q^0 < \vec{q}^2$ whether or not q^0 is imaginary.

We now can drop $i\epsilon$ and express the quantity

$$\Pi^{*ab}(q) = \Pi^{*ab}(\vec{q}, q^0) = \Pi^{*ab}(\vec{q}, iq^4) \tag{4.157}$$

either as

$$\begin{aligned}\Pi^{*ab}(q) = \frac{-4ie^2}{(2\pi)^4} \int_0^1 dx \int d^4p \, \frac{1}{\left[p^2 + m^2 - i\epsilon + x(1-x)\, q^2\right]^2} \\ \times \Big[-(p+xq)^a (p-(1-x)q)^b + (p+xq)\cdot(p-(1-x)q)\, \eta^{ab} \\ - (p+xq)^b (p-(1-x)q)^a + m^2 \eta^{ab} \Big]\end{aligned} \tag{4.158}$$

or with $p^4 = -ip^0$ real and $d^4p_e = dp^1 dp^2 dp^3 dp^4$ as

$$\begin{aligned}\Pi^{*ab}(q) = \frac{4e^2}{(2\pi)^4} \int_0^1 dx \int d^4p_e \, \frac{1}{\left[p^2 + m^2 + x(1-x)\, q^2\right]^2} \\ \times \Big[-(p+xq)^a (p-(1-x)q)^b + (p+xq)\cdot(p-(1-x)q)\, \delta^{ab} \\ - (p+xq)^b (p-(1-x)q)^a + m^2 \delta^{ab} \Big]\end{aligned} \tag{4.159}$$

in which all Lorentz signs are $+1$ and $q^4 = -iq^0$ is imaginary if q is real.

The next step is to say how we regularize the integral over d^4p. Since the 1970s, the method of choice has been **dimensional regularization** because it preserves Lorentz and gauge invariance. In dimensional regularization, the p integration is taken to be over an arbitrary number d of spacetime dimensions.

The denominator of the integral (4.159) is a function of p^2 which is taken to be the square of the radius of a sphere in d dimensions. Terms in the numerator that are odd powers of p^a vanish. We set

$$p^a p^p = \frac{p^2\, \delta^{ab}}{d}. \tag{4.160}$$

The area of a sphere of unit radius in d dimensions is

$$A_d = \frac{2\pi^{d/2}}{\Gamma(d/2)} \tag{4.161}$$

as we can verify by computing an integral of the gaussian $\exp(-x^2)$ in d

dimensions using both rectangular and spherical coordinates.

$$\begin{aligned}\int e^{-x^2}\,d^dx &= \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)^d = \pi^{d/2}\\ &= A_d\int_0^\infty r^{d-1}e^{-r^2}\,dr = A_d\,\frac{1}{2}\,\Gamma(d/2).\end{aligned} \tag{4.162}$$

The integral (4.159) converges as long as d is complex or a fraction or anything but an even integer. Nonetheless, the divergencies in that integral reappear as powers of $1/(d-4)$ as $d\to 4$. We'll use the counterterms (4.136) in L_2 to cancel them.

The area of a sphere of radius $k=\sqrt{p^2}$ in d dimensions is $A_d\,k^{d-1}$. So using the formulas (4.160 and 4.161) of dimensional regularization, we can write the integral (4.159) as

$$\begin{aligned}\Pi^{*ab}(q) = \frac{4e^2A_d}{(2\pi)^4}\int_0^1 dx\int_0^\infty k^{d-1}dk\,\frac{1}{\left[k^2+m^2+x(1-x)\,q^2\right]^2}\\ \times\Big[-2\frac{k^2\delta^{ab}}{d}+2q^aq^bx(1-x)+[k^2-x(1-x)q^2]\delta^{ab}+m^2\delta^{ab}\Big].\end{aligned} \tag{4.163}$$

The needed integrals are

$$\int_0^\infty \frac{k^{d-1}}{(k^2+c^2)^2}\,dk = \tfrac{1}{2}\,(c^2)^{d/2-2}\,\Gamma(d/2)\,\Gamma(2-d/2) \tag{4.164}$$

$$\int_0^\infty \frac{k^{d+1}}{(k^2+c^2)^2}\,dk = \tfrac{1}{2}\,(c^2)^{d/2-1}\,\Gamma(1+d/2)\,\Gamma(1-d/2). \tag{4.165}$$

Using these integrals, we get

$$\begin{aligned}\Pi^{*ab}(q) = \frac{2e^2A_d}{(2\pi)^4}\int_0^1 dx\Bigg[\Big(1-\frac{2}{d}\Big)\delta^{ab}(c^2)^{d/2-1}\,\Gamma(1+d/2)\,\Gamma(1-d/2)\\ +\Big[(2q^aq^b-q^2\delta^{ab})x(1-x)+m^2\delta^{ab}\Big](c^2)^{d/2-2}\Gamma(d/2)\Gamma(2-d/2)\Bigg]\end{aligned} \tag{4.166}$$

in which

$$c^2 = m^2+x(1-x)\,q^2. \tag{4.167}$$

The identity

$$\Gamma(z+1) = z\,\Gamma(z) \tag{4.168}$$

implies that

$$\Gamma(2-d/2) = (1-d/2)\,\Gamma(1-d/2), \tag{4.169}$$

and that

$$\Gamma(1+d/2) = (d/2)\,\Gamma(d/2), \tag{4.170}$$

so

$$\Gamma(1+d/2)\,\Gamma(1-d/2) = (2/d-1)^{-1}\Gamma(d/2)\,\Gamma(2-d/2). \tag{4.171}$$

So we can combine terms in the integral (4.172)

$$\begin{aligned}\Pi^{*ab}(q) = \frac{2e^2 A_d}{(2\pi)^4}\Gamma(d/2)\Gamma(2-d/2)\int_0^1 dx\Bigg[&-\delta^{ab}c^2 \\ &+\Big[(2q^a q^b - q^2\delta^{ab})x(1-x) + m^2\delta^{ab}\Big]\Bigg](c^2)^{d/2-2}.\end{aligned} \tag{4.172}$$

Using the value (4.167) of c^2, we have

$$-\delta^{ab}c^2 + (2q^a q^b - q^2\delta^{ab})x(1-x) + m^2\delta^{ab} = 2(q^a q^b - q^2\delta^{ab})x(1-x). \tag{4.173}$$

Thus we find

$$\begin{aligned}\Pi^{*ab}(q) = &\frac{4e^2 A_d}{(2\pi)^4}\Gamma(d/2)\Gamma(2-d/2)(q^a q^b - q^2\delta^{ab}) \\ &\times\int_0^1 dx x(1-x)[m^2 + x(1-x)\,q^2]^{d/2-2}.\end{aligned} \tag{4.174}$$

Current conservation and gauge invariance imply that

$$q_a\,\Pi^{*ab}(q) = 0. \tag{4.175}$$

One of the advantages of dimensional regularization is that it preserves the vanishing of this quantity

$$q_a\,\Pi^{*ab}(q) \propto q_a(q^a q^b - q^2\delta^{ab}) = q^2 q^b - q^2 q^b = 0. \tag{4.176}$$

This works because current conservation and the dimension of spacetime have nothing to do with each other:

The vanishing of $q_a\,\Pi^{*ab}(q)$ follows from the conservation

$$\partial_a J^a(x) = \partial_a[\psi(x)\gamma^a\psi(x)] = 0 \tag{4.177}$$

of the current

$$J^a(x) = \psi(x)\gamma^a\psi(x) \tag{4.178}$$

and from the neutrality of the current which is due to the fact that ψ lowers the charge while $\bar{\psi}$ raises it.

Apart from a constant f, the quantity $\Pi^{*ab}(q)$ is by (4.142)

$$\Pi^{*ab}(q) = f \int d^4x d^4y e^{iq(y-x)} \langle 0|\mathcal{T}[J^a(x)J^b(y)]|0\rangle. \tag{4.179}$$

So

$$\begin{aligned} q_a \Pi^{*ab}(q) &= f \int d^4x d^4y \, q_a \, e^{iq(y-x)} \langle 0|\mathcal{T}[J^a(x)J^b(y)]|0\rangle \\ &= f \int d^4x d^4y \, e^{iq(y-x)} (i\partial_a) \, \langle 0|\mathcal{T}[J^a(x)J^b(y)]|0\rangle \\ &= f \int d^4x d^4y \, e^{iq(y-x)} \langle 0|\mathcal{T}[i\partial_a J^a(x)J^b(y)]|0\rangle \\ &\quad + f \int d^4x d^4y \, i \, \delta(x^0 - y^0) \, e^{iq(y-x)} \langle 0|[J^0(x), J^b(y)]|0\rangle \\ &= f \int d^4x d^4y \, i \, \delta(x^0 - y^0) \, e^{iq(y-x)} \langle 0|[J^0(x), J^b(y)]|0\rangle \end{aligned} \tag{4.180}$$

since the current is conserved. But the equal-time commutator of the charge $J^0(x)$ with any operator is proportional to the charge of that operator because $J^0(x)$ generates the $U(1)$ rotation of the symmetry that conserves charge. So since currents are neutral, the commutator vanishes

$$\delta(x^0 - y^0) \, e^{iq(y-x)} [J^0(x), J^b(y)] = 0, \tag{4.181}$$

and we have

$$q_a \Pi^{*ab}(q) = 0. \tag{4.182}$$

The gamma function $\Gamma(2 - d/2)$ in $\Pi^{*ab}(q)$ (4.174) blows up as $d \to 4$. So we bring in the contribution to $\Pi^{*ab}(q)$ arising from the counterterm $-(Z_3 - 1)F_{ab}F^{ab}$. This quadratic form in derivatives of A_a and A_b adds to $\Pi^{*ab}(q)$ the term

$$\Pi^{*ab}(q)_{Z_3} = -(Z_3 - 1)(q^2\delta^{ab} - q^a q^b) \tag{4.183}$$

in euclidian space. So to order e^2, $\Pi^{*ab}(q)$ is

$$\Pi^{*ab}(q) = (q^2\delta^{ab} - q^a q^b)\pi(q^2) \tag{4.184}$$

where

$$\begin{aligned} \pi(q) = &- \frac{4e^2 A_d}{(2\pi)^4} \Gamma(d/2)\Gamma(2 - d/2) \int_0^1 dx x(1-x)[m^2 + x(1-x)\, q^2]^{d/2-2} \\ &- (Z_3 - 1). \end{aligned} \tag{4.185}$$

So we can use Z_3 to cancel the poles in the gamma functions.

The quantity $\Pi^{*ab}(q)$ is a correction to the photon propagator. And $\Pi^{*ab}(0)$ is a correction to the mass of the photon which we want to remain at zero. So we set

$$\Pi^{*ab}(0) = 0. \tag{4.186}$$

This means that set set

$$Z_3 = 1 - \frac{4e^2 A_d}{(2\pi)^4}\Gamma(d/2)\Gamma(2-d/2)\,(m^2)^{d/2-2}\int_0^1 dx x(1-x), \tag{4.187}$$

so to order e^2

$$\begin{aligned}\pi(q) = &-\frac{4e^2 A_d}{(2\pi)^4}\Gamma(d/2)\Gamma(2-d/2)\int_0^1 dx x(1-x)\\ &\times\Big[[m^2 + x(1-x)\,q^2]^{d/2-2} - (m^2)^{d/2-2}\Big].\end{aligned} \tag{4.188}$$

As we let $d \to 4$, the tricky term is

$$\begin{aligned}[m^2 + x(1-x)\,q^2]^{d/2-2} - (m^2)^{d/2-2} &= \exp\Big[(d/2-2)\ln[m^2 + x(1-x)\,q^2]\Big]\\ &\quad - \exp\Big[(d/2-2)\ln(m^2)]\Big]\\ &= (d/2-2)\ln[1 + x(1-x)\,q^2/m^2].\end{aligned} \tag{4.189}$$

The final result is

$$\pi(q^2) = \frac{e^2}{2\pi^2}\int_0^1 x(1-x)\ln\left[1 + x(1-x)\,q^2/m^2\right]dx. \tag{4.190}$$

The scattering of two particles of charges e_1 and e_2 that exchange momentum $q = p_1 - p_1' = p_2' - p_2$ is

$$\begin{aligned}S = &\frac{-ie_1e_2}{4\pi^2}\delta^4(p_1' + p_2' - p_1 - p_2)\bar{u}_1'\gamma^a u_1\,\bar{u}_2'\gamma^b u_2\\ &\times\left[\frac{\eta_{ab}}{q^2} + \frac{[q^2\eta_{ab} - q_a q_b]\pi(q^2)}{(q^2)^2}\right].\end{aligned} \tag{4.191}$$

The spinors obey the Dirac equation in momentum space (1.264)

$$(i\,p^c\gamma_c + m)u(p,s) = 0 \tag{4.192}$$

so

$$p_{1a}\bar{u}_1'\gamma^a u_1 = im\bar{u}_1'\gamma^a u_1. \tag{4.193}$$

The adjoint of Dirac's equation is

$$u^\dagger(p,s)(-i\,p^c\gamma_c^\dagger + m) = 0. \tag{4.194}$$

The spatial gammas are hermitian while γ^0 is antihermitian, so

$$iu^\dagger(p,s)(-i\,p^c\gamma_c^\dagger + m)\gamma^0 = iu^\dagger(p,s)\gamma^0(i\,p^c\gamma_c + m) = \bar{u}(p,s)(i\,p^c\gamma_c + m) = 0.$$

Thus

$$p'_{1a}\bar{u}'_1\gamma^a u_1 = -im\bar{u}'_1\gamma^a u_1. \tag{4.195}$$

Subtracting, we get

$$q_a\bar{u}'_1\gamma^a u_1 = (p'_{1a} - p_{1a})\bar{u}'_1\gamma^a u_1 = (-im - (-im))\bar{u}'_1\gamma^a u_1 = 0. \tag{4.196}$$

So the amplitude (4.191) simplifies to

$$S = \frac{-ie_1e_2}{4\pi^2 q^2}[1 + \pi(q^2)]\delta^4(p'_1 + p'_2 - p_1 - p_2)\bar{u}'_1\gamma^a u_1\,\bar{u}'_2\gamma_a u_2. \tag{4.197}$$

For small $\vec{p}_1$ and $\vec{p}_2$,

$$\bar{u}'_1\gamma^0 u_1 = iu'^\dagger_1(\gamma^0)^2 u_1 = -iu'^\dagger_1 u_1 \approx -i\delta_{s'_1 s_1} \quad \text{and} \quad iu'^\dagger_1(\gamma^0)^2 u_1 \approx -i\delta_{s'_2 s_2}$$

while

$$\bar{u}'_1\vec{\gamma}u_1 = 0 \quad \text{and} \quad \bar{u}'_2\vec{\gamma}u_2 = 0.$$

So the amplitude is approximately

$$S = \frac{-ie_1e_2}{4\pi^2\vec{q}^2}[1 + \pi(\vec{q}^2)]\delta^4(p'_1 + p'_2 - p_1 - p_2)\,\delta_{s'_1 s_1}\delta_{s'_2 s_2}. \tag{4.198}$$

The lowest, Born, approximation to the nonrelativistic scattering due to a spin-independent potential $V(|\vec{x}_1 - \vec{x}_2|)$ which is

$$S_B = -2\pi i\delta(E'_1 + E'_2 - E_1 - E_2)\,T_B \tag{4.199}$$

where

$$T_B = \frac{\delta_{s'_1 s_1}\delta_{s'_2 s_2}}{(2\pi)^6}\int d^3x_1 d^3x_2 V(|\vec{x}_1 - \vec{x}_2|)e^{-i\vec{p}'_1\cdot\vec{x}_1}e^{-i\vec{p}'_2\cdot\vec{x}_2}e^{i\vec{p}_1\cdot\vec{x}_1}e^{i\vec{p}_2\cdot\vec{x}_2}.$$

We set $\vec{r} = \vec{x}_1 - \vec{x}_2$ and get

$$S_B = \frac{-i}{4\pi^2}\,\delta^4(p'_1 + p'_2 - p_1 - p_2)\delta_{s'_1 s_1}\delta_{s'_2 s_2}\int V(r)e^{-i\vec{q}\cdot\vec{r}}\,d^3r. \tag{4.200}$$

Comparing this formula with (4.198), we have

$$\int V(r)e^{-i\vec{q}\cdot\vec{r}}\,d^3r = e_1e_2\,\frac{1 + \pi(\vec{q}^2)}{\vec{q}^2}$$

or

$$V(r) = \frac{e_1e_2}{(2\pi)^3}\int e^{i\vec{q}\cdot\vec{r}}\left[\frac{1 + \pi(\vec{q}^2)}{\vec{q}^2}\right]. \tag{4.201}$$

This is the potential that the extended charge distribution

$$c(\vec{r}) = \delta^3(\vec{r}) + \frac{1}{2(2\pi)^3} \int \pi(\vec{q}^2)\, e^{i\vec{q}\cdot\vec{r}}\, d^3q, \tag{4.202}$$

which is normalized to unity

$$\int c(\vec{r})\, d^3r = 1 + \frac{\pi(0)}{2} = 1, \tag{4.203}$$

would produce if separated from itself by $\vec{r}$

$$V(r) = e_1 e_2 \int d^3x \int d^3y \frac{c(\vec{x})c(\vec{y})}{4\pi|\vec{x} - \vec{y} + \vec{r}|}. \tag{4.204}$$

Apart from a delta function, which we may write as

$$\begin{aligned} \delta^3(\vec{r}) &= \frac{1}{(2\pi)^3} \int_0^\infty e^{iqr\cos\theta}\, 2\pi q^2 dq d\cos\theta \\ &= \frac{1}{(2\pi)^2} \int_0^\infty \left(\frac{e^{iqr} - e^{iqr}}{ir} \right) q dq, \end{aligned} \tag{4.205}$$

the isotropic charge distribution $c'(r) = c(r) - \delta^3(\vec{r})$ is

$$\begin{aligned} c'(r) &= \frac{1}{2(2\pi)^3} \int_0^\infty \pi(q^2)\, e^{iqr\cos\theta}\, 2\pi q^2 dq d\cos\theta \\ &= \frac{1}{(2\pi)^2 r} \int \pi(q^2)\, \sin(qr)\, q\, dq \\ &= \frac{e^2}{8\pi^4 r} \int_0^\infty q dq \int_0^1 dx x(1-x) \ln\left[1 + \frac{x(1-x)\, q^2}{m^2}\right] \sin(qr) \\ &= \frac{-ie^2}{16\pi^4 r} \int_{-\infty}^\infty e^{iqr} q dq \int_0^1 dx x(1-x) \ln\left[1 + \frac{x(1-x)\, q^2}{m^2}\right]. \end{aligned} \tag{4.206}$$

Vacuum polarization (4.190) shifts the energy of an atomic state with wave function $\psi(\boldsymbol{r})$ by

$$\Delta E = \int \Delta V(\boldsymbol{r}) |\psi(\boldsymbol{r})|^2\, d^3r \tag{4.207}$$

where (SW 11.2.28)

$$\Delta V(r) = \Delta V(\boldsymbol{r}) = \frac{e_1 e_2}{(2\pi)^3} \int d^3q e^{i\boldsymbol{q}\cdot\boldsymbol{r}} \frac{\pi(\boldsymbol{q}^2)}{\boldsymbol{q}^2}. \tag{4.208}$$

For hydrogen-like electrons in states with $\ell = 0$ and wave function at $r = 0$

$$\psi(0) = \frac{2}{\sqrt{4\pi}} \left(\frac{Z\alpha m}{n} \right)^{3/2} \tag{4.209}$$

the shift is (SW 11.2.42)

$$\Delta E = \; = -\frac{4Z^4\alpha^5 m}{15\pi n^3} \tag{4.210}$$

and is called the Uehling effect. In the 2s state of hydrogen it is -1.122×10^{-7} eV.

4.12 Magnetic moment of the electron

The one-loop correction to the interaction of an electron of momentum p with a photon of momentum $p'-p$ (SW's figure 11.4(d)) is the logarithmically divergent integral (SW 11.3.1)

$$\begin{aligned}\Gamma^a(p',p) = \int d^4k\big[e\gamma^b(2\pi)^4\big]\Big[\frac{-i}{(2\pi)^4}\frac{-i(\not p'-\not k)+m}{(p'-k)^2+m^2-i\epsilon}\Big]\gamma^a \\ \times\Big[\frac{-i}{(2\pi)^4}\frac{-i(\not p-\not k)+m}{(p-k)^2+m^2-i\epsilon}\Big]\big[e\gamma_b(2\pi)^4\big]\Big[\frac{-i}{(2\pi)^4}\frac{1}{k^2-i\epsilon}\Big].\end{aligned} \tag{4.211}$$

The first step is to use Feynman's trick

$$\frac{1}{ABC} = 2\int_0^1 dx\int_0^x dy\frac{1}{\big[Ay+B(x-y)+C(1-x)\big]^3} \tag{4.212}$$

to write the three denominators as

$$\begin{aligned}\frac{1}{(p'-k)^2+m^2-i\epsilon}\frac{1}{(p-k)^2+m^2-i\epsilon}\frac{1}{k^2-i\epsilon} = 2\int_0^1 dx\int_0^x dy \\ \times\frac{1}{\Big[[(p'-k)^2+m^2-i\epsilon]y+[(p-k)^2+m^2-i\epsilon](x-y)+(k^2-i\epsilon)(1-x)\Big]^3} \\ = 2\int_0^1 dx\int_0^x dy\frac{1}{\Big[[k-p'y-p(x-y)]^2+m^2x^2+q^2y(x-y)-i\epsilon\Big]^3}.\end{aligned} \tag{4.213}$$

We now assume that we have made the integral sufficiently finite that we can shift k to $k+p'y+p(x-y)$. The integral (4.211) for $\Gamma(p',p)$ then becomes SW's (11.3.4)

$$\begin{aligned}\Gamma^a(p',p) = \frac{2ie^2}{(2\pi)^4}\int_0^1 dx\int_0^x dy\int d^4k\Big[\gamma^b\Big[-i[\not p'(1-y)-\not k-\not p(x-y)]+m\Big] \\ \times\gamma^a\Big[-i[\not p(1-x+y)-\not k-\not p'y]+m\Big]\gamma_b\Big] \\ \Big/\Big[k^2+m^2x^2+q^2y(x-y)-i\epsilon\Big]^3.\end{aligned} \tag{4.214}$$

The next step is a Wick rotation that sets $k^0 = ik^4$ and makes the denominator a function of $k^2 = \boldsymbol{k}^2 + (k^4)^2$. The $O(4)$ symmetry of the denominator lets us drop terms in the numerator that are odd under reflections. Also since $\Gamma(n) = (n-1)!$, our formula (4.161) for the area of the unit sphere in d dimensions gives for $d = 4$

$$A_4 = \frac{2\pi^{4/2}}{\Gamma(4/2)} = 2\pi^2. \tag{4.215}$$

The integral (4.214) in terms of $d^4k = id^4k_e = 2i\pi^2 k^3 dk$ now is

$$\begin{aligned}\Gamma^a(p',p) = \frac{-4\pi^2 e^2}{(2\pi)^4}\int_0^1 dx \int_0^x dy \int_0^\infty k^3 dk \Big[&- \frac{k^2\gamma^b\gamma^c\gamma^a\gamma_c\gamma_b}{4} \\ &+ \gamma^b\Big[- i[\not{p}'(1-y) - \not{p}(x-y)] + m\Big]\gamma^a \\ &\times \Big[- i[\not{p}(1-x+y) - \not{p}'y] + m\Big]\gamma_b\Big] \\ &\Big/ \Big[k^2 + m^2x^2 + q^2y(x-y)\Big]^3 .\end{aligned} \tag{4.216}$$

Dirac's spinors obey his equation (1.264) in momentum space

$$\bar{u}'(i\not{p}' + m) = 0 \quad \text{and} \quad (i\not{p} + m)u = 0. \tag{4.217}$$

So we may (eventually) write the vertex correction (4.216) as

$$\begin{aligned}\bar{u}'\Gamma^a(p',p)u = \frac{-4\pi^2 e^2}{(2\pi)^4}\int_0^1 dx \int_0^x dy \int_0^\infty k^3 dk \\ \times \bar{u}'\Big[\gamma^a\Big[- k^2 + 2m^2(x^2 - 4x + 2) + 2q^2[y(x-y) + 1 - x]\Big] \\ + 4imp'^a(y - x + xy) + 4imp^a(x^2 - xy - y)\Big]u \\ \Big/ \Big[k^2 + m^2x^2 + q^2y(x-y)\Big]^3 .\end{aligned} \tag{4.218}$$

Under the reflection $y \to x - y$, the factor $y - x + xy$, which multiplies p'^a, becomes $x - y - x + x(x-y) = x^2 - xy - y$, which multiplies p^a. This reflection maps the lower half of the triangle $\{(x,y)|0 \le x \le 1; 0 \le y \le x\}$ into its upper half. The jacobian is unity. So we can replace each factor by the average of the two factors

$$-\frac{1}{2}x(1-x) = \frac{1}{2}(y - x + xy + x^2 - xy - y). \tag{4.219}$$

The vertex (4.218 then reduces to

$$\bar{u}'\Gamma^a(p',p)u = \frac{-4\pi^2 e^2}{(2\pi)^4}\int_0^1 dx\int_0^x dy\int_0^\infty k^3 dk \tag{4.220}$$
$$\times\,\bar{u}'\Big[\gamma^a\Big[-k^2+2m^2(x^2-4x+2)+2q^2[y(x-y)+1-x]\Big]$$
$$+-2im(p'^a+p^a)x(1-x)\Big]u\Big/\Big[k^2+m^2x^2+q^2y(x-y)\Big]^3.$$

This term is a correction to a term that involves $\langle p',s'|A_a(x)J^a(x)|p,s\rangle$ and we ignore the electromagnetic field $A_a(x)$ except for its momentum. So we are dealing with a matrix element of a current $J^a(x)$

$$\langle p',s'|J^a(x)|p,s\rangle = \langle p',s'|e^{-iP\cdot x}J^a(0)e^{iP\cdot x}|p,s\rangle = e^{i(p-p')\cdot x}\langle p',s'|J^a(0)|p,s\rangle. \tag{4.221}$$

Since the current $J^a(x)$ is conserved

$$0 = \partial_a J^a(x) \tag{4.222}$$

it follows that

$$\begin{aligned}0 &= \langle p',s'|\partial_a J^a(x)|p,s\rangle = \partial_a\langle p',s'|J^a(x)|p,s\rangle\\ &= \partial_a e^{i(p-p')\cdot x}\langle p',s'|J^a(0)|p,s\rangle = i(p-p')_a\langle p',s'|J^a(0)|p,s\rangle.\end{aligned} \tag{4.223}$$

Does the amplitude (4.220) obey this rule of current conservation? To check this, we note that since Dirac's spinors obey his equation (4.217 or 4.237), we see that the γ^a term does

$$i(p-p')_a\bar{u}'\gamma^a u = \bar{u}'(i\not{p}-i\not{p}')u = \bar{u}'(m-m)u = 0. \tag{4.224}$$

So does the $p'-p$ term:

$$(p-p')_a\bar{u}'(p'^a-p^a)u = \bar{u}'(-m^2+m^2)u = 0. \tag{4.225}$$

If we integrate the charge density $J^0(x)$ over all space, we get the charge operator Q

$$Q = \int J^0(\boldsymbol{x},t)\,d^3x. \tag{4.226}$$

Because the current is conserved (4.222), the charge operator Q has a vanishing time derivative

$$\begin{aligned}\frac{\partial Q}{\partial t} &= \int \partial_0 J^0(\boldsymbol{x},t)\,d^3x = -\int \nabla\cdot\boldsymbol{J}(\boldsymbol{x},t)\,d^3x\\ &= -\int \boldsymbol{J}(\boldsymbol{x},t)\cdot d\sigma = 0\end{aligned} \tag{4.227}$$

as long as the integral over the surface σ vanishes. (Massless scalar fields could pose a problem here.) The matrix element of the charge operator is an integral of the matrix element of the charge density, which is the time component of the current (4.221),

$$\begin{aligned}\langle p', s'|Q|p, s\rangle &= \langle p', s'| \int J^0(\boldsymbol{x}, t)\, d^3x|p, s\rangle \\ &= \int e^{i(p-p')\cdot x} d^3x \langle p', s'|J^0(0)\, d^3x|p, s\rangle \\ &= (2\pi)^3 \delta^3(\boldsymbol{p} - \boldsymbol{p}')\langle p', s'|J^0(0)\, d^3x|p, s\rangle. \end{aligned} \tag{4.228}$$

The state $|p, s\rangle$ is a state of one electron (of momentum $\boldsymbol{p}$ and spin s in the z direction), so it has charge $-e$. Therefore

$$Q|p, s\rangle = -e|p, s\rangle, \tag{4.229}$$

and so

$$\langle p's'|Q|p, s\rangle = -e\langle p's'|p, s\rangle = -e\delta^3(\boldsymbol{p} - \boldsymbol{p}')\delta_{ss'}. \tag{4.230}$$

Comparing this equation with the matrix element (4.228) of the charge operator, we find that

$$\langle p', s'|J^0(0)\, d^3x|p, s\rangle = -\frac{e}{(2\pi)^3}\,\delta_{ss'}. \tag{4.231}$$

SW writes the matrix element of a current J^a of a particle of charge $-e$ as (10.6.9)

$$\langle p', s'|J^a(0)|p, s\rangle = -\frac{ie}{(2\pi)^3}\bar{u}(p', s')\Gamma^a(p', p)u(p, s). \tag{4.232}$$

Lorentz invariance requires the last factor here to be

$$\begin{aligned}\bar{u}(p', s')\Gamma^a(p', p)u(p, s) = \bar{u}(p', s')\Big[\gamma^a F(q^2) - \frac{i}{2m}(p + p')^a G(q^2) \\ + \frac{(p - p')^a}{2m} H(q^2)\Big] u(p, s). \end{aligned} \tag{4.233}$$

The hermiticity of the current implies that $F, G, \& H$ are real functions of $q^2 = (p' - p)^2$. Charge conservation (4.223) requires that $H(q^2) = 0$. So the vertex correction (4.220) is the sum of two **form factors** $F(q^2)$ and $G(q^2)$

$$\bar{u}'\Gamma^a(p', p)u = \bar{u}'\Big[\gamma^a F(q^2) - \frac{i}{2m}(p + p')^a G(q^2)\Big] u. \tag{4.234}$$

Letting $p' \to p$ in this equation and in the matrix element (4.232), we find

$$\begin{aligned}\langle p, s'|J^a(0)|p, s\rangle &= -\frac{ie}{(2\pi)^3}\bar{u}(p, s')\Gamma^a(p, p)u(p, s) \\ &= -\frac{ie}{(2\pi)^3}\bar{u}(p, s')\Big[\gamma^a F(0) - \frac{i}{m}p^a G(0)\Big]u(p, s).\end{aligned} \tag{4.235}$$

Since Dirac's gamma matrices are defined by the anticommutation relation $\{\gamma^a, \gamma^b\} = 2\eta^{ab}$, it follows that

$$\{\gamma^a, i\not{p} + m\} = \{\gamma^a, ip_b\gamma^b + m\} = 2ip_b\eta^{ab} + 2m\gamma^a = 2ip^a + 2m\gamma^a. \tag{4.236}$$

Dirac's spinors obey his equation

$$\bar{u}'(i\not{p}' + m) = 0 \quad \text{and} \quad (i\not{p} + m)u = 0. \tag{4.237}$$

So between spinors of the same momentum, the anticommutation relation (4.236) gives

$$\bar{u}(p, s')\{\gamma^a, i\not{p} + m\}u(p, s) = 0 = \bar{u}(p, s')(2ip^a + 2m\gamma^a)u(p, s)$$

or

$$\bar{u}(p, s')\gamma^a u(p, s) = -\frac{ip^a}{m}\,\bar{u}(p, s')u(p, s). \tag{4.238}$$

Using the formulas (1.273) for the spinors

$$\begin{aligned}u(\boldsymbol{p}, s) &= \frac{m - i\not{p}}{\sqrt{2p^0(p^0 + m)}}\,u(\mathbf{0}, s) \\ v(\boldsymbol{p}, s) &= \frac{m + i\not{p}}{\sqrt{2p^0(p^0 + m)}}\,v(\mathbf{0}, s),\end{aligned} \tag{4.239}$$

we can compute the inner product

$$\begin{aligned}\bar{u}(\boldsymbol{p}, s')u(\boldsymbol{p}, s) &= iu^\dagger(\boldsymbol{p}, s')\gamma^0 u(\boldsymbol{p}, s) \\ &= \frac{i}{2p^0(p^0 + m)}\,u^\dagger(\mathbf{0}, s')(m + ip_a\gamma^{a\dagger})\gamma^0(m - i\not{p})u(\mathbf{0}, s) \\ &= \frac{i}{2p^0(p^0 + m)}\,u^\dagger(\mathbf{0}, s')\gamma^0(m - ip_a\gamma^a)(m - i\not{p})u(\mathbf{0}, s) \\ &= \frac{1}{2p^0(p^0 + m)}\,\bar{u}(\mathbf{0}, s')(m - i\not{p})(m - i\not{p})u(\mathbf{0}, s).\end{aligned} \tag{4.240}$$

The formula (4.238) for the terms linear in gamma tells us that

$$\bar{u}(\mathbf{0}, s')p_a\gamma^a u(\mathbf{0}, s) = -ip_a\delta^a_0\bar{u}(\mathbf{0}, s')u(\mathbf{0}, s) = ip^0\bar{u}(\mathbf{0}, s')u(\mathbf{0}, s). \tag{4.241}$$

Also the anticommutation relation $\{\gamma^a, \gamma^b\} = 2\eta^{ab}$ tells us that

$$\not{p}\not{p} = p_a p_b \gamma^a \gamma^b = \frac{1}{2} p_a p_b \{\gamma^a, \gamma^b\} = p_a p_b \eta^{ab} = -m^2. \tag{4.242}$$

So we find that the inner product is

$$\bar{u}(\boldsymbol{p}, s')u(\boldsymbol{p}, s) = \frac{\bar{u}(\mathbf{0}, s')u(\mathbf{0}, s)}{2p^0(p^0 + m)}\,(2m^2 + 2mp^0) = \frac{m}{p^0}\bar{u}(\mathbf{0}, s')u(\mathbf{0}, s). \tag{4.243}$$

The zero-momentum spinors are

$$\begin{aligned} u(\vec{0}, m = \tfrac{1}{2}) &= \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\1\\0\end{bmatrix} \quad \text{and} \quad u(\vec{0}, m = -\tfrac{1}{2}) = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\0\\1\end{bmatrix}, \\ v(\vec{0}, m = \tfrac{1}{2}) &= \frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\0\\-1\end{bmatrix} \quad \text{and} \quad v(\vec{0}, m = -\tfrac{1}{2}) = \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\0\\1\\0\end{bmatrix}. \end{aligned} \tag{4.244}$$

So the inner product (4.243) is

$$\bar{u}(\boldsymbol{p}, s')u(\boldsymbol{p}, s) = \frac{m}{p^0}\,\delta_{ss'}. \tag{4.245}$$

Incidentally, SW's pseudounitarity condition (1.209, SW's 5.4.32)

$$\beta D^\dagger(\Lambda)\beta = D^{-1}(\Lambda) \tag{4.246}$$

leads to a much shorter derivation of this (4.245) result. Using the formulas (4.238 & 4.245), we can write the expression (4.235) for a matrix element of $J^a(0)$ as

$$\begin{aligned} \langle p, s'|J^a(0)|p, s\rangle &= -\frac{ie}{(2\pi)^3}\bar{u}(p, s')\Big[-\frac{ip^a}{m}F(0) - \frac{i}{m}p^a G(0)\Big]u(p, s) \\ &= -\frac{ep^a}{(2\pi)^3 m}\bar{u}(p, s')\Big[F(0) + G(0)\Big]u(p, s) \\ &= -\frac{e}{(2\pi)^3}\frac{p^a}{p^0}\delta_{ss'}\big[F(0) + G(0)\big]. \end{aligned} \tag{4.247}$$

Setting $a = 0$ and using the expression (4.231) for the matrix element of the charge density

$$\langle p', s'|J^0(0)\, d^3x|p, s\rangle = -\frac{e}{(2\pi)^3}\,\delta_{ss'}, \tag{4.248}$$

we get the normalization condition

$$F(0) + G(0) = 1 \tag{4.249}$$

for these quantities when computed with counterterms.

We can use Dirac's equation (4.237) and the anticommutation relation (4.236) to get

$$\begin{aligned}\bar{u}'[i(p+p')^a + 2m\gamma^a]u &= \bar{u}'[i(p+p')^a - i\not{p}'\gamma^a - i\gamma^a\not{p}]u \\ &= \bar{u}'\big[\tfrac{1}{2}i\{\gamma^a,\not{p}'\} + \tfrac{1}{2}i\{\gamma^a,\not{p}\} - i\not{p}'\gamma^a - i\gamma^a\not{p}\big]u \\ &= \bar{u}'\big(\tfrac{1}{2}i[\gamma^a,\gamma^b](p'-p)_b\big)u. \end{aligned} \tag{4.250}$$

We now can write the coefficient of the first term in the vertex (4.234)

$$\bar{u}'\Gamma^a(p',p)u = \bar{u}'\Big[\gamma^a F(q^2) - \frac{i}{2m}(p+p')^a G(q^2)\Big]u \tag{4.251}$$

as

$$\begin{aligned}\bar{u}'\Gamma^a(p',p)u = &-\frac{i}{2m}\,\bar{u}'\Big[(p+p')^a\Big(F(q^2)+G(q^2)\Big) \\ &-\frac{1}{2}[\gamma^a,\gamma^b](p'-p)_b F(q^2)\Big]u. \end{aligned} \tag{4.252}$$

The generators of rotations and boosts in the Dirac representation of the Lorentz group are (1.175) for spatial values of i and j

$$\begin{aligned}\mathcal{J}^{ij} &= -\frac{i}{4}[\gamma^i,\gamma^j] = -\frac{i}{4}\left[-i\begin{pmatrix}0 & \sigma_i\\ -\sigma_i & 0\end{pmatrix}, -i\begin{pmatrix}0 & \sigma_j\\ -\sigma_j & 0\end{pmatrix}\right] \\ &= \frac{1}{2}\epsilon_{ijk}\begin{pmatrix}\sigma_k & 0\\ 0 & \sigma_k\end{pmatrix}\end{aligned} \tag{4.253}$$

and

$$\begin{aligned}\mathcal{J}^{i0} &= -\frac{i}{4}[\gamma^i,\gamma^0] = -\frac{i}{4}\left[-i\begin{pmatrix}0 & \sigma_i\\ -\sigma_i & 0\end{pmatrix}, -i\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right] \\ &= \frac{i}{2}\begin{pmatrix}\sigma_i & 0\\ 0 & -\sigma_i\end{pmatrix}.\end{aligned} \tag{4.254}$$

So their $\boldsymbol{p} = \boldsymbol{p}' = 0$ matrix elements between the spinors (4.244) are

$$\begin{aligned}&\bar{u}(0,s')[\gamma^i,\gamma^j]u(0,s) = \; = 4i\epsilon_{ijk}\left(J_k^{(1/2)}\right)_{s's} = 2i\epsilon_{ijk}\sigma_{ks's} \\ &\bar{u}(0,s')[\gamma^i,\gamma^0]u(0,s) = 0.\end{aligned} \tag{4.255}$$

So for small momenta and to lowest order in $\boldsymbol{p}$ and $\boldsymbol{p}'$, the vertex (4.252) is

$$\begin{aligned}\bar{u}(\boldsymbol{p}',s')\boldsymbol{\Gamma}(p',p)u(\boldsymbol{p},s) &\approx \frac{1}{m}\left[(\boldsymbol{p}-\boldsymbol{p}')\times\boldsymbol{J}^{(1/2)}_{k,s's}\right]F(0) \\ &= \frac{1}{2m}\left[(\boldsymbol{p}-\boldsymbol{p}')\times\boldsymbol{\sigma}_{s's}\right]F(0).\end{aligned} \tag{4.256}$$

In weak, static fields, the interaction hamiltonian is

$$H' = -\int \boldsymbol{J}(\boldsymbol{x}) \cdot \boldsymbol{A}(\boldsymbol{x})\, d^3x. \tag{4.257}$$

So the interaction is

$$\begin{aligned}\langle p', s'|H'|p, s\rangle &= \frac{ieF(0)}{(2\pi)^3 2m} \int e^{i(\boldsymbol{p}-\boldsymbol{p}')\cdot\boldsymbol{x}} \boldsymbol{A}(\boldsymbol{x}) \cdot (\boldsymbol{p}-\boldsymbol{p}') \times \boldsymbol{\sigma}_{s's}\, d^3x \\ &= \frac{eF(0)}{(2\pi)^3 2m} \int e^{i(\boldsymbol{p}-\boldsymbol{p}')\cdot\boldsymbol{x}} \boldsymbol{B}(\boldsymbol{x}) \cdot \boldsymbol{\sigma}_{s's}\, d^3x \end{aligned} \tag{4.258}$$

where $\boldsymbol{B}(\boldsymbol{x}) = \nabla \times \boldsymbol{A}(\boldsymbol{x})$ is the magnetic field. So for a slowly varying magnetic field,

$$\langle p', s'|H'|p, s\rangle = \frac{eF(0)}{2m} \boldsymbol{B} \cdot \boldsymbol{\sigma}_{s's}\, \delta^3(\boldsymbol{p}-\boldsymbol{p}'). \tag{4.259}$$

So the magnetic moment is

$$\mu = -\frac{e}{2m} F(0). \tag{4.260}$$

The form factor $G(q^2)$ is

$$G(q^2) = -\frac{4\pi^2 e^2}{(2\pi)^4} \int_0^1 dx \int_0^x dy \int_0^\infty k^3 dk \frac{4m^2 x(1-x)}{\Big[k^2 + m^2x^2 + q^2 y(x-y)\Big]^3}. \tag{4.261}$$

The k integral is finite:

$$G(q^2) = -\frac{4\pi^2 e^2}{(2\pi)^4} \int_0^1 dx \int_0^x dy \frac{4m^2 x(1-x)}{4\Big[m^2x^2 + q^2 y(x-y)\Big]} \tag{4.262}$$

$$= -\frac{e^2 m^2}{4\pi^2} \int_0^1 dx \int_0^x dy \frac{x(1-x)}{m^2x^2 + q^2 y(x-y)}. \tag{4.263}$$

The magnetic moment of the electron to order e^3 is

$$\mu = -\frac{e}{2m}\big(1 - G(0)\big) \tag{4.264}$$

where

$$\begin{aligned} G(0) &= -\frac{e^2m^2}{4\pi^2}\int_0^1 dx\int_0^x dy\,\frac{x(1-x)}{m^2x^2} \\ &= -\frac{e^2m^2}{4\pi^2}\int_0^1 dx\int_0^x dy\,\frac{(1-x)}{m^2x} \\ &= -\frac{e^2m^2}{4\pi^2}\int_0^1 dx\,\frac{(1-x)}{m^2} = -\frac{e^2}{4\pi^2}\int_0^1 dx\,(1-x) \\ &= -\frac{e^2}{8\pi^2}. \end{aligned} \tag{4.265}$$

So the absolute value of the magnetic moment of the electron to order e^3 is

$$\mu = \frac{e}{2m}\big(1+\frac{e^2}{8\pi^2}\big) = \frac{e}{2m}\big(1+\frac{\alpha}{2\pi}\big) = 1.001161\,\frac{e}{2m} \tag{4.266}$$

first calculated by Julian Schwinger in 1948. The 2018 experimental value by Gabrielse et al. (PRA 83, 052122 (2011)) is

$$\mu_{exp} = (1.00115965218073 \pm 0.00000000000028)\,\frac{e}{2m} \tag{4.267}$$

in which 73 could be 73 ± 28. Kinoshita and others have done QED calculations to 5 loops. Their best estimate (as of 2012) including all standard-model effects is (arXiv:1205.5368)

$$\mu_{th} = 1.00115965218178\,\frac{e}{2m}. \tag{4.268}$$

The standard model value of $g-2$ agrees with experiment to 11 decimal digits

$$\mu_{SM} = \mu_{exp} = 1.00115965218\,\frac{e}{2m}. \tag{4.269}$$

Most of the theoretical uncertainty is due hadronic effects and to uncertainty in the experimental value of the fine-structure constant α.

4.13 Charge form factor of electron

If we add in the contribution due to vacuum polaritzation, then the charge form factor of electron is

$$\begin{aligned} F(q^2) = \pi(q^2) &+ \frac{4\pi^2e^2}{(2\pi)^4}\int_0^1 dx\int_0^x dy\int_0^\infty k^3dk \\ &\times \frac{k^2-2m^2(x^2-4x+2)-2q^2[y(x-y)+1-x]}{\Big[k^2+m^2x^2+q^2y(x-y)\Big]^3} \end{aligned} \tag{4.270}$$

which diverges logarithmically. Let's take the value of this quantity at $q^2 = 0$ from experiment. Then the charge form factor is

$$F(q^2) = F(0) + \frac{4\pi^2 e^2}{(2\pi)^4} \int_0^1 dx \int_0^x dy \int_0^\infty k^3 dk$$
$$\times \left[\frac{k^2 - 2m^2(x^2 - 4x + 2) - 2q^2[y(x-y) + 1 - x]}{\Big[k^2 + m^2x^2 + q^2y(x-y)\Big]^3} \right.$$
$$\left. - \frac{k^2 - 2m^2(x^2 - 4x + 2)}{\Big[k^2 + m^2x^2\Big]^3} \right]. \tag{4.271}$$

The k integral gives

$$F(q^2) = F(0) + \frac{4\pi^2 e^2}{(2\pi)^4} \int_0^1 dx \int_0^x dy \tag{4.272}$$
$$- \Big[q^2\big(- x^3 + 2y^2 - 2xy(1+2y) + x^2(1+4y)\big)$$
$$- \big(m^2x^2 + q^2x^2(x-y)y\big) \ln(m^2x^2)$$
$$+ x^2[m^2x^2 + q^2(x-y)y] \ln[m^2x^2 + q^2(x-y)y] \Big]$$
$$\Big/ \Big[2x^2(m^2x^2 + q^2(x-y)y) \Big] \tag{4.273}$$

5

Nonabelian gauge theory

5.1 Yang and Mills invent local nonabelian symmetry

The action of a Yang-Mills theory is unchanged when a spacetime-dependent unitary matrix $U(x) = \exp(-it_a\,\theta^a(x))$ maps a vector $\psi(x)$ of matter fields to $\psi'_i(x) = U_{ij}(x)\psi_j(x)$. The symmetry $\psi^\dagger(x)U^\dagger(x)U(x)\psi(x) = \psi^\dagger(x)\psi(x)$ is obvious, but how can kinetic terms like $\partial_i\psi^\dagger\,\partial^i\psi$ be made invariant?

Yang and Mills introduced matrices $A_i = t_a\,A_i^a$ of gauge fields in which the hermitian matrices t_a are the generators of the Lie algebra of a compact gauge group and so obey the commutation relation

$$[t_a, t_b] = i f_{abc} t_c \tag{5.1}$$

in which the f_{abc} are real, totally antisymmetric **structure constants**. More importantly, they replaced ordinary derivatives ∂_i by **covariant derivatives**

$$D_{i\alpha\beta} = \partial_i\delta_{\alpha\beta} + A_{i\alpha\beta} \equiv \partial_i\delta_{\alpha\beta} + t_{a\alpha\beta}\,A_i^a \tag{5.2}$$

and required that covariant derivatives of fields transform like fields so that $(D\psi)' = UD\psi$ or

$$D'\psi' = \left(\partial_i + A'_i\right)U\psi = \left(\partial_i U + U\partial_i + A'_i U\right)\psi = U\left(\partial_i + A_i\right)\psi. \tag{5.3}$$

The nonabelian gauge field transforms as

$$A'_i(x) = U(x)A_i(x)U^\dagger(x) - \left(\partial_i U(x)\right)U^\dagger(x). \tag{5.4}$$

In full detail, this is

$$A'_{i\alpha\delta}(x) = U_{\alpha\beta}(x)A_{i\beta\gamma}(x)(U^\dagger)_{\gamma\delta}(x) - \left(\partial_i U_{\alpha\beta}(x)\right)(U^\dagger)_{\beta\delta}(x). \tag{5.5}$$

The nonabelian Faraday tensor is

$$F_{ik} = [D_i, D_k] = \partial_i A_k - \partial_k A_i + [A_i, A_k] \tag{5.6}$$

in which both F and A are matrices in the Lie algebra. The nonabelian Faraday tensor transforms as

$$F'_{ik}(x) = U(x)F_{ik}U^{-1}(x) = U(x)[D_i, D_k]U^{-1}(x). \tag{5.7}$$

The trace $\mathrm{Tr}[F_{ik}F^{ik}]$ of the Lorentz-invariant product $F_{ik}F^{ik}$ is both Lorentz invariant and gauge invariant. The nonabelian generalization of the abelian action density

$$L = -\frac{1}{4}F_{jk}F^{jk} - \bar{\psi}[\gamma^k(\partial_k + ieA_k) + m]\psi \tag{5.8}$$

is

$$L = -\frac{1}{4}\mathrm{Tr}\big[F_{jk}F^{jk}\big] - \bar{\psi}[\gamma^k(\partial_k + ieA_k) + m]\psi. \tag{5.9}$$

In more detail, the Fermi action is

$$\bar{\psi}[\gamma^k(\partial_k + ieA_k) + m]\psi = \bar{\psi}_{\ell\alpha}\big[\gamma^k_{\ell\ell'}(\partial_k\delta_{\alpha\beta} + iet_{a\alpha\beta}A^a_k) + m\delta_{\ell\ell'}\delta_{\alpha\beta}\big]\psi_{\ell'\beta}. \tag{5.10}$$

Use of matrix notation and of summation conventions is necessary.

5.2 $SU(3)$

The gauge group of quantum chromodynamics is $SU(3)$ which has eight generators. The Gell-Mann matrices are

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$
$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \text{and} \quad \lambda_8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \tag{5.11}$$

The generators t_a of the 3×3 defining representation of $SU(3)$ are these Gell-Mann matrices divided by 2

$$t_a = \lambda_a/2 \tag{5.12}$$

(Murray Gell-Mann, 1929–).

The eight generators t_a are orthogonal with $k = 1/2$

$$\mathrm{Tr}\,(t_a t_b) = \frac{1}{2}\delta_{ab} \tag{5.13}$$

and satisfy the commutation relation

$$[t_a, t_b] = i f_{abc}\, t_c. \tag{5.14}$$

The trace formula

$$f^c{}_{ab} = -\frac{i}{k}\,\mathrm{Tr}\left([t_a, t_b]\, t_c^\dagger\right). \tag{5.15}$$

gives us the ***SU*(3) structure constants** as

$$f_{abc} = -2i\,\mathrm{Tr}\,([t_a, t_b]t_c)\,. \tag{5.16}$$

They are real and totally antisymmetric with $f_{123} = 1$, $f_{458} = f_{678} = \sqrt{3}/2$, and $f_{147} = -f_{156} = f_{246} = f_{257} = f_{345} = -f_{367} = 1/2$.

While no two generators of $SU(2)$ commute, two generators of $SU(3)$ do. In the representation (5.11,5.12), t_3 and t_8 are diagonal and so commute

$$[t_3, t_8] = 0. \tag{5.17}$$

They generate the **Cartan subalgebra** of $SU(3)$. The generators defined by Eqs.(5.12 & 5.11) give us the 3×3 representation

$$D(\alpha) = \exp\,(i\alpha_a t_a) \tag{5.18}$$

in which the sum $a = 1, 2, \ldots 8$ is over the eight generators t_a. This representation acts on complex 3-vectors and is called the **3**.

Note that if

$$D(\alpha_1)D(\alpha_2) = D(\alpha_3) \tag{5.19}$$

then the complex conjugates of these matrices obey the same multiplication rule

$$D^*(\alpha_1)D^*(\alpha_2) = D^*(\alpha_3) \tag{5.20}$$

and so form another representation of $SU(3)$. It turns out that (unlike in $SU(2)$) this representation is inequivalent to the **3**; it is the $\overline{\mathbf{3}}$.

There are three quarks with masses less than about 100 MeV/c^2—the u, d, and s quarks. The other three quarks c, b, and t are more massive; $m_c = 1.28$ GeV, $m_b = 4.18$ GeV, and $m_t = 173.1$ GeV. Nobody knows why. Gell-Mann and Zweig suggested that the low-energy strong interactions were approximately invariant under unitary transformations of the three light quarks, which they represented by a **3**, and of the three light antiquarks, which they represented by a $\overline{\mathbf{3}}$. They imagined that the eight

light pseudoscalar mesons, that is, the three pions π^-, π^0, π^+, the neutral η, and the four kaons K^0, K^+, K^- $\overline{K}^0$, were composed of a quark and an antiquark. So they should transform as the tensor product

$$\mathbf{3} \otimes \overline{\mathbf{3}} = \mathbf{8} \oplus \mathbf{1}. \tag{5.21}$$

They put the eight pseudoscalar mesons into an **8**.

They imagined that the eight light baryons — the two nucleons N and P, the three sigmas Σ^-, Σ^0, Σ^+, the neutral lambda Λ, and the two cascades Ξ^- and Ξ^0 were each made of three quarks. They should transform as the tensor product

$$\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = \mathbf{10} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{1}. \tag{5.22}$$

They put the eight light baryons into one of these **8**'s. When they were writing these papers, there were nine spin-3/2 resonances with masses somewhat heavier than 1200 MeV/c^2 — four Δ's, three Σ^*'s, and two Ξ^*'s. They put these into the **10** and predicted the tenth and its mass. In 1964, a tenth spin-3/2 resonance, the Ω^-, was found with a mass close to their prediction of 1680 MeV/c^2, and by 1973 an MIT-SLAC team had discovered quarks inside protons and neutrons. (George Zweig, 1937–)

For a given quark, say the up quark, the action of quantum chromodynamics is

$$L = -\frac{1}{4}\mathrm{Tr}\big[F_{jk}F^{jk}\big] - \bar{\psi}[\gamma^k(\partial_k + ieA_k) + m]\psi \tag{5.23}$$

in which the index c on ψ_c takes on the values $1, 2, 3$ which we may think of as red, green, and blue. The Faraday tensor is a 3×3 matrix

$$F_{ik} = [D_i, D_k] = \partial_i A_k - \partial_k A_i + [A_i, A_k]. \tag{5.24}$$

The trace term in the QCD action (5.23) is

$$\mathrm{Tr}\big[F_{jk}F^{jk}\big] = \mathrm{Tr}\Big[\Big(\partial_i A_k - \partial_k A_i + [A_i, A_k]\Big)\Big(\partial^i A^k - \partial^k A^i + [A^i, A^k]\Big)\Big] \tag{5.25}$$

Quarks and gluons are confined in color-singlet particles. This effect is robust and mysterious.

6
Standard model

6.1 Quantum chromodynamics

If to a pure $SU(3)$ gauge theory we add massless quarks in the fundamental or defining representation, then we get a theory of the strong interactions called **quantum chromodynamics** or QCD. Thus, let ψ be a complex 3-vector of Dirac fields

$$\psi = \begin{pmatrix} \psi_r \\ \psi_g \\ \psi_b \end{pmatrix} \tag{6.1}$$

(so 12 fields in all). This complex 12-vector could represent u or "up" quarks. In terms of the covariant derivative

$$D_\mu = I\partial_\mu + A_\mu(x) = I\partial_\mu + ig\sum_{b=1}^{8} t^b A_\mu^b(x), \tag{6.2}$$

the action density is

$$L = \frac{1}{2g^2}\,\mathrm{Tr}\,[F_{\mu\nu}F^{\mu\nu}] - \overline{\psi}\,(\gamma^\mu D_\mu + m)\,\psi \tag{6.3}$$

where

$$F_{\mu\nu} = [D_\mu, D_\nu]. \tag{6.4}$$

Nonperturbative effects are supposed to "confine" the quarks and massless gluons. There are 6 known "flavors" of quarks—u, d, c, s, t, b.

6.2 Abelian Higgs mechanism

Suppose $\phi = (\phi_1 + i\phi_2)/\sqrt{2}$ is a complex scalar field with action density

$$L = -\,\partial_a\phi^*\partial^a\phi + \mu^2\phi^*\phi - \lambda(\phi^*\phi)^2. \tag{6.5}$$

The minimum of the potential $V = -\mu^2\phi^*\phi + \lambda(\phi^*\phi)^2$ is given by the equation

$$0 = \frac{\partial V}{\partial|\phi|^2} = -\mu^2 + 2\lambda|\phi|^2. \tag{6.6}$$

It is a circle with

$$|\phi|^2 = \frac{\mu^2}{2\lambda} \equiv \frac{v^2}{2}. \tag{6.7}$$

As is customary, we pick one asymmetrical minimum at $\phi = \mu/\sqrt{2\lambda}$, so the mean values in the vacuum are

$$\langle 0|\phi_1|0\rangle = v = \sqrt{\frac{\mu^2}{\lambda}} \quad \text{and} \quad \langle 0|\phi_2|0\rangle = 0. \tag{6.8}$$

The $U(1)$ symmetry has been spontaneously broken. So

$$\phi = \frac{v}{\sqrt{2}} + \frac{\phi_1}{\sqrt{2}} + i\frac{\phi_2}{\sqrt{2}} \tag{6.9}$$

where now ϕ_1 is the departure of the real part of the field from its mean value v. So the potential V is

$$\begin{aligned} V &= -\mu^2\phi^*\phi + \lambda(\phi^*\phi)^2 \\ &= -\frac{1}{2}\mu^2\left[(v+\phi_1)^2 + (\phi_2)^2\right] + \frac{\lambda}{4}\left[(v+\phi_1)^2 + (\phi_2)^2\right]^2 \\ &= -\frac{\mu^4}{4\lambda} + \mu^2\phi_1^2 + \ldots \end{aligned} \tag{6.10}$$

where the dots denote cubic and quartic in ϕ_1 and ϕ_2. The quadratic part of the action determines the properties of the particles of the physical fields. The cubic and higher-order terms tell us how the particles interact with each other.

The first part of the last homework problem is to find the masses of the particles of this theory.

Scalar particles of zero mass that arise from spontaneous symmetry breaking are called Goldstone bosons.

Now we give the theory a local $U(1)$ symmetry by adding an abelian gauge field to the action density

$$L = -(D_a\phi)^* D^a\phi + \mu^2\phi^*\phi - \lambda(\phi^*\phi)^2 - \frac{1}{4}F_{ab}F^{ab} \tag{6.11}$$

in which

$$D_a = \partial_a + ieA_a \quad \text{and} \quad F_{ab} = [D_a, D_b]. \tag{6.12}$$

Once again the field assumes a mean value in the vacuum, let us say (6.9). But we write the field ϕ as

$$\phi = \frac{1}{\sqrt{2}}(v + \phi_1 + i\phi_2). \tag{6.13}$$

The term $-(D_a\phi)^* D^a\phi$ then is

$$\begin{aligned} -(D_a\phi)^* D^a\phi = & -(\partial_a\phi^* - ieA_a\phi^*)(\partial^a\phi + ieA^a\phi) \\ = & -\frac{1}{2}[\partial_a\phi_1 - i\partial_a\phi_2 - ieA_a(v + \phi_1 - i\phi_2)] \\ & \times [(\partial^a\phi_1 + i\partial^a\phi_2 + ieA^a(v + \phi_1 + i\phi_2)] \\ = & -\frac{1}{2}(\partial^a\phi_1 - eA^a\phi_2)(\partial_a\phi_1 - eA_a\phi_2) \\ & -\frac{1}{2}(\partial^a\phi_2 + eA^a(v + \phi_1))(\partial_a\phi_2 + eA_a(v + \phi_1)). \end{aligned} \tag{6.14}$$

So we set

$$B_a = A_a + \frac{1}{ev}\partial_a\phi_2 \quad \text{or} \quad A_a = B_a - \frac{1}{ev}\partial_a\phi_2 \tag{6.15}$$

and find that

$$\begin{aligned} -(D_a\phi)^* D^a\phi = & -\frac{1}{2}(\partial^a\phi_1 - eB^a\phi_2 + \frac{\phi_2}{v}\partial^a\phi_2)(\partial_a\phi_1 - eB_a\phi_2 + \frac{\phi_2}{v}\partial_a\phi_2) \\ & -\frac{1}{2}(evB^a + eB^a\phi_1 - \frac{\phi_1}{v}\partial^a\phi_2)(evB_a + eB_a\phi_1 - \frac{\phi_1}{v}\partial_a\phi_2) \\ & -\frac{1}{2}\left(\partial^a\phi_1\partial_a\phi_1 + e^2v^2B^aB_a\right) + \dots \end{aligned} \tag{6.16}$$

in which the dots denote cubic and quartic terms.

The second part of the last homework problem is to find the masses of the particles of this theory.

6.3 Higgs's mechanism

The local gauge group of the Glashow-Salam-Weinberg electroweak theory is $SU(2)_\ell \times U(1)$. What's weird is that the $SU(2)_\ell$ symmetry applies only to the left-handed quarks and leptons and to the Higgs boson, a complex doublet (or 2-vector) of scalar fields H.

Let's leave out the fermions for the moment, and focus just on the Higgs and the gauge fields. The gauge transformation is

$$\begin{aligned} H'(x) &= U(x)H(x) \\ A'_\mu(x) &= U(x)A_\mu(x)U^\dagger(x) + (\partial_\mu U(x))U^\dagger(x) \end{aligned} \tag{6.17}$$

in which the 2×2 unitary matrix $U(x)$ is

$$U(x) = \exp\left[i\,g\,\frac{\sigma^a}{2}\,\alpha^a(x) + i\,g'\,\frac{Y\,I}{2}\,\beta(x)\right]. \tag{6.18}$$

The generators here are the 3 Pauli matrices and the matrix $I/2$, where I is the 2×2 identity matrix.

The action density of the theory (without the fermions) is

$$L = -(D_\mu H)^\dagger D^\mu H + \frac{1}{4k}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) + m^2|H|^2 - \lambda|H|^4 \tag{6.19}$$

in which the covariant derivative for the Higgs doublet is

$$D_\mu H = \left(I\partial_\mu + i\,g\,\frac{\sigma^a}{2}\,A^a_\mu + i\,g'\,\frac{Y\,I}{2}\,B_\mu\right)H. \tag{6.20}$$

The potential energy of the Higgs field is

$$V = -m^2|H|^2 + \lambda|H|^4. \tag{6.21}$$

Its minimum is where

$$0 = \frac{\partial V}{\partial |H|^2} = 2\lambda|H|^2 - m^2. \tag{6.22}$$

So

$$|H| = \frac{m}{\sqrt{2\lambda}} = \frac{v}{\sqrt{2}}. \tag{6.23}$$

By making an $SU(2)_\ell \times U(1)$ gauge transformation, we can transform this mean value to

$$H_0 = \langle 0|H(x)|0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\ v\end{pmatrix}. \tag{6.24}$$

This transformation is called "going to unitary gauge."

In this gauge, the Higgs potential is

$$V(v) = -\frac{1}{2}m^2v^2 + \frac{1}{4}\lambda v^4, \tag{6.25}$$

and its second derivative is

$$V''(v) = 3\lambda v^2 - m^2 = 2m^2 = m_H^2. \tag{6.26}$$

The mass of the Higgs then is

$$m_H = \sqrt{2}\,m = \sqrt{2\,\lambda}\,v. \tag{6.27}$$

Experiments at LEP2 and at the LHC have revealed value of v to be

$$v = 246.22 \quad \text{GeV}. \tag{6.28}$$

In 2012, the Higgs boson was discovered at the LHC. Its mass has been measured to be

$$m_H = 125.18 \pm 0.16 \quad \text{GeV}. \tag{6.29}$$

The self coupling λ therefore is

$$\lambda = \frac{m_H^2}{2\,v^2} = \frac{1}{2}\left(\frac{125.18}{246.22}\right)^2 = 0.12924. \tag{6.30}$$

After spontaneous symmetry breaking, mass terms for the gauge bosons emerge from the kinetic action of the Higgs doublet $-(D_\mu H)^\dagger D^\mu H$. Since the generators of a compact group like $SU(2)_\ell \times U(1)$ are hermitian, the part of the kinetic action that contains the mass terms is

$$L_m = -H^\dagger\left(-i\,g\,\frac{\sigma^a}{2}\,A^a_\mu - i\,g'\,\frac{Y\,I}{2}\,B_\mu\right)\left(i\,g\,\frac{\sigma^a}{2}\,A^a_\mu + i\,g'\,\frac{Y\,I}{2}\,B_\mu\right)H. \tag{6.31}$$

In unitary gauge (6.24), these mass terms are

$$\begin{aligned}
L_m =& -\frac{1}{2}(0,v)\left(g\,\frac{\sigma^a}{2}\,A^a_\mu + g'\,\frac{I}{2}\,B_\mu\right)\left(g\,\frac{\sigma^a}{2}\,A^\mu_a + g'\,\frac{I}{2}\,B^\mu\right)\begin{pmatrix}0\\ v\end{pmatrix}\\
=& -\frac{1}{8}(0,v)\begin{pmatrix} gA^3_\mu + g'B_\mu & g(A^1_\mu - iA^2_\mu)\\ g(A^1_\mu + iA^2_\mu) & -gA^3_\mu + g'B_\mu\end{pmatrix}\\
&\times\begin{pmatrix} gA^\mu_3 + g'B^\mu & g(A^\mu_1 - iA^\mu_2)\\ g(A^\mu_1 + iA^\mu_2) & -gA^\mu_3 + g'B^\mu\end{pmatrix}\begin{pmatrix}0\\ v\end{pmatrix}\\
=& -\frac{v^2}{8}\left(g(A^1_\mu + iA^2_\mu),\quad -gA^3_\mu + g'B_\mu\right)\begin{pmatrix} g(A^\mu_1 - iA^\mu_2)\\ -gA^\mu_3 + g'B^\mu\end{pmatrix}\\
=& -\frac{v^2}{8}\left[g^2\left(A^1_\mu A^\mu_1 + A^2_\mu A^\mu_2\right) + \left(-gA^3_\mu + g'B_\mu\right)\left(-gA^\mu_3 + g'B^\mu\right)\right].
\end{aligned} \tag{6.32}$$

The normalized complex, charged gauge bosons are

$$W^\pm_\mu = \frac{1}{\sqrt{2}}\left(A^1_\mu \mp iA^2_\mu\right) \tag{6.33}$$

and the normalized neutral ones are

$$Z_\mu = \frac{1}{\sqrt{g^2 + g'^2}}\left(gA^3_\mu - g'B_\mu\right) \tag{6.34}$$

and the photon, which is the normalized linear combination orthogonal to Z_μ

$$A_\mu = \frac{1}{\sqrt{g^2 + g'^2}}\left(g'A^3_\mu + g\,B_\mu\right). \tag{6.35}$$

It remains massless. In terms of these properly normalized fields, the mass terms are

$$L_M = -\frac{g^2 v^2}{4} W_\mu^- W^{+\mu} - \frac{(g^2 + g'^2) v^2}{8} Z_\mu Z^\mu. \tag{6.36}$$

So the W^+ and the W^- get the same mass

$$M_W = g\,\frac{v}{2} = 80.370 \pm 0.012 \text{ GeV}/c^2. \tag{6.37}$$

while the Z (also called the Z^0) has mass

$$M_Z = \sqrt{g^2 + g'^2}\,\frac{v}{2} = 91.1876 \pm 0.0021 \text{ GeV}/c^2. \tag{6.38}$$

The accuracy and precision of the mass of the Z boson is an order of magnitude higher than that of the W boson because of measurements made in the Large Electron Collider.

It is the photon. The fine-structure constant is

$$\alpha = \frac{e^2}{4\pi\hbar c} = 1/137.035\ 999\ 139(31) \approx 1/137.036. \tag{6.39}$$

Why do three gauge bosons become massive? Because there are three Goldstone bosons corresponding to three ways of moving $\langle 0|H|0\rangle$ without changing the Higgs potential. Why does one gauge boson stay massless? Because one linear combination of the generators of $SU_L(2) \otimes U(1)$ maps $\langle 0|H|0\rangle$ to zero.

In terms of these mass eigenstates, the original gauge bosons are

$$\begin{aligned} A_\mu^1 &= \frac{1}{\sqrt{2}}\left(W_\mu^+ + W_\mu^-\right) \\ A_\mu^2 &= \frac{1}{i\sqrt{2}}\left(W_\mu^- - W_\mu^+\right) \\ A_\mu^3 &= \frac{1}{\sqrt{g^2 + g'^2}}\left(g' A_\mu + g Z_\mu\right) \\ B_\mu &= \frac{1}{\sqrt{g^2 + g'^2}}\left(g A_\mu - g' Z_\mu\right). \end{aligned} \tag{6.40}$$

The 4×4 matrices

$$P_\ell = \frac{1}{2}\left(1 + \gamma^5\right) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad P_r = \frac{1}{2}\left(1 - \gamma^5\right) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{6.41}$$

project out the upper and lower two components of a 4-component Dirac field. The $SU(2)_\ell$ gauge fields $\vec{A}_\mu$ interact only with the upper two components $\psi_\ell = P_\ell \psi$, while the $U(1)_Y$ gauge field B_μ interacts with all four components of a Dirac field.

Thus the covariant derivative for a fermion of coupling g' to the $U(1)$ gauge field B_μ with hypercharge Y and coupling g to the $SU(2)_\ell$ gauge fields is

$$\begin{aligned} D_\mu &= I\partial_\mu + i\,g\,\frac{\sigma^a}{2}\,A^a_\mu\,P_\ell + i\,g'\,Y\,I\,B_\mu \\ &= I\partial_\mu + i\,g\,\left[\frac{\sigma_1}{2}\frac{1}{\sqrt{2}}\left(W^+_\mu + W^-_\mu\right) + \frac{\sigma_2}{2}\frac{1}{i\sqrt{2}}\left(W^-_\mu - W^+_\mu\right)\right. \\ &\quad \left. + \frac{\sigma_3}{2}\frac{1}{\sqrt{g^2+g'^2}}\left(g'A_\mu + gZ_\mu\right)\right] P_\ell + i\,g'\,Y\,\frac{I}{\sqrt{g^2+g'^2}}\left(gA_\mu - g'Z_\mu\right). \end{aligned} \tag{6.42}$$

The $SU(2)_\ell$ generators are

$$\vec{T} = \frac{1}{2}\vec{\sigma}, \quad T^\pm = \frac{1}{2}\left(\sigma_1 \pm i\sigma_2\right) \quad \text{and} \quad T^3 = \frac{\sigma_3}{2}. \tag{6.43}$$

In terms of them, the covariant derivative is

$$\begin{aligned} D_\mu = I\partial_\mu &+ i\,\frac{g}{\sqrt{2}}\,\left(W^+_\mu T^+ + W^-_\mu T^-\right) P_\ell + i\frac{1}{\sqrt{g^2+g'^2}} Z_\mu\left(g^2 T^3 P_\ell - g'^2 Y I\right) \\ &+\, i\,\frac{gg'}{\sqrt{g^2+g'^2}}\,A_\mu\left(T^3 P_\ell + Y I\right) \end{aligned} \tag{6.44}$$

Equivalently, the left and right covariant derivatives are

$$\begin{aligned} D_{\mu\ell} = \Big[I\partial_\mu &+ i\,\frac{g}{\sqrt{2}}\,\left(W^+_\mu T^+ + W^-_\mu T^-\right) + i\frac{1}{\sqrt{g^2+g'^2}} Z_\mu\left(g^2 T^3 - g'^2 Y I\right) \\ &+\, i\,\frac{gg'}{\sqrt{g^2+g'^2}}\,A_\mu\left(T^3 + Y I\right)\Big] P_\ell \end{aligned} \tag{6.45}$$

and

$$D_{\mu r} = \Big[I\partial_\mu - i\frac{g'^2 Y I}{\sqrt{g^2+g'^2}} Z_\mu + \, i\,\frac{gg' Y I}{\sqrt{g^2+g'^2}}\,A_\mu\Big] P_r. \tag{6.46}$$

The interaction strength or coupling constant of the photon A_μ is

$$e = \frac{gg'}{\sqrt{g^2+g'^2}} > 0. \tag{6.47}$$

The charge operator is

$$Q = T^3_\ell + Y I. \tag{6.48}$$

When acting on the doublet

$$E_\ell = \begin{pmatrix} \nu_e \\ e \end{pmatrix} \tag{6.49}$$

to which we assign $Y = -1/2$, the charge Q gives 0 as the charge of the neutrino and -1 as the charge of the electron. The photon-lepton term then is

$$\begin{aligned}\frac{gg'}{\sqrt{g^2+g'^2}}\,A_\mu\left(T^3+YI\right)E_\ell &= e\,A_\mu\,\left(T^3+YI\right)E_\ell \\ &= e\,A_\mu\begin{pmatrix}0 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\nu_e\\ e\end{pmatrix} = -e\,A_\mu\,e\end{aligned} \tag{6.50}$$

in which the first e is the absolute value of the charge (6.47) of the electron and the second is the field of the electron.

The **weak mixing angle** θ_w is defined by

$$\begin{pmatrix}Z\\ A\end{pmatrix} = \begin{pmatrix}\cos\theta_w & -\sin\theta_w\\ \sin\theta_w & \cos\theta_w\end{pmatrix}\begin{pmatrix}A^3\\ B\end{pmatrix}. \tag{6.51}$$

Equations (6.34 and 6.35) identify these trigonometric values as

$$\cos\theta_w = \frac{g}{\sqrt{g^2+g'^2}} \quad \text{and} \quad \sin\theta_w = \frac{g'}{\sqrt{g^2+g'^2}}. \tag{6.52}$$

Since the charge is $Q = T^3 + YI$, the hypercharge $YI = Q - T^3$, and so fields couple to the Z as

$$\frac{1}{\sqrt{g^2+g'^2}}Z_\mu\left[g^2T^3 - g'^2Y\right] = \frac{1}{\sqrt{g^2+g'^2}}Z_\mu\left[(g^2+g'^2)T^3 - g'^2Q\right] \tag{6.53}$$

and to the photon as

$$\frac{gg'}{\sqrt{g^2+g'^2}}\,A_\mu\left(T^3+Y\right) = \frac{gg'}{\sqrt{g^2+g'^2}}\,A_\mu\,Q. \tag{6.54}$$

We also have

$$\frac{g^2+g'^2}{\sqrt{g^2+g'^2}} = \sqrt{g^2+g'^2} = \frac{g}{\cos\theta_w} \tag{6.55}$$

and

$$\frac{g'^2}{\sqrt{g^2+g'^2}} = \sqrt{g^2+g'^2}\,\frac{g'^2}{g^2+g'^2} = \frac{g}{\cos\theta_w}\,\sin^2\theta_w. \tag{6.56}$$

So the coupling to the Z is

$$\frac{1}{\sqrt{g^2+g'^2}}Z_\mu\left[(g^2+g'^2)T^3 - g'^2Q\right] = \frac{g}{\cos\theta_w}Z_\mu\left(T^3 - \sin^2\theta_w\,Q\right) \tag{6.57}$$

and, if we set

$$e = g\,\sin\theta_w, \tag{6.58}$$

then the coupling to the photon A is

$$\frac{gg'}{\sqrt{g^2+g'^2}}\,A_\mu\,Q = g\,\sin\theta_w\,A_\mu\,Q = e\,A_\mu\,Q. \tag{6.59}$$

In these terms, the covariant derivative (6.44) is

$$\begin{aligned} D_\mu &= I\partial_\mu + i\,\frac{g}{\sqrt{2}}\,\left(W_\mu^+T^+ + W_\mu^-T^-\right)P_\ell + i\frac{1}{\sqrt{g^2+g'^2}}Z_\mu\left(g^2T^3P_\ell - g'^2Y\right) \\ &\quad + i\,\frac{gg'}{\sqrt{g^2+g'^2}}\,A_\mu\left(T^3P_\ell + Y\right) \\ &= I\partial_\mu + i\,\frac{g}{\sqrt{2}}\,\left(W_\mu^+T^+ + W_\mu^-T^-\right)P_\ell + i\frac{g}{\cos\theta_w}Z_\mu\left(T^3P_\ell - \sin^2\theta_w\,Q\right) \\ &\quad + ie\,A_\mu\,Q \end{aligned} \tag{6.60}$$

in which the generators $T^\pm$ and T^3 are those of the representation to which the fields they act on belong. When acting on **left-handed** fermions, they are half the Pauli matrices, $\boldsymbol{T} = \frac{1}{2}\boldsymbol{\sigma}$. When acting on **right-handed** fermions, they are zero, $\boldsymbol{T} = \mathbf{0}$, and so the explicit appearance of P_ℓ is unnecessary. Since $g = e/\sin\theta_w$, the couplings involve one new parameter θ_w.

Our mass formulas (6.37 and 6.38) for the W and the Z show that their masses are related by

$$M_W = g\,\frac{v}{2} = \frac{g}{\sqrt{g^2+g'^2}}\,\sqrt{g^2+g'^2}\,\frac{v}{2} = \cos\theta_w\,\sqrt{g^2+g'^2}\,\frac{v}{2} = \cos\theta_w\,M_Z. \tag{6.61}$$

Experiments have determined the masses and shown that at the mass of the Z and in the $\overline{MS}$ convention

$$\sin^2\theta_w = 0.23122(4) \quad \text{or} \quad \theta_w = 0.50163 \tag{6.62}$$

and that

$$v = 246.22\ \text{GeV}. \tag{6.63}$$

6.4 Quark and lepton interactions

The right-handed fermions u_r, d_r, e_r, and $\nu_{e,r}$ are singlets under $SU_L(2)\otimes U_Y(1)$. So they have $T^3 = 0$. The definition (6.48) of the charge Q

$$Q = T^3 + YI \tag{6.64}$$

then implies that

$$Y_r = Q_r. \tag{6.65}$$

That is, $Y_{\nu_{e,r}} = 0$, $Y_{e_r} = -1$, $Y_{u_r} = 2/3$, and $Y_{d_r} = -1/3$.

The left-handed fermions are in doublets

$$E_\ell = \begin{pmatrix} \nu_e \\ e^- \end{pmatrix} \quad \text{and} \quad Q_\ell = \begin{pmatrix} u \\ d \end{pmatrix} \tag{6.66}$$

with $T^3 = \pm 1/2$. So the choices $Y_E = -1/2$ and $Y_Q = 1/6$ and the definition (6.48) of the charge Q give the right charges:

$$QE_\ell = Q \begin{pmatrix} \nu_e \\ e^- \end{pmatrix} = \begin{pmatrix} 0 \\ -e^- \end{pmatrix} \quad \text{and} \quad QQ_\ell = Q \begin{pmatrix} u \\ d \end{pmatrix} = \begin{pmatrix} 2u/3 \\ -d/3 \end{pmatrix}. \tag{6.67}$$

Fermion-gauge-boson interactions are due to the covariant derivative (6.60) acting on either the left- or right-handed fields. On right-handed fermions, the covariant derivative is just

$$\begin{aligned} D^r_\mu &= I\partial_\mu + i\frac{g}{\cos\theta_w} Z_\mu \left(-\sin^2\theta_w\, Q\right) + ie\, A_\mu\, Q \\ &= I\partial_\mu - ig\,\frac{\sin^2\theta_w}{\cos\theta_w}\, Z_\mu\, Q + ie\, A_\mu\, Q \\ &= I\partial_\mu - ie\,\frac{\sin\theta_w}{\cos\theta_w}\, Z_\mu\, Q + ie\, A_\mu\, Q = I\partial_\mu - ie\,\tan\theta_w\, Z_\mu\, Q + ie\, A_\mu\, Q. \end{aligned} \tag{6.68}$$

So the covariant derivative of a neutral right-handed fermion is just the ordinary derivative.

On left-handed fermions, the covariant derivative is

$$\begin{aligned} D^\ell_\mu &= I\partial_\mu + i\frac{g}{\sqrt{2}}\left(W^+_\mu T^+ + W^-_\mu T^-\right) + i\frac{g}{\cos\theta_w} Z_\mu \left(T^3 - \sin^2\theta_w\, Q\right) + ieA_\mu Q \\ &= I\partial_\mu + i\,\frac{e}{\sqrt{2}\,\sin\theta_w}\,\left(W^+_\mu T^+ + W^-_\mu T^-\right) + i\frac{e}{\cos\theta_w} Z_\mu \left(\frac{T^3}{\sin\theta_w} - \sin\theta_w\, Q\right) \\ &\quad + ie\, A_\mu\, Q. \end{aligned} \tag{6.69}$$

For the first **family** or **generation** of quarks and leptons, the kinetic action density is

$$L_k = -\overline{E}_\ell \not{D}^\ell E_\ell - \overline{E}_r \not{D}^r E_r - \overline{Q}_\ell \not{D}^\ell Q_\ell - \overline{Q}_r \not{D}^r Q_r \tag{6.70}$$

in which $\not{D} \equiv \gamma^\mu D_\mu$. The 4×4 matrix $\gamma_5 = \gamma^5$ plays the role of a fifth (spatial) gamma matrix $\gamma^4 = \gamma_5$ in 5-dimensional space-time in the sense that the anticommutator

$$\{\gamma^a, \gamma^b\} = 2\eta^{ab} \tag{6.71}$$

in which η is the 5×5 diagonal matrix with $\eta^{00}=-1$ and $\eta^{aa}=1$ for $a=1$, 2, 3, 4. In Weinberg's notation, γ_5 is

$$\gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{6.72}$$

The combinations

$$P_\ell = \frac{1}{2}\,(1+\gamma_5) \quad \text{and} \quad P_r = \frac{1}{2}\,(1-\gamma_5) \tag{6.73}$$

are projection operators onto the left- and right-handed fields. That is,

$$P_\ell Q = Q_\ell \quad \text{and} \quad P_\ell Q_\ell = Q_\ell \tag{6.74}$$

with a similar equation for P_r. We can write L_k as

$$\begin{aligned} L_k =& -\overline{E}\not{D}^\ell P_\ell E - \overline{E}\not{D}^r P_r E - \overline{Q}\not{D}^\ell P_\ell Q - \overline{Q}\not{D}^r P_r Q \\ =& -\left[\overline{E}\not{D}^\ell P_\ell E + \overline{E}\not{D}^r P_r E + \overline{Q}\not{D}^\ell P_\ell Q + \overline{Q}\not{D}^r P_r Q\right]. \end{aligned} \tag{6.75}$$

Homework set 4, problem 1: Show that

$$\overline{E}_\ell \not{D}^\ell E_\ell = [\tfrac{1}{2}(1+\gamma_5)E]^\dagger i\gamma^0 \not{D}^\ell \tfrac{1}{2}(1+\gamma_5)E = \overline{E}\not{D}^\ell \tfrac{1}{2}(1+\gamma_5)E. \tag{6.76}$$

Recall that in Weinberg's notation

$$\overline{\psi} = \psi^\dagger i\gamma^0 = \psi^\dagger\beta = \psi^\dagger \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \tag{6.77}$$

in which I is the 2×2 identity matrix.

6.5 Quark and Lepton Masses

The Higgs mechanism also gives masses to the fermions, but somewhat arbitrarily. Dirac's action density (4.56) has as its mass term

$$-m\,\overline{\psi}\psi = -im\,\psi^\dagger\gamma^0\psi = -im\,\psi^\dagger\gamma^0(P_\ell+P_r)\psi = -im\,\psi^\dagger\gamma^0(P_\ell^2+P_r^2)\psi. \tag{6.78}$$

Since $\{\gamma^0,\gamma^5\}=0$, this mass term is

$$\begin{aligned} -m\,\overline{\psi}\psi =& -im\,\psi^\dagger P_r\gamma^0 P_\ell\psi - im\,\psi^\dagger P_\ell\gamma^0 P_r\psi \\ =& -im\,(P_r\psi)^\dagger\gamma^0 P_\ell\psi - im\,(P_\ell\psi)^\dagger\gamma^0 P_r\psi \\ =& -im\,\psi_r^\dagger\gamma^0\psi_\ell - im\,\psi_\ell^\dagger\gamma^0\psi_r = -m\,\overline{\psi}_r\psi_\ell - m\,\overline{\psi}_\ell\psi_r. \end{aligned} \tag{6.79}$$

Incidentally, because the fields ψ_ℓ and ψ_r are independent, we can redefine them as

$$\psi'_\ell = e^{i\theta}\,\psi_\ell \quad \text{and} \quad \psi'_r = e^{i\phi}\,\psi_r \tag{6.80}$$

at will. Such a redefinition changes the mass term to

$$-\,m'\,\overline{\psi}_r\psi_\ell - m'^*\,\overline{\psi}_\ell\psi_r = -m\,e^{i(\theta-\phi)}\,\overline{\psi}_r\psi_\ell - m\,e^{-i(\theta-\phi)}\,\overline{\psi}_\ell\psi_r. \tag{6.81}$$

So the phase of a Dirac mass term has no significance.

The definition (6.77) of $\overline{\psi}$ shows that the Dirac mass term is

$$-\,m\overline{\psi}\psi = -m\,\psi^\dagger\beta\psi = -m\left(\psi_\ell^\dagger, \psi_r^\dagger\right)\begin{pmatrix}0 & I\\ I & 0\end{pmatrix}\begin{pmatrix}\psi_\ell\\ \psi_r\end{pmatrix} = -m\left(\psi_\ell^\dagger\psi_r + \psi_r^\dagger\psi_\ell\right). \tag{6.82}$$

These mass terms are invariant under the Lorentz transformations

$$\psi_\ell' = \exp(-\boldsymbol{z}\cdot\boldsymbol{\sigma})\,\psi_\ell \quad \text{and} \quad \psi_r' = \exp(\boldsymbol{z}^*\cdot\boldsymbol{\sigma})\,\psi_r \tag{6.83}$$

because

$$\begin{aligned}\psi_\ell'^\dagger\,\psi_r' &= \psi_\ell^\dagger\exp(-\boldsymbol{z}^*\cdot\boldsymbol{\sigma})\exp(\boldsymbol{z}^*\cdot\boldsymbol{\sigma})\,\psi_r = \psi_\ell^\dagger\,\psi_r\\ \psi_r'^\dagger\,\psi_\ell' &= \psi_r^\dagger\exp(\boldsymbol{z}\cdot\boldsymbol{\sigma})\exp(-\boldsymbol{z}\cdot\boldsymbol{\sigma})\,\psi_\ell = \psi_r^\dagger\,\psi_\ell.\end{aligned} \tag{6.84}$$

They are *not* invariant under rigid, let alone local, $SU(2)_\ell$ transformations. But we can make them invariant by using the Higgs field $H(x)$. For instance, the quantity $Q_\ell^\dagger H d_r$ is invariant under local $SU(2)_\ell$ transformations. In unitary gauge, its mean value in the vacuum is

$$\langle 0|Q_\ell^\dagger H d_r|0\rangle = \frac{1}{\sqrt{2}}\,v\,\langle 0|d_\ell^\dagger d_r|0\rangle. \tag{6.85}$$

So the term

$$-\,c_d\,Q_\ell^\dagger H d_r - c_d^*\,d_r^\dagger H^\dagger Q_\ell \tag{6.86}$$

is invariant, and in the vacuum it is

$$\langle 0| - c_d\,Q_\ell^\dagger H d_r - c_d^*\,d_r^\dagger H^\dagger Q_\ell|0\rangle = \frac{1}{\sqrt{2}}\,v\,\langle 0| - c_d\,d_\ell^\dagger d_r - c_d^*\,d_r^\dagger d_\ell|0\rangle \tag{6.87}$$

which gives to the d quark the mass

$$m_d = \frac{|c_d|}{\sqrt{2}}\,v. \tag{6.88}$$

Note that we must add one new parameter c_d to get one new mass m_d. This parameter c_d is complex in general, but the mass m_d depends only upon the absolute value and not upon its phase of c_d.

Similarly, the term

$$-\,c_e\,E_\ell^\dagger H e_r - c_e^*\,e_r^\dagger H^\dagger E_\ell \tag{6.89}$$

is invariant, and in the vacuum it is

$$\langle 0| - c_e\, E_\ell^\dagger H e_r - c_e^*\, e_r^\dagger H^\dagger Q_\ell |0\rangle = \frac{1}{\sqrt{2}}\, v\, \langle 0| - c_e\, e_\ell^\dagger e_r - c_e^*\, e_r^\dagger e_\ell |0\rangle \tag{6.90}$$

which gives to the electron the mass

$$m_e = \frac{|c_e|}{\sqrt{2}}\, v. \tag{6.91}$$

Again, we must add one new (complex) parameter c_e to get one new mass m_e.

The mass of the up quark requires a new trick. The Higgs field H transforms under $SU(2)_\ell \times U(1)$ as

$$H'(x) = \exp\left[i\, g\, \frac{\sigma^a}{2}\, \alpha^a(x) + i\, g'\, \frac{I}{2}\, \beta(x)\right] H(x). \tag{6.92}$$

If for clarity, we leave aside the $U(1)$ part for the moment, then the Higgs field H transforms under $SU(2)_\ell$ as

$$H'(x) = \exp\left[i\, g\, \frac{\sigma^a}{2}\, \alpha^a(x)\right] H(x). \tag{6.93}$$

Let us use H^* to be the complex column vector whose components are $H_1^\dagger$ and $H_2^\dagger$. How does $\sigma_2 H^*$ transform under $SU(2)_\ell$? Suppressing our explicit mention of the space-time dependence and using the asterisk to mean hermitian conjugation when applied to operators but complex conjugation when applied to matrices and vectors, we have, since σ_2 is imaginary with $\sigma_2^2 = I$ while σ_1 and σ_3 are real,

$$\begin{aligned}(\sigma_2 H^*)' &= \sigma_2 \left[\exp\left(i\, g\, \frac{\sigma^a}{2}\, \alpha^a\right) H\right]^* = \sigma_2 \exp\left(-i\, g\, \frac{\sigma_a^*}{2}\, \alpha^a\right) H^* \\ &= \sigma_2 \exp\left(-i\, g\, \frac{\sigma_a^*}{2}\, \alpha^a\right) \sigma_2 \sigma_2\, H^* = \exp\left(i\, g\, \frac{\sigma^a}{2}\, \alpha^a\right) \sigma_2\, H^*.\end{aligned} \tag{6.94}$$

Thus, the term

$$-\, c_u Q_\ell^\dagger \sigma_2 H^* u_r - c_u^* u_r^\dagger H^\mathsf{T} \sigma_2 Q_\ell \tag{6.95}$$

is invariant under $SU(2)_\ell$. In the vacuum of the unitary gauge, it is

$$\begin{aligned}\langle 0| - c_u Q_\ell^\dagger \sigma_2 H^* u_r - c_u^* u_r^\dagger H^\mathsf{T} \sigma_2 Q_\ell |0\rangle &= \frac{1}{\sqrt{2}}\, v \langle 0| i c_u u_\ell^\dagger u_r - i c_u^* u_r^\dagger u_\ell |0\rangle \\ &= \frac{1}{\sqrt{2}}\, v \langle 0| i c_u u_\ell^\dagger u_r - i c_u^* u_r^\dagger u_\ell |0\rangle\end{aligned} \tag{6.96}$$

which gives the up quark the mass

$$m_u = \frac{|c_u|}{\sqrt{2}}\, v. \tag{6.97}$$

But there are three families of generations of quarks and leptons on which the gauge fields act simply:

$$F_1 = \begin{pmatrix} u \\ d \\ \nu_e \\ e \end{pmatrix}', \quad F_2 = \begin{pmatrix} c \\ s \\ \nu_\mu \\ \mu \end{pmatrix}', \quad \text{and} \quad F_3 = \begin{pmatrix} t \\ b \\ \nu_\tau \\ \tau \end{pmatrix}'. \tag{6.98}$$

The quark and lepton flavor families are

$$\begin{aligned} Q'_1 &= \begin{pmatrix} u \\ d \end{pmatrix}', \quad Q'_2 = \begin{pmatrix} c \\ s \end{pmatrix}', \quad \text{and} \quad Q'_3 = \begin{pmatrix} t \\ b \end{pmatrix}'; \\ E'_1 &= \begin{pmatrix} \nu_e \\ e \end{pmatrix}', \quad E'_2 = \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}', \quad \text{and} \quad E'_3 = \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}'. \end{aligned} \tag{6.99}$$

These are called the **flavor** eigenstates or more properly flavor eigenfields, designated here with primes. They are the ones on which the $W^\pm$ act simply. The weak interactions use $W^-_\mu T^-$ to map the flavor *up* fields $u'_1 = u'$, $u'_2 = c'$, $u'_3 = t'$ into the flavor *down* fields $d'_1 = d'$, $d'_2 = s'$, $d'_3 = b'$, and $W^+_\mu T^+$ to map the flavor *down* fields d'_i into the flavor *up* fields u'_i.

The action density

$$\sum_{i,j=1}^{3} -c_{dij}\, Q'^\dagger_{\ell i} H d'_{rj} - c^*_{dij}\, d'^\dagger_{rj} H^\dagger Q'_{\ell i} \tag{6.100}$$

gives for the d', s', and b' quarks the mixed mass terms

$$\frac{v}{\sqrt{2}} \sum_{i,j=1}^{3} -c_{dij}\, d'^\dagger_{\ell i} d'_{rj} - c^*_{dij}\, d'^\dagger_{rj} d'_{\ell i} = \frac{v}{\sqrt{2}} \sum_{i,j=1}^{3} -c_{dij}\, d'^\dagger_{\ell i} d'_{rj} - c^*_{dij}\, d'^\dagger_{rj} d'_{\ell i}. \tag{6.101}$$

The 3×3 mass matrix M_d with entries

$$[M_d]_{ij} = \frac{v}{\sqrt{2}}\, c_{dij} \tag{6.102}$$

need have no special properties. It need not be hermitian because for each i and j, the term (6.101) is hermitian. But every 3×3 complex matrix has a **singular-value decomposition**

$$M_d = L_d \Sigma_d R_d^\dagger \tag{6.103}$$

in which L_d and R_d are 3×3 unitary matrices, and Σ_d is a 3×3 diagonal matrix with nonincreasing positive singular values on its main diagonal.

The singular value decomposition works for any $N \times M$ (real or) complex matrix. Every complex $M \times N$ rectangular matrix A is the product of an $M \times M$ unitary matrix U, an $M \times N$ rectangular matrix Σ that is zero except on its main diagonal which consists of its nonnegative singular values S_k, and an $N \times N$ unitary matrix $V^\dagger$

$$A = U\,\Sigma\,V^\dagger. \tag{6.104}$$

This singular-value decomposition is a key theorem of matrix algebra. One can use the *Matlab* command "[U,S,V] = svd(A)" to perform the svd $A = USV^\dagger$.

The singular values of Σ_d are the masses m_b, m_s, and m_d:

$$\Sigma_d = \begin{pmatrix} m_b & 0 & 0 \\ 0 & m_s & 0 \\ 0 & 0 & m_d \end{pmatrix}. \tag{6.105}$$

So the mass eigenfields of the left and right down-quark fields are

$$d_{ri} = R^\dagger_{dij} d'_{rj} \quad \text{and} \quad d^\dagger_{\ell i} = d'^\dagger_{\ell j} L_{dji} \quad \text{or} \quad d_{\ell i} = L^\dagger_{dij} d_{\ell j}. \tag{6.106}$$

The inverse relations are

$$d'_{ri} = R_{dij} d_{rj} \quad \text{and} \quad d'^\dagger_{\ell i} = d^\dagger_{\ell j} L^\dagger_{dji} \quad \text{or} \quad d'_{\ell i} = L_{dij} d_{\ell j} \tag{6.107}$$

or in matrix notation

$$\boldsymbol{d}'_r = R_d\,\boldsymbol{d}_r, \quad \boldsymbol{d}'^\dagger_r = \boldsymbol{d}^\dagger_r\,R^\dagger_d, \quad \boldsymbol{d}'^\dagger_\ell = \boldsymbol{d}^\dagger_\ell L^\dagger_d, \quad \text{and} \quad \boldsymbol{d}'_\ell = L_d\,\boldsymbol{d}_\ell \tag{6.108}$$

in which

$$\boldsymbol{d}_\ell = \begin{pmatrix} b \\ s \\ d \end{pmatrix}_\ell \tag{6.109}$$

are the down-quark fields of definite masses.

Similarly, the *up* quark action density

$$\sum_{i,j=1}^{3} -c_{uij} Q'^\dagger_{\ell i} \sigma_2 H^* u'_{rj} - c^*_{uij} u'^\dagger_{rj} H^\mathsf{T} \sigma_2 Q'_{\ell i} \tag{6.110}$$

gives for the three known families the mixed mass terms

$$\frac{iv}{\sqrt{2}} \sum_{i,j=1}^{3} c_{uij} u'^\dagger_{\ell i}\, u'_{rj} - c^*_{uij} u'^\dagger_{rj}\, u'_{\ell i} = \frac{iv}{\sqrt{2}} \sum_{i,j=1}^{3} c_{uij} u'^\dagger_{\ell i}\, u'_{rj} - c^*_{uij} u'^\dagger_{rj}\, u'_{\ell i}. \tag{6.111}$$

The 3×3 mass matrix M_u with entries

$$[M_u]_{ij} = \frac{iv}{\sqrt{2}}\, c_{uij} \tag{6.112}$$

need have no special properties. It need not be hermitian because for each i and j, the term (6.111) is hermitian. But every 3×3 complex matrix M_u has a singular value decomposition

$$M_u = L_u \Sigma_u R_u^\dagger \tag{6.113}$$

in which L_u and R_u are 3×3 unitary matrices, and Σ_u is a 3×3 diagonal matrix with nonincreasing positive singular values on its main diagonal. These singular values are the masses m_t, m_c, and m_u:

$$\Sigma_u = \begin{pmatrix} m_t & 0 & 0 \\ 0 & m_c & 0 \\ 0 & 0 & m_u \end{pmatrix}. \tag{6.114}$$

So the mass eigenfields of the left and right up-quark fields are

$$u_{ri} = R_{uij}^\dagger u'_{rj} \quad \text{and} \quad u_{\ell i}^\dagger = u_{\ell j}'^\dagger L_{uji} \quad \text{or} \quad u_{\ell i} = L_{uij}^\dagger u'_{\ell j}. \tag{6.115}$$

The inverse relations are

$$u'_{ri} = R_{uij} u_{rj} \quad \text{and} \quad u_{\ell i}'^\dagger = u_{\ell j}^\dagger L_{uji}^\dagger \quad \text{or} \quad u'_{\ell i} = L_{uij} u_{\ell j} \tag{6.116}$$

or in matrix notation

$$\boldsymbol{u}'_r = R_u\, \boldsymbol{u}_r, \quad \boldsymbol{u}_r'^\dagger = \boldsymbol{u}_r^\dagger\, R_u^\dagger \quad \boldsymbol{u}_\ell'^\dagger = \boldsymbol{u}_\ell^\dagger L_u^\dagger, \quad \text{and} \quad \boldsymbol{u}'_\ell = L_u \boldsymbol{u}_\ell \tag{6.117}$$

in which

$$\boldsymbol{u}_\ell = \begin{pmatrix} t \\ c \\ u \end{pmatrix}_\ell \tag{6.118}$$

are the up-quark fields of definite masses.

6.6 CKM matrix

We will use the labels u, c, t and d, s, b for the states that are eigenstates of the quadratic part of the hamiltonian after the Higgs mechanism has given a mean value to the real part of the neutral Higgs boson in the unitary gauge. The u, c, t quarks have the same charge $2e/3 > 0$ and the same $T^3 = 1/2$, so they all have the same electroweak interactions. Similarly, the d, s, b quarks have the same charge $-e/3 < 0$ and the same $T^3 = -1/2$, so they also all have the same electroweak interactions.

The right-handed covariant derivative (6.68)

$$D^r_\mu = I\partial_\mu - ie\,\tan\theta_w\,Z_\mu\,Q + ie\,A_\mu\,Q \tag{6.119}$$

just sends the fields of these mass eigenstates into themselves multiplied by their charge and either a Z or a photon. That is,

$$\begin{aligned} \boldsymbol{u}'^\dagger_r D^r_\mu \boldsymbol{u}'_r &= \boldsymbol{u}^\dagger_r R^\dagger_u D^r_\mu R_u \boldsymbol{u}_r = \boldsymbol{u}^\dagger_r D^r_\mu \boldsymbol{u}_r \\ \boldsymbol{d}'^\dagger_r D^r_\mu \boldsymbol{d}'_r &= \boldsymbol{d}^\dagger_r R^\dagger_d D^r_\mu R_d \boldsymbol{d}_r = \boldsymbol{d}^\dagger_r D^r_\mu \boldsymbol{d}_r \end{aligned} \tag{6.120}$$

In these terms, the interactions of the Z and the photon with the right-handed fields are diagonal both in mass and in flavor.

But the left-handed covariant derivative (6.69) is

$$\begin{aligned} D^\ell_\mu = I\partial_\mu &+ i\,\frac{e}{\sqrt{2}\,\sin\theta_w}\,\left(W^+_\mu T^+ + W^-_\mu T^-\right) \\ &+ i\frac{e}{\cos\theta_w} Z_\mu \left(\frac{T^3}{\sin\theta_w} - \sin\theta_w\,Q\right) + ie\,A_\mu\,Q. \end{aligned} \tag{6.121}$$

So we have

$$\begin{pmatrix} \boldsymbol{u}'^\dagger_\ell & \boldsymbol{d}'^\dagger_\ell \end{pmatrix} D^\ell_\mu \begin{pmatrix} \boldsymbol{u}'_\ell \\ \boldsymbol{d}'_\ell \end{pmatrix} = \begin{pmatrix} \boldsymbol{u}^\dagger_\ell L^\dagger_u & \boldsymbol{d}^\dagger_\ell L^\dagger_d \end{pmatrix} D^\ell_\mu \begin{pmatrix} L_u\,\boldsymbol{u}_\ell \\ L_d\,\boldsymbol{d}_\ell \end{pmatrix}. \tag{6.122}$$

Some of the unitary matrices just give unity, $L^\dagger_u L_u = I$ and $L^\dagger_d L_d = I$ like $R^\dagger_u R_u = I$ and $R^\dagger_d R_d = I$ in the right-handed covariant derivatives (6.120). Thus the interactions of the Z and the photon with the both the right-handed fields and with the left-handed fields are diagonal both in mass and in flavor. The Z and the photon do not mediate top-to-charm or charm-to-up or $\mu^- \to e^- + \gamma$ decays. Also, the Higgs mass terms are diagonal, so the neutral Higgs boson can't mediate such processes. **Thus, in the standard model, there are no flavor-changing neutral-currents.**

The only changes are in the nonzero parts of $T^\pm$ which become

$$T^+_{\text{CKM}} = \begin{pmatrix} 0 & L^\dagger_u L_d \\ 0 & 0 \end{pmatrix} \equiv \begin{pmatrix} 0 & V \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad T^-_{\text{CKM}} = \begin{pmatrix} 0 & 0 \\ L^\dagger_d L_u & 0 \end{pmatrix} \equiv \begin{pmatrix} 0 & 0 \\ V^\dagger & 0 \end{pmatrix} \tag{6.123}$$

in which the unitary matrix $V = L^\dagger_u L_d$ is the CKM matrix (Nicola Cabibbo, Makoto Kobayashi, and Toshihide Maskawa). The left-handed covariant derivative on the mass eigenfields then is

$$\begin{aligned} D^\ell_\mu = I\partial_\mu &+ i\,\frac{e}{\sqrt{2}\,\sin\theta_w}\,\left(W^+_\mu T^+_{\text{CKM}} + W^-_\mu T^-_{\text{CKM}}\right) \\ &+ i\frac{e}{\cos\theta_w} Z_\mu \left(\frac{T^3}{\sin\theta_w} - \sin\theta_w\,Q\right) + ie\,A_\mu\,Q. \end{aligned} \tag{6.124}$$

It has a second part that acts more or less like the right-handed covariant derivative, but the first part uses $W_\mu^- T^-$ to map the *up* fields u, c, t into linear combinations of the *down* fields d, s, b and $W_\mu^+ T^+$ to map the *down* fields into linear combinations of the *up* fields. The $W_\mu^\pm$ terms are sensitive to the CKM matrix $V = L_u^\dagger L_d$. We write them suggestively as

$$\begin{pmatrix} u & c & t & d & s & b \end{pmatrix}^\dagger \begin{pmatrix} 0 & V W_\mu^+ \\ V^\dagger W_\mu^- & 0 \end{pmatrix} \begin{pmatrix} u \\ c \\ t \\ d \\ s \\ b \end{pmatrix} \tag{6.125}$$

$$= \begin{pmatrix} u & c & t & V \begin{pmatrix} d \\ s \\ b \end{pmatrix} \end{pmatrix}^\dagger \begin{pmatrix} 0 & W_\mu^+ \\ W_\mu^- & 0 \end{pmatrix} \begin{pmatrix} u \\ c \\ t \\ V \begin{pmatrix} d \\ s \\ b \end{pmatrix} \end{pmatrix}. \tag{6.126}$$

By choosing the phases of the six quark fields, that is, $u(x) \to e^{i\theta_u} u(x)$ $\ldots b(x) \to e^{i\theta_b} b(x)$, one may make the CKM matrix $L_u^\dagger L_d$ real apart from a single phase. The existence of that phase probably is the cause of most of the breakdown of CP invariance that Fitch and Cronin and others have observed since 1964. The magnitudes of the elements of the CKM matrix V are

$$V = \begin{pmatrix} |V_{ud}| & |V_{us}| & |V_{ub}| \\ |V_{cd}| & |V_{cs}| & |V_{cb}| \\ |V_{td}| & |V_{ts}| & |V_{tb}| \end{pmatrix} = \begin{pmatrix} 0.97427 & 0.22536 & 0.00355 \\ 0.22522 & 0.97343 & 0.0414 \\ 0.00886 & 0.0405 & 0.99914 \end{pmatrix}. \tag{6.127}$$

Although there is only one phase $\exp(i\delta)$ in the CKM matrix V, the experimental constraints on this phase often are expressed in terms of the angles α, β, and γ defined as

$$\begin{aligned} \alpha &= \arg\left[-V_{td}V_{tb}^* / \left(V_{ud}V_{ub}^*\right)\right] \\ \beta &= \arg\left[-V_{cd}V_{cb}^* / \left(V_{td}V_{tb}^*\right)\right] \\ \gamma &= \arg\left[-V_{ud}V_{ub}^* / \left(V_{cd}V_{cb}^*\right)\right]. \end{aligned} \tag{6.128}$$

If V is unitary, then $\alpha + \beta + \gamma = 180°$. From $B \to \pi\pi$, $\rho\pi$, and $\rho\rho$ decays, the limits on the angle α are roughly

$$\alpha = (85.4 \pm 4)°. \tag{6.129}$$

From $B^\pm \to DK^\pm$ decays, the limits on the angle γ are roughly

$$\gamma = (68.0 \pm 8)^\circ. \tag{6.130}$$

So the angle β is about 26.6°.

One of the quark-Higgs interactions is

$$\begin{aligned} -c_{dij} Q'^\dagger_{\ell i} H d'_{rj} &= -\frac{\sqrt{2}}{v} \boldsymbol{Q}'^\dagger_\ell M_d \boldsymbol{d}'_r H = -\frac{\sqrt{2}}{v} \boldsymbol{Q}'^\dagger_\ell L_d \Sigma_d R_d^\dagger \boldsymbol{d}'_r H \\ &= -\frac{\sqrt{2}}{v} \boldsymbol{Q}^\dagger_\ell \Sigma_d \boldsymbol{d}_r H = -\frac{\sqrt{2}}{v} \boldsymbol{Q}^\dagger_\ell \begin{pmatrix} 0 \\ (v+h)/\sqrt{2} \end{pmatrix} \Sigma_d \boldsymbol{d}_r \\ &= -\boldsymbol{d}^\dagger_\ell \left(1 + \frac{h}{v}\right) \Sigma_d \boldsymbol{d}_r = -m_{di} d^\dagger_{\ell i} \left(1 + \frac{h}{v}\right) d_{ri}. \end{aligned} \tag{6.131}$$

A similar term describes the coupling of the *up* quarks to the Higgs

$$-m_{ui} u^\dagger_{\ell i} \left(1 + \frac{h}{v}\right) u_{ri}. \tag{6.132}$$

Thus, the rate of quark-antiquark to Higgs is proportional to the mass of the quark in the standard model.

6.7 Lepton Masses

We can treat the leptons just like the quarks. The *up* leptons are the flavor neutrinos ν'_e, ν'_μ, and ν'_τ, and the *down* leptons are the flavor charged leptons e', μ', and τ'. The action density

$$\sum_{i,j=1}^{3} -c_{eij}\, E'^\dagger_{\ell i} H e'_{rj} - c^*_{eij}\, e'^\dagger_{rj} H^\dagger E'_{\ell i} \tag{6.133}$$

gives for the e', μ', and τ' the mixed mass terms

$$\frac{v}{\sqrt{2}} \sum_{i,j=1}^{3} -c_{eij}\, e'^\dagger_{\ell i} e'_{rj} - c^*_{eij}\, e'^\dagger_{rj} e'_{\ell i}. \tag{6.134}$$

The 3×3 mass matrix M_e with entries

$$[M_e]_{ij} = \frac{v}{\sqrt{2}}\, c_{eij} \tag{6.135}$$

has a singular value decomposition

$$M_e = L_e \Sigma_e R_e^\dagger \tag{6.136}$$

in which L_e and R_e are 3×3 unitary matrices, and Σ_e is a 3×3 diagonal matrix with nonincreasing positive singular values m_τ, m_μ, and m_e on its main diagonal.

6.8 Before and after symmetry breaking

Before spontaneous symmetry breaking, all the fields of the standard model are massless, and the local symmetry under $SU(2)_\ell \otimes U(1)$ is exact. Under these gauge transformations, the left-handed electron and neutrino fields are rotated among themselves. If e'_ℓ is a linear combination of itself and of $\nu'_{e\ell}$, then these two fields, e'_ℓ and $\nu'_{e\ell}$, must be of the same kind. The left-handed electron field is a Dirac field. Thus, the left-handed neutrino field also must be a Dirac field. This makes sense because before symmetry breaking, all the fields are massless, and so there is no problem combining two Majorana fields of the same mass, namely zero, into one Dirac field. Thus, there are three flavor Dirac neutrino fields $\nu'_{e\ell}$, $\nu'_{\mu\ell}$, and $\nu'_{\tau\ell}$.

A massless left-handed neutrino field ν'_ℓ satisfies the two-component Dirac equation

$$(\partial_0 I - \boldsymbol{\nabla} \cdot \boldsymbol{\sigma})\, \nu'_\ell(x) = 0 \tag{6.137}$$

which in momentum space is

$$(E + \boldsymbol{p} \cdot \boldsymbol{\sigma})\, \nu'_\ell(p) = 0. \tag{6.138}$$

Since the angular momentum is $\boldsymbol{J} = \boldsymbol{\sigma}/2$, and $E = |\boldsymbol{p}|$, we have

$$\hat{\boldsymbol{p}} \cdot \boldsymbol{J}\, \nu'_\ell(p) = -\frac{1}{2}\, \nu'_\ell(p). \tag{6.139}$$

The left-handed neutrino field ν'_ℓ annihilates neutrinos of negative helicity and creates antineutrinos of positive helicity.

Since the neutrinos are massive, there may be right-handed neutrino fields. As for the *up* quarks, we can use them to make an action density

$$\sum_{i,j=1}^{3} -c_{\nu ij} E'^{\dagger}_{\ell i} \sigma_2 H^* \nu'_{rj} - c^*_{\nu ij} \nu'^{\dagger}_{rj} H^{\mathsf{T}} \sigma_2 E'_{\ell i} \tag{6.140}$$

that is invariant under $SU(2)_\ell \otimes U(1)$ and that gives for the neutrinos the mixed mass terms

$$\sum_{i,j=1}^{3} \frac{iv}{\sqrt{2}} \left(c_{\nu ij} \nu'^{\dagger}_{\ell i} \nu'_{rj} - c^*_{\nu ij} \nu'^{\dagger}_{rj} \nu'_{\ell i}\right). \tag{6.141}$$

The 3×3 mass matrix M_ν with entries

$$[M_\nu]_{ij} = \frac{iv}{\sqrt{2}}\, c_{\nu ij} \tag{6.142}$$

has a singular value decomposition

$$M_\nu = L_\nu \Sigma_\nu R^\dagger_\nu \tag{6.143}$$

in which L_ν and R_ν are 3×3 unitary matrices, and Σ_ν is a 3×3 diagonal matrix with nonincreasing positive singular values m_{ν_τ}, m_{ν_μ}, and m_{ν_e} on its main diagonal (here, I have assumed that the neutrino masses mimic those of the charged leptons and quarks, rising with family number). The neutrino CKM matrix then would be $L_\nu^\dagger L_e$, but since we are accustomed to treating the charged leptons as flavor and mass eigenfields, we apply the neutrino CKM matrix to the neutrinos rather than to the charged leptons. Thus the neutrino CKM matrix is

$$V_\nu = L_e^\dagger L_\nu. \tag{6.144}$$

By choosing the phases of the six lepton fields, we can make the neutrino CKM matrix real except for CP-breaking phases. If the neutrinos are Dirac fields, then there is one such phase; if not, there are three.

So far, I have assumed that the mass terms for the neutrinos are the usual Dirac mass terms. However, the right-handed Majorana neutrino fields $\boldsymbol{\nu}_r'$ are not affected by the $SU(2)_\ell \otimes U(1)$.

Note that a gauge transformation between e and ν_e rotates the operators $a(p, s, e)$ and $a(p, s, \nu_e)$ into each other. This rotation makes sense only when the two particles have the same mass. In the standard model, such a gauge transformation makes sense only before symmetry breaking when all the particles are massless. Moreover, only when the particles are massless can one say that they are left- or right-handed. While the particles are massless, the operator $a(p, -)$ annihilates a particle of negative helicity and occurs only in a left-handed field, while the operator $a(p, +)$ annihilates a particle of positive helicity and occurs only in a right-handed field. But when the particles are massive, the operator $a(p, \frac{1}{2})$ annihilates a particle that is spin up and occurs in both left-handed and right-handed fields. So a symmetry transformation that acted on the operator $a(p, \frac{1}{2})$ would change both left-handed and right-handed fields.

The left-handed fields of the neutrino and electron are

$$\nu_{e,\ell}(x) = \int u(p, -) \frac{a_1(p, -, \nu_e) + i a_2(p, -, \nu_e)}{\sqrt{2}} e^{ipx} \tag{6.145}$$

$$+ v(p, +) \frac{a_1^\dagger(p, +, \nu_e) + i a_2^\dagger(p, +, \nu_e)}{\sqrt{2}} e^{-ipx} \frac{d^3p}{(2\pi)^{3/2}} \tag{6.146}$$

$$e_\ell(x) = \int u(p, -) \frac{a_1(p, -, e) + i a_2(p, -, e)}{\sqrt{2}} e^{ipx} \tag{6.147}$$

$$+ v(p, +) \frac{a_1^\dagger(p, +, e) + i a_2^\dagger(p, +, e)}{\sqrt{2}} e^{-ipx} \frac{d^3p}{(2\pi)^{3/2}} \tag{6.148}$$

where $(p, -, \nu_e)$ means momentum p, spin down, and electron flavor, and

$(p, +, \nu_e)$ means momentum p, spin up, and electron flavor. These fields satisfy equations like (6.137–6.139) apart from their interactions with other fields. Since a gauge transformation maps the fields $\nu_{e,\ell}(x)$ and $e_\ell(x)$ into each other, we know that when all the fields are massless, before symmetry breaking, there are (for each momentum) at least two neutrino and antineutrino states

$$\frac{1}{\sqrt{2}}\left[a_1^\dagger(p,-,\nu_e) - i a_2^\dagger(p,-,\nu_e)\right]|0\rangle \tag{6.149}$$

$$\frac{1}{\sqrt{2}}\left[a_1^\dagger(p,+,\nu_e) + i a_2^\dagger(p,+,\nu_e)\right]|0\rangle \tag{6.150}$$

for each of the three flavors, $f = e, \mu, \tau$. So there are at least six neutrino (and antineutrino) states.

The right-handed electron field exists and interacts with gauge bosons and other fields. So there are 12 electron states $a_i^\dagger(p, \pm, e_f)|0\rangle$ for $i = 1$ and 2 and for the three flavors, $f = e, \mu, \tau$. We don't know yet whether a right-handed neutrino field exists or interacts with other fields. So there may be only 6 neutrino states or as many as 12.

Neutrino oscillations tell us that neutrinos have masses. If there are 12 neutrino states, then there can be three massive Dirac neutrinos analagous to the e, μ, and τ or six massive Majorana neutrinos or some intermediate combination. If there are only 6, then there can be 3 massive Majorana neutrinos.

The Majorana mass terms for the right-handed neutrino fields are

$$\sum_{ij=1}^{6} \frac{1}{2}\left[i m_{ij} \nu'^{\mathsf{T}}{}_{ri}\, \sigma_2\, \nu'_{rj} + (i m_{ij} \nu'^{\mathsf{T}}{}_{ri}\, \sigma_2\, \nu'_{rj})^\dagger\right]. \tag{6.151}$$

They are Lorentz invariant because under the Lorentz transformations (6.83)

$$\begin{aligned} \nu''^{\mathsf{T}}{}_r \sigma_2 \nu''_r &= \nu'^{\mathsf{T}}{}_r \exp(\boldsymbol{z}^* \cdot \boldsymbol{\sigma}^{\mathsf{T}}) \sigma_2 \exp(\boldsymbol{z}^* \cdot \boldsymbol{\sigma})\, \nu'_r \\ &= \nu'^{\mathsf{T}}{}_r\, \sigma_2 \exp(-\boldsymbol{z}^* \cdot \boldsymbol{\sigma}) \exp(\boldsymbol{z}^* \cdot \boldsymbol{\sigma})\, \nu'_r = \nu'^{\mathsf{T}}{}_r\, \sigma_2\, \nu'_r. \end{aligned} \tag{6.152}$$

The Majorana mass terms (6.151) are unrelated to the scale v of the Higgs field's mean value. One can show that the complex matrix m_{ij} is symmetric. One then must combine the mass matrix in (6.151) with the mass matrix M_ν in (6.142). The resulting mass matrix will have a singular-value decomposition with six singular values that would be the masses of the "physical" neutrinos. If these six masses are equal in pairs, then the three pairs would form three Dirac neutrinos.

Whether or not there are right-handed neutrinos, we can make Majorana mass terms like $\nu_\ell{}^{\mathsf{T}} \sigma_2 \nu_\ell$, which are Lorentz invariant but not invariant under

$SU_\ell(2)$ or $U_Y(1)$. We can make them gauge invariant by using a triplet $\vec{\phi} = \sigma_i \phi_i$ of Higgs fields that transforms as $\vec{\phi}' \cdot \vec{\sigma} = g(\vec{\phi} \cdot \vec{\sigma})g^\dagger$ for $g \in SU_\ell(2)$ and that carries a value of $Y = -1$. Then if σ_2 has Lorentz indices and σ_2' has $SU_\ell(2)$ indices, the term

$$E_\ell{}^\mathsf{T} \sigma_2 \sigma_2' (\vec{\phi} \cdot \vec{\sigma}) E_\ell \tag{6.153}$$

is both Lorentz invariant and gauge invariant. If the potential $V(\vec{\phi})$ has minima at $\vec{\phi} \neq 0$, then this term violates lepton number and gives a Majorana mass to the neutrino. Another possibility is to say that at higher energies a theory with new fields of very high mass Λ plays a role, and that when one path-integrates over these heavy fields, one is left with an effective, nonrenormalizable term in the action

$$-\frac{(H^\dagger E_\ell)^2}{\Lambda} \tag{6.154}$$

which gives a Majorana mass to ν_ℓ.

Models with both right-handed and left-handed neutrinos are easier to think about, but only experiments can tell us whether right-handed neutrinos exist.

What is known experimentally is that there are at least three masses that satisfy

$$\begin{aligned}
\left|\Delta m_{21}^2\right| &\equiv \left|m_2^2 - m_1^2\right| = (7.53 \pm 0.18) \times 10^{-5}\,\mathrm{eV}^2 \\
\left|\Delta m_{32}^2\right| &\equiv \left|m_3^2 - m_2^2\right| = (2.44 \pm 0.06) \times 10^{-3}\,\mathrm{eV}^2 \quad \text{normal mass hierarchy} \\
\left|\Delta m_{32}^2\right| &\equiv \left|m_3^2 - m_2^2\right| = (2.52 \pm 0.07) \times 10^{-3}\,\mathrm{eV}^2 \quad \text{inverted mass hierarchy.}
\end{aligned} \tag{6.155}$$

If the neutrinos are Dirac particles, then they have a CKM matrix like that of the quarks with one CP-violating phase. But whereas one chooses to make the mass and flavor eigenfields the same for the up quarks u, c, t, for the leptons one makes the mass and flavor eigenfields the same for the down or charged leptons e, μ, τ. So the neutrino CKM matrix actually is $V = L_e^\dagger L_\nu$. If they are three Majorana particles, then their CKM matrix has two extra CP-violating phases α_{12} and α_{31}. A common convention for the

neutrino CKM matrix is

$$V = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_{23} & \sin\theta_{23} \\ 0 & -\sin\theta_{23} & \cos\theta_{23} \end{pmatrix} \begin{pmatrix} \cos\theta_{13} & 0 & \sin\theta_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -\sin\theta_{13}e^{i\delta} & 0 & \cos\theta_{13} \end{pmatrix} \tag{6.156}$$

$$\times \begin{pmatrix} \cos\theta_{12} & \sin\theta_{12} & 0 \\ -\sin\theta_{12} & \cos\theta_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\alpha_{12}/2} & 0 \\ 0 & 0 & e^{i\alpha_{31}/2} \end{pmatrix}. \tag{6.157}$$

This convention without the last 3×3 matrix also is used for the quark CKM matrix. The current estimates are

$$\sin^2(2\theta_{12}) = 0.846 \pm 0.021 \tag{6.158}$$

$$\sin^2(2\theta_{23}) = 0.999\,{}^{+0.001}_{-0.018} \quad \text{normal mass hierarchy} \tag{6.159}$$

$$\sin^2(2\theta_{23}) = 1.000\,{}^{+0.000}_{-0.017} \quad \text{inverted mass hierarchy} \tag{6.160}$$

$$\sin^2(2\theta_{13}) = 0.093 \pm 0.008. \tag{6.161}$$

Two of these are big angles: $2\theta_{12} \approx 2\theta_{23} = \pi/2 \pm n\pi$. In the normal hierarchy, the lightest neutrino is about 2/3 electron, 1/6 muon, and 1/6 tau; the very slightly heavier neutrino is about 1/3 electron, 1/3 muon, and 1/3 tau; and the much heavier heavier neutrino is about 1/6 electron, 5/12 muon, and 5/12 tau.

6.9 The Seesaw Mechanism

Why are the neutrino masses so light? Suppose we wish to find the eigenvalues of the real, symmetric mass matrix

$$\mathcal{M} = \begin{pmatrix} 0 & m \\ m & M \end{pmatrix} \tag{6.162}$$

in which m is an ordinary mass and M is a huge mass. The eigenvalues μ of this hermitian mass matrix satisfy $\det(\mathcal{M} - \mu I) = \mu(\mu - M) - m^2 = 0$ with solutions $\mu_\pm = \left(M \pm \sqrt{M^2 + 4m^2}\right)/2$. The larger mass $\mu_+ \approx M + m^2/M$ is approximately the huge mass M and the smaller mass $\mu_- \approx -m^2/M$ is tiny. The physical mass of a fermion is the absolute value of its mass parameter, here m^2/M.

The product of the two eigenvalues is the constant $\mu_+\mu_- = \det\mathcal{M} = -m^2$ so as μ_- goes down, μ_+ must go up. In 1975, Gell-Mann, Ramond, Slansky, and Jerry Stephenson invented this "**seesaw**" mechanism as an explanation

of why neutrinos have such small masses, less than 1 eV/c^2. If $mc^2 = 10$ MeV, and $\mu_- c^2 \approx 0.01$ eV, which is a plausible light-neutrino mass, then the rest energy of the huge mass would be $Mc^2 = 10^7$ GeV. This huge mass would be one of the six neutrino masses and would point at new physics, beyond the standard model. Yet the small masses of the neutrinos may be related to the weakness of their interactions.

Before leaving the subject of fermion masses, let's look more closely at Dirac and Majorana mass terms. A Dirac field is a linear combination of two Majorana fields of the same mass

$$\psi = \frac{1}{\sqrt{2}} \begin{pmatrix} L + i\ell \\ R + ir \end{pmatrix} \tag{6.163}$$

in which L and ℓ are two-component left-handed spinors, and R and r are two-component right-handed spinors. The Dirac mass term

$$\begin{aligned} m\,\overline{\psi}\psi &= i\,m\,\psi^\dagger\gamma^0\psi = m\,\psi^\dagger \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \psi \\ &= m\frac{1}{2}\left(L^\dagger - i\ell^\dagger, \quad R^\dagger - ir^\dagger\right) \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} L + i\ell \\ R + ir \end{pmatrix} \\ &= m\frac{1}{2}\left(L^\dagger - i\ell^\dagger, \quad R^\dagger - ir^\dagger\right) \begin{pmatrix} R + ir \\ L + i\ell \end{pmatrix} \\ &= m\frac{1}{2}\left[\left(L^\dagger - i\ell^\dagger\right)(R + ir) + \left(R^\dagger - ir^\dagger\right)(L + i\ell)\right] \\ &= m\,\frac{1}{2}\left(R^\dagger - ir^\dagger\right)(L + i\ell) + \text{h.c.}, \end{aligned} \tag{6.164}$$

in which h.c. means hermitian conjugate, gives mass m to the particle and antiparticle of the Dirac field ψ.

We may set

$$R = i\sigma_2\,L^* \iff L = -i\sigma_2\,R^* \tag{6.165}$$

$$r = i\sigma_2\,\ell^* \iff \ell = -i\sigma_2\,r^* \tag{6.166}$$

or equivalently

$$\begin{pmatrix} R_1 \\ R_2 \end{pmatrix} = \begin{pmatrix} L_2^\dagger \\ -L_1^\dagger \end{pmatrix} \iff \begin{pmatrix} L_1 \\ L_2 \end{pmatrix} = \begin{pmatrix} -R_2^\dagger \\ R_1^\dagger \end{pmatrix} \tag{6.167}$$

$$\begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = \begin{pmatrix} \ell_2^\dagger \\ -\ell_1^\dagger \end{pmatrix} \iff \begin{pmatrix} \ell_1 \\ \ell_2 \end{pmatrix} = \begin{pmatrix} -r_2^\dagger \\ r_1^\dagger \end{pmatrix} \tag{6.168}$$

which are the Majorana conditions (1.252 & 1.253). Since $R^\dagger = -iL^\mathsf{T}\sigma_2$, we

can write the Dirac mass term (6.164) in terms of left-handed fields as

$$m\,\overline{\psi}\psi = \frac{1}{2}\,m\,\left(-i\,L^{\mathsf{T}} - \ell^{\mathsf{T}}\right)\sigma_2\,(L + i\ell) + \text{h.c.} \tag{6.169}$$

$$= \frac{1}{2}\,m\,\left(L^{\mathsf{T}} - i\ell^{\mathsf{T}}\right)(-i\,\sigma_2)\,(L + i\ell) + \text{h.c.} \tag{6.170}$$

$$= \frac{1}{2}\,m\,\begin{pmatrix} L_1 - i\ell_1, & L_2 - i\ell_2 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} L_1 + i\ell_1 \\ L_2 + i\ell_2 \end{pmatrix} + \text{h.c.} \tag{6.171}$$

$$= \frac{1}{2}\,m\,\begin{pmatrix} L_1 - i\ell_1, & L_2 - i\ell_2 \end{pmatrix}\begin{pmatrix} -L_2 - i\ell_2 \\ L_1 + i\ell_1 \end{pmatrix} + \text{h.c.} \tag{6.172}$$

$$= \frac{1}{2}\,m\,(L_1 - i\ell_1)\,(-L_2 - i\ell_2) + (L_2 - i\ell_2)\,(L_1 + i\ell_1) + \text{h.c.} \tag{6.173}$$

The fermion fields anticommute, so the Dirac mass term is

$$m\,\overline{\psi}\psi = \frac{1}{2}\,m\,(-2L_1L_2 - 2\ell_1\ell_2) + \text{h.c.} = -m\,(L_1L_2 + \ell_1\ell_2) + \text{h.c.}, \tag{6.174}$$

and it says that the fields L and ℓ have the same mass m, as they must if they are to form a Dirac field.

Since $L^\dagger = iR^{\mathsf{T}}\,\sigma_2$, we also can write the Dirac mass term in terms of the right-handed fields as

$$m\,\overline{\psi}\psi = \frac{1}{2}\,m\,\left(R^{\mathsf{T}} - ir^{\mathsf{T}}\right)i\sigma_2\,(R + ir) + \text{h.c.} \tag{6.175}$$

$$= \frac{1}{2}\,m\,\begin{pmatrix} R_1 - ir_1, & R_2 - ir_2 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} R_1 + ir_1 \\ R_2 + ir_2 \end{pmatrix} + \text{h.c.} \tag{6.176}$$

$$= m\,(R_1R_2 + r_1r_2) + \text{h.c.} \tag{6.177}$$

So the fields R and r have the same mass m, as they must if they are to form a Dirac field.

The Majorana mass term for a right-handed field r of mass m evidently is

$$m\,r_1\,r_2 + \text{h.c.} \tag{6.178}$$

6.10 Neutrino Oscillations

The phase difference $\Delta\phi$ between two highly relativistic neutrinos of momentum p going a distance L in a time $t \approx L$ varies with their masses m_1 and m_2 as

$$\Delta\phi = t\,\Delta E = \frac{LE}{p}\Delta E = \frac{LE}{p}\left(\sqrt{p^2 + m_1^2} - \sqrt{p^2 + m_2^2}\right) \tag{6.179}$$

in natural units. We can approximate this phase by using the first two terms of the binomial expansion of the square roots with $y = 1$ and $x = m_i^2/p^2$

$$\Delta\phi = LE\left(\sqrt{1 + m_1^2/p^2} - \sqrt{1 + m_2^2/p^2}\right) \approx \frac{LE\Delta m^2}{p^2} \approx \frac{L\Delta m^2}{E} \tag{6.180}$$

or in ordinary units $\Delta\phi \approx L\Delta m^2 c^3/(\hbar E)$.

7

Path integrals

7.1 Path integrals and Richard Feynman

Since Richard Feynman invented them over 70 years ago, path integrals have been used with increasing frequency in high-energy and condensed-matter physics, in optics and biophysics, and even in finance. Feynman used them to express the amplitude for a process as a sum of all the ways the process could occur each weighted by an exponential of its classical action $\exp(iS/\hbar)$. Others have used them to compute partition functions and to study the QCD vacuum. (Richard Feynman, 1918–1988)

7.2 Gaussian integrals and Trotter's formula

Path integrals are based upon the gaussian integral (**??**) which holds for real $a \neq 0$ and real b

$$\int_{-\infty}^{\infty} e^{iax^2+2ibx}\, dx = \sqrt{\frac{i\pi}{a}}\; e^{-ib^2/a} \tag{7.1}$$

and upon the gaussian integral (**??**)

$$\int_{-\infty}^{\infty} e^{-ax^2+2ibx}\, dx = \sqrt{\frac{\pi}{a}}\; e^{-b^2/a} \tag{7.2}$$

which holds both for $\operatorname{Re} a > 0$ and also for $\operatorname{Re} a = 0$ with b real and $\operatorname{Im} a \neq 0$.

The extension of the integral formula (7.1) to any $n \times n$ real symmetric nonsingular matrix s_{jk} and any real vector c_j is (exercises **??** & **??**)

$$\int_{-\infty}^{\infty} e^{is_{jk}x_jx_k+2ic_jx_j}\, dx_1 \ldots dx_n = \sqrt{\frac{(i\pi)^n}{\det s}}\; e^{-ic_j(s^{-1})_{jk}c_k} \tag{7.3}$$

in which $\det a$ is the determinant of the matrix a, a^{-1} is its inverse, and sums over the repeated indices j and k from 1 to n are understood. One may similarly extend the gaussian integral (7.2) to any positive symmetric $n \times n$ matrix s_{jk} and any vector c_j (exercises **??** & **??**)

$$\int_{-\infty}^{\infty} e^{-s_{jk}x_jx_k+2ic_jx_j}\, dx_1 \ldots dx_n = \sqrt{\frac{\pi^n}{\det s}}\; e^{-c_j(s^{-1})_{jk}c_k}. \tag{7.4}$$

Path integrals also are based upon Trotter's product formula (Trotter, 1959; Kato, 1978)

$$e^{a+b} = \lim_{n\to\infty} \left(e^{a/n}\, e^{b/n}\right)^n \tag{7.5}$$

both sides of which are symmetrically ordered and obviously equal when $ab = ba$.

Separating a given hamiltonian $H = K + V$ into a kinetic part K and a potential part V, we can use Trotter's formula to write the time-evolution operator $e^{-itH/\hbar}$ as

$$e^{-it(K+V)/\hbar} = \lim_{n\to\infty} \left(e^{-itK/(n\hbar)}\, e^{-itV/(n\hbar)}\right)^n \tag{7.6}$$

and the Boltzmann operator $e^{-\beta H}$ as

$$e^{-\beta(K+V)} = \lim_{n\to\infty} \left(e^{-\beta K/n}\, e^{-\beta V/n}\right)^n. \tag{7.7}$$

7.3 Path integrals in quantum mechanics

Path integrals can represent matrix elements of the time-evolution operator $\exp(-i(t_b - t_a)H/\hbar)$ in which H is the hamiltonian. For a particle of mass m moving nonrelativistically in one dimension in a potential $V(q)$, the hamiltonian is

$$H = \frac{p^2}{2m} + V(q). \tag{7.8}$$

The position and momentum operators q and p obey the commutation relation $[q, p] = i\hbar$. Their eigenstates $|q'\rangle$ and $|p'\rangle$ have eigenvalues q' and p' for all real numbers q' and p'

$$q\,|q'\rangle = q'\,|q'\rangle \quad \text{and} \quad p\,|p'\rangle = p'\,|p'\rangle. \tag{7.9}$$

These eigenstates are complete. Their outer products $|q'\rangle\langle q'|$ and $|p'\rangle\langle p'|$

provide expansions for the identity operator I and have inner products (**??**) that are phases

$$I = \int_{-\infty}^{\infty} |q'\rangle\langle q'|\, dq' = \int_{-\infty}^{\infty} |p'\rangle\langle p'|\, dp' \quad \text{and} \quad \langle q'|p'\rangle = \frac{e^{iq'p'/\hbar}}{\sqrt{2\pi\hbar}}. \tag{7.10}$$

Setting $\epsilon = (t_b - t_a)/n$ and writing the hamiltonian (7.8) over $\hbar$ as $H/\hbar = p^2/(2m\hbar) + V/\hbar = k + v$, we can write Trotter's formula (7.6) for the time-evolution operator as the limit as $n \to \infty$ of n factors of $e^{-i\epsilon k}e^{-i\epsilon v}$

$$e^{-i(t_b-t_a)(k+v)} = e^{-i\epsilon k}\, e^{-i\epsilon v}\; e^{-i\epsilon k}\, e^{-i\epsilon v} \cdots e^{-i\epsilon k}\, e^{-i\epsilon v}\; e^{-i\epsilon k}\, e^{-i\epsilon v}. \tag{7.11}$$

The advantage of using Trotter's formula is that we now can evaluate the matrix element $\langle q_1|e^{-i\epsilon k}\, e^{-i\epsilon v}|q_a\rangle$ between eigenstates $|q_a\rangle$ and $|q_1\rangle$ of the position operator q by inserting the momentum-state expansion (7.10) of the identity operator I between these two exponentials

$$\langle q_1|e^{-i\epsilon k}\, e^{-i\epsilon v}|q_a\rangle = \langle q_1|\, e^{-i\epsilon p^2/(2m\hbar)} \int_{-\infty}^{\infty} |p'\rangle\langle p'|\, dp'\; e^{-i\epsilon\, V(q)/\hbar}|q_a\rangle \tag{7.12}$$

and using the eigenvalue formulas (7.9)

$$\langle q_1|e^{-i\epsilon k}\, e^{-i\epsilon v}|q_a\rangle = \int_{-\infty}^{\infty} e^{-i\epsilon p'^2/(2m\hbar)}\, \langle q_1|p'\rangle\, e^{-i\epsilon V(q_a)/\hbar}\, \langle p'|q_a\rangle\, dp'. \tag{7.13}$$

Now using the formula (7.10) for the inner product $\langle q_1|p'\rangle$ and the complex conjugate of that formula for $\langle p'|q_a\rangle$, we get

$$\langle q_1|e^{-i\epsilon k}\, e^{-i\epsilon v}|q_a\rangle = e^{-i\epsilon\, V(q_a)/\hbar} \int_{-\infty}^{\infty} e^{-i\epsilon p'^2/(2m\hbar)}\, e^{i(q_1-q_a)p'/\hbar}\, \frac{dp'}{2\pi\hbar}. \tag{7.14}$$

In this integral, the momenta that are important are very high, being of order $\sqrt{m\hbar/\epsilon}$ which diverges as $\epsilon \to 0$; nonetheless, the integral converges.

If we adopt the suggestive notation $q_1 - q_a = \epsilon\, \dot{q}_a$ and use the gaussian integral (7.1) with $a = -\epsilon/(2m\hbar)$, $x = p$, and $b = \epsilon\dot{q}/(2\hbar)$

$$\int_{-\infty}^{\infty} \exp\left(-\, i\, \epsilon\, \frac{p^2}{2m\hbar} + i\, \epsilon\, \frac{\dot{q}\, p}{\hbar} \right) \frac{dp}{2\pi\hbar} \;=\; \sqrt{\frac{m}{2\pi i\epsilon\hbar}}\, \exp\left(i\, \frac{\epsilon}{\hbar}\, \frac{m\dot{q}^2}{2} \right), \tag{7.15}$$

then we find

$$\begin{aligned} \langle q_1|e^{-i\epsilon k}\, e^{-i\epsilon v}|q_a\rangle &= \frac{1}{2\pi\hbar}\, e^{-\epsilon\, V(q_a)} \int_{-\infty}^{\infty} \exp\left(-\, i\frac{\epsilon p'^2}{2m\hbar} + i\frac{\epsilon\, \dot{q}_a\, p'}{\hbar} \right) dp' \\ &= \left(\frac{m}{2\pi i\hbar\epsilon} \right)^{1/2} \exp\left[i\, \frac{\epsilon}{\hbar} \left(\frac{m\dot{q}_a^2}{2} - V(q_a) \right) \right]. \end{aligned} \tag{7.16}$$

The dependence of the amplitude $\langle q_1|e^{-i\epsilon k}\, e^{-i\epsilon v}|q_a\rangle$ upon q_1 is hidden in the formula $\dot{q}_a = (q_1 - q_a)/\epsilon$.

The next step is to use the position-state expansion (7.10) of the identity operator to link two of these matrix elements together

$$\langle q_2|\big(e^{-i\epsilon k}\,e^{-i\epsilon v}\big)^2|q_a\rangle = \int_{-\infty}^{\infty}\langle q_2|\,e^{-i\epsilon k}\,e^{-i\epsilon v}\,|q_1\rangle\langle q_1|\,e^{-i\epsilon k}\,e^{-i\epsilon v}\,|q_a\rangle\,dq_1 \tag{7.17}$$

$$= \frac{m}{2\pi i\hbar\epsilon}\int_{-\infty}^{\infty}\exp\left[i\frac{\epsilon}{\hbar}\left(\frac{m\dot{q}_1^2}{2} - V(q_1) + \frac{m\dot{q}_a^2}{2} - V(q_a)\right)\right]dq_1$$

where now $\dot{q}_1 = (q_2 - q_1)/\epsilon$.

By stitching together $n = (t_b - t_a)/\epsilon$ time intervals each of length ϵ and letting $n \to \infty$, we get

$$\begin{aligned}\langle q_b|e^{-ni\epsilon H/\hbar}|q_a\rangle &= \int\langle q_b|e^{-i\epsilon k}\,e^{-i\epsilon v}|q_{n-1}\rangle\cdots\langle q_1|e^{-i\epsilon k}\,e^{-i\epsilon v}|q_a\rangle\,dq_{n-1}\cdots dq_1\\ &= \left(\frac{m}{2\pi i\hbar\epsilon}\right)^{n/2}\int\exp\left[i\frac{\epsilon}{\hbar}\sum_{j=0}^{n-1}\frac{m\dot{q}_j^2}{2} - V(q_j)\right]dq_{n-1}\cdots dq_1\\ &= \left(\frac{m}{2\pi i\hbar\epsilon}\right)^{n/2}\int\exp\left(i\frac{\epsilon}{\hbar}\sum_{j=0}^{n-1}L_j\right)dq_{n-1}\cdots dq_1 \qquad (7.18)\end{aligned}$$

in which $L_j = m\dot{q}_j^2/2 - V(q_j)$ is the lagrangian of the jth interval, and the q_j integrals run from $-\infty$ to ∞. In the limit $\epsilon \to 0$ with $n\epsilon = (t_b - t_a)/\epsilon$, this multiple integral is an integral over all paths $q(t)$ that go from q_a, t_a to q_b, t_b

$$\langle q_b|e^{-i\,(t_b-t_a)\,H/\hbar}|q_a\rangle = \int e^{iS[q]/\hbar}\,Dq \tag{7.19}$$

in which each path is weighted by the phase of its classical action

$$S[q] = \int_{t_a}^{t_b}L(\dot{q},q)\,dt = \int_{t_a}^{t_b}\left(\frac{m\dot{q}(t)^2}{2} - V(q(t))\right)dt \tag{7.20}$$

in units of $\hbar$ and $Dq = (mn/(2\pi i\hbar(t_b - t_a)))^{n/2}\,dq_{n-1}\ldots dq_1$.

If we multiply the path-integral (7.19) for $\langle q_b|e^{-i\,(t_b-t_a)\,H/\hbar}|q_a\rangle$ from the left by $|q_b\rangle$ and from the right by $\langle q_a|$ and integrate over q_a and q_b as in the resolution (7.10) of the identity operator, then we can write the time-evolution operator as an integral over all paths from t_a to t_b

$$e^{-i\,(t_b-t_a)\,H/\hbar} = \int|q_b\rangle\,e^{iS[q]/\hbar}\,\langle q_a|\,Dq\,dq_a\,dq_b \tag{7.21}$$

with $Dq = (mn/(2\pi i\hbar(t_b - t_a)))^{n/2}\,dq_{n-1}\ldots dq_1$ and $S[q]$ the action (7.20).

The path integral for a particle moving in three-dimensional space is

$$\langle \boldsymbol{q}_b | e^{-i(t_b-t_a)H/\hbar} | \boldsymbol{q}_a \rangle = \int \exp\left[\frac{i}{\hbar}\int_{t_a}^{t_b} \tfrac{1}{2}\, m\, \dot{\boldsymbol{q}}^2(t) - V(\boldsymbol{q}(t))\, dt\right] D\boldsymbol{q} \tag{7.22}$$

where $D\boldsymbol{q} = (mn/(2\pi i\hbar(t_b - t_a)))^{3n/2}\, d\boldsymbol{q}_{n-1} \ldots d\boldsymbol{q}_1$.

Let us first consider macroscopic processes whose actions are large compared to $\hbar$. Apart from the factor $D\boldsymbol{q}$, the amplitude (7.22) is a sum of phases $e^{iS[\boldsymbol{q}]/\hbar}$ one for each path from $\boldsymbol{q}_a, t_a$ to $\boldsymbol{q}_b, t_b$. When is this amplitude big? When is it small? Suppose there is a path $\boldsymbol{q}_c(t)$ from $\boldsymbol{q}_a, t_a$ to $\boldsymbol{q}_b, t_b$ that obeys the classical equation of motion (**??**–**??**)

$$\frac{\delta S[\boldsymbol{q}_c]}{\delta \boldsymbol{q}_{jc}} = m\ddot{\boldsymbol{q}}_{jc} + V'(\boldsymbol{q}_c) = 0. \tag{7.23}$$

Its action may be minimal. It certainly is stationary: a path $\boldsymbol{q}_c(t)+\delta\boldsymbol{q}(t)$ that differs from $\boldsymbol{q}_c(t)$ by a small detour $\delta\boldsymbol{q}(t)$ has an action $S[\boldsymbol{q}_c+\delta\boldsymbol{q}]$ that differs from $S[\boldsymbol{q}_c]$ only by terms of second order and higher in $\delta\boldsymbol{q}$. Thus a classical path has infinitely many neighboring paths whose actions differ only by integrals of $(\delta\boldsymbol{q})^n$, $n \geq 2$, and so have the same action to within a small fraction of $\hbar$. These paths add with nearly the same phase to the path integral (7.22) and so make a huge contribution to the amplitude $\langle \boldsymbol{q}_b | e^{-i(t_b-t_a)H/\hbar} | \boldsymbol{q}_a \rangle$. But if no classical path goes from $\boldsymbol{q}_a, t_a$ to $\boldsymbol{q}_b, t_b$, then the nonclassical, nonstationary paths that go from $\boldsymbol{q}_a, t_a$ to $\boldsymbol{q}_b, t_b$ have actions that differ from each other by large multiples of $\hbar$. These amplitudes cancel each other, and their sum, which is amplitude for going from $\boldsymbol{q}_a, t_a$ to $\boldsymbol{q}_b, t_b$, is small. **Thus the path-integral formula for an amplitude in quantum mechanics explains why macroscopic processes are described by the principle of stationary action** (section **??**).

What about microscopic processes whose actions are tiny compared to $\hbar$? The path integral (7.22) gives large amplitudes for all microscopic processes. On very small scales, anything can happen that doesn't break a conservation law.

The path integral for two or more particles $\{\boldsymbol{q}\} = \{\boldsymbol{q}_1, \ldots, \boldsymbol{q}_k\}$ interacting with a potential $V(\{\boldsymbol{q}\})$ is

$$\langle \{\boldsymbol{q}\}_b | e^{-i(t_b-t_a)H/\hbar} | \{\boldsymbol{q}\}_a \rangle = \int e^{iS[\{q\}]/\hbar}\, D\{\boldsymbol{q}\} \tag{7.24}$$

where

$$S[\{\boldsymbol{q}\}] = \int_{t_a}^{t_b} \left[\frac{m_1 \dot{\boldsymbol{q}}_1^2(t)}{2} + \cdots + \frac{m_k \dot{\boldsymbol{q}}_k^2(t)}{2} - V(\{\boldsymbol{q}(t)\})\right] dt \tag{7.25}$$

and $D\{\boldsymbol{q}\} = D\boldsymbol{q}_1 \cdots D\boldsymbol{q}_k$.

Example 7.1 (A free particle) For a free particle, the potential is zero, and the path integral (7.19, 7.20) is the $\epsilon \to 0, n \to \infty$ limit of

$$\langle q_b|e^{-i\,t\,H/\hbar}|q_a\rangle = \left(\frac{m}{2\pi i\hbar\epsilon}\right)^{n/2} \times \int \exp\left[i\frac{m\epsilon}{2\hbar}\left(\frac{(q_b-q_{n-1})^2}{\epsilon^2}+\cdots+\frac{(q_1-q_a)^2}{\epsilon^2}\right)\right] dq_{n-1}\cdots dq_1. \tag{7.26}$$

The q_1 integral is by the gaussian formula (7.1)

$$\frac{m}{2\pi i\hbar\epsilon}\int e^{im[(q_2-q_1)^2+(q_1-q_a)^2]/(2\hbar\epsilon)}dq_1 = \sqrt{\frac{m}{2\pi i\hbar 2\epsilon}}\,e^{im(q_2-q_a)^2/(2\hbar 2\epsilon)}. \tag{7.27}$$

The q_2 integral is (exercise **??**)

$$\frac{m}{2\sqrt{2}\,\pi i\hbar\epsilon}\int e^{im[(q_3-q_2)^2+(q_2-q_a)^2/2]/(2\hbar\epsilon)}dq_2 = \sqrt{\frac{m}{2\pi i\hbar 3\epsilon}}\,e^{im(q_3-q_a)^2/(2\hbar 3\epsilon)}. \tag{7.28}$$

Doing all $n-1$ integrals (7.26) in this way and setting $n\epsilon = t_b - t_a$, we get

$$\begin{aligned}\langle q_b|\,e^{-i\,(t_b-t_a)\,H/\hbar}\,|q_a\rangle &= \sqrt{\frac{m}{2\pi i\hbar n\epsilon}}\,\exp\left[\frac{im(q_b-q_a)^2}{2\hbar n\epsilon}\right]\\ &= \sqrt{\frac{m}{2\pi i\hbar(t_b-t_a)}}\,\exp\left[\frac{im(q_b-q_a)^2}{2\hbar(t_b-t_a)}\right].\end{aligned} \tag{7.29}$$

The path integral (7.26) is perfectly convergent even though the velocities $\dot{q}_j = (q_{j+1}-q_j)/\epsilon$ that are important are very high, being of order $\sqrt{\hbar/(m\epsilon)}$.

It is easier to compute this amplitude (7.29) by using the outer products (7.10) (exercise **??**).

In three dimensions, the amplitude to go from $\boldsymbol{q}_a, t_a$ to $\boldsymbol{q}_b, t_b$ is

$$\langle \boldsymbol{q}_t|e^{-i(t_b-t_a)H/\hbar}|\boldsymbol{q}_0\rangle = \left(\frac{m}{2\pi i\hbar(t_b-t_a)}\right)^{3/2}\exp\left[\frac{im(\boldsymbol{q}_b-\boldsymbol{q}_a)^2}{2\hbar(t_b-t_a)}\right]. \tag{7.30}$$

□

7.4 Path integrals for quadratic actions

If a path $q(t) = q_c(t) + x(t)$ differs from a classical path $q_c(t)$ by a detour $x(t)$ that vanishes at the endpoints $x(t_a) = 0 = x(t_b)$ so that both paths go from q_a, t_a to q_b, t_b, then the difference $S[q_c + x] - S[q_c]$ in their actions vanishes to first order in the detour $x(t)$ (section **??**). Thus the actions of

the two paths differ by a time integral of quadratic and higher powers of the detour $x(t)$

$$
\begin{aligned}
S[q_c + x] &= \int_{t_a}^{t_b} \tfrac{1}{2} m \dot{q}(t)^2 - V(q(t))\, dt \\
&= \int_{t_a}^{t_b} \tfrac{1}{2} m \left(\dot{q}_c(t) + \dot{x}(t)\right)^2 - V(q_c(t) + x(t))\, dt \\
&= \int_{t_a}^{t_b} \left[\frac{m}{2} \dot{q}_c^2 + m \dot{q}_c \dot{x} + \frac{m}{2} \dot{x}^2 - V(q_c) - V'(q_c) x - \frac{V''(q_c)}{2} x^2 \right. \\
&\qquad \left. - \frac{V'''(q_c)}{6}\, x^3 - \frac{V''''(q_c)}{24}\, x^4 - \dots \right] dt \qquad (7.31) \\
&= \int_{t_a}^{t_b} \left[\frac{m}{2} \dot{q}_c^2 - V(q_c) \right] dt + \int_{t_a}^{t_b} \left[\frac{m}{2} \dot{x}^2 - \frac{V''(q_c)}{2} x^2 \right. \\
&\qquad \left. - \frac{V'''(q_c)}{6}\, x^3 - \frac{V''''(q_c)}{24}\, x^4 - \dots \right] dt \\
&= S[q_c] + \Delta S[q_c, x]
\end{aligned}
$$

in which $S[q_c]$ is the action of the classical path, and the detour $x(t)$ is a loop that goes from $x(t_a) = 0$ to $x(t_b) = 0$.

If the potential $V(q)$ is quadratic in the position q, then the third V''' and higher derivatives of the potential vanish, and the second derivative is a constant $V''(q_c(t)) = V''$. In this quadratic case, the correction $\Delta S[q_c, x]$ depends only on the time interval $t_b - t_a$ and on $\hbar$, m, and V''

$$
\Delta S[q_c, x] = \Delta S[x] = \int_{t_a}^{t_b} \left[\tfrac{1}{2} m\, \dot{x}^2(t) - \tfrac{1}{2} V''\, x^2(t) \right] dt. \qquad (7.32)
$$

It is independent of the classical path.

Thus for quadratic actions, the path integral (7.19) is an exponential of the action $S[q_c]$ of the classical path multiplied by a function $f(t_b - t_a, \hbar, m, V'')$ of the time interval $t_b - t_a$ and of $\hbar$, m, and V''

$$
\begin{aligned}
\langle q_b | e^{-i(t_b - t_a)H/\hbar} | q_a \rangle &= \int e^{iS[q]/\hbar} Dq = \int e^{i(S[q_c] + \Delta S[x])/\hbar} Dq \\
&= e^{iS[q_c]/\hbar} \int e^{i\Delta S[x])/\hbar}\, Dx \qquad (7.33) \\
&= f(t_b - t_a, \hbar, m, V'')\, e^{iS[q_c]/\hbar}.
\end{aligned}
$$

The function $f = f(t_b - t_a, \hbar, m, V'')$ is the limit as $n \to \infty$ of the $(n-1)$-

dimensional integral

$$f = \left[\frac{mn}{2\pi i\hbar(t_b - t_a)}\right]^{n/2} \int e^{i\Delta S[x])/\hbar} dx_{n-1} \dots dx_1 \tag{7.34}$$

where

$$\Delta S[x] = \frac{t_b - t_a}{n} \sum_{j=1}^{n} \frac{1}{2} m \frac{(x_j - x_{j-1})^2}{[(t_b - t_a)/n]^2} - \frac{1}{2} V'' x_j^2 \tag{7.35}$$

and $x_n = 0 = x_0$.

More generally, the path integral for any quadratic action of the form

$$S[q] = \int_{t_a}^{t_b} u\, \dot{q}^2(t) + v\, q(t)\dot{q}(t) + w\, q^2(t) + s(t)\, \dot{q}(t) + j(t)\, q(t)\, dt \tag{7.36}$$

is (exercise **??**)

$$\langle q_b | e^{-i(t_b - t_a)H/\hbar} | q_a \rangle = f(t_a, t_b, \hbar, u, v, w)\, e^{iS[q_c]/\hbar}. \tag{7.37}$$

The dependence of the amplitude upon $s(t)$ and $j(t)$ is contained in the classical action $S[q_c]$.

These formulas (7.33–7.37) may be generalized to any number of particles with coordinates $\{\boldsymbol{q}\} = \{\boldsymbol{q}^1, \dots, \boldsymbol{q}^k\}$ moving nonrelativistically in a space of multiple dimensions as long as the action is quadratic in the $\{\boldsymbol{q}\}$'s and their velocities $\{\dot{\boldsymbol{q}}\}$. The amplitude is then an exponential of the action $S[\{\boldsymbol{q}\}_c]$ of the classical path multiplied by a function $f(t_a, t_b, \hbar, \dots)$ that is independent of the classical path q_c

$$\langle \{\boldsymbol{q}\}_b | e^{-i(t_b - t_a)H/\hbar} | \{\boldsymbol{q}\}_a \rangle = f(t_a, t_b, \hbar, \dots)\, e^{iS[\{\boldsymbol{q}\}_c]/\hbar}. \tag{7.38}$$

Example 7.2 (A free particle) The classical path of a free particle going from $\boldsymbol{q}_a$ at time t_a to $\boldsymbol{q}_b$ at time t_b is

$$\boldsymbol{q}_c(t) = \boldsymbol{q}_a + \frac{t - t_a}{t_b - t_a} (\boldsymbol{q}_b - \boldsymbol{q}_a). \tag{7.39}$$

Its action is

$$S[\boldsymbol{q}_c] = \int_{t_a}^{t_b} \frac{1}{2} m\, \dot{\boldsymbol{q}}_c^2\, dt = \frac{m(\boldsymbol{q}_b - \boldsymbol{q}_a)^2}{2(t_b - t_a)} \tag{7.40}$$

and for this case our quadratic-potential formula (7.38) is

$$\langle \boldsymbol{q}_b | e^{-i(t_b - t_a)H/\hbar} | \boldsymbol{q}_a \rangle = f(t_b - t_a, \hbar, m) \exp\left[i \frac{m(\boldsymbol{q}_b - \boldsymbol{q}_a)^2}{2\hbar(t_b - t_a)}\right] \tag{7.41}$$

which agrees with our explicit calculation (7.30) when $f(t_b - t_a, \hbar, m) = [m/(2\pi i\hbar(t_b - t_a))]^{3/2}$. □

Example 7.3 (Bohm-Aharonov effect) From the formula (**??**) for the action of a relativistic particle of mass m and charge e, it follows (exercise **??**) that the action a nonrelativistic particle in an electromagnetic field with no scalar potential is

$$S = \int_{t_a}^{t_b} \left[\tfrac{1}{2} m \dot{\boldsymbol{q}}^2 + e \boldsymbol{A} \cdot \dot{\boldsymbol{q}}\right] dt = \int_{\boldsymbol{q}_a}^{\boldsymbol{q}_b} \left[\tfrac{1}{2} m \dot{\boldsymbol{q}} + e\, \boldsymbol{A}\right] \cdot \boldsymbol{dq}\,. \tag{7.42}$$

Since this action is quadratic in $\dot{\boldsymbol{q}}$, the amplitude for a particle to go from $\boldsymbol{q}_a$ at t_a to $\boldsymbol{q}_b$ at t_b is an exponential of the classical action

$$\langle \boldsymbol{q}_b | e^{-i(t_b - t_a)H/\hbar} | \boldsymbol{q}_a \rangle = f(t_b - t_a, \hbar, m, e)\, e^{iS[\boldsymbol{q}_c]/\hbar} \tag{7.43}$$

multiplied by a function $f(t_b - t_a, \hbar, m, e)$ that is independent of the path $\boldsymbol{q}_c$. A beam of such particles goes horizontally past but not through a vertical pipe in which a vertical magnetic field is confined. The particles can go both ways around the pipe of cross-sectional area S but do not enter it. The difference in the phases of the amplitudes for the two paths is a loop integral from the source to the detector around the pipe and back to the source

$$\oint \left[\frac{m\dot{\boldsymbol{q}}}{2} + e\, \boldsymbol{A}\right] \cdot \frac{\boldsymbol{dq}}{\hbar} = \oint \frac{m\dot{\boldsymbol{q}} \cdot \boldsymbol{dq}}{2\hbar} + \frac{e}{\hbar} \int_S \boldsymbol{B} \cdot \boldsymbol{dS} = \oint \frac{m\dot{\boldsymbol{q}} \cdot \boldsymbol{dq}}{2\hbar} + \frac{e\Phi}{\hbar} \tag{7.44}$$

in which Φ is the magnetic flux through the cylinder. □

Example 7.4 (Harmonic oscillator) The action

$$S = \int_{t_a}^{t_b} \frac{1}{2} m \dot{q}^2(t) - \frac{1}{2} m \omega^2 q^2(t)\, dt \tag{7.45}$$

of a harmonic oscillator is quadratic in q and $\dot{q}$. So apart from a factor f, its path integral (7.33–7.35) is an exponential

$$\langle q_b | e^{-i(t_b - t_a)H/\hbar} | q_a \rangle = f\, e^{iS[q_c]/\hbar} \tag{7.46}$$

of the action $S[q_c]$ (exercise **??**)

$$S[q_c] = \frac{m\omega \left[\left(q_a^2 + q_b^2\right) \cos(\omega(t_b - t_a)) - 2 q_a q_b\right]}{2 \sin(\omega(t_b - t_a))} \tag{7.47}$$

of the classical path

$$q_c(t) = q_a \cos \omega(t - t_a) + \frac{q_b - q_a \cos \omega(t_b - t_a)}{\sin \omega(t_b - t_a)} \sin \omega(t - t_a) \tag{7.48}$$

that runs from q_a, t_a to q_b, t_b and obeys the classical equation of motion $m\ddot{q}_c(t) = -\omega^2 q_c(t)$.

The factor f is a function $f(t_b - t_a, \hbar, m, m\omega^2)$ of the time interval and the

parameters of the oscillator. It is the $n \to \infty$ limit of the $(n-1)$-dimensional integral (7.34)

$$f = \left[\frac{mn}{2\pi i\hbar(t_b - t_a)}\right]^{n/2} \int e^{i\Delta S[x])/\hbar}\, dx_{n-1} \ldots dx_1 \tag{7.49}$$

over all loops that run from 0 to 0 in time $t_b - t_a$ in which the quadratic correction to the classical action is (7.35)

$$\Delta S[x] = \frac{t_b - t_a}{n} \sum_{j=1}^{n} \frac{1}{2} m \frac{(x_j - x_{j-1})^2}{[(t_b - t_a)/n]^2} - \frac{1}{2} m\omega^2\, x_j^2, \tag{7.50}$$

and $x_n = 0 = x_0$.

Setting $t_b - t_a = T$, we use the many-variable imaginary gaussian integral (7.3) to write f as

$$f = \left[\frac{mn}{2\pi i\hbar T}\right]^{n/2} \int e^{i a_{jk} x_j x_k} dx_{n-1} \ldots dx_1 = \left[\frac{mn}{2\pi i\hbar T}\right]^{n/2} \sqrt{\frac{(i\pi)^{n-1}}{\det a}} \tag{7.51}$$

in which the quadratic form $a_{jk} x_j x_k$ is

$$\frac{nm}{\hbar T} \sum_{j=1}^{n} \left[- x_j x_{j-1} + \frac{1}{2}(x_j^2 + x_{j-1}^2) - \frac{(\omega T)^2}{2n^2} x_j^2 \right] \tag{7.52}$$

which has no linear term because $x_0 = x_n = 0$.

The $(n-1)$-dimensional square matrix a is a tridiagonal Toeplitz matrix

$$a = \frac{nm}{2\hbar T} \begin{pmatrix} y & -1 & 0 & 0 & \cdots \\ -1 & y & -1 & 0 & \cdots \\ 0 & -1 & y & -1 & \cdots \\ 0 & 0 & -1 & y & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}. \tag{7.53}$$

Apart from the factor $nm/(2\hbar T)$, the matrix $a = (nm/(2\hbar T))\, C_{n-1}(y)$ is a tridiagonal matrix $C_{n-1}(y)$ whose off-diagonal elements are -1 and whose diagonal elements are

$$y = 2 - \frac{(\omega T)^2}{n^2}. \tag{7.54}$$

Their determinants $|C_n(y)| = \det C_n(y)$ obey (exercise **??**) the recursion relation

$$|C_{n+1}(y)| = y\, |C_n(y)| - |C_{n-1}(y)| \tag{7.55}$$

and have the initial values $|C_1(y)| = y$ and $|C_2(y)| = y^2 - 1$. The trigonometric functions $U_n(y) = \sin[(n+1)\theta]/\sin\theta$ with $y = 2\cos\theta$ obey the same recursion relation and have the same initial values (exercise **??**), so

$$|C_n(y)| = \frac{\sin(n+1)\theta}{\sin\theta}. \tag{7.56}$$

Since for large n

$$\theta = \arccos(y/2) = \arccos\left(1 - \frac{\omega^2 t^2}{2n^2}\right) \approx \frac{\omega T}{n}, \tag{7.57}$$

the determinant of the matrix a is

$$\begin{aligned} \det a &= \left(\frac{nm}{2\hbar T}\right)^{n-1} |C_{n-1}(y)| = \left(\frac{nm}{2\hbar T}\right)^{n-1} \frac{\sin n\theta}{\sin\theta} \\ &\approx \left(\frac{nm}{2\hbar T}\right)^{n-1} \frac{\sin(\omega T)}{\sin(\omega T/n)} \approx \left(\frac{nm}{2\hbar T}\right)^{n-1} \frac{n\sin\omega T}{\omega T}. \end{aligned} \tag{7.58}$$

Thus the factor f is

$$\begin{aligned} f &= \left[\frac{mn}{2\pi i\hbar T}\right]^{n/2} \sqrt{\frac{(i\pi)^{n-1}}{\det a}} = \left[\frac{mn}{2\pi i\hbar T}\right]^{n/2} \sqrt{\left(\frac{2\pi i\hbar T}{nm}\right)^{n-1} \frac{\omega T}{n\sin\omega T}} \\ &= \sqrt{\frac{m\omega}{2\pi i\hbar\sin\omega T}}. \end{aligned} \tag{7.59}$$

The amplitude (7.46) is then an exponential of the action $S[q_c]$ (7.47) of the classical path (7.48) multiplied by this factor f

$$\begin{aligned} \langle q_b|e^{-i(t_b-t_a)H/\hbar}|q_a\rangle &= \sqrt{\frac{m\omega}{2\pi i\hbar\sin\omega(t_b-t_a)}} \\ &\quad \times \exp\left\{\frac{i}{\hbar}\frac{m\omega\left[\left(q_a^2+q_b^2\right)\cos(\omega(t_b-t_a)) - 2q_aq_b\right]}{2\sin(\omega(t_b-t_a))}\right\}. \end{aligned} \tag{7.60}$$

□

As these examples (7.2 & 7.4) suggest, path integrals are as mathematically well defined as ordinary integrals.

7.5 Path integrals in statistical mechanics

At the imaginary time $t = -i\hbar\beta = -i\hbar/(kT)$, the time-evolution operator $e^{-itH/\hbar}$ becomes the **Boltzmann operator** $e^{-\beta H}$ whose trace is the

partition function $Z(\beta)$ at inverse energy $\beta = 1/(kT)$

$$Z(\beta) = \mathrm{Tr}\big(e^{-\beta H}\big) = \sum_n \langle n|e^{-\beta H}|n\rangle \tag{7.61}$$

in which the states $|n\rangle$ form a complete orthonormal set, $k = 8.617 \times 10^{-5}$ eV/K is Boltzmann's constant, and T is the absolute temperature. Partition functions are important in statistical mechanics and quantum field theory.

Since the Boltzmann operator $e^{-\beta H}$ is the time-evolution operator $e^{-itH/\hbar}$ at the imaginary time $t = -i\hbar\beta$, we can write it as a path integral by imitating the derivation of the preceding section (7.3). We will use the same hamiltonian $H = p^2/(2m) + V(q)$ and the operators q and p which have complete sets of eigenstates (7.9) that satisfy (7.10).

Changing our definitions of ϵ, k, and v to $\epsilon = \beta/n$, $k = \beta p^2/(2m)$, and $v = \beta V(q)$, we can write Trotter's formula (7.7) for the Boltzmann operator as the $n \to \infty$ limit of n factors of $e^{-\epsilon k}\, e^{-\epsilon v}$

$$e^{-\beta H} = e^{-\epsilon k}\, e^{-\epsilon v}\; e^{-\epsilon k}\, e^{-\epsilon v} \cdots e^{-\epsilon k}\, e^{-\epsilon v}\; e^{-\epsilon k}\, e^{-\epsilon v}. \tag{7.62}$$

To evaluate the matrix element $\langle q_1|e^{-\epsilon k}\, e^{-\epsilon v}|q_a\rangle$, we insert the identity operator $\langle q_1|e^{-\epsilon k}\, I\, e^{-\epsilon v}|q_a\rangle$ as an integral (7.10) over outer products $|p'\rangle\langle p'|$ of momentum eigenstates and use the inner products $\langle q_1|p'\rangle = e^{iq_1p'/\hbar}/\sqrt{2\pi\hbar}$ and $\langle p'|q_a\rangle = e^{-iq_ap'/\hbar}/\sqrt{2\pi\hbar}$

$$\begin{aligned}\langle q_1|e^{-\epsilon k}\, e^{-\epsilon v}|q_a\rangle &= \int_{-\infty}^{\infty} \langle q_1|e^{-\epsilon p^2/(2m)}|p'\rangle\langle p'|e^{-\epsilon V(q)}|q_a\rangle\, dp' \\ &= e^{-\epsilon\, V(q_a)} \int_{-\infty}^{\infty} e^{-\epsilon p'^2/(2m)}\; e^{ip'(q_1-q_a)/\hbar}\, \frac{dp'}{2\pi\hbar}.\end{aligned} \tag{7.63}$$

If we adopt the suggestive notation $q_1 - q_a = \epsilon\hbar\,\dot{q}_a$ and use the gaussian integral (7.2) with $a = \epsilon/(2m)$, $x = p$, and $b = \epsilon\dot{q}/2$

$$\int_{-\infty}^{\infty} \exp\left(-\epsilon\,\frac{p^2}{2m} + i\epsilon\,\dot{q}\,p\right)\frac{dp}{2\pi\hbar} = \sqrt{\frac{m}{2\pi\epsilon\hbar^2}}\;\exp\left(-\epsilon\,\frac{m\dot{q}^2}{2}\right), \tag{7.64}$$

then we find

$$\begin{aligned}\langle q_1|e^{-\epsilon k}\, e^{-\epsilon v}|q_a\rangle &= e^{-\epsilon\, V(q_a)} \int_{-\infty}^{\infty} \exp\left(-\epsilon\,\frac{p'^2}{2m} + i\,\epsilon\,p'\,\dot{q}_a\right)\frac{dp'}{2\pi\hbar} \\ &= \left(\frac{m}{2\pi\hbar^2\epsilon}\right)^{1/2} \exp\left[-\epsilon\left(\frac{m\,\dot{q}_a^2}{2} + V(q_a)\right)\right]\end{aligned} \tag{7.65}$$

in which q_1 is hidden in the formula $q_1 - q_a = \hbar\,\epsilon\,\dot{q}_a$.

The next step is to link two of these matrix elements together

$$\langle q_2|\big(e^{-\epsilon k}\, e^{-\epsilon v}\big)^2|q_a\rangle = \int_{-\infty}^{\infty}\langle q_2|e^{-\epsilon k}\, e^{-\epsilon v}|q_1\rangle\langle q_1|e^{-\epsilon k}\, e^{-\epsilon v}|q_a\rangle dq_1 \tag{7.66}$$
$$= \frac{m}{2\pi\hbar^2\epsilon}\int_{-\infty}^{\infty}\exp\left\{-\epsilon\left[\frac{m\,\dot{q}_1^2}{2}+V(q_1)+\frac{m\,\dot{q}_0^2}{2}+V(q_a)\right]\right\}dq_1.$$

Passing from 2 to n and suppressing some integral signs, we get

$$\langle q_b|e^{-n\epsilon H}\,|q_a\rangle = \int_{-\infty}^{\infty}\langle q_b|e^{-\epsilon k}\, e^{-\epsilon v}|q_{n-1}\rangle\cdots\langle q_1|e^{-\epsilon k}\, e^{-\epsilon v}|q_a\rangle\, dq_{n-1}\ldots dq_1$$
$$= \left(\tfrac{m}{2\pi\hbar^2\epsilon}\right)^{n/2}\int_{-\infty}^{\infty}\exp\left[-\epsilon\sum_{j=0}^{n-1}\left(\tfrac{m\,\dot{q}_j^2}{2}+V(q_j)\right)\right]dq_{n-1}\ldots dq_1.$$

Setting $du = \hbar\epsilon = \hbar\beta/n$ and taking the limit $n\to\infty$, we find that the matrix element $\langle q_b|e^{-\beta H}|q_a\rangle$ is the path integral

$$\langle q_b|e^{-\beta H}|q_a\rangle = \int e^{-S_e[q]/\hbar}\, Dq \tag{7.67}$$

in which each path is weighted by its euclidian action

$$S_e[q] = \int_0^{\hbar\beta}\frac{m\dot{q}^2(u)}{2}+V(q(u))\;du, \tag{7.68}$$

$\dot{q}$ is the derivative of the coordinate $q(u)$ with respect to euclidian time $u=\hbar\beta$, and $Dq \equiv (n\,m/2\pi\,\hbar^2\beta)^{n/2}\,dq_{n-1}\ldots dq_1$.

A derivation identical to the one that led from (7.62) to (7.68) leads in a more elaborate notation to

$$\langle q_b|e^{-(\beta_b-\beta_a)H}|q_a\rangle = \int e^{-S_e[q]/\hbar}\, Dq \tag{7.69}$$

in which each path is weighted by its euclidian action

$$S_e[q] = \int_{\hbar\beta_a}^{\hbar\beta_b}\frac{m\dot{q}^2(u)}{2}+V(q(u))\,du, \tag{7.70}$$

and $\dot{q}$ and Dq are the same as in (7.68).

If we multiply the path integral (7.69) from the left by $|q_b\rangle$ and from the right by $\langle q_a|$ and integrate over q_a and q_b as in the resolution (7.10) of the identity operator, then we can write the Boltzmann operator as an integral over all paths from t_a to t_b

$$e^{-(\beta_b-\beta_a)\,H} = \int |q_b\rangle\, e^{-S_e[q]/\hbar}\,\langle q_a|\, Dq\, dq_a\, dq_b \tag{7.71}$$

with $Dq = (mn/(2\pi i\hbar(t_b-t_a)))^{n/2}\,dq_{n-1}\ldots dq_1$ and $S_e[q]$ the action (7.70).

To get the partition function $Z(\beta)$, we set $q_b = q_a \equiv q_n$ and integrate over all n q's letting $n \to \infty$

$$\begin{aligned} Z(\beta) = \operatorname{Tr} e^{-\beta H} &= \int \langle q_n | e^{-\beta H} | q_n \rangle \, dq_n \\ &= \int \exp\left[-\frac{1}{\hbar} \int_0^{\hbar\beta} \frac{m\dot{q}^2(u)}{2} + V(q(u)) \, du \right] Dq \end{aligned} \tag{7.72}$$

where $Dq \equiv (n\, m/2\pi\, \hbar^2\, \beta)^{n/2}\, dq_n \ldots dq_1$. We sum over all loops $q(u)$ that go from $q(0) = q_n$ at euclidian time 0 to $q(\hbar\beta) = q_n$ at euclidian time $\hbar\beta$.

In the low-temperature limit, $T \to 0$ and $\beta \to \infty$, the Boltzmann operator $\exp(-\beta H)$ projects out the ground state $|E_0\rangle$ of the system

$$\lim_{\beta\to\infty} e^{-\beta H} = \lim_{\beta\to\infty} \sum_n e^{-\beta E_n} |E_n\rangle\langle E_n| = e^{-\beta E_0} |E_0\rangle\langle E_0|. \tag{7.73}$$

The maximum-entropy **density operator** (section **??**, example **??**) is the Boltzmann operator $e^{-\beta H}$ divided by its trace $Z(\beta)$

$$\rho = \frac{e^{-\beta H}}{\operatorname{Tr}(e^{-\beta H})} = \frac{e^{-\beta H}}{Z(\beta)}. \tag{7.74}$$

Its matrix elements are matrix elements of Boltzmann operator (7.68) divided by the partition function (7.72)

$$\langle q_b | \rho | q_a \rangle = \frac{\langle q_b | e^{-\beta H} | q_a \rangle}{Z(\beta)}. \tag{7.75}$$

In three dimensions with $\dot{\boldsymbol{q}}(u) = d\boldsymbol{q}(u)/du$, the $\boldsymbol{q}_a, \boldsymbol{q}_b$ matrix element of the Boltzmann operator is the analog of equation (7.68) (exercise **??**)

$$\langle \boldsymbol{q}_b | e^{-\beta H} | \boldsymbol{q}_a \rangle = \int \exp\left[-\frac{1}{\hbar} \int_0^{\hbar\beta} \frac{m\dot{\boldsymbol{q}}^2(u)}{2} + V(\boldsymbol{q}(u)) \, du \right] D\boldsymbol{q} \tag{7.76}$$

where $Dq \equiv (n\, m/2\pi\, \hbar^2 \beta)^{3n/2}\, d\boldsymbol{q}_{n-1} \ldots d\boldsymbol{q}_1$, and the partition function is the integral over all loops that go from $q_0 = q_n$ to q_n in time $\hbar\beta$

$$Z(\beta) = \int \exp\left[-\frac{1}{\hbar} \int_0^{\hbar\beta} \frac{m\dot{\boldsymbol{q}}^2(u)}{2} + V(\boldsymbol{q}(u)) \, du \right] D\boldsymbol{q} \tag{7.77}$$

where now $D\boldsymbol{q} \equiv (n\, m/2\pi\, \hbar^2\, \beta)^{3n/2}\, d\boldsymbol{q}_n \ldots d\boldsymbol{q}_1$.

Because the Boltzmann operator $e^{-\beta H}$ is the time-evolution operator $e^{-itH/\hbar}$ at the imaginary time $t = -iu = -i\hbar\beta = -i\hbar/(kT)$, the path integrals of statistical mechanics are called **euclidian path integrals**.

Example 7.5 (Density operator for a free particle) For a free particle, the matrix element of the Boltzmann operator $e^{-\beta H}$ is the $n = \beta/\epsilon \to \infty$ limit of the integral

$$\langle q_b|e^{-\beta H}|q_a\rangle = \left(\frac{m}{2\pi\hbar^2\epsilon}\right)^{n/2} \tag{7.78}$$

$$\times\int \exp\left[-\frac{m(q_b-q_{n-1})^2}{2\hbar^2\epsilon}\cdots-\frac{(q_1-q_a)^2}{2\hbar^2\epsilon}\right]dq_{n-1}\cdots dq_1. \tag{7.79}$$

The formula (7.2) gives for the q_1 integral

$$\left(\frac{m}{2\pi\hbar^2\epsilon}\right)^{1/2}\int e^{-[m(q_2-q_1)^2+m(q_1-q_a)^2]/(2\hbar^2\epsilon)}dq_1 = \frac{e^{-m(q_2-q_a)^2/(2\hbar^2 2\epsilon)}}{\sqrt{2}}. \tag{7.80}$$

The q_2 integral is (exercise **??**)

$$\left(\frac{m}{4\pi\hbar^2\epsilon}\right)^{1/2}\int e^{-m(q_3-q_2)^2/(2\hbar^2\epsilon)-m(q_2-q_a)^2]/(4\hbar^2\epsilon)}dq_2 = \frac{e^{-m(q_3-q_a)^2/(2\hbar^2 3\epsilon)}}{\sqrt{3}}. \tag{7.81}$$

All $n-1$ integrations give

$$\langle q_b|e^{-\beta H}|q_a\rangle = \sqrt{\frac{m}{2\pi\hbar^2\epsilon}}\,\frac{e^{-m(q_b-q_a)^2/(2\hbar^2 n\epsilon)}}{\sqrt{n}} = \sqrt{\frac{m}{2\pi\hbar^2\beta}}\,e^{-m(q_b-q_a)^2/(2\hbar^2\beta)}. \tag{7.82}$$

The partition function is the integral of this matrix element over $q_a = q_b$

$$Z(\beta) = \left(\frac{m}{2\pi\hbar^2\beta}\right)^{1/2}\int dq_a = \left(\frac{m}{2\pi\hbar^2\beta}\right)^{1/2} L \tag{7.83}$$

where L is the (infinite) 1-dimensional volume of the system. The q_b, q_a matrix element of the maximum-entropy density operator is

$$\langle q_b|\rho|q_a\rangle = \frac{e^{-m(\boldsymbol{q}_b-\boldsymbol{q}_a)^2/(2\hbar^2\beta)}}{L}. \tag{7.84}$$

In 3 dimensions, equations (7.82 & 7.83) are

$$\langle \boldsymbol{q}_b|e^{-\beta H}|\boldsymbol{q}_a\rangle = \left(\frac{mkT}{2\pi\hbar^2}\right)^{3/2} e^{-m(\boldsymbol{q}_b-\boldsymbol{q}_a)^2/(2\hbar^2\beta)} \text{ and } Z(\beta) = \left(\frac{mkT}{2\pi\hbar^2}\right)^{3/2} L^3. \tag{7.85}$$

□

Example 7.6 (Partition function at high temperatures) At high temperatures, the time $\hbar\beta = \hbar/(kT)$ is very short, and the density operator (7.85)

for a free particle shows that free paths are damped and limited to distances of order $\hbar/\sqrt{mkT}$. We thus can approximate the path integral (7.77) for the partition function by replacing the potential $V(q(u))$ by $V(q_n)$ and then using the free-particle matrix element (7.85)

$$Z(\beta) \approx \int e^{-\beta V(q_n)} \exp\left[-\frac{1}{\hbar}\int_0^{\hbar\beta} \frac{m\dot{\boldsymbol{q}}^2(u)}{2}\, du\right] D\boldsymbol{q} \tag{7.86}$$
$$= \int e^{-\beta V(q_n)} \langle \boldsymbol{q}_n | e^{-\beta H} | \boldsymbol{q}_n \rangle \, dq_n = \left(\frac{mkT}{2\pi\hbar^2}\right)^{3/2} \int e^{-\beta V(q_n)}\, dq_n.$$

□

7.6 Boltzmann path integrals for quadratic actions

Apart from the factor $Dq \equiv (n\, m/2\pi\, \hbar^2\beta)^{n/2}\, dq_{n-1} \ldots dq_1$, the euclidian path integral

$$\langle q_b | e^{-\beta H} | q_a \rangle = \int \exp\left[-\frac{1}{\hbar}\int_0^{\hbar\beta} \frac{m\dot{q}^2(u)}{2} + V(q(u))\, du\right] Dq \tag{7.87}$$

is a sum of positive terms $e^{-S_e[q]/\hbar}$ one for each path from $q_a, 0$ to q_b, β. If a path from $q_a, 0$ to q_b, β obeys the classical euclidian equation of motion

$$m\, \frac{d^2 q_{ce}}{du^2} = m\, \ddot{q}_{ce} = V'(q_{ce}) \tag{7.88}$$

then its euclidian action

$$S_e[q] = \int_0^{\hbar\beta} \frac{m\, \dot{q}^2(u)}{2} + V(u)\, du \tag{7.89}$$

is stationary and may be minimal. So we can approximate the euclidian action $S_e[q_{ce} + x]$ as we approximated the action $S[q_c + x]$ in section 7.4. The euclidian action $S_e[q_{ce} + x]$ of an arbitrary path from $q_a, 0$ to q_b, β is the stationary euclidian action $S_e[q_{ce}]$ plus a u-integral of quadratic and higher powers of the detour x which goes from $x(0) = 0$ to $x(\hbar\beta) = 0$

$$S_e[q_{ce} + x] = \int_0^{\hbar\beta} \left[\frac{m}{2}\dot{q}_{ce}^2 + V(q_{ce})\right] du + \int_0^{\hbar\beta} \left[\frac{m}{2}\dot{x}^2 + \frac{V''(q_{ce})}{2} x^2 + \frac{V'''(q_{ce})}{6}\, x^3 + \frac{V''''(q_{ce})}{24}\, x^4 + \ldots\right] du$$
$$= S_e[q_{ce}] + \Delta S_e[q_{ce}, x], \tag{7.90}$$

and the path integral for the matrix element $\langle q_b|e^{-\beta H}|q_a\rangle$ is

$$\langle q_b|e^{-\beta H}|q_a\rangle = e^{-S_e[q_{ce}]/\hbar}\int e^{-\Delta S_e[q_{ce},x]/\hbar}Dx \tag{7.91}$$

as $n\to\infty$ where $Dx = (n\,m/2\pi\,\hbar^2\beta)^{n/2}\,dq_{n-1}\ldots dq_1$ in the limit $n\to\infty$.

If the action is quadratic in q and $\dot{q}$, then the integral $\Delta S_e[q_{ce},x]$ over the detour x is a gaussian path integral that is independent of the path q_{ce} and so is a function f only of the parameters β, m, $\hbar$, and V''

$$\begin{aligned}\langle q_b|e^{-\beta H}|q_a\rangle &= e^{-S_e[q_{ce}]/\hbar}\int e^{-\Delta S_e[x]/\hbar}Dx\\ &= f(\beta,\hbar,m,V'')\;e^{-S_e[q_{ce}]/\hbar}\end{aligned} \tag{7.92}$$

where

$$\begin{aligned}f(\beta,\hbar,m,V'') &= \left[\frac{mn}{2\pi\hbar^2\beta}\right]^{n/2}\int e^{-\Delta S_e[x]/\hbar}dx_{n-1}\ldots dx_1,\\ \Delta S_e[x] &= \frac{\hbar\beta}{n}\sum_{j=1}^{n}\frac{m}{2\hbar^2}\frac{(x_j-x_{j-1})^2}{(\beta/n)^2}+\frac{1}{2}V''\,x_j^2,\end{aligned} \tag{7.93}$$

and $x_n = 0 = x_0$.

Example 7.7 (Density operator for the harmonic oscillator) The path $q_{ce}(\beta)$ that satisfies the classical euclidian equation of motion (7.88)

$$\ddot{q}_{ce}(u) = \frac{d^2q_{ce}(u)}{du^2} = \omega^2 q_{ce}(u) \tag{7.94}$$

and goes from $q_a, 0$ to $q_b, \hbar\beta$ is

$$q_{ce}(u) = \frac{\sinh(\omega u)\,q_b + \sinh[\omega(\hbar\beta-u)]\,q_a}{\sinh(\hbar\omega\beta)}. \tag{7.95}$$

Its euclidian action is (exercise **??**)

$$\begin{aligned}S_e[q_{ce}] &= \int_0^{\hbar\beta}\frac{m\dot{q}_{ce}^2(u)}{2}+\frac{m\omega^2q_{ce}^2(u))}{2}\;du\\ &= \frac{m\omega}{2\hbar\sinh(\hbar\omega\beta)}\left[\cosh(\hbar\omega\beta)(q_a^2+q_b^2)-2q_aq_b\right].\end{aligned} \tag{7.96}$$

Since $V''=m\omega^2$, our formulas (7.92 & 7.93) for quadratic actions give as the matrix element

$$\langle q_b|e^{-\beta H}|q_a\rangle = f(\beta,\hbar,m,m\omega^2)\;e^{-S_e[q_{ce}]/\hbar} \tag{7.97}$$

in which

$$f(\beta,\hbar,m,m\omega^2) = \left[\frac{mn}{2\pi\hbar^2\beta}\right]^{n/2}\int e^{-\Delta S_e[x]/\hbar)}\,dx_{n-1}\ldots dx_1,$$
$$\Delta S_e[x] = \frac{\hbar\beta}{n}\sum_{j=1}^{n}\frac{m}{2\hbar^2}\frac{(x_j-x_{j-1})^2}{(\beta/n)^2}+\frac{m\omega^2\,x_j^2}{2}, \tag{7.98}$$

and $x_n = 0 = x_0$. We can do this integral by using the formula (7.4) for a many variable real gaussian integral

$$f = \left[\frac{mn}{2\pi\hbar^2 B}\right]^{n/2}\int e^{-a_{jk}x_jx_k}dx_{n-1}\ldots dx_1 = \left[\frac{mn}{2\pi\hbar^2 B}\right]^{n/2}\sqrt{\frac{(\pi)^{n-1}}{\det a}} \tag{7.99}$$

in which the positive quadratic form $a_{jk}x_jx_k$ is

$$\frac{nm}{2\hbar^2 B}\sum_{j=1}^{n}\left[-2x_jx_{j-1}+x_j^2+x_{j-1}^2+\frac{(\hbar\omega B)^2}{n^2}x_j^2\right] \tag{7.100}$$

which has no linear term because $x_0 = x_n = 0$.

The matrix a is $(nm/(2\hbar^2 B))\,C_{n-1}(y)$ in which $C_{n-1}(y)$ is a square, tridiagonal, $(n-1)$-dimensional matrix whose off-diagonal elements are -1 and whose diagonal elements are $y = 2+(\hbar\omega\beta)^2/n^2$. The determinants $|C_n(y)|$ obey the recursion relation $|C_{n+1}(y)| = y\,|C_n(y)| - |C_{n-1}(y)|$ and have the initial values $C_1(y) = y$ and $C_2(y) = y^2-1$. So do the hyperbolic functions $\sinh(n+1)\theta/\sinh\theta$ with $y = 2\cosh\theta$. So we set $C_n(y) = \sinh(n+1)\theta/\sinh\theta$ with $\theta = \operatorname{arccosh}(y/2)$. We then get as the matrix element (7.97)

$$\langle q_b|e^{-\beta H}|q_a\rangle = \sqrt{\frac{m\omega}{2\pi\hbar\sinh(\hbar\omega\beta)}}\exp\left[-\frac{m\omega[\cosh(\hbar\omega\beta)(q_a^2+q_b^2)-2q_aq_b]}{2\hbar\sinh(\hbar\omega\beta)}\right]. \tag{7.101}$$

The partition function is the integral over q_a of this matrix element for $q_b = q_a$

$$Z(\beta) = \sqrt{\frac{m\omega}{2\pi\hbar\sinh(\hbar\omega\beta)}}\int\exp\left[-\frac{m\omega[\cosh(\hbar\omega\beta)-1]q_a^2}{\hbar\sinh(\hbar\omega\beta)}\right]dq_a$$
$$= \frac{1}{\sqrt{2[\cosh(\hbar\omega\beta)-1]}}. \tag{7.102}$$

The matrix elements of the maximum-entropy density operator (7.74) are

$$\langle q_b|\rho|q_a\rangle = \frac{\langle q_b|e^{-\beta H}|q_a\rangle}{Z(\beta)} \tag{7.103}$$
$$= \sqrt{\frac{m\omega[\cosh(\hbar\omega\beta)-1]}{\pi\hbar\sinh(\hbar\omega\beta)}}\exp\left[-\frac{m\omega[\cosh(\hbar\omega\beta)(q_a^2+q_b^2)-2q_aq_b]}{2\hbar\sinh(\hbar\omega\beta)}\right]$$

which reveals the ground-state wave functions

$$\lim_{\beta\to\infty} \langle q_b|\rho|q_a\rangle = \langle q_b|0\rangle\langle 0|q_a\rangle = \sqrt{\frac{m\omega}{\pi\hbar}}\; e^{-m\omega(q_a^2+q_b^2)/(2\hbar)}. \tag{7.104}$$

The partition function gives us the ground-state energy

$$\lim_{\beta\to\infty} Z(\beta) = \lim_{\beta\to\infty} \frac{1}{\sqrt{2[\cosh(\hbar\omega\beta)-1]}} = e^{-\beta E_0} = e^{-\beta\hbar\omega/2}. \tag{7.105}$$

□

7.7 Mean values of time-ordered products

In the Heisenberg picture, the position operator at time t is

$$q(t) = e^{itH/\hbar} q\, e^{-itH/\hbar} \tag{7.106}$$

in which $q = q(0)$ is the position operator at time $t = 0$ or equivalently the position operator in the Schrödinger picture. The position operator q at the imaginary time $t = -iu = -i\hbar\beta = -i\hbar/(kT)$ is the euclidian position operator

$$q_e(u) = q_e(\hbar\beta) = e^{uH/\hbar} q\, e^{-uH/\hbar}. \tag{7.107}$$

The time-ordered product of two position operators is

$$\mathcal{T}[q(t_1)q(t_2)] = \left\{ \begin{array}{ll} q(t_1)\, q(t_2) & \text{if } t_1 \geq t_2 \\ q(t_2)\, q(t_1) & \text{if } t_2 \geq t_1 \end{array} \right\} = q(t_>)\, q(t_<) \tag{7.108}$$

in which $t_>$ is the later and $t_<$ the earlier of the two times t_1 and t_2. Similarly, the time-ordered product of two euclidian position operators at euclidian times $u_1 = \hbar\beta_1$ and $u_2 = \hbar\beta_2$ is

$$\mathcal{T}[q_e(u_1)q_e(u_2)] = \left\{ \begin{array}{ll} q_e(u_1)\, q_e(u_2) & \text{if } u_1 \geq u_2 \\ q_e(u_2)\, q_e(u_1) & \text{if } u_2 \geq u_1 \end{array} \right\} = q_e(u_>)\, q_e(u_<). \tag{7.109}$$

The matrix element of the time-ordered product (7.108) of two position operators and two exponentials $e^{-itH/\hbar}$ between states $|a\rangle$ and $|b\rangle$ is

$$\begin{aligned} \langle b|e^{-itH/\hbar}\mathcal{T}[q(t_1)q(t_2)]e^{-itH/\hbar}|a\rangle &= \langle b|e^{-itH/\hbar} q(t_>)q(t_<) e^{-itH/\hbar}|a\rangle \\ &= \langle b|e^{-i(t-t_>)H/\hbar} q\, e^{-i(t_>-t_<)H/\hbar} q\, e^{-i(t+t_<)H/\hbar}|a\rangle. \end{aligned} \tag{7.110}$$

We use the path-integral formula (7.21) for each of the exponentials on the right-hand side of this equation and find (exercise **??**)

$$\langle b|e^{-itH/\hbar}\mathcal{T}[q(t_1)q(t_2)]e^{-itH/\hbar}|a\rangle = \int \langle b|q_b\rangle q(t_1)q(t_2)e^{iS[q]/\hbar}\langle q_a|a\rangle Dq \tag{7.111}$$

in which the integral is over all paths that run from $-t$ to t. This equation simplifies if the states $|a\rangle$ and $|b\rangle$ are eigenstates of H with eigenvalues E_m and E_n

$$e^{-it(E_n+E_m)/\hbar}\langle n|\mathcal{T}[q(t_1)q(t_2)]|m\rangle = \int \langle n|q_b\rangle q(t_1)q(t_2)e^{iS[q]/\hbar}\langle q_a|m\rangle Dq. \tag{7.112}$$

By setting $n = m$ and omitting the time-ordered product, we get

$$e^{-2itE_n/\hbar} = \int \langle n|q_b\rangle e^{iS[q]/\hbar}\langle q_a|n\rangle Dq. \tag{7.113}$$

The ratio of (7.112) with $n = m$ to (7.113) is

$$\langle n|\mathcal{T}[q(t_1)q(t_2)]|n\rangle = \frac{\displaystyle\int \langle n|q_b\rangle q(t_1)q(t_2)e^{iS[q]/\hbar}\langle q_a|n\rangle\, Dq}{\displaystyle\int \langle n|q_b\rangle e^{iS[q]/\hbar}\langle q_a|n\rangle\, Dq} \tag{7.114}$$

in which the integrations are over all paths that go from $-t \le t_<$ to $t \ge t_>$. The mean value of the time-ordered product of k position operators is

$$\langle n|\mathcal{T}[q(t_1)\cdots q(t_k)]|n\rangle = \frac{\displaystyle\int \langle n|q_b\rangle q(t_1)\cdots q(t_k)e^{iS[q]/\hbar}\langle q_a|n\rangle\, Dq}{\displaystyle\int \langle n|q_b\rangle e^{iS[q]/\hbar}\langle q_a|n\rangle\, Dq} \tag{7.115}$$

in which the integrations are over all paths that go from some time before $t_1, \ldots, t_k$ to some time them.

We may perform the same operations on the euclidian position operators by replacing t by $-iu = -i\hbar\beta$. A matrix element of the euclidian time-ordered product (7.109) between two states is

$$\begin{aligned}\langle b|e^{-uH/\hbar}\mathcal{T}[q_e(u_1)q_e(u_2)]e^{-uH/\hbar}|a\rangle &= \langle b|e^{-uH/\hbar}q_e(u_>)q_e(u_<)e^{-uH/\hbar}|a\rangle \\ &= \langle b|e^{-(u-u_>)H/\hbar}q\, e^{-(u_>-u_<)H/\hbar}q\, e^{-(u+u_<)H/\hbar}|a\rangle.\end{aligned} \tag{7.116}$$

As $u \to \infty$, the exponential $e^{-uH/\hbar}$ projects (7.73) states in onto the ground state $|0\rangle$ which is an eigenstate of H with energy E_0. So we replace the

arbitrary states in (7.116) with the ground state and use the path-integral formula (7.71 for the last three exponentials of (7.116)

$$e^{-2uE_0/\hbar}\langle 0|\mathcal{T}[q_e(u_1)q_e(u_2)]|0\rangle = \int \langle 0|q_b\rangle q(u_1)q(u_2)e^{-S_e[q]/\hbar}\langle q_a|0\rangle Dq. \tag{7.117}$$

The same equation without the time-ordered product is

$$e^{-2uE_0/\hbar}\langle 0|0\rangle = e^{-2uE_0/\hbar} = \int \langle 0|q_b\rangle e^{-S_e[q]/\hbar}\langle q_a|0\rangle Dq. \tag{7.118}$$

The ratio of the last two equations is

$$\langle 0|\mathcal{T}[q_e(u_1)q_e(u_2)]|0\rangle = \frac{\displaystyle\int \langle 0|q_b\rangle q(u_1)q(u_2)e^{-S_e[q]/\hbar}\langle q_a|0\rangle Dq}{\displaystyle\int \langle 0|q_b\rangle e^{-S_e[q]/\hbar}\langle q_a|0\rangle Dq} \tag{7.119}$$

in which the integration is over all paths from $u = -\infty$ to $u = \infty$. The mean value in the ground state of the time-ordered product of k euclidian position operators is

$$\langle 0|\mathcal{T}[q_e(u_1)\cdots q_e(u_k)]|0\rangle = \frac{\displaystyle\int \langle 0|q_b\rangle\, q(u_1)\cdots q(u_k)\, e^{-S_e[q]/\hbar}\langle q_a|0\rangle Dq}{\displaystyle\int \langle 0|q_b\rangle e^{-S_e[q]/\hbar}\langle q_a|0\rangle Dq}. \tag{7.120}$$

7.8 Quantum field theory

Quantum mechanics imposes upon n coordinates q_i and conjugate momenta p_k the equal-time commutation relations

$$[q_i, p_k] = i\,\hbar\,\delta_{i,k} \quad \text{and} \quad [q_i, q_k] = [p_i, p_k] = 0. \tag{7.121}$$

In the theory of a single spinless quantum field, a coordinate $q_{\boldsymbol{x}} \equiv \phi(\boldsymbol{x})$ and a conjugate momentum $p_{\boldsymbol{x}} \equiv \pi(\boldsymbol{x})$ are associated with each point $\boldsymbol{x}$ of space. The operators $\phi(\boldsymbol{x})$ and $\pi(\boldsymbol{x})$ obey the commutation relations

$$\begin{aligned}[\phi(\boldsymbol{x}), \pi(\boldsymbol{x}')] &= i\,\hbar\,\delta(\boldsymbol{x} - \boldsymbol{x}') \\ [\phi(\boldsymbol{x}), \phi(\boldsymbol{x}')] &= [\pi(\boldsymbol{x}), \pi(\boldsymbol{x}')] = 0\end{aligned} \tag{7.122}$$

inherited from quantum mechanics.

To make path integrals, we will replace space by a 3-dimensional lattice of

points $\boldsymbol{x} = a(i,j,k) = (ai, aj, ak)$ and eventually let the distance a between adjacent points go to zero. On this lattice and at equal times $t = 0$, the field operators obey discrete forms of the commutation relations (7.122)

$$\begin{aligned}
&[\phi(a(i,j,k)), \pi(a(\ell,m,n))] = i\,\frac{\hbar}{a^3}\,\delta_{i,\ell}\,\delta_{j,m}\,\delta_{k,n} \\
&[\phi(a(i,j,k)), \phi(a(\ell,m,n))] = [\pi(a(i,j,k)), \pi(a(\ell,m,n))] = 0.
\end{aligned} \tag{7.123}$$

The vanishing commutators imply that the field and the momenta have "simultaneous" eigenvalues

$$\phi(a(i,j,k))|\phi'\rangle = \phi'(a(i,j,k))|\phi'\rangle \text{ and } \pi(a(i,j,k))|\pi'\rangle = \pi'(a(i,j,k))|\pi'\rangle \tag{7.124}$$

for all lattice points $a(i,j,k)$. Their inner products are

$$\langle\phi'|\pi'\rangle = \prod_{i,j,k} \sqrt{\frac{a^3}{2\pi\hbar}}\; e^{ia^3\phi'(a(i,j,k))\pi'(a(i,j,k))/\hbar}. \tag{7.125}$$

These states are complete

$$\int |\phi'\rangle\langle\phi'| \prod_{i,j,k} d\phi'(a(i,j,k)) = I = \int |\pi'\rangle\langle\pi'| \prod_{i,j,k} d\pi'(a(i,j,k)) \tag{7.126}$$

and orthonormal

$$\langle\phi'|\phi''\rangle = \prod_{i,j,k} \delta(\phi'(a(i,j,k)) - \phi''(a(i,j,k))) \tag{7.127}$$

with a similar equation for $\langle\pi'|\pi''\rangle$.

The hamiltonian for a free field of mass m is

$$H = \tfrac{1}{2}\int \pi^2 + c^2(\nabla\phi)^2 + \frac{m^2c^4}{\hbar^2}\phi^2\, d^3x = \frac{a^3}{2}\sum_v \pi_v^2 + c^2(\nabla\phi_v)^2 + \frac{m^2c^4}{\hbar^2}\phi_v^2 \tag{7.128}$$

where $v = a(i,j,k)$, $\pi_v = \pi(a(i,j,k))$, $\phi_v = \phi(a(i,j,k))$, and the square of the lattice gradient $(\nabla\phi_v)^2$ is

$$\begin{aligned}
\big[(\phi(a(i+1,j,k)) - \phi(a(i,j,k)))^2 + (\phi(a(i,j+1,k)) - \phi(a(i,j,k)))^2 \\
+ (\phi(a(i,j,k+1)) - \phi(a(i,j,k)))^2\big]/a^2.
\end{aligned} \tag{7.129}$$

Other fields or terms, such as $c^3\phi^4/\hbar$, can be added to this hamiltonian.

To simplify the appearance of the equations in the rest of this chapter, I will mostly use **natural units** in which $\hbar = c = 1$. To convert the value of a physical quantity from natural units to universal units, one multiplies or divides its natural-unit value by suitable factors of $\hbar$ and c until one gets the right dimensions. For instance, if $V = 1/m$ is the value of a time in

natural units, where m is a mass, then the time you want is $T = \hbar/(mc^2)$. If $V = 1/m$ is supposed to be a length, then the needed length is $L = \hbar/(mc)$.

We set $K = a^3 \sum_v \pi_v^2/2$ and $V = (a^3/2)\sum_v (\nabla\phi_v)^2 + m^2\phi_v^2 + P(\phi_v)$ in which $P(\phi_v)$ represents the self-interactions of the field. With $\epsilon = (t_b - t_a)/n$, Trotter's product formula (7.6) is the $n \to \infty$ limit of

$$e^{-i(t_b-t_a)(K+V)} = \left(e^{-i(t_b-t_a)K/n} e^{-i(t_b-t_a)V/n}\right)^n = \left(e^{-i\epsilon K} e^{-i\epsilon V}\right)^n. \quad (7.130)$$

We insert I in the form (7.126) between $e^{-i\epsilon K}$ and $e^{-i\epsilon V}$

$$\langle\phi_1|e^{-i\epsilon K}\, e^{-i\epsilon V}|\phi_a\rangle = \langle\phi_1|e^{-i\epsilon K} \int |\pi'\rangle\langle\pi'| \prod_v d\pi'_v e^{-i\epsilon V}|\phi_a\rangle \quad (7.131)$$

and use the eigenstate formula (7.124)

$$\langle\phi_1|e^{-i\epsilon K}\, e^{-i\epsilon V}|\phi_a\rangle = e^{-i\epsilon V(\phi_a)} \int e^{-i\epsilon K(\pi')} \langle\phi_1|\pi'\rangle\langle\pi'|\phi_a\rangle \prod_v d\pi'_v. \quad (7.132)$$

The inner product formula (7.125) now gives

$$\langle\phi_1|e^{-i\epsilon K}\, e^{-i\epsilon V}|\phi_a\rangle = e^{-i\epsilon V(\phi_a)} \prod_v \left[\int \frac{a^3 d\pi'_v}{2\pi} e^{a^3[-i\epsilon\pi_v^2/2 + i(\phi_{1v}-\phi_{av})\pi'_v]}\right]. \quad (7.133)$$

We again adopt the suggestive notation $\dot\phi_a = (\phi_1 - \phi_a)/\epsilon$ and use the gaussian integral (7.1) to find

$$\langle\phi_1|e^{-i\epsilon K}\, e^{-i\epsilon V}|\phi_a\rangle = \prod_v \left[\left(\frac{a^3}{2\pi i\epsilon}\right)^{1/2} e^{i\epsilon a^3[\dot\phi_{av}^2 - (\nabla\phi_{av})^2 - m^2\phi_{av}^2 - P(\phi_v)]/2}\right]. \quad (7.134)$$

The product of $n = (t_b - t_a)/\epsilon$ such time intervals is

$$\langle\phi_b|e^{-i(t_b-t_a)H}|\phi_a\rangle = \prod_v \left[\left(\frac{a^3 n}{2\pi i(t_b - t_a)}\right)^{n/2} \int e^{iS_v} D\phi_v\right] \quad (7.135)$$

in which

$$S_v = \frac{t_b - t_a}{n}\frac{a^3}{2}\sum_{j=0}^{n-1}\left[\dot\phi_{jv}^2 - (\nabla\phi_{jv})^2 - m^2\phi_{jv}^2 - P(\phi_v)\right] \quad (7.136)$$

$\dot\phi_{jv} = n(\phi_{j+1,v} - \phi_{j,v})/(t_b - t_a)$, and $D\phi_v = d\phi_{n-1,v}\cdots d\phi_{1,v}$.

The amplitude $\langle\phi_b|e^{-i(t_b-t_a)H}|\phi_a\rangle$ is the integral over all fields that go from $\phi_a(\boldsymbol{x})$ at t_a to $\phi_b(\boldsymbol{x})$ at t_b each weighted by an exponential

$$\langle\phi_b|e^{-i(t_b-t_a)H}|\phi_a\rangle = \int e^{iS[\phi]}\, D\phi \quad (7.137)$$

of its action

$$S[\phi] = \int_{t_a}^{t_b} dt \int d^3x \, \frac{1}{2} \left[\dot{\phi}^2 - (\nabla\phi)^2 - m^2\phi^2 - P(\phi) \right] \tag{7.138}$$

in which $D\phi$ is the $n \to \infty$ limit of the product over all spatial vertices v

$$D\phi = \prod_v \left[\left(\frac{a^3 n}{2\pi i (t_b - t_a)} \right)^{n/2} d\phi_{n-1,v} \cdots d\phi_{1,v} \right]. \tag{7.139}$$

Equivalently, the time-evolution operator is

$$e^{-i(t_b - t_a)H} = \int |\phi_b\rangle e^{iS[\phi]} \langle \phi_a| \, D\phi \, D\phi_a D\phi_b \tag{7.140}$$

in which $D\phi_a D\phi_b = \prod_v d\phi_{a,v} d\phi_{b,v}$ is an integral over the initial and final states.

As in quantum mechanics (section 7.4), the path integral for an action that is quadratic in the fields is an exponential of the action of a stationary process times a function of the times and of the other parameters in the action

$$\langle \phi_b | e^{-i(t_b - t_a)H} | \phi_a \rangle = \int e^{iS[\phi]} \, D\phi = f(t_a, t_b, \dots) \, e^{iS[\phi_c]} \tag{7.141}$$

in which $S[\phi_c]$ is the action of the process that goes from $\phi(\boldsymbol{x}, t_a) = \phi_a(\boldsymbol{x})$ to $\phi(\boldsymbol{x}, t_b) = \phi_b(\boldsymbol{x})$ and obeys the classical equations of motion, and the function f is a path integral over all fields that go from $\phi(\boldsymbol{x}, t_a) = 0$ to $\phi(\boldsymbol{x}, t_b) = 0$.

Example 7.8 (A stationary process) The field

$$\phi(\boldsymbol{x}, t) = \int e^{i\boldsymbol{k}\cdot\boldsymbol{x}} [a(\boldsymbol{k}) \cos\omega t + b(\boldsymbol{k}) \sin\omega t] \, d^3k \tag{7.142}$$

with $\omega = \sqrt{\boldsymbol{k}^2 + m^2}$ makes the action (7.138) for $P = 0$ stationary because it is a solution of the equation of motion $\nabla^2\phi - \ddot{\phi} - m^2\phi = 0$. In terms of the Fourier transforms

$$\tilde{\phi}(\boldsymbol{k}, t_a) = \int e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} \, \phi(\boldsymbol{x}, t_a) \, \frac{d^3x}{(2\pi)^3} \quad \text{and} \quad \tilde{\phi}(\boldsymbol{k}, t_b) = \int e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} \, \phi(\boldsymbol{x}, t_b) \, \frac{d^3x}{(2\pi)^3}, \tag{7.143}$$

the solution that goes from $\phi(\boldsymbol{x}, t_a)$ to $\phi(\boldsymbol{x}, t_b)$ is

$$\phi(\boldsymbol{x}, t) = \int e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \, \frac{\sin\omega(t_b - t) \, \tilde{\phi}(\boldsymbol{k}, t_a) + \sin\omega(t - t_a) \, \tilde{\phi}(\boldsymbol{k}, t_b)}{\sin\omega(t_b - t_a)} \, d^3k. \tag{7.144}$$

The solution that evolves from $\phi(\boldsymbol{x}, t_a)$ and $\dot{\phi}(\boldsymbol{x}, t_a)$ is

$$\phi(\boldsymbol{x}, t) = \int e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \left[\cos\omega(t - t_a)\, \tilde{\phi}(\boldsymbol{k}, t_a) + \frac{\sin\omega(t - t_a)}{\omega}\, \tilde{\dot{\phi}}(\boldsymbol{k}, t_a)\right] d^3k \tag{7.145}$$

in which the Fourier transform $\tilde{\dot{\phi}}(\boldsymbol{k}, t_a)$ is defined as in (7.143). □

Like a position operator (7.106), a field at time t is defined as

$$\phi(\boldsymbol{x}, t) = e^{itH/\hbar}\phi(\boldsymbol{x})e^{-itH/\hbar} \tag{7.146}$$

in which $\phi(\boldsymbol{x}) = \phi(\boldsymbol{x}, 0)$ is the field at time zero, which obeys the commutation relations (7.122). The time-ordered product of several fields is their product with newer (later time) fields standing to the left of older (earlier time) fields as in the definition (7.108). The logic (7.110–7.115) of the derivation of the path-formulas for time-ordered products of position operators applies directly to field operators. One finds (exercise **??**) for the mean value of the time-ordered product of two fields in an energy eigenstate $|n\rangle$

$$\langle n|\mathcal{T}[\phi(x_1)\phi(x_2)]|n\rangle = \frac{\displaystyle\int \langle n|\phi_b\rangle\phi(x_1)\phi(x_2)e^{iS[\phi]/\hbar}\langle\phi_a|n\rangle\, D\phi}{\displaystyle\int \langle n|\phi_b\rangle e^{iS[\phi]/\hbar}\langle\phi_a|n\rangle\, D\phi} \tag{7.147}$$

in which the integrations are over all paths that go from before t_1 and t_2 to after both times. The analogous result for several fields is (exercise **??**)

$$\langle n|\mathcal{T}[\phi(x_1)\cdots\phi(x_k)]|n\rangle = \frac{\displaystyle\int \langle n|\phi_b\rangle\phi(x_1)\cdots\phi(x_k)e^{iS[\phi]/\hbar}\langle\phi_a|n\rangle\, D\phi}{\displaystyle\int \langle n|\phi_b\rangle e^{iS[\phi]/\hbar}\langle\phi_a|n\rangle\, D\phi} \tag{7.148}$$

in which the integrations are over all paths that go from before the times $t_1, \ldots, t_k$ to after them.

7.9 Finite-temperature field theory

Since the Boltzmann operator $e^{-\beta H} = e^{-H/(kT)}$ is the time evolution operator $e^{-itH/\hbar}$ at the imaginary time $t = -i\hbar\beta = -i\hbar/(kT)$, the formulas of finite-temperature field theory are those of quantum field theory with t replaced by $-iu = -i\hbar\beta = -i\hbar/(kT)$.

If as in section 7.8, we use as our hamiltonian $H = K + V$ where K and V are sums over all lattice vertices $v = a(i, j, k) = (ai, aj, ak)$ of the cubes of volume a^3 times the squared momentum and potential terms

$$H = K + V = \frac{a^3}{2} \sum_v \pi_v^2 + \frac{a^3}{2} \sum_v (\nabla \phi_v)^2 + m^2 \phi_v^2 + P(\phi_v). \tag{7.149}$$

A matrix element of the first term of the Trotter product formula (7.7)

$$e^{-\beta(K+V)} = \lim_{n\to\infty} \left(e^{-\beta K/n}\, e^{-\beta V/n} \right)^n \tag{7.150}$$

is the imaginary-time version of (7.133) with $\epsilon = \hbar\beta/n$

$$\langle \phi_1 | e^{-\epsilon K}\, e^{-\epsilon V} | \phi_a \rangle = e^{-\epsilon V(\phi_a)} \prod_v \left[\int \frac{a^3 d\pi'_v}{2\pi} e^{a^3[-\epsilon \pi_v^2/2 + i(\phi_{1v} - \phi_{av})\pi'_v]} \right]. \tag{7.151}$$

Setting $\dot\phi_{av} = (\phi_{1v} - \phi_{av})/\epsilon$, we find, instead of (7.134)

$$\langle \phi_1 | e^{-\epsilon K}\, e^{-\epsilon V} | \phi_a \rangle = \prod_v \left[\left(\frac{a^3}{2\pi\epsilon} \right)^{1/2} e^{-\epsilon a^3 [\dot\phi_{av}^2 + (\nabla\phi_{av})^2 + m^2\phi_{av}^2 + P(\phi_v)]/2} \right]. \tag{7.152}$$

The product of $n = \hbar\beta/\epsilon$ such inverse-temperature intervals is

$$\langle \phi_b | e^{-\beta H} | \phi_a \rangle = \prod_v \left[\left(\frac{a^3 n}{2\pi\beta} \right)^{n/2} \int e^{-S_{ev}} D\phi_v \right] \tag{7.153}$$

in which the euclidian action is

$$S_{ev} = \frac{\beta}{n} \frac{a^3}{2} \sum_{j=0}^{n-1} \left[\dot\phi_{jv}^2 + (\nabla\phi_{jv})^2 + m^2 \phi_{jv}^2 + P(\phi_v) \right] \tag{7.154}$$

$\dot\phi_{jv} = n(\phi_{j+1,v} - \phi_{j,v})/\beta$, and $D\phi_v = d\phi_{n-1,v} \cdots d\phi_{1,v}$.

The amplitude $\langle \phi_b | e^{-(\beta_b - \beta_a)H} | \phi_a \rangle$ is the integral over all fields that go from $\phi_a(\boldsymbol{x})$ at β_a to $\phi_b(\boldsymbol{x})$ at β_b each weighted by an exponential

$$\langle \phi_b | e^{-(\beta_b - \beta_a)H} | \phi_a \rangle = \int e^{-S_e[\phi]}\, D\phi \tag{7.155}$$

of its euclidian action

$$S_e[\phi] = \int_{\beta_a}^{\beta_b} du \int d^3x\, \frac{1}{2} \left[\dot\phi^2 + (\nabla\phi)^2 + m^2\phi^2 + P(\phi) \right] \tag{7.156}$$

in which $D\phi$ is the $n \to \infty$ limit of the product over all spatial vertices v

$$D\phi = \prod_v \left[\left(\frac{a^3 n}{2\pi(\beta_b - \beta_a)} \right)^{n/2} d\phi_{n-1,v} \cdots d\phi_{1,v} \right]. \tag{7.157}$$

Equivalently, the Boltzmann operator is

$$e^{-(\beta_b - \beta_a)H} = \int |\phi_b\rangle e^{-S_e[\phi]} \langle \phi_a| \, D\phi \, D\phi_a D\phi_b \tag{7.158}$$

in which $D\phi_a D\phi_b = \prod_v d\phi_{a,v} d\phi_{b,v}$ is an integral over the initial and final states.

The trace of the Boltzmann operator is the partition function

$$Z(\beta) = \mathrm{Tr}(e^{-\beta H}) = \int e^{-S_e[\phi]} \langle \phi_a | \phi_b \rangle \, D\phi \, D\phi_a D\phi_b = \int e^{-S_e[\phi]} \, D\phi \, D\phi_a \tag{7.159}$$

which is an integral over all fields that go back to themselves in euclidian time β.

Like a position operator (7.107), a field at an imaginary time $t = -iu = -i\hbar\beta$ is defined as

$$\phi_e(\boldsymbol{x}, u) = \phi_e(\boldsymbol{x}, \hbar\beta) = e^{uH/\hbar} \phi(\boldsymbol{x}) \, e^{-uH/\hbar}. \tag{7.160}$$

in which $\phi(\boldsymbol{x}) = \phi(\boldsymbol{x}, 0) = \phi_e(\boldsymbol{x}, 0)$ is the field at time zero, which obeys the commutation relations (7.122). The euclidian-time-ordered product of several fields is their product with newer (higher $u = \hbar\beta$) fields standing to the left of older (lower $u = \hbar\beta$) fields as in the definition (7.109).

The euclidian path integrals for the mean values of euclidian-time-ordered-products of fields are similar to those (7.161 & 7.148) for ordinary time-ordered-products. The euclidian-time-ordered-product of the fields $\phi(x_j) = \phi(\boldsymbol{x}_j, u_j)$ is the path integral

$$\langle n | \mathcal{T}[\phi_e(x_1)\phi_e(x_2)] | n \rangle = \frac{\int \langle n | \phi_b \rangle \phi(x_1)\phi(x_2) e^{-S_e[\phi]/\hbar} \langle \phi_a | n \rangle \, D\phi}{\int \langle n | \phi_b \rangle e^{-S_e[\phi]/\hbar} \langle \phi_a | n \rangle \, D\phi} \tag{7.161}$$

in which the integrations are over all paths that go from before u_1 and u_2 to after both euclidian times. The analogous result for several fields is

$$\langle n | \mathcal{T}[\phi_e(x_1) \cdots \phi_e(x_k)] | n \rangle = \frac{\int \langle n | \phi_b \rangle \phi(x_1) \cdots \phi(x_k) e^{-S_e[\phi]/\hbar} \langle \phi_a | n \rangle \, D\phi}{\int \langle n | \phi_b \rangle e^{-S_e[\phi]/\hbar} \langle \phi_a | n \rangle \, D\phi} \tag{7.162}$$

in which the integrations are over all paths that go from before the times $u_1, \ldots, u_k$ to after them.

A distinctive feature of these formulas is that in the low-temperature $\beta = 1/(kT) \to \infty$ limit, the Boltzmann operator is a multiple of an outer product $|0\rangle\langle 0|$ of the ground-state kets, $e^{-\beta H} \to e^{-\beta E_0}|0\rangle\langle 0|$. In this limit, the integrations are over all fields that run from $u = -\infty$ to $u = \infty$ and the energy eigenstates are the ground state of the theory

$$\langle 0|\mathcal{T}[\phi_e(x_1)\cdots\phi_e(x_k)]|0\rangle = \frac{\displaystyle\int \langle 0|\phi_b\rangle\phi(x_1)\cdots\phi(x_k)e^{-S_e[q]/\hbar}\langle\phi_a|0\rangle\, D\phi}{\displaystyle\int \langle 0|\phi_b\rangle e^{-S_e[q]/\hbar}\langle\phi_a|0\rangle\, D\phi}. \tag{7.163}$$

Formulas like this one are used in lattice gauge theory.

7.10 Perturbation theory

Field theories with hamiltonians that are quadratic in their fields like

$$H_0 = \int \tfrac{1}{2}\left[\pi^2(x) + (\nabla\phi(x))^2 + m^2\phi^2(x)\right] d^3x \tag{7.164}$$

are soluble. Their fields evolve in time as

$$\phi(\boldsymbol{x}, t) = e^{itH_0}\phi(\boldsymbol{x}, 0)e^{-itH_0}. \tag{7.165}$$

The mean value in the ground state of H_0 of a time-ordered product of these fields is a ratio (7.148) of path integrals

$$\langle 0|\mathcal{T}\left[\phi(x_1)\ldots\phi(x_k)\right]|0\rangle = \frac{\displaystyle\int \langle 0|\phi_b\rangle\, \phi(x_1)\ldots\phi(x_n)\, e^{iS_0[\phi]}\langle\phi_a|0\rangle\, D\phi}{\displaystyle\int \langle 0|\phi_b\rangle\, e^{iS_0[\phi]}\langle\phi_a|0\rangle\, D\phi} \tag{7.166}$$

in which the action $S_0[\phi]$ is quadratic in the field ϕ

$$\begin{aligned} S_0[\phi] &= \int \tfrac{1}{2}\left[\dot\phi^2(x) - (\nabla\phi(x))^2 - m^2\phi^2(x)\right] d^4x \\ &= \int \tfrac{1}{2}\left[-\partial_a\phi(x)\partial^a\phi(x) - m^2\phi^2(x)\right] d^4x \end{aligned} \tag{7.167}$$

and the integrations are over all fields that run from ϕ_a at a time before the times $t_1, \ldots, t_k$ to ϕ_b at a time after $t_1, \ldots, t_k$. The path integrals in the ratio (7.166) are gaussian and doable.

The Fourier transforms

$$\tilde{\phi}(p) = \int e^{-ipx} \phi(x)\, d^4x \quad \text{and} \quad \phi(x) = \int e^{ipx} \tilde{\phi}(p)\, \frac{d^4p}{(2\pi)^4} \tag{7.168}$$

turn the spacetime derivatives in the action into a quadratic form

$$S_0[\phi] = -\tfrac{1}{2} \int |\tilde{\phi}(p)|^2\, (p^2 + m^2)\, \frac{d^4p}{(2\pi)^4} \tag{7.169}$$

in which $p^2 = \boldsymbol{p}^2 - p^{02}$ and $\tilde{\phi}(-p) = \tilde{\phi}^*(p)$ by (**??**) since the field ϕ is real.

The initial $\langle \phi_a | 0 \rangle$ and final $\langle 0 | \phi_b \rangle$ wave functions produce the $i\epsilon$ in the Feynman propagator (2.24). Although its exact form doesn't matter here, the wave function $\langle \phi | 0 \rangle$ of the ground state of H_0 is the exponential (**??**)

$$\langle \phi | 0 \rangle = c \exp \left[-\frac{1}{2} \int |\tilde{\phi}(\boldsymbol{p})|^2 \sqrt{\boldsymbol{p}^2 + m^2}\, \frac{d^3p}{(2\pi)^3} \right] \tag{7.170}$$

in which $\tilde{\phi}(\boldsymbol{p})$ is the spatial Fourier transform of the eigenvalue $\phi(\boldsymbol{x})$

$$\tilde{\phi}(\boldsymbol{p}) = \int e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}\, \phi(\boldsymbol{x})\, d^3x \tag{7.171}$$

and c is a normalization factor that will cancel in ratios of path integrals.

Apart from $-2i \ln c$ which we will not keep track of, the wave functions $\langle \phi_a | 0 \rangle$ and $\langle 0 | \phi_b \rangle$ add to the action $S_0[\phi]$ the term

$$\Delta S_0[\phi] = \frac{i}{2} \int \sqrt{\boldsymbol{p}^2 + m^2} \left(|\tilde{\phi}(\boldsymbol{p}, t)|^2 + |\tilde{\phi}(\boldsymbol{p}, -t)|^2 \right) \frac{d^3p}{(2\pi)^3} \tag{7.172}$$

in which we envision taking the limit $t \to \infty$ with $\phi(\boldsymbol{x}, t) = \phi_b(\boldsymbol{x})$ and $\phi(\boldsymbol{x}, -t) = \phi_a(\boldsymbol{x})$. The identity (Weinberg, 1995, pp. 386–388)

$$f(+\infty) + f(-\infty) = \lim_{\epsilon \to 0+} \epsilon \int_{-\infty}^{\infty} f(t)\, e^{-\epsilon|t|}\, dt \tag{7.173}$$

(exercise **??**) allows us to write $\Delta S_0[\phi]$ as

$$\Delta S_0[\phi] = \lim_{\epsilon \to 0+} \frac{i\epsilon}{2} \int \sqrt{\boldsymbol{p}^2 + m^2} \int_{-\infty}^{\infty} |\tilde{\phi}(\boldsymbol{p}, t)|^2\, e^{-\epsilon|t|}\, dt\, \frac{d^3p}{(2\pi)^3}. \tag{7.174}$$

To first order in ϵ, the change in the action is (exercise **??**)

$$\begin{aligned} \Delta S_0[\phi] &= \lim_{\epsilon \to 0+} \frac{i\epsilon}{2} \int \sqrt{\boldsymbol{p}^2 + m^2} \int_{-\infty}^{\infty} |\tilde{\phi}(\boldsymbol{p}, t)|^2\, dt\, \frac{d^3p}{(2\pi)^3} \\ &= \lim_{\epsilon \to 0+} \frac{i\epsilon}{2} \int \sqrt{\boldsymbol{p}^2 + m^2}\, |\tilde{\phi}(p)|^2\, \frac{d^4p}{(2\pi)^4}. \end{aligned} \tag{7.175}$$

Thus the modified action is

$$S_0[\phi,\epsilon] = S_0[\phi] + \Delta S_0[\phi] = -\frac{1}{2}\int |\tilde{\phi}(p)|^2 \left(p^2 + m^2 - i\epsilon\sqrt{\boldsymbol{p}^2 + m^2}\right) \frac{d^4p}{(2\pi)^4}$$
$$= -\frac{1}{2}\int |\tilde{\phi}(p)|^2 \left(p^2 + m^2 - i\epsilon\right) \frac{d^4p}{(2\pi)^4} \tag{7.176}$$

since the square root is positive. In terms of the modified action, our formula (7.166) for the time-ordered product is the ratio

$$\langle 0|\mathcal{T}\left[\phi(x_1)\ldots\phi(x_n)\right]|0\rangle = \frac{\int \phi(x_1)\ldots\phi(x_n)\, e^{iS_0[\phi,\epsilon]}\, D\phi}{\int e^{iS_0[\phi,\epsilon]}\, D\phi}. \tag{7.177}$$

We can use this formula (7.177) to express the mean value in the vacuum $|0\rangle$ of the time-ordered exponential of a spacetime integral of $j(x)\phi(x)$, in which $j(x)$ is a classical (c-number, external) current, as the ratio

$$Z_0[j] \equiv \langle 0|\,\mathcal{T}\left\{\exp\left[i\int j(x)\,\phi(x)\,d^4x\right]\right\}|0\rangle$$
$$= \frac{\int \exp\left[i\int j(x)\,\phi(x)\,d^4x\right] e^{iS_0[\phi,\epsilon]}\, D\phi}{\int e^{iS_0[\phi,\epsilon]}\, D\phi}. \tag{7.178}$$

Since the state $|0\rangle$ is normalized, the mean value $Z_0[0]$ is unity,

$$Z_0[0] = 1. \tag{7.179}$$

If we absorb the current into the action

$$S_0[\phi,\epsilon,j] = S_0[\phi,\epsilon] + \int j(x)\,\phi(x)\,d^4x \tag{7.180}$$

then in terms of the current's Fourier transform

$$\tilde{j}(p) = \int e^{-ipx}\, j(x)\, d^4x \tag{7.181}$$

the modified action $S_0[\phi,\epsilon,j]$ is (exercise **??**)

$$S_0[\phi,\epsilon,j] = -\tfrac{1}{2}\int \left[|\tilde{\phi}(p)|^2\left(p^2 + m^2 - i\epsilon\right) - \tilde{j}^*(p)\tilde{\phi}(p) - \tilde{\phi}^*(p)\tilde{j}(p)\right] \frac{d^4p}{(2\pi)^4}. \tag{7.182}$$

Changing variables to

$$\tilde{\psi}(p) = \tilde{\phi}(p) - \tilde{j}(p)/(p^2 + m^2 - i\epsilon) \tag{7.183}$$

we write the action $S_0[\phi, \epsilon, j]$ as (exercise **??**)

$$\begin{aligned} S_0[\phi, \epsilon, j] = & -\tfrac{1}{2} \int \left[|\tilde{\psi}(p)|^2 \left(p^2 + m^2 - i\epsilon\right) - \frac{\tilde{j}^*(p)\tilde{j}(p)}{(p^2 + m^2 - i\epsilon)} \right] \frac{d^4p}{(2\pi)^4} \\ = & \, S_0[\psi, \epsilon] + \tfrac{1}{2} \int \left[\frac{\tilde{j}^*(p)\tilde{j}(p)}{(p^2 + m^2 - i\epsilon)} \right] \frac{d^4p}{(2\pi)^4}. \end{aligned} \tag{7.184}$$

And since $D\phi = D\psi$, our formula (7.178) gives simply (exercise **??**)

$$Z_0[j] = \exp\left(\frac{i}{2} \int \frac{|\tilde{j}(p)|^2}{p^2 + m^2 - i\epsilon} \frac{d^4p}{(2\pi)^4} \right). \tag{7.185}$$

Going back to position space, one finds (exercise **??**)

$$Z_0[j] = \exp\left[\frac{i}{2} \int j(x)\, \Delta(x - x')\, j(x')\, d^4x\, d^4x' \right] \tag{7.186}$$

in which $\Delta(x - x')$ is Feynman's **propagator** (2.24)

$$\Delta(x - x') = \int \frac{e^{ip(x-x')}}{p^2 + m^2 - i\epsilon} \frac{d^4p}{(2\pi)^4}. \tag{7.187}$$

The functional derivative (chapter **??**) of $Z_0[j]$, defined by (7.178), is

$$\frac{1}{i} \frac{\delta Z_0[j]}{\delta j(x)} = \langle 0| \mathcal{T} \left[\phi(x) \exp\left(i \int j(x')\phi(x') d^4x' \right) \right] |0\rangle \tag{7.188}$$

while that of equation (7.186) is

$$\frac{1}{i} \frac{\delta Z_0[j]}{\delta j(x)} = Z_0[j] \int \Delta(x - x')\, j(x')\, d^4x'. \tag{7.189}$$

Thus the second functional derivative of $Z_0[j]$ evaluated at $j = 0$ gives

$$\langle 0| \mathcal{T} \left[\phi(x)\phi(x') \right] |0\rangle = \frac{1}{i^2} \frac{\delta^2 Z_0[j]}{\delta j(x) \delta j(x')} \bigg|_{j=0} = -i\, \Delta(x - x'). \tag{7.190}$$

Similarly, one may show (exercise **??**) that

$$\begin{aligned} \langle 0| \mathcal{T} \left[\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) \right] |0\rangle = & \frac{1}{i^4} \frac{\delta^4 Z_0[j]}{\delta j(x_1)\delta j(x_2)\delta j(x_3)\delta j(x_4)} \bigg|_{j=0} \\ = & -\Delta(x_1 - x_2)\Delta(x_3 - x_4) - \Delta(x_1 - x_3)\Delta(x_2 - x_4) \\ & - \Delta(x_1 - x_4)\Delta(x_2 - x_3). \end{aligned} \tag{7.191}$$

Suppose now that we add a potential $V = P(\phi)$ to the free hamiltonian (7.164). Scattering amplitudes are matrix elements of the time-ordered exponential $\mathcal{T} \exp\left[-i \int P(\phi)\, d^4x \right]$ (Weinberg, 1995, p. 260) Our formula (7.177)

for the mean value in the ground state $|0\rangle$ of the free hamiltonian H_0 of any time-ordered product of fields leads us to

$$\langle 0|\mathcal{T}\left\{\exp\left[-i\int P(\phi)\,d^4x\right]\right\}|0\rangle = \frac{\displaystyle\int \exp\left[-i\int P(\phi)\,d^4x\right]\,e^{iS_0[\phi,\epsilon]}\,D\phi}{\displaystyle\int e^{iS_0[\phi,\epsilon]}\,D\phi}. \tag{7.192}$$

Using (7.190 & 7.191), we can cast this expression into the magical form

$$\langle 0|\mathcal{T}\left\{\exp\left[-i\int P(\phi)\,d^4x\right]\right\}|0\rangle = \exp\left[-i\int P\left(\frac{\delta}{i\delta j(x)}\right)\,d^4x\right]Z_0[j]\Big|_{j=0}. \tag{7.193}$$

The generalization of the path-integral formula (7.177) to the ground state $|\Omega\rangle$ of an interacting theory with action S is

$$\langle\Omega|\mathcal{T}\left[\phi(x_1)\ldots\phi(x_n)\right]|\Omega\rangle = \frac{\displaystyle\int \phi(x_1)\ldots\phi(x_n)\,e^{iS[\phi,\epsilon]}\,D\phi}{\displaystyle\int e^{iS[\phi,\epsilon]}\,D\phi} \tag{7.194}$$

in which a term like $i\epsilon\phi^2$ is added to make the modified action $S[\phi,\epsilon]$.

These are some of the techniques one uses to make states of incoming and outgoing particles and to compute scattering amplitudes (Weinberg, 1995, 1996; Srednicki, 2007; Zee, 2010).

7.11 Application to quantum electrodynamics

In the Coulomb gauge $\boldsymbol{\nabla}\cdot\boldsymbol{A}=0$, the QED hamiltonian is

$$H = H_m + \int\left[\tfrac{1}{2}\boldsymbol{\pi}^2 + \tfrac{1}{2}(\boldsymbol{\nabla}\times\boldsymbol{A})^2 - \boldsymbol{A}\cdot\boldsymbol{j}\right]\,d^3x + V_C \tag{7.195}$$

in which H_m is the matter hamiltonian, and V_C is the Coulomb term

$$V_C = \frac{1}{2}\int\frac{j^0(\boldsymbol{x},t)\,j^0(\boldsymbol{y},t)}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}\,d^3x\,d^3y. \tag{7.196}$$

The operators $\boldsymbol{A}$ and $\boldsymbol{\pi}$ are canonically conjugate, but they satisfy the Coulomb-gauge conditions

$$\boldsymbol{\nabla}\cdot\boldsymbol{A}=0 \quad\text{and}\quad \boldsymbol{\nabla}\cdot\boldsymbol{\pi}=0. \tag{7.197}$$

One may show (Weinberg, 1995, pp. 413–418) that in this theory, the

analog of equation (7.194) is

$$\langle\Omega|\mathcal{T}\left[\mathcal{O}_1\ldots\mathcal{O}_n\right]|\Omega\rangle = \frac{\displaystyle\int \mathcal{O}_1\ldots\mathcal{O}_n\, e^{iS_C}\,\delta[\boldsymbol{\nabla}\cdot\boldsymbol{A}]\,D\boldsymbol{A}\,D\psi}{\displaystyle\int e^{iS_C}\,\delta[\boldsymbol{\nabla}\cdot\boldsymbol{A}]\,D\boldsymbol{A}\,D\psi} \tag{7.198}$$

in which the Coulomb-gauge action is

$$S_C = \int \tfrac{1}{2}\dot{\boldsymbol{A}}^2 - \tfrac{1}{2}(\boldsymbol{\nabla}\times\boldsymbol{A})^2 + \boldsymbol{A}\cdot\boldsymbol{j} + \mathcal{L}_m\; d^4x \;-\int V_C\,dt \tag{7.199}$$

and the functional delta function

$$\delta[\boldsymbol{\nabla}\cdot\boldsymbol{A}] = \prod_x \delta(\boldsymbol{\nabla}\cdot\boldsymbol{A}(x)) \tag{7.200}$$

enforces the Coulomb-gauge condition. The term $\mathcal{L}_m$ is the action density of the matter field ψ.

Tricks are available. We introduce a new field $A^0(x)$ and consider the factor

$$F = \int \exp\left[i\int \frac{1}{2}\left(\boldsymbol{\nabla}A^0 + \boldsymbol{\nabla}\triangle^{-1}j^0\right)^2\, d^4x\right]\, DA^0 \tag{7.201}$$

which is just a *number* independent of the charge density j^0 since we can cancel the j^0 term by shifting A^0. By $\triangle^{-1}$, we mean $-1/4\pi|\boldsymbol{x}-\boldsymbol{y}|$. By integrating by parts, we can write the number F as (exercise **??**)

$$\begin{aligned} F &= \int \exp\left[i\int \tfrac{1}{2}\left(\boldsymbol{\nabla}A^0\right)^2 - A^0j^0 - \tfrac{1}{2}\,j^0\triangle^{-1}j^0\; d^4x\right]\, DA^0 \\ &= \int \exp\left[i\int \tfrac{1}{2}\left(\boldsymbol{\nabla}A^0\right)^2 - A^0j^0\; d^4x + i\int V_C\,dt\right]\, DA^0. \end{aligned} \tag{7.202}$$

So when we multiply the numerator and denominator of the amplitude (7.198) by F, the awkward Coulomb term V_C cancels, and we get

$$\langle\Omega|\mathcal{T}\left[\mathcal{O}_1\ldots\mathcal{O}_n\right]|\Omega\rangle = \frac{\displaystyle\int \mathcal{O}_1\ldots\mathcal{O}_n\, e^{iS'}\,\delta[\boldsymbol{\nabla}\cdot\boldsymbol{A}]\,DA\,D\psi}{\displaystyle\int e^{iS'}\,\delta[\boldsymbol{\nabla}\cdot\boldsymbol{A}]\,DA\,D\psi} \tag{7.203}$$

where now DA includes all four components A^μ and

$$S' = \int \tfrac{1}{2}\,\dot{\boldsymbol{A}}^2 - \tfrac{1}{2}\,(\boldsymbol{\nabla}\times\boldsymbol{A})^2 + \tfrac{1}{2}\left(\boldsymbol{\nabla}A^0\right)^2 + \boldsymbol{A}\cdot\boldsymbol{j} - A^0j^0 + \mathcal{L}_m\,d^4x. \tag{7.204}$$

Since the delta-functional $\delta[\boldsymbol{\nabla}\cdot\boldsymbol{A}]$ enforces the Coulomb-gauge condition, we can add to the action S' the term $(\boldsymbol{\nabla}\cdot\dot{\boldsymbol{A}})\,A^0$ which is $-\dot{\boldsymbol{A}}\cdot\boldsymbol{\nabla}A^0$ after

we integrate by parts and drop the surface term. This extra term makes the action gauge invariant

$$\begin{aligned} S &= \int \tfrac{1}{2}\,(\dot{\boldsymbol{A}} - \boldsymbol{\nabla} \boldsymbol{A}^{\mathbf{0}})^2 - \tfrac{1}{2}\,(\boldsymbol{\nabla} \times \boldsymbol{A})^2 + \boldsymbol{A} \cdot \boldsymbol{j} - A^0 j^0 + \mathcal{L}_m \, d^4x \\ &= \int -\tfrac{1}{4}\, F_{ab}\, F^{ab} + A^b j_b + \mathcal{L}_m \; d^4x. \end{aligned} \tag{7.205}$$

Thus at this point we have

$$\langle \Omega | \mathcal{T}\,[\mathcal{O}_1 \ldots \mathcal{O}_n]\,|\Omega\rangle = \frac{\displaystyle\int \mathcal{O}_1 \ldots \mathcal{O}_n \, e^{iS}\, \delta[\boldsymbol{\nabla} \cdot \boldsymbol{A}]\, DA\, D\psi}{\displaystyle\int e^{iS}\, \delta[\boldsymbol{\nabla} \cdot \boldsymbol{A}]\, DA\, D\psi} \tag{7.206}$$

in which S is the gauge-invariant action (7.205), and the integral is over all fields. The only relic of the Coulomb gauge is the gauge-fixing delta functional $\delta[\boldsymbol{\nabla} \cdot \boldsymbol{A}]$.

We now make the gauge transformation

$$A'_b(x) = A_b(x) + \partial_b \Lambda(x) \quad \text{and} \quad \psi'(x) = e^{iq\Lambda(x)} \psi(x) \tag{7.207}$$

in the numerator and also, using a different gauge transformation Λ', in the denominator of the ratio (7.206) of path integrals. Since we are integrating over all gauge fields, these gauge transformations merely change the order of integration in the numerator and denominator of that ratio. They are like replacing $\int_{-\infty}^{\infty} f(x)\, dx$ by $\int_{-\infty}^{\infty} f(y)\, dy$. They change nothing, and so

$$\langle \Omega | \mathcal{T}\,[\mathcal{O}_1 \ldots \mathcal{O}_n]\,|\Omega\rangle = \langle \Omega | \mathcal{T}\,[\mathcal{O}_1 \ldots \mathcal{O}_n]\,|\Omega\rangle' \tag{7.208}$$

in which the prime refers to the gauge transformations (7.207) Λ and Λ'.

We've seen that the action S is gauge invariant. So is the measure $DA\, D\psi$. We now restrict ourselves to operators $\mathcal{O}_1 \ldots \mathcal{O}_n$ that are *gauge invariant*. So in the right-hand side of equation (7.208), the replacement of the fields by their gauge transforms affects only the term $\delta[\boldsymbol{\nabla} \cdot \boldsymbol{A}]$ that enforces the Coulomb-gauge condition

$$\langle \Omega | \mathcal{T}\,[\mathcal{O}_1 \ldots \mathcal{O}_n]\,|\Omega\rangle = \frac{\displaystyle\int \mathcal{O}_1 \ldots \mathcal{O}_n \, e^{iS}\, \delta[\boldsymbol{\nabla} \cdot \boldsymbol{A} + \triangle\Lambda]\, DA\, D\psi}{\displaystyle\int e^{iS}\, \delta[\boldsymbol{\nabla} \cdot \boldsymbol{A} + \triangle\Lambda']\, DA\, D\psi}. \tag{7.209}$$

We now have two choices. If we integrate over all gauge functions $\Lambda(x)$ and $\Lambda'(x)$ in both the numerator and the denominator of this ratio (7.209), then apart from over-all constants that cancel, the mean value in the vacuum of

the time-ordered product is the ratio

$$\langle\Omega|\mathcal{T}\left[\mathcal{O}_1\ldots\mathcal{O}_n\right]|\Omega\rangle = \frac{\displaystyle\int \mathcal{O}_1\ldots\mathcal{O}_n\, e^{iS}\, DA\, D\psi}{\displaystyle\int e^{iS}\, DA\, D\psi} \tag{7.210}$$

in which we integrate over all matter fields, gauge fields, and gauges. That is, **we do not fix the gauge**.

The analogous formula for the euclidian time-ordered product is

$$\langle\Omega|\mathcal{T}\left[\mathcal{O}_{e,1}\ldots\mathcal{O}_{e,n}\right]|\Omega\rangle = \frac{\displaystyle\int \mathcal{O}_1\ldots\mathcal{O}_n\, e^{-S_e}\, DA\, D\psi}{\displaystyle\int e^{-S_e}\, DA\, D\psi} \tag{7.211}$$

in which the euclidian action S_e is the spacetime integral of the energy density. This formula is quite general; it holds in nonabelian gauge theories and is important in lattice gauge theory.

Our second choice is to multiply the numerator and the denominator of the ratio (7.209) by the exponential $\exp[-i\frac{1}{2}\alpha\int(\triangle\Lambda)^2\, d^4x]$ and then integrate over $\Lambda(x)$ in the numerator and over $\Lambda'(x)$ in the denominator. This operation just multiplies the numerator and denominator by the same constant factor, which cancels. But if before integrating over all gauge transformations, we shift Λ so that $\triangle\Lambda$ changes to $\triangle\Lambda - \dot{A}^0$, then the exponential factor is $\exp[-i\frac{1}{2}\alpha\int(\dot{A}^0 - \triangle\Lambda)^2\, d^4x]$. Now when we integrate over $\Lambda(x)$, the delta function $\delta(\boldsymbol{\nabla}\cdot\boldsymbol{A} + \triangle\Lambda)$ replaces $\triangle\Lambda$ by $-\boldsymbol{\nabla}\cdot\boldsymbol{A}$ in the inserted exponential, converting it to $\exp[-i\frac{1}{2}\alpha\int(\dot{A}^0 + \boldsymbol{\nabla}\cdot\boldsymbol{A})^2\, d^4x]$. This term changes the gauge-invariant action (7.205) to the gauge-fixed action

$$S_\alpha = \int -\frac{1}{4}F_{ab}\, F^{ab} - \frac{\alpha}{2}(\partial_b A^b)^2 + A^b j_b + \mathcal{L}_m\; d^4x. \tag{7.212}$$

This Lorentz-invariant, gauge-fixed action is much easier to use than the Coulomb-gauge action (7.199) with the Coulomb potential (7.196). We can use it to compute scattering amplitudes perturbatively. The mean value of a time-ordered product of operators in the ground state $|0\rangle$ of the free theory is

$$\langle 0|\mathcal{T}\left[\mathcal{O}_1\ldots\mathcal{O}_n\right]|0\rangle = \frac{\displaystyle\int \mathcal{O}_1\ldots\mathcal{O}_n\, e^{iS_\alpha}\, DA\, D\psi}{\displaystyle\int e^{iS_\alpha}\, DA\, D\psi}. \tag{7.213}$$

By following steps analogous to those the led to (7.187), one may show

(exercise **??**) that in Feynman's gauge, $\alpha = 1$, the photon propagator is

$$\langle 0|\mathcal{T}\left[A_\mu(x)A_\nu(y)\right]|0\rangle = -i\triangle_{\mu\nu}(x-y) = -i\int \frac{\eta_{\mu\nu}}{q^2 - i\epsilon}\, e^{iq\cdot(x-y)}\, \frac{d^4q}{(2\pi)^4}. \tag{7.214}$$

7.12 Fermionic path integrals

In our brief introduction (**??**–**??**) and (**??**–**??**), to Grassmann variables, we learned that because

$$\theta^2 = 0 \tag{7.215}$$

the most general function $f(\theta)$ of a single Grassmann variable θ is

$$f(\theta) = a + b\,\theta. \tag{7.216}$$

So a complete integral table consists of the integral of this linear function

$$\int f(\theta)\, d\theta = \int a + b\,\theta\, d\theta = a\int d\theta + b\int \theta\, d\theta. \tag{7.217}$$

This equation has two unknowns, the integral $\int d\theta$ of unity and the integral $\int \theta\, d\theta$ of θ. We choose them so that the integral of $f(\theta + \zeta)$

$$\int f(\theta+\zeta)\, d\theta = \int a + b\,(\theta+\zeta)\, d\theta = (a + b\,\zeta)\int d\theta + b\int \theta\, d\theta \tag{7.218}$$

is the same as the integral (7.217) of $f(\theta)$. Thus the integral $\int d\theta$ of unity must vanish, while the integral $\int \theta\, d\theta$ of θ can be any constant, which we choose to be unity. Our complete table of integrals is then

$$\int d\theta = 0 \quad \text{and} \quad \int \theta\, d\theta = 1. \tag{7.219}$$

The anticommutation relations for a fermionic degree of freedom ψ are

$$\{\psi, \psi^\dagger\} \equiv \psi\,\psi^\dagger + \psi^\dagger\psi = 1 \quad \text{and} \quad \{\psi,\psi\} = \{\psi^\dagger, \psi^\dagger\} = 0. \tag{7.220}$$

Because ψ has $\psi^\dagger$, it is conventional to introduce a variable $\theta^* = \theta^\dagger$ that anti-commutes with itself and with θ

$$\{\theta^*, \theta^*\} = \{\theta^*, \theta\} = \{\theta, \theta\} = 0. \tag{7.221}$$

The logic that led to (7.219) now gives

$$\int d\theta^* = 0 \quad \text{and} \quad \int \theta^*\, d\theta^* = 1. \tag{7.222}$$

We define the reference state $|0\rangle$ as $|0\rangle \equiv \psi|s\rangle$ for a state $|s\rangle$ that is not annihilated by ψ. Since $\psi^2 = 0$, the operator ψ annihilates the state $|0\rangle$

$$\psi|0\rangle = \psi^2|s\rangle = 0. \tag{7.223}$$

The effect of the operator ψ on the state

$$|\theta\rangle = \exp\left(\psi^\dagger\theta - \tfrac{1}{2}\theta^*\theta\right)|0\rangle = \left(1 + \psi^\dagger\theta - \tfrac{1}{2}\theta^*\theta\right)|0\rangle \tag{7.224}$$

is

$$\psi|\theta\rangle = \psi(1 + \psi^\dagger\theta - \tfrac{1}{2}\theta^*\theta)|0\rangle = \psi\psi^\dagger\theta|0\rangle = (1 - \psi^\dagger\psi)\theta|0\rangle = \theta|0\rangle \tag{7.225}$$

while that of θ on $|\theta\rangle$ is

$$\theta|\theta\rangle = \theta(1 + \psi^\dagger\theta - \tfrac{1}{2}\theta^*\theta)|0\rangle = \theta|0\rangle. \tag{7.226}$$

The state $|\theta\rangle$ therefore is an eigenstate of ψ with eigenvalue θ

$$\psi|\theta\rangle = \theta|\theta\rangle. \tag{7.227}$$

The bra corresponding to the ket $|\zeta\rangle$ is

$$\langle\zeta| = \langle 0|\left(1 + \zeta^*\psi - \frac{1}{2}\zeta^*\zeta\right) \tag{7.228}$$

and the inner product $\langle\zeta|\theta\rangle$ is (exercise **??**)

$$\begin{aligned}
\langle\zeta|\theta\rangle &= \langle 0|\left(1 + \zeta^*\psi - \frac{1}{2}\zeta^*\zeta\right)\left(1 + \psi^\dagger\theta - \frac{1}{2}\theta^*\theta\right)|0\rangle \\
&= \langle 0|1 + \zeta^*\psi\psi^\dagger\theta - \frac{1}{2}\zeta^*\zeta - \frac{1}{2}\theta^*\theta + \frac{1}{4}\zeta^*\zeta\theta^*\theta|0\rangle \\
&= \langle 0|1 + \zeta^*\theta - \frac{1}{2}\zeta^*\zeta - \frac{1}{2}\theta^*\theta + \frac{1}{4}\zeta^*\zeta\theta^*\theta|0\rangle \\
&= \exp\left[\zeta^*\theta - \frac{1}{2}\left(\zeta^*\zeta + \theta^*\theta\right)\right].
\end{aligned} \tag{7.229}$$

Example 7.9 (A gaussian integral) For *any* number c, we can compute the integral of $\exp(c\,\theta^*\theta)$ by expanding the exponential

$$\int e^{c\,\theta^*\theta}\,d\theta^* d\theta = \int (1 + c\,\theta^*\theta)\,d\theta^* d\theta = \int (1 - c\,\theta\,\theta^*)\,d\theta^* d\theta = -c. \tag{7.230}$$

□

The identity operator for the space of states

$$c|0\rangle + d|1\rangle \equiv c|0\rangle + d\psi^\dagger|0\rangle \tag{7.231}$$

is (exercise **??**) the integral

$$I = \int |\theta\rangle\langle\theta|\, d\theta^* d\theta = |0\rangle\langle 0| + |1\rangle\langle 1| \tag{7.232}$$

in which the differentials anti-commute with each other and with other fermionic variables: $\{d\theta, d\theta^*\} = 0$, $\{d\theta, \theta\} = 0$, $\{d\theta, \psi\} = 0$, and so forth.

The case of several Grassmann variables $\theta_1, \theta_2, \ldots, \theta_n$ and several Fermi operators $\psi_1, \psi_2, \ldots, \psi_n$ is similar. The θ_k anticommute among themselves

$$\{\theta_i, \theta_j\} = \{\theta_i, \theta_j^*\} = \{\theta_i^*, \theta_j^*\} = 0 \tag{7.233}$$

while the ψ_k satisfy

$$\{\psi_k, \psi_\ell^\dagger\} = \delta_{k\ell} \quad \text{and} \quad \{\psi_k, \psi_l\} = \{\psi_k^\dagger, \psi_\ell^\dagger\} = 0. \tag{7.234}$$

The reference state $|0\rangle$ is

$$|0\rangle = \left(\prod_{k=1}^{n} \psi_k\right) |s\rangle \tag{7.235}$$

in which $|s\rangle$ is any state not annihilated by any ψ_k (so the resulting $|0\rangle$ isn't zero). The direct-product state

$$|\theta\rangle \equiv \exp\left(\sum_{k=1}^{n} \psi_k^\dagger \theta_k - \frac{1}{2}\theta_k^* \theta_k\right) |0\rangle = \left[\prod_{k=1}^{n} \left(1 + \psi_k^\dagger \theta_k - \frac{1}{2}\theta_k^* \theta_k\right)\right] |0\rangle \tag{7.236}$$

is (exercise **??**) a simultaneous eigenstate of each ψ_k

$$\psi_k |\theta\rangle = \theta_k |\theta\rangle. \tag{7.237}$$

It follows that

$$\psi_\ell \psi_k |\theta\rangle = \psi_\ell \theta_k |\theta\rangle = -\theta_k \psi_\ell |\theta\rangle = -\theta_k \theta_\ell |\theta\rangle = \theta_\ell \theta_k |\theta\rangle \tag{7.238}$$

and so too $\psi_k \psi_\ell |\theta\rangle = \theta_k \theta_\ell |\theta\rangle$. Since the ψ's anticommute, their eigenvalues must also

$$\theta_\ell \theta_k |\theta\rangle = \psi_\ell \psi_k |\theta\rangle = -\psi_k \psi_\ell |\theta\rangle = -\theta_k \theta_\ell |\theta\rangle \tag{7.239}$$

(even if the eigenvalues commuted with the ψ's in which case we'd have $\psi_\ell \psi_k |\theta\rangle = \theta_k \theta_\ell |\theta\rangle = -\psi_k \psi_\ell |\theta\rangle = -\theta_\ell \theta_k |\theta\rangle$).

The inner product $\langle\zeta|\theta\rangle$ is

$$\begin{aligned}\langle\zeta|\theta\rangle &= \langle 0|\left[\prod_{k=1}^{n}(1+\zeta_k^*\psi_k-\frac{1}{2}\zeta_k^*\zeta_k)\right]\left[\prod_{\ell=1}^{n}(1+\psi_\ell^\dagger\theta_\ell-\frac{1}{2}\theta_\ell^*\theta_\ell)\right]|0\rangle \\ &= \exp\left[\sum_{k=1}^{n}\zeta_k^*\theta_k-\frac{1}{2}\left(\zeta_k^*\zeta_k+\theta_k^*\theta_k\right)\right] = e^{\zeta^\dagger\theta-(\zeta^\dagger\zeta+\theta^\dagger\theta)/2}. \qquad (7.240)\end{aligned}$$

The identity operator is

$$I = \int |\theta\rangle\langle\theta| \prod_{k=1}^{n} d\theta_k^* d\theta_k. \qquad (7.241)$$

Example 7.10 (Gaussian Grassmann Integral) For *any* 2×2 matrix A, we may compute the gaussian integral

$$g(A) = \int e^{-\theta^\dagger A\theta}\, d\theta_1^* d\theta_1 d\theta_2^* d\theta_2 \qquad (7.242)$$

by expanding the exponential. The only terms that survive are the ones that have exactly one of each of the four variables θ_1, θ_2, θ_1^*, and θ_2^*. Thus, the integral is the determinant of the matrix A

$$\begin{aligned}g(A) &= \int \frac{1}{2}\left(\theta_k^* A_{k\ell}\theta_\ell\right)^2\, d\theta_1^* d\theta_1 d\theta_2^* d\theta_2 \\ &= \int \left(\theta_1^* A_{11}\theta_1\, \theta_2^* A_{22}\theta_2 + \theta_1^* A_{12}\theta_2\, \theta_2^* A_{21}\theta_1\right)\, d\theta_1^* d\theta_1 d\theta_2^* d\theta_2 \\ &= A_{11}A_{22} - A_{12}A_{21} = \det A. \qquad (7.243)\end{aligned}$$

The natural generalization to n dimensions

$$\int e^{-\theta^\dagger A\theta} \prod_{k=1}^{n} d\theta_k^* d\theta_k = \det A \qquad (7.244)$$

is true for *any* $n\times n$ matrix A. If A is invertible, then the invariance of

Grassmann integrals under translations implies that

$$\begin{aligned}\int e^{-\theta^\dagger A\theta+\theta^\dagger\zeta+\zeta^\dagger\theta}\prod_{k=1}^n d\theta_k^* d\theta_k &= \int e^{-\theta^\dagger A(\theta+A^{-1}\zeta)+\theta^\dagger\zeta+\zeta^\dagger(\theta+A^{-1}\zeta)}\prod_{k=1}^n d\theta_k^* d\theta_k \\ &= \int e^{-\theta^\dagger A\theta+\zeta^\dagger\theta+\zeta^\dagger A^{-1}\zeta}\prod_{k=1}^n d\theta_k^* d\theta_k \\ &= \int e^{-(\theta^\dagger+\zeta^\dagger A^{-1})A\theta+\zeta^\dagger\theta+\zeta^\dagger A^{-1}\zeta}\prod_{k=1}^n d\theta_k^* d\theta_k \\ &= \int e^{-\theta^\dagger A\theta+\zeta^\dagger A^{-1}\zeta}\prod_{k=1}^n d\theta_k^* d\theta_k \\ &= \det A\; e^{\zeta^\dagger A^{-1}\zeta}. \end{aligned} \tag{7.245}$$

The values of θ and $\theta^\dagger$ that make the argument $-\theta^\dagger A\theta+\theta^\dagger\zeta+\zeta^\dagger\theta$ of the exponential stationary are $\overline{\theta} = A^{-1}\zeta$ and $\overline{\theta^\dagger} = \zeta^\dagger A^{-1}$. So a gaussian Grassmann integral is equal to its exponential evaluated at its stationary point, apart from a prefactor involving the determinant $\det A$. This result is a fermionic echo of the bosonic result (**??**). □

One may further extend these definitions to a Grassmann field $\chi_m(x)$ and an associated Dirac field $\psi_m(x)$. The $\chi_m(x)$'s anticommute among themselves and with all fermionic variables at all points of spacetime

$$\{\chi_m(x), \chi_n(x')\} = \{\chi_m^*(x), \chi_n(x')\} = \{\chi_m^*(x), \chi_n^*(x')\} = 0 \tag{7.246}$$

and the Dirac field $\psi_m(x)$ obeys the equal-time anticommutation relations

$$\begin{aligned}&\{\psi_m(\boldsymbol{x},t), \psi_n^\dagger(\boldsymbol{x'},t)\} = \delta_{mn}\,\delta(\boldsymbol{x}-\boldsymbol{x'}) \quad (n,m=1,\ldots,4) \\ &\{\psi_m(\boldsymbol{x},t), \psi_n(\boldsymbol{x'},t)\} = \{\psi_m^\dagger(\boldsymbol{x},t), \psi_n^\dagger(\boldsymbol{x'},t)\} = 0.\end{aligned} \tag{7.247}$$

As in (7.235), we use eigenstates of the field ψ at $t=0$. If $|0\rangle$ is defined in terms of a state $|s\rangle$ that is not annihilated by any $\psi_m(\boldsymbol{x},0)$ as

$$|0\rangle = \left[\prod_{m,\boldsymbol{x}} \psi_m(\boldsymbol{x},0)\right]|s\rangle \tag{7.248}$$

then (exercise **??**) the state

$$\begin{aligned}|\chi\rangle &= \exp\left(\int \sum_m \psi_m^\dagger(\boldsymbol{x},0)\,\chi_m(\boldsymbol{x}) - \frac{1}{2}\chi_m^*(\boldsymbol{x})\chi_m(\boldsymbol{x})\,d^3x\right)|0\rangle \\ &= \exp\left(\int \psi^\dagger\chi - \tfrac{1}{2}\chi^\dagger\chi\,d^3x\right)|0\rangle\end{aligned} \tag{7.249}$$

is an eigenstate of the operator $\psi_m(\boldsymbol{x},0)$ with eigenvalue $\chi_m(\boldsymbol{x})$

$$\psi_m(\boldsymbol{x},0)|\chi\rangle = \chi_m(\boldsymbol{x})|\chi\rangle. \tag{7.250}$$

The inner product of two such states is (exercise **??**)

$$\langle\chi'|\chi\rangle = \exp\left[\int \chi'^\dagger\chi - \frac{1}{2}\chi'^\dagger\chi' - \frac{1}{2}\chi^\dagger\chi\; d^3x\right]. \tag{7.251}$$

The identity operator is the integral

$$I = \int |\chi\rangle\langle\chi|\, D\chi^* D\chi \tag{7.252}$$

in which

$$D\chi^* D\chi \equiv \prod_{m,\boldsymbol{x}} d\chi_m^*(\boldsymbol{x}) d\chi_m(\boldsymbol{x}). \tag{7.253}$$

The hamiltonian for a free Dirac field ψ of mass m is the spatial integral

$$H_0 = \int \overline{\psi}\,(\boldsymbol{\gamma}\cdot\boldsymbol{\nabla} + m)\,\psi\; d^3x \tag{7.254}$$

in which $\overline{\psi} \equiv i\psi^\dagger\gamma^0$ and the gamma matrices (**??**) satisfy

$$\{\gamma^a, \gamma^b\} = 2\,\eta^{ab} \tag{7.255}$$

where η is the 4×4 diagonal matrix with entries $(-1,1,1,1)$. Since $\psi|\chi\rangle = \chi|\chi\rangle$ and $\langle\chi'|\psi^\dagger = \langle\chi'|\chi'^\dagger$, the quantity $\langle\chi'|\exp(-i\epsilon H_0)|\chi\rangle$ is by (7.251)

$$\begin{aligned}\langle\chi'|e^{-i\epsilon H_0}|\chi\rangle &= \langle\chi'|\chi\rangle \exp\left[-i\epsilon\int \overline{\chi}'\,(\boldsymbol{\gamma}\cdot\boldsymbol{\nabla} + m)\,\chi\; d^3x\right] \qquad (7.256)\\ &= \exp\left[\int \tfrac{1}{2}(\chi'^\dagger - \chi^\dagger)\chi - \tfrac{1}{2}\chi'^\dagger(\chi'-\chi) - i\epsilon\overline{\chi}'(\boldsymbol{\gamma}\cdot\boldsymbol{\nabla} + m)\chi d^3x\right]\\ &= \exp\left\{\epsilon\int\left[\tfrac{1}{2}\dot{\chi}^\dagger\chi - \tfrac{1}{2}\chi'^\dagger\dot{\chi} - i\overline{\chi}'\,(\boldsymbol{\gamma}\cdot\boldsymbol{\nabla} + m)\,\chi\right]\; d^3x\right\}\end{aligned}$$

in which $\chi'^\dagger - \chi^\dagger = \epsilon\dot{\chi}^\dagger$ and $\chi' - \chi = \epsilon\dot{\chi}$. Everything within the square brackets is multiplied by ϵ, so we may replace $\chi'^\dagger$ by $\chi^\dagger$ and $\overline{\chi}'$ by $\overline{\chi}$ so as to write to first order in ϵ

$$\langle\chi'|e^{-i\epsilon H_0}|\chi\rangle = \exp\left[\epsilon\int \tfrac{1}{2}\dot{\chi}^\dagger\chi - \tfrac{1}{2}\chi^\dagger\dot{\chi} - i\overline{\chi}\,(\boldsymbol{\gamma}\cdot\boldsymbol{\nabla} + m)\,\chi\; d^3x\right] \tag{7.257}$$

in which the dependence upon χ' is through the time derivatives.

Putting together $n = 2t/\epsilon$ such matrix elements, integrating over all intermediate-state dyadics $|\chi\rangle\langle\chi|$, and using our formula (7.252), we find

$$\langle\chi_t|e^{-2itH_0}|\chi_{-t}\rangle = \int \exp\left[\int \tfrac{1}{2}\dot{\chi}^\dagger\chi - \tfrac{1}{2}\chi^\dagger\dot{\chi} - i\overline{\chi}\,(\boldsymbol{\gamma}\cdot\boldsymbol{\nabla} + m)\,\chi d^4x\right] D\chi^* D\chi. \tag{7.258}$$

Integrating $\dot{\chi}^\dagger\chi$ by parts and dropping the surface term, we get

$$\langle\chi_t|e^{-2itH_0}|\chi_{-t}\rangle = \int \exp\left[\int -\chi^\dagger\dot{\chi} - i\overline{\chi}\,(\boldsymbol{\gamma}\cdot\boldsymbol{\nabla} + m)\,\chi\, d^4x\right] D\chi^* D\chi. \tag{7.259}$$

Since $-\chi^\dagger\dot{\chi} = -i\overline{\chi}\gamma^0\dot{\chi}$, the argument of the exponential is

$$i\int -\overline{\chi}\gamma^0\dot{\chi} - \overline{\chi}\,(\boldsymbol{\gamma}\cdot\boldsymbol{\nabla} + m)\,\chi\; d^4x = i\int -\overline{\chi}\,(\gamma^\mu\partial_\mu + m)\,\chi\; d^4x. \tag{7.260}$$

We then have

$$\langle\chi_t|e^{-2itH_0}|\chi_{-t}\rangle = \int \exp\left(i\int \mathcal{L}_0(\chi)\, d^4x\right) D\chi^* D\chi \tag{7.261}$$

in which $\mathcal{L}_0(\chi) = -\overline{\chi}\,(\gamma^\mu\partial_\mu + m)\,\chi$ is the action density (4.56) for a free Dirac field. Thus the amplitude is a path integral with phases given by the classical action $S_0[\chi]$

$$\langle\chi_t|e^{-2itH_0}|\chi_{-t}\rangle = \int e^{i\int\mathcal{L}_0(\chi)\, d^4x} D\chi^* D\chi = \int e^{iS_0[\chi]} D\chi^* D\chi \tag{7.262}$$

and the integral is over all fields that go from $\chi(\boldsymbol{x}, -t) = \chi_{-t}(\boldsymbol{x})$ to $\chi(\boldsymbol{x}, t) = \chi_t(\boldsymbol{x})$. Any normalization factor will cancel in ratios of such integrals.

Since Fermi fields anticommute, their time-ordered product has an extra minus sign

$$\mathcal{T}\left[\overline{\psi}(x_1)\psi(x_2)\right] = \theta(x_1^0 - x_2^0)\,\overline{\psi}(x_1)\,\psi(x_2) - \theta(x_2^0 - x_1^0)\,\psi(x_2)\,\overline{\psi}(x_1). \tag{7.263}$$

The logic behind our formulas (7.148) and (7.166) for the time-ordered product of bosonic fields now leads to an expression for the time-ordered product of $2n$ Dirac fields (with $D\chi''$ and $D\chi'$ and so forth suppressed)

$$\langle 0|\mathcal{T}\left[\overline{\psi}(x_1)\ldots\psi(x_{2n})\right]|0\rangle = \frac{\displaystyle\int \langle 0|\chi''\rangle\,\overline{\chi}(x_1)\ldots\chi(x_{2n})\, e^{iS_0[\chi]}\langle\chi'|0\rangle\, D\chi^* D\chi}{\displaystyle\int \langle 0|\chi''\rangle\, e^{iS_0[\chi]}\langle\chi'|0\rangle\, D\chi^* D\chi}. \tag{7.264}$$

As in (7.177), the effect of the inner products $\langle 0|\chi''\rangle$ and $\langle\chi'|0\rangle$ is to insert

ϵ-terms which modify the Dirac propagators

$$\langle 0|\mathcal{T}\left[\overline{\psi}(x_1)\ldots\psi(x_{2n})\right]|0\rangle = \frac{\displaystyle\int \overline{\chi}(x_1)\ldots\chi(x_{2n})\, e^{iS_0[\chi,\epsilon]}\, D\chi^* D\chi}{\displaystyle\int e^{iS_0[\chi,\epsilon]}\, D\chi^* D\chi}. \tag{7.265}$$

Imitating (7.178), we introduce a Grassmann external current $\zeta(x)$ and define a fermionic analog of $Z_0[j]$

$$Z_0[\zeta] \equiv \langle 0|\,\mathcal{T}\left[e^{\int \overline{\zeta}\psi+\overline{\psi}\zeta\, d^4x}\right]|0\rangle = \frac{\displaystyle\int e^{\int \overline{\zeta}\chi+\overline{\chi}\zeta\, d^4x} e^{iS_0[\chi,\epsilon]} D\chi^* D\chi}{\displaystyle\int e^{iS_0[\chi,\epsilon]} D\chi^* D\chi}. \tag{7.266}$$

Example 7.11 (Feynman's fermion propagator) Since

$$\begin{aligned} i\left(\gamma^\mu\partial_\mu + m\right)\Delta(x-y) &\equiv i\left(\gamma^\mu\partial_\mu + m\right)\int \frac{d^4p}{(2\pi)^4} e^{ip(x-y)} \frac{-i\left(-i\gamma^\nu p_\nu + m\right)}{p^2+m^2-i\epsilon} \\ &= \int \frac{d^4p}{(2\pi)^4} e^{ip(x-y)} \left(i\gamma^\mu p_\mu + m\right) \frac{\left(-i\gamma^\nu p_\nu + m\right)}{p^2+m^2-i\epsilon} \\ &= \int \frac{d^4p}{(2\pi)^4} e^{ip(x-y)} \frac{p^2+m^2}{p^2+m^2-i\epsilon} = \delta^4(x-y), \end{aligned} \tag{7.267}$$

the function $\Delta(x-y)$ is the inverse of the differential operator $i(\gamma^\mu\partial_\mu + m)$. Thus the Grassmann identity (7.245) implies that $Z_0[\zeta]$ is

$$\begin{aligned} \langle 0|\,\mathcal{T}\left[e^{\int \overline{\zeta}\psi+\overline{\psi}\zeta\, d^4x}\right]|0\rangle &= \frac{\displaystyle\int e^{\int [\overline{\zeta}\chi+\overline{\chi}\zeta - i\,\overline{\chi}(\gamma^\mu\partial_\mu+m)\chi] d^4x} D\chi^* D\chi}{\displaystyle\int e^{iS_0[\chi,\epsilon]} D\chi^* D\chi} \\ &= \exp\left[\int \overline{\zeta}(x)\Delta(x-y)\zeta(y)\, d^4x d^4y\right]. \end{aligned} \tag{7.268}$$

Differentiating we get

$$\langle 0|\mathcal{T}\left[\psi(x)\overline{\psi}(y)\right]|0\rangle = \Delta(x-y) = -i\int \frac{d^4p}{(2\pi)^4} e^{ip(x-y)} \frac{-i\gamma^\nu p_\nu + m}{p^2+m^2-i\epsilon}. \tag{7.269}$$

□

7.13 Application to nonabelian gauge theories

The action of a generic non-abelian gauge theory is

$$S = \int -\tfrac{1}{4} F_{a\mu\nu} F_a^{\mu\nu} - \overline{\psi}\,(\gamma^\mu D_\mu + m)\,\psi\; d^4x \tag{7.270}$$

in which the Maxwell field is

$$F_{a\mu\nu} \equiv \partial_\mu A_{a\nu} - \partial_\nu A_{a\mu} + g\, f_{abc}\, A_{b\mu}\, A_{c\nu} \tag{7.271}$$

and the covariant derivative is

$$D_\mu \psi \equiv \partial_\mu \psi - ig\, t_a\, A_{a\mu}\, \psi. \tag{7.272}$$

Here g is a coupling constant, f_{abc} is a structure constant (**??**), and t_a is a generator (**??**) of the Lie algebra (section **??**) of the gauge group.

One may show (Weinberg, 1996, pp. 14–18) that the analog of equation (7.206) for quantum electrodynamics is

$$\langle\Omega|\mathcal{T}\,[\mathcal{O}_1 \ldots \mathcal{O}_n]\,|\Omega\rangle = \frac{\displaystyle\int \mathcal{O}_1 \ldots \mathcal{O}_n\, e^{iS}\, \delta[A_{a3}]\, DA\, D\psi}{\displaystyle\int e^{iS}\, \delta[A_{a3}]\, DA\, D\psi} \tag{7.273}$$

in which the functional delta function

$$\delta[A_{a3}] \equiv \prod_{x,b} \delta(A_{a3}(x)) \tag{7.274}$$

enforces the axial-gauge condition, and $D\psi$ stands for $D\psi^* D\psi$.

Initially, physicists had trouble computing nonabelian amplitudes beyond the lowest order of perturbation theory. Then DeWitt showed how to compute to second order (DeWitt, 1967), and Faddeev and Popov, using path integrals, showed how to compute to all orders (Faddeev and Popov, 1967).

7.14 Faddeev-Popov trick

The path-integral tricks of Faddeev and Popov are described in (Weinberg, 1996, pp. 19–27). We will use gauge-fixing functions $G_a(x)$ to impose a gauge condition on our non-abelian gauge fields $A_\mu^a(x)$. For instance, we can use $G_a(x) = A_a^3(x)$ to impose an axial gauge or $G_a(x) = i\partial_\mu A_a^\mu(x)$ to impose a Lorentz-invariant gauge.

Under an infinitesimal gauge transformation (**??**)

$$A_{a\mu}^\lambda = A_{a\mu} - \partial_\mu \lambda_a - g\, f_{abc}\, A_{b\mu}\, \lambda_c \tag{7.275}$$

the gauge fields change, and so the gauge-fixing functions $G_b(x)$, which depend upon them, also change. The jacobian J of that change at $\lambda = 0$ is

$$J = \det\left(\frac{\delta G_a^\lambda(x)}{\delta\lambda_b(y)}\right)\bigg|_{\lambda=0} \equiv \frac{DG^\lambda}{D\lambda}\bigg|_{\lambda=0} \tag{7.276}$$

and it typically involves the delta function $\delta^4(x-y)$.

Let $B[G]$ be any functional of the gauge-fixing functions $G_b(x)$ such as

$$B[G] = \prod_{x,a}\delta(G_a(x)) = \prod_{x,a}\delta(A_a^3(x)) \tag{7.277}$$

in an axial gauge or

$$B[G] = \exp\left[\frac{i}{2}\int (G_a(x))^2\, d^4x\right] = \exp\left[-\frac{i}{2}\int (\partial_\mu A_a^\mu(x))^2\, d^4x\right] \tag{7.278}$$

in a Lorentz-invariant gauge.

We want to understand functional integrals like (7.273)

$$\langle\Omega|\mathcal{T}\left[\mathcal{O}_1\ldots\mathcal{O}_n\right]|\Omega\rangle = \frac{\displaystyle\int \mathcal{O}_1\ldots\mathcal{O}_n\, e^{iS}\, B[G]\, J\, DA\, D\psi}{\displaystyle\int e^{iS}\, B[G]\, J\, DA\, D\psi} \tag{7.279}$$

in which the operators $\mathcal{O}_k$, the action functional $S[A]$, and the differentials $DAD\psi$ (but not the gauge-fixing functional $B[G]$ or the Jacobian J) are gauge invariant. The axial-gauge formula (7.273) is a simple example in which $B[G] = \delta[A_{a3}]$ enforces the axial-gauge condition $A_{a3}(x) = 0$ and the determinant $J = \det\left(\delta_{ab}\partial_3\delta(x-y)\right)$ is a constant that cancels.

If we translate the gauge fields by gauge transformations Λ and Λ', then the ratio (7.279) does not change

$$\langle\Omega|\mathcal{T}\left[\mathcal{O}_1\ldots\mathcal{O}_n\right]|\Omega\rangle = \frac{\displaystyle\int \mathcal{O}_1^\Lambda\ldots\mathcal{O}_n^\Lambda\, e^{iS^\Lambda}\, B[G^\Lambda]\, J^\Lambda\, DA^\Lambda\, D\psi^\Lambda}{\displaystyle\int e^{iS^{\Lambda'}}\, B[G^{\Lambda'}]\, J^{\Lambda'}\, DA^{\Lambda'}\, D\psi^{\Lambda'}} \tag{7.280}$$

any more than $\int f(y)\,dy$ is different from $\int f(x)\,dx$. Since the operators $\mathcal{O}_k$, the action functional $S[A]$, and the differentials $DAD\psi$ are gauge invariant, most of the Λ-dependence goes away

$$\langle\Omega|\mathcal{T}\left[\mathcal{O}_1\ldots\mathcal{O}_n\right]|\Omega\rangle = \frac{\displaystyle\int \mathcal{O}_1\ldots\mathcal{O}_n\, e^{iS}\, B[G^\Lambda]\, J^\Lambda\, DA\, D\psi}{\displaystyle\int e^{iS}\, B[G^{\Lambda'}]\, J^{\Lambda'}\, DA\, D\psi}. \tag{7.281}$$

Let $\Lambda\lambda$ be a gauge transformation Λ followed by an infinitesimal gauge

transformation λ. The jacobian J^Λ is a determinant of a product of matrices which is a product of their determinants

$$\begin{aligned} J^\Lambda &= \det\left(\frac{\delta G_a^{\Lambda\lambda}(x)}{\delta\lambda_b(y)}\right)\bigg|_{\lambda=0} = \det\left(\int \frac{\delta G_a^{\Lambda\lambda}(x)}{\delta\Lambda\lambda_c(z)}\,\frac{\delta\Lambda\lambda_c(z)}{\delta\lambda_b(y)}\,d^4z\right)\bigg|_{\lambda=0} \\ &= \det\left(\frac{\delta G_a^{\Lambda\lambda}(x)}{\delta\Lambda\lambda_c(z)}\right)\bigg|_{\lambda=0} \det\left(\frac{\delta\Lambda\lambda_c(z)}{\delta\lambda_b(y)}\right)\bigg|_{\lambda=0} \\ &= \det\left(\frac{\delta G_a^{\Lambda}(x)}{\delta\Lambda_c(z)}\right) \det\left(\frac{\delta\Lambda\lambda_c(z)}{\delta\lambda_b(y)}\right)\bigg|_{\lambda=0} \equiv \frac{DG^\Lambda}{D\Lambda}\,\frac{D\Lambda\lambda}{D\lambda}\bigg|_{\lambda=0}. \end{aligned} \tag{7.282}$$

Now we integrate over the gauge transformations Λ (and Λ') with weight function $\rho(\Lambda) = (D\Lambda\lambda/D\lambda|_{\lambda=0})^{-1}$ and find, since the ratio (7.281) is Λ-independent

$$\begin{aligned} \langle\Omega|\mathcal{T}\left[\mathcal{O}_1\ldots\mathcal{O}_n\right]|\Omega\rangle &= \frac{\displaystyle\int \mathcal{O}_1\ldots\mathcal{O}_n\, e^{iS}\, B[G^\Lambda]\,\frac{DG^\Lambda}{D\Lambda}\,D\Lambda\,DA\,D\psi}{\displaystyle\int e^{iS}\, B[G^{\Lambda'}]\,\frac{DG^{\Lambda'}}{D\Lambda'}\,D\Lambda'\,DA\,D\psi} \\ &= \frac{\displaystyle\int \mathcal{O}_1\ldots\mathcal{O}_n\, e^{iS}\, B[G^\Lambda]\,DG^\Lambda\,DA\,D\psi}{\displaystyle\int e^{iS}\, B[G^\Lambda]\,DG^\Lambda\,DA\,D\psi} \\ &= \frac{\displaystyle\int \mathcal{O}_1\ldots\mathcal{O}_n\, e^{iS}\,DA\,D\psi}{\displaystyle\int e^{iS}\,DA\,D\psi}. \end{aligned} \tag{7.283}$$

Thus the mean-value in the vacuum of a time-ordered product of gauge-invariant operators is a ratio of path integrals over all gauge fields without any gauge fixing. No matter what gauge condition G or gauge-fixing functional $B[G]$ we use, the resulting gauge-fixed ratio (7.279) is equal to the ratio (7.283) of path integrals over all gauge fields without any gauge fixing. All gauge-fixed ratios (7.279) give the same time-ordered products, and so we can use whatever gauge condition G or gauge-fixing functional $B[G]$ is most convenient.

The analogous formula for the euclidian time-ordered product is

$$\langle\Omega|\mathcal{T}_e\left[\mathcal{O}_1\ldots\mathcal{O}_n\right]|\Omega\rangle = \frac{\displaystyle\int \mathcal{O}_1\ldots\mathcal{O}_n\, e^{-S_e}\,DA\,D\psi}{\displaystyle\int e^{-S_e}\,DA\,D\psi} \tag{7.284}$$

where the euclidian action S_e is the spacetime integral of the energy density. This formula is the basis for lattice gauge theory.

The path-integral formulas (7.210 & 7.211) derived for quantum electrodynamics therefore also apply to nonabelian gauge theories.

7.15 Ghosts

Faddeev and Popov showed how to do perturbative calculations in which one does fix the gauge. To continue our description of their tricks, we return to the gauge-fixed expression (7.279) for the time-ordered product

$$\langle\Omega|\mathcal{T}\left[\mathcal{O}_1\ldots\mathcal{O}_n\right]|\Omega\rangle = \frac{\int \mathcal{O}_1\ldots\mathcal{O}_n\, e^{iS}\, B[G]\, J\, DA\, D\psi}{\int e^{iS}\, B[G]\, J\, DA\, D\psi} \tag{7.285}$$

set $G_b(x) = -i\partial_\mu A^\mu_b(x)$ and use (7.278) as the gauge-fixing functional $B[G]$

$$B[G] = \exp\left[\frac{i}{2}\int (G_a(x))^2\, d^4x\right] = \exp\left[-\frac{i}{2}\int (\partial_\mu A^\mu_a(x))^2\, d^4x\right]. \tag{7.286}$$

This functional adds to the action density the term $-(\partial_\mu A^\mu_a)^2/2$ which leads to a gauge-field propagator like the photon's (7.214)

$$\langle 0|\mathcal{T}\left[A^a_\mu(x)A^b_\nu(y)\right]|0\rangle = -i\delta_{ab}\triangle_{\mu\nu}(x-y) = -i\int \frac{\eta_{\mu\nu}\delta_{ab}}{q^2-i\epsilon}\, e^{iq\cdot(x-y)}\, \frac{d^4q}{(2\pi)^4}. \tag{7.287}$$

What about the determinant J? Under an infinitesimal gauge transformation (7.275), the gauge field becomes

$$A^\lambda_{a\mu} = A_{a\mu} - \partial_\mu\lambda_a - g\, f_{abc}\, A_{b\mu}\, \lambda_c \tag{7.288}$$

and so $G^\lambda_a(x) = i\partial^\mu A^\lambda_{a\mu}(x)$ is

$$G^\lambda_a(x) = i\partial^\mu A_{a\mu}(x) + i\partial^\mu \int [-\delta_{ac}\partial_\mu - g\, f_{abc}A_{b\mu}(x)]\, \delta^4(x-y)\lambda_c(y)\, d^4y. \tag{7.289}$$

The jacobian J then is the determinant (7.276) of the matrix

$$\left(\frac{\delta G^\lambda_a(x)}{\delta\lambda_c(y)}\right)\bigg|_{\lambda=0} = -i\delta_{ac}\,\Box\,\delta^4(x-y) - ig\, f_{abc}\,\frac{\partial}{\partial x^\mu}\left[A^\mu_b(x)\delta^4(x-y)\right] \tag{7.290}$$

that is

$$J = \det\left(-i\delta_{ac}\,\Box\,\delta^4(x-y) - ig\, f_{abc}\,\frac{\partial}{\partial x^\mu}\left[A^\mu_b(x)\delta^4(x-y)\right]\right). \tag{7.291}$$

But we've seen (7.244) that a determinant can be written as a fermionic path integral

$$\det A = \int e^{-\theta^\dagger A\,\theta} \prod_{k=1}^{n} d\theta_k^* d\theta_k. \tag{7.292}$$

So we can write the jacobian J as

$$J = \int \exp\left[\int i\omega_a^* \Box \omega_a + ig f_{abc} \omega_a^* \partial_\mu (A_b^\mu \omega_c)\, d^4x\right] D\omega^* D\omega \tag{7.293}$$

which contributes the terms $-\partial_\mu \omega_a^* \partial^\mu \omega_a$ and

$$-\,\partial_\mu \omega_a^*\, g\, f_{abc}\, A_b^\mu \omega_c = \partial_\mu \omega_a^*\, g\, f_{abc}\, A_c^\mu \omega_b \tag{7.294}$$

to the action density.

Thus we can do perturbation theory by using the modified action density

$$\mathcal{L}' = -\tfrac{1}{4} F_{a\mu\nu} F_a^{\mu\nu} - \tfrac{1}{2}\left(\partial_\mu A_a^\mu\right)^2 - \partial_\mu \omega_a^* \partial^\mu \omega_a + \partial_\mu \omega_a^*\, g\, f_{abc}\, A_c^\mu \omega_b - \overline{\psi}\left(\not{D} + m\right)\psi \tag{7.295}$$

in which $\not{D} \equiv \gamma^\mu D_\mu = \gamma^\mu(\partial_\mu - igt^a A_{a\mu})$. The **ghost** field ω is a mathematical device, not a physical field describing real particles, which would be spinless fermions violating the spin-statistics theorem (example **??**).

7.16 Integrating over the momenta

When a hamiltonian is quadratic in the momenta like (7.128 & 10.8), one easily integrates over its momentum and converts it into its lagrangian. If, however, the hamiltonian is a more complicated function of the momenta, one usually can't path-integrate over the momenta analytically. The partition function is then a path integral over both ϕ and π

$$Z(\beta) = \int \exp\left\{\int_0^\beta \int \left[i\dot{\phi}(x)\pi(x) - H(\phi,\pi)\right] dt d^3x\right\} D\phi\, D\pi. \tag{7.296}$$

This exponential is not positive, and so is not a probability distribution for ϕ and π. The Monte Carlo methods of chapter **??** are designed for probability distributions, not for distributions that assume negative or complex values. This is one aspect of the **sign problem**.

The integral over the momentum

$$P[\phi] = \int \exp\left\{\int_0^\beta \int \left[i\dot{\phi}(x)\pi(x) - H(\phi,\pi)\right] dt d^3x\right\} D\pi \tag{7.297}$$

is positive, however, and is a probability distribution. So one can numerically integrate over the momentum, make a look-up table for $P[\phi]$, and then apply

the usual Monte Carlo method to the probability functional $P[\phi]$ (Amdahl and Cahill, 2016). Programs that do this are in the repository Path_integrals at `github.com/kevinecahill`.

8

Landau theory of phase transitions

8.1 First- and second-order phase transitions

In a **first-order phase transition**, a physical quantity, called an **order parameter**, changes discontinuously. Often there is a point where the discontinuous phase transition becomes continuous. Such a point is called a **critical point**. The continuous phase transition is called a **second-order phase transition**.

Binary alloys, fluids, and helium-4 exhibit phase transitions.

The archetypical example is the ferromagnet. In an idealized ferromagnet with spins parallel or antiparallel to a given axis, the order parameter is the total magnetization M along this axis. At low temperatures, an external magnetic field H will favor one spin direction. At $H = 0$ the states will be in equilibrium, but small changes in H will make M change discontinuously. There is a first-order phase transition at $H = 0$ at low temperatures.

At slightly higher temperatures, the spins fluctuate and the mean value of $|M|$ decreases. At the **critical temperature** T_c, tiny changes in H no longer make M change discontinuously. At $T > T_c$, the phase transition is continuous and second order.

Landau used a Gibbs free-energy potential G which obeys

$$\left.\frac{\partial G}{\partial M}\right|_T = -H. \tag{8.1}$$

He worked at $T \approx T_c$ and $M \approx 0$ and expanded $G(M)$ at $H = 0$ as

$$G(M) = A(T) + B(T)M^2 + C(T)M^4 + \dots \tag{8.2}$$

keeping only even terms because of the symmetry under $M \to -M$. So at

$H = 0$ equation (8.1) gives

$$0 = -H = \left.\frac{\partial G}{\partial M}\right|_T = 2B(T)M + 4C(T)M^3. \tag{8.3}$$

If B and C are positive, the only solution is $M = 0$. But if $C > 0$ and if for $T < T_c$ it happens that $B < 0$, then we get

$$2CM^2 + B = 0 \quad \text{or} \quad M = \sqrt{-B/2C}. \tag{8.4}$$

The approximations

$$B(T) \approx b(T - T_c) \quad \text{and} \quad C(T) = c \tag{8.5}$$

give

$$M = \pm\sqrt{(b/2c)(T_c - T)} \quad \text{for} \quad T < Tc \tag{8.6}$$

while $M = 0$ for $T > T_c$.

The minimum of the potential

$$G(M, H) = A(T) + B(T)M^2 + C(T)M^4 - HM \tag{8.7}$$

as a function of M at constant T gives equation (8.3).

One may represent the magnetization in terms of a spin density $s(\boldsymbol{x})$ as

$$M = \int s(\boldsymbol{x})\, d^3x. \tag{8.8}$$

Gibb's free energy then becomes

$$G = \int \frac{1}{2}(\nabla s)^2 + b(T - T_c)s^2 + cs^4 - Hs\, d^3x. \tag{8.9}$$

The spin density that minimizes the free energy must make G stationary

$$0 = \frac{\delta G}{\delta s(\boldsymbol{x})} = -\nabla^2 s(\boldsymbol{x}) + 2b(T - T_c)s(\boldsymbol{x}) + 4cs^3(\boldsymbol{x}) - H(\boldsymbol{x}). \tag{8.10}$$

For $T > T_c$, the macroscopic magnetization vanishes and so the spin density s is small. So we drop the nonlinear term and get

$$\left(-\nabla^2 + 2b(T - T_c)\right) s(\boldsymbol{x}) = H(\boldsymbol{x}). \tag{8.11}$$

The spin density $D(\boldsymbol{x})$ generated by $H(\boldsymbol{x}) = H_0\delta^3(\boldsymbol{x})$ is the Green's function of the differential equation

$$\left(-\nabla^2 + 2b(T - T_c)\right) D(\boldsymbol{x}) = H_0\, \delta(\boldsymbol{x}). \tag{8.12}$$

Expressing both sides as Fourier transforms, we get for $T > T_c$

$$\begin{aligned} D(\boldsymbol{x}) &= \int \frac{d^3k}{(2\pi)^3} \frac{H_0 e^{i\boldsymbol{k}\cdot\boldsymbol{x}}}{\boldsymbol{k}^2 + 2b(T-T_c)} \\ &= \frac{H_0}{4\pi r}\, e^{-r/\xi} \end{aligned} \tag{8.13}$$

where the **correlation length**

$$\xi = [2b(T-T_c)]^{-1/2} \tag{8.14}$$

which diverges as $T \to T_c$ indicating the onset of large-scale fluctuations.

Different classes of systems have different **critical exponents**. For example, when $T < T_c$, the magnetization can go as

$$M \approx (T_c - T)^{\beta} \tag{8.15}$$

in which instead of $\beta = 1/2$ one has $\beta = 0.313$ for some fluids and some 3D magnets.

The susceptibility $\chi = (\partial M/\partial H)|_H = 0$ diverges as $|T - T_c|^{-\gamma}$. We differentiate the equation (8.3) with respect to H

$$0 = -H = \left.\frac{\partial G}{\partial M}\right|_T = 2B(T)M + 4C(T)M^3 \tag{8.16}$$

getting

$$-1 = 2B\chi + 4C3M^2\chi. \tag{8.17}$$

So

$$\chi = \frac{-1}{2B + 12CM^2}. \tag{8.18}$$

Setting $B = b(T - T_c)$, $C = c$, and $M = \pm\sqrt{(b/2c)(T_c - T)}$, we get

$$\chi = \frac{-1}{2b(T-T_c) + 6b(T_c - T)} = \frac{-1}{4b(T_c - T)} = \frac{1}{4b(T - T_c)} \tag{8.19}$$

which says $\gamma = 1$.

9

Effective field theories and gravity

9.1 Effective field theories

Suppose a field ϕ whose mass M is huge compared to accessible energies interacts with a field ψ of a low-energy theory such as the standard model

$$L_\phi = -\tfrac{1}{2}\partial_a\phi(x)\,\partial^a\phi(x) - \tfrac{1}{2}M^2\phi^2(x) + g\,\overline{\psi}(x)\psi(x)\phi(x). \tag{9.1}$$

Compared to the mass term M^2, the derivative terms $\partial_a\phi\,\partial^a\phi$ contribute little to the low-energy path integral. So we drop them and represent the effect of the heavy field ϕ as $L_0 = -\frac{1}{2}M^2\phi^2 + g\,\overline{\psi}\psi\phi$. Completing the square

$$L_0 = -\tfrac{1}{2}M^2\left(\phi - \frac{g}{M^2}\overline{\psi}\psi\right)^2 + \frac{g^2}{2M^2}\,(\overline{\psi}\psi)^2 \tag{9.2}$$

and shifting ϕ by $g\overline{\psi}\psi/M^2$, we evaluate the path integral over ϕ as

$$\int \exp\left[i\int -\tfrac{1}{2}M^2\phi^2 + \frac{g^2}{2M^2}(\overline{\psi}\psi)^2\,d^4x\right]D\phi = \exp\left[i\int \frac{g^2}{2M^2}(\overline{\psi}\psi)^2\,d^4x\right]$$

apart from a field-independent factor. The net effect of heavy field ϕ is thus to add to the low-energy theory a new interaction

$$L_\phi(\psi) = \frac{g^2}{2M^2}(\overline{\psi}\psi)^2 \tag{9.3}$$

which is small because M^2 is large.

If a gauge boson Y_a of huge mass M interacts with a spin-one-half field ψ as

$$\begin{aligned} L_0 &= -\tfrac{1}{2}M^2Y_aY^a + ig\overline{\psi}\gamma^a\psi Y_a \qquad (9.4)\\ &= -\tfrac{1}{2}M^2\left(Y_a - \frac{ig}{M^2}\overline{\psi}\gamma_a\psi\right)\left(Y^a - \frac{ig}{M^2}\overline{\psi}\gamma^a\psi\right) - \frac{g^2}{2M^2}\,\overline{\psi}\gamma^a\psi\,\overline{\psi}\gamma_a\psi \end{aligned}$$

then the new low-energy interaction is

$$S_Y = -\frac{g^2}{2M^2}\,\overline{\psi}\gamma^a\psi\,\overline{\psi}\gamma_a\psi. \tag{9.5}$$

The low-energy theory of the weak interactions by Fermi and by Gell-Mann and Feynman is an example, but with $\gamma^a \to \gamma^a(1+\gamma^5)$.

One way of explaining the lightness of the masses of the (known) neutrinos is to say that new a field of very high mass M plays a role, and that when one path-integrates over this heavy field, one is left with an effective, nonrenormalizable term in the action

$$\frac{g^2}{M}\overline{L}\sigma_2 H^* H^{\mathsf{T}}\sigma_2 L \tag{9.6}$$

which gives a tiny mass to ν_ℓ. Here's how this can work: take as part of the action density of the high-energy theory

$$L_\psi = -\overline{\psi}(\partial\!\!\!/ + M)\psi + g\,\overline{\psi}H^{\mathsf{T}}\sigma_2 L + g\,\overline{L}\sigma_2 H^*\psi \tag{9.7}$$

where M is huge. Drop $\partial\!\!\!/$ and complete the square:

$$L_{\psi 0} = -M\left(\overline{\psi} - \frac{g}{M}\overline{L}\sigma_2 H^*\right)\left(\psi - \frac{g}{M}H^{\mathsf{T}}\sigma_2 L\right) + \frac{g^2}{M}\overline{L}\sigma_2 H^* H^{\mathsf{T}}\sigma_2 L. \tag{9.8}$$

The path integral over the field ψ of mass M yields a field-independent constant and leaves in the action the term

$$L_\nu = \frac{g^2}{M}\overline{L}\sigma_2 H^* H^{\mathsf{T}}\sigma_2 L = -\frac{g^2}{M}(\bar{\nu}h^{*0} - \bar{e}h^-)(h^+ e - h^0\nu). \tag{9.9}$$

The mean value of the Higgs field in the vacuum yields the tiny mass term

$$L_{\nu m} = \frac{g^2 v^2}{2M}\overline{\nu}\nu. \tag{9.10}$$

If the neutrino is a Dirac field, then this is a Dirac mass term. If the neutrino is a Majorana field, then this is a Majorana mass term. In both cases, the mass is intrinsically tiny

$$m_\nu \sim \frac{g^2 v^2}{2M}. \tag{9.11}$$

9.2 Is gravity an effective field theory?

The action of the gravitational field is

$$S_E = \frac{c^3}{16\pi G}\int R\sqrt{g}\;d^4x = \frac{c^3}{16\pi G}\int g^{ik}\,R_{ik}\sqrt{g}\;d^4x. \tag{9.12}$$

The action of fields whose energy-momentum tensor is T^{ik} interacting with gravity is

$$S = \frac{c^3}{16\pi G}\int g^{ik}\, R_{ik}\, \sqrt{g}\; d^4x + \frac{1}{2c}\int T^{ik} g_{ik}\sqrt{g}\, d^4x. \tag{9.13}$$

The action of a free scalar field of mass m in flat spacetime is

$$S_{\phi f} = \frac{1}{2}\int \left(\dot{\phi}^2 - c^2(\nabla\phi)^2 - \frac{m^2c^4}{\hbar^2}\phi^2\right) d^4x \tag{9.14}$$

where $d^4x = cdtd^3x$ and a dot means a time derivative. In curved spacetime and in natural units, the action is

$$S_\phi = -\frac{1}{2}\int \left(\partial^i\phi\partial^k\phi\, g_{ik} + m^2\phi^2\right)\sqrt{g}\, d^4x. \tag{9.15}$$

The total action is

$$S = \frac{c^3}{16\pi G}\int R\sqrt{g}\; d^4x - \frac{1}{2}\int \left(\partial^i\phi\partial^k\phi\, g_{ik} + m^2\phi^2\right)\sqrt{g}\, d^4x. \tag{9.16}$$

The change in $\sqrt{g} \equiv \sqrt{|\det g_{ik}|}$ is

$$\delta\sqrt{g} = -\frac{1}{2}\frac{\delta\det g_{ik}}{\sqrt{g}} = \frac{1}{2}\sqrt{g}\, g^{ik}\delta g_{ki} = -\frac{1}{2}\sqrt{g}\; g_{ik}\,\delta g^{ik}. \tag{9.17}$$

The change in the inverse of the metric is

$$\delta g^{ik} = -g^{ij}g^{\ell k}\delta g_{j\ell}. \tag{9.18}$$

So when the metric changes, the action (9.15 changes by

$$\begin{aligned}
\delta S_\phi &= -\frac{1}{2}\int \left[\partial^i\phi\partial^k\phi\; (\sqrt{g}\,\delta g_{ik} + g_{ik}\,\delta\sqrt{g}) + m^2\phi^2\,\delta\sqrt{g}\right]\; d^4x \\
&= -\frac{1}{2}\int \left[\partial^i\phi\partial^k\phi\left(\sqrt{g}\,\delta g_{ik} + g_{ik}\,\tfrac{1}{2}\sqrt{g}\, g^{j\ell}\delta g_{\ell j}\right) + m^2\phi^2\,\tfrac{1}{2}\sqrt{g}\, g^{ik}\delta g_{ki}\right] d^4x \\
&= -\frac{1}{2}\int \left[\partial^i\phi\partial^k\phi\left(\delta g_{ki} + g_{ik}\,\tfrac{1}{2}\, g^{j\ell}\delta g_{\ell j}\right) + m^2\phi^2\,\tfrac{1}{2}\, g^{ik}\delta g_{ki}\right]\sqrt{g}\, d^4x \\
&= -\frac{1}{2}\int \left[\partial^i\phi\partial^k\phi + \left(\tfrac{1}{2}\partial^j\phi\partial^\ell\phi\, g_{j\ell} + \tfrac{1}{2}m^2\phi^2\right)g^{ik}\right]\delta g_{ki}\sqrt{g}\, d^4x \\
&= \frac{1}{2}\int \left[Lg^{ik} - \partial^i\phi\partial^k\phi\right]\delta g_{ki}\sqrt{g}\, d^4x = \frac{1}{2}\int T^{ik}\,\delta g_{ki}\sqrt{g}\, d^4x
\end{aligned} \tag{9.19}$$

in which the energy-momentum tensor is

$$T^{ik} = Lg^{ik} - \partial^i\phi\,\partial^k\phi. \tag{9.20}$$

Equivalently, in terms of the change in the inverse metric (9.18), the change in the scalar action is

$$\delta S_\phi = -\frac{1}{2}\int T_{ik}\,\delta g^{ki}\,\sqrt{g}\,d^4x \tag{9.21}$$

The variation of the gravitational action is

$$\delta S_E = \frac{c^3}{16\pi G}\int\Big(R_{ik} - \frac{1}{2}\,g_{ik}R\Big)\,\sqrt{g}\;\delta g^{ik}\,d^4x. \tag{9.22}$$

The variation of the whole action must vanish

$$0 = \frac{c^3}{16\pi G}\int\Big(R_{ik} - \frac{1}{2}\,g_{ik}R\Big)\,\sqrt{g}\;\delta g^{ik}\,d^4x - \frac{1}{2c}\int T_{ik}\,\delta g^{ki}\,\sqrt{g}\,d^4x. \tag{9.23}$$

Thus Einstein's equations are

$$R_{ik} - \frac{1}{2}\,g_{ik}\,R = \frac{8\pi G}{c^4}T_{ik}. \tag{9.24}$$

In terms of the Planck mass $m_P = \sqrt{\hbar c/G}$, the way the world works is

$$R_{ik} - \frac{1}{2}\,g_{ik}\,R = \frac{8\pi\hbar}{c^3 m_P^2}T_{ik} \tag{9.25}$$

or in natural units

$$R_{ik} - \frac{1}{2}\,g_{ik}\,R = \frac{8\pi}{m_P^2}T_{ik}. \tag{9.26}$$

The total action is proportional to

$$\begin{aligned}\hat{S} &= \int R\,\sqrt{g}\;d^4x - 8\pi G\int\Big(\partial^i\phi\partial^k\phi\,g_{ik} + m^2\phi^2\Big)\,\sqrt{g}\,d^4x\\ &= \int\Big[R - \frac{8\pi}{m_P^2}\Big(\partial^i\phi\,\partial^k\phi\,g_{ik} + m^2\phi^2\Big)\Big]\sqrt{g}\,d^4x.\end{aligned} \tag{9.27}$$

For a massless scalar field, this is simply

$$\hat{S}_0 = \int\Big(R - \frac{8\pi}{m_P^2}\partial^i\phi\,\partial^k\phi\,g_{ik}\Big)\sqrt{g}\,d^4x. \tag{9.28}$$

The second term in this action $\hat{S}_0$ could arise in a path integral over a gauge field A_i whose mass is $M = m_P$ the Planck mass

$$\begin{aligned}\int\exp\Big\{i\int\Big[-\frac{1}{2}m_P^2\Big(A_i - \frac{4\sqrt{\pi}}{m_P^2}\partial_i\phi\Big)\Big(A^i - \frac{4\sqrt{\pi}}{m_P^2}\partial^i\phi\Big)\\ +\frac{8\pi}{m_P^2}\partial^i\phi\,\partial^k\phi\,g_{ik}\Big]\sqrt{g}\,d^4x\Big\}DA\end{aligned} \tag{9.29}$$

which might arise from an interaction such as

$$L = -\tfrac{1}{2} m_P^2 \, A_i A^i + 4\sqrt{\pi} A_i \partial^i \phi. \tag{9.30}$$

Let's try this again. The usual action is

$$S = \int \sqrt{g}\, d^4x \left[\frac{c^3}{16\pi G} R - \frac{1}{2} \left(\phi^{,i} \phi_{,i} + m^2 \phi^2 \right) \right] \tag{9.31}$$

where $1/G = M^2/(\hbar c)$ in which $M = m_P$ is the Planck mass. So

$$S = \int \sqrt{g}\, d^4x \left[\frac{c^2 M^2}{16\pi\hbar} R - \frac{1}{2} \left(\phi^{,i} \phi_{,i} + m^2 \phi^2 \right) \right]. \tag{9.32}$$

9.3 Quantization of Fields in Curved Space

Some good references for these ideas are:

Einstein's Gravity in a Nutshell by Zee
Quantum Fields in Curved Space by Birrell and Davies,
Conformal Field Theory by Di Francesco, Mathieu, and Sénéchal

Let's focus on scalar fields for simplicity. We usually expand a scalar field in flat space as Fourier might have (1.56)

$$\phi(x) = \int \frac{d^3p}{\sqrt{(2\pi)^3 2p^0}} \left[a(p)\, e^{ip\cdot x} + a^\dagger(p)\, e^{-ip\cdot x} \right]. \tag{9.33}$$

The field ϕ obeys the Klein-Gordon equation

$$(\nabla^2 - \partial_0^2 - m^2)\, \phi(x) \equiv (\Box - m^2)\, \phi(x) = 0 \tag{9.34}$$

because the flat-space modes, which have $p^2 = -m^2$,

$$f_p(x) = e^{ip\cdot x} \tag{9.35}$$

do

$$(\nabla^2 - \partial_0^2 - m^2)\, f_p(x) \equiv (\Box - m^2)\, f_p(x) = 0. \tag{9.36}$$

In terms of these functions $f_p(x)$, the field is

$$\phi(x) = \int \frac{d^3p}{\sqrt{(2\pi)^3 2p^0}} \left[a(p)\, f_p(x) + a^\dagger(p)\, f_p^*(x) \right]. \tag{9.37}$$

In terms of a discrete set of modes f_n, it is

$$\phi(x) = \sum_n \left[a_n(p)\, f_n(x) + a_n^\dagger\, f_n^*(x) \right]. \tag{9.38}$$

The action for a scalar field in a space described by the metric g_{ik} is

$$S = -\frac{1}{2}\int \sqrt{g}\, d^4x \Big[g^{ik}\phi_{,i}\phi_{,k} + \big(m^2 + \xi R\big)\phi^2 \Big] \tag{9.39}$$

in which R is the scalar curvature, ξ is a constant, commas denote derivatives as in $\phi_{,k} = \partial_k \phi$, g^{ij} is the inverse of the metric g_{ij}, and g is the absolute value of the determinant of the metric g_{ij}. If the spacetime metric is g_{ij}, then instead of (9.34), the field ϕ obeys the covariant Klein-Gordon equation

$$\partial_i \Big[\sqrt{g}\, g^{ij}\, \partial_j\, \phi(x) \Big] - \big(m^2 + \xi R\big)\sqrt{g}\, \phi(x) = 0. \tag{9.40}$$

However, to simplify what follows, we will now set $\xi = 0$.

To quantize in the new coordinates or in the gravitational field of the metric g_{ij}, we need solutions $f'_n(x)$ of the curved-space equation (9.40)

$$\partial_i \Big[\sqrt{g}\, g^{ij}\, \partial_j\, f'_n(x) \Big] - m^2 \sqrt{g}\, f'_n(x) = 0 \tag{9.41}$$

which we label with primes to distinguish them from the flat-space solutions (9.35). We use these solutions to expand the field in terms of curved-space annihilation and creation operators, which we also label with primes

$$\phi(x) = \sum_n \Big[a'_n(p)\, f'_n(x) + a'^{\dagger}_n\, f'^{*}_n(x) \Big]. \tag{9.42}$$

The flat-space modes obey the orthonormality relations

$$\begin{aligned} (f_p, f_q) &= i \int d^3x \Big[f^*_p(x)\, \partial_t f_q(x) - \big(\partial_t f^*_p(x)\big) f_q(x) \Big] \\ &= i \int d^3x \Big[e^{-ipx}\, (-iq^0) e^{iqx} - ip^0 e^{-ipx}\, e^{iqx} \Big] \\ &= 2p^0 (2\pi)^3 \delta^3(\boldsymbol{p} - \boldsymbol{q}), \end{aligned} \tag{9.43}$$

$$(f^*_p, f^*_q) = -\, 2p^0 (2\pi)^3 \delta^3(\boldsymbol{p} - \boldsymbol{q}) \quad \text{and} \quad (f_p, f^*_q) = 0. \tag{9.44}$$

In terms of discrete modes, the flat-space scalar product is

$$(f_n, f_m) = i \int d^3x \Big[f^*_n(x)\, \partial_t f_m(x) - \big(\partial_t f^*_n(x)\big) f_m(x) \Big] = \delta_{nm} \tag{9.45}$$

and its orthonormality relations are

$$(f_n, f_m) = \delta_{nm}, \quad (f^*_n, f^*_m) = -\, \delta_{nm} \quad \text{and} \quad (f_n, f^*_m) = 0. \tag{9.46}$$

The scalar product (9.45) is a special case of more general curved-space scalar product

$$(f, g) = i \int_S \sqrt{g_S}\; d^3S\, v^a \Big[f^*(x)\, \partial_a g(x) - \big(\partial_a f^*(x)\big) g(x) \Big] \tag{9.47}$$

in which the integral is over a spacelike surface S with a future-pointing timelike vector v^a, and g_S is the absolute value of the spatial part of the metric g_{ik}. This more general scalar product (9.47) is hermitian

$$(f, g)^* = (g, f). \tag{9.48}$$

It also satisfies the rule

$$(f, g)^* = -(f^*, g^*). \tag{9.49}$$

One may use Gauss's theorem to show (Hawking and Ellis, 1973, section 2.8) that this inner product is independent of S and v. The curved-space modes $f_n(x)$ obey orthonormality relations

$$(f'_n, f'_m) = \delta_{nm}, \quad (f'^*_n, f'^*_m) = -\delta_{nm}, \quad \text{and} \quad (f'_n, f'^*_m) = 0 \tag{9.50}$$

like those (9.46) of the flat-space modes.

The flat-space modes $f_n(x) = e^{ip_n x}$ naturally describe particles of momentum p_n in flat Minkowski space. In curved space, however, there are in general no equally natural modes. We can consider other complete sets of modes $f''_n(x)$ that are solutions of the curved-space Klein-Gordon equation (9.41) and obey the orthonormality relations (9.50). We can use any of these complete sets of mode functions to expand a scalar field $\phi(x)$

$$\begin{aligned}
\phi(x) &= \sum_n a_n f_n(x) + a_n^\dagger f_n^*(x) \\
\phi(x) &= \sum_n a'_n f'_n(x) + a'^\dagger_n f'^*_n(x) \\
\phi(x) &= \sum_n a''_n f''_n(x) + a''^\dagger_n f''^*_n(x).
\end{aligned} \tag{9.51}$$

The curved-space orthonormality relations (9.50) imply that

$$\begin{aligned}
(f_\ell, \phi) &= \sum_n a_n (f_\ell, f_n) + a_n^\dagger (f_\ell, f_n^*) = a_\ell \\
(f'_\ell, \phi) &= \sum_n a'_n (f'_\ell, f'_n) + a'^\dagger_n (f'_\ell, f'^*_n) = a'_\ell \\
(f''_\ell, \phi) &= \sum_n a''_n (f''_\ell, f''_n) + a''^\dagger_n (f''_\ell, f''^*_n) = a''_\ell
\end{aligned} \tag{9.52}$$

and that

$$
\begin{aligned}
(f_\ell^*, \phi) &= \sum_n a_n (f_\ell^*, f_n) + a_n^\dagger (f_\ell^*, f_n^*) = -a_\ell^\dagger \\
(f_\ell'^*, \phi) &= \sum_n a_n' (f_\ell'^*, f_n') + a_n'^\dagger (f_\ell'^*, f_n'^*) = -a_\ell'^\dagger \\
(f_\ell''^*, \phi) &= \sum_n a_n'' (f_\ell''^*, f_n'') + a_n''^\dagger (f_\ell''^*, f_n''^*) = -a_\ell''^\dagger .
\end{aligned}
\tag{9.53}
$$

The completeness of the mode functions lets us expand them in terms of each other. Suppressing the spacetime argument x, we have

$$
f_j' = \sum_i \left(\alpha_{ji} f_i + \beta_{ji}\, f_i^* \right). \tag{9.54}
$$

The curved-space orthonormality relations (9.50) let us identify these Bogoliubov coefficients

$$
\begin{aligned}
(f_\ell, f_j') &= \sum_i \Big[\alpha_{ji} (f_\ell, f_i) + \beta_{ji}\, (f_\ell, f_i^*) \Big] = \alpha_{j\ell} \\
(f_\ell^*, f_j') &= \sum_i \Big[\alpha_{ji} (f_\ell^*, f_i) + \beta_{ji}\, (f_\ell^*, f_i^*) \Big] = -\beta_{j\ell}.
\end{aligned}
\tag{9.55}
$$

To find the inverse relations, we use the completeness of the mode functions f_i' to expand the f_j's

$$
f_j = \sum_i \left(c_{ji} f_i' + d_{ji}\, f_i'^* \right) \tag{9.56}
$$

and then use the orthonormality relations (9.50) to form the inner products

$$
\begin{aligned}
(f_\ell', f_j) &= \sum_i \Big[c_{ji} (f_\ell', f_i') + d_{ji}\, (f_\ell', f_i'^*) \Big] = c_{j\ell} \\
(f_\ell'^*, f_j) &= \sum_i \Big[c_{ji} (f_\ell'^*, f_i') + d_{ji}\, (f_\ell'^*, f_i'^*) \Big] = -d_{j\ell}.
\end{aligned}
\tag{9.57}
$$

The hermiticity (9.48) of the scalar product tells us that the $c_{j\ell}$'s are related to the α's

$$
c_{j\ell} = (f_j, f_\ell')^* = \alpha_{\ell j}^*. \tag{9.58}
$$

The hermiticity (9.48) of the scalar product and the rule (9.49) show that

$$
d_{j\ell} = -(f_\ell'^*, f_j) = -(f_j, f_\ell'^*)^* = (f_j^*, f_\ell') = -\beta_{\ell j}. \tag{9.59}
$$

So the inverse relation (9.56) is

$$
f_j = \sum_i \left(\alpha_{ij}^* f_i' - \beta_{ij}\, f_i'^* \right). \tag{9.60}
$$

The formulas (9.54 & 9.60) that relate the mode functions of different metrics are known as Bogoliubov transformations.

The vacuum state for a given metric is the state that is mapped to zero by all the annihilation operators. Our formulas (9.53) for the annihilation and creation operators

$$\begin{aligned} a_\ell &= (f_\ell, \phi) \quad \text{and} \quad a'_\ell = (f'_\ell, \phi) \\ a_\ell^\dagger &= -(f_\ell^*, \phi) \quad \text{and} \quad a_\ell'^\dagger = -(f_\ell'^*, \phi) \end{aligned} \tag{9.61}$$

let us express the annihilation and creation operators for one metric in terms of those for a different metric. Thus, using our formula (9.54) for the f''s in terms of the f's, we find

$$a'_j = (f'_j, \phi) = \sum_i \Big[\alpha_{ji}\,(f_i, \phi) + \beta_{ji}\,(f_i^*, \phi)\Big] = \sum_i \Big(\alpha_{ji}\, a_i - \beta_{ji}\, a_i^\dagger\Big). \tag{9.62}$$

Our formula (9.60) for the f's in terms of the f''s gives us the inverse relation

$$a_j = (f_j, \phi) = \sum_i \Big[\alpha_{ij}^*\,(f'_i, \phi) - \beta_{ij}\,(f_i'^*, \phi)\Big] = \sum_i \Big(\alpha_{ij}^*\, a'_i + \beta_{ij}\, a_i'^\dagger\Big). \tag{9.63}$$

The annihilation operators a'_j define the vacuum state $|0\rangle'$ by the rules

$$a'_j\,|0\rangle' = 0 \tag{9.64}$$

for all modes j. Thus our formula (9.63) for a_j says that

$$a_j\,|0\rangle' = \sum_i \Big(\alpha_{ij}^*\, a'_i + \beta_{ij}\, a_i'^\dagger\Big)|0\rangle' = \sum_i \beta_{ij}\, a_i'^\dagger|0\rangle' \tag{9.65}$$

The adjoint of this equation is

$$'\langle 0|a_j^\dagger = '\langle 0| \sum_i a'_i \beta_{ij}^*. \tag{9.66}$$

Thus the mean value of the number operator $a_j^\dagger a_j$ in the $|0\rangle'$ vacuum is

$$'\langle 0|a_j^\dagger a_j\,|0\rangle' = '\langle 0| \sum_i a'_i \beta_{ij}^* \sum_k \beta_{kj}\, a_k'^\dagger|0\rangle'. \tag{9.67}$$

The commutation relations

$$[a'_i, a_k'^\dagger] = \delta_{ik} \tag{9.68}$$

and the definition $a'_j\,|0\rangle' = 0$ (9.64) of the vacuum $|0\rangle'$ imply that the average number (9.67) of particles of mode j in the (normalized) vacuum $|0\rangle'$ is

$$'\langle 0|a_j^\dagger a_j\,|0\rangle' = '\langle 0| \sum_{ik} \beta_{ij}^* \beta_{kj}\, \delta_{ik}|0\rangle' = \sum_i \beta_{ij}^* \beta_{ij}. \tag{9.69}$$

It follows that the vacuum of one metric contains particles of the other metric unless the Bogoliubov matrix

$$\beta_{j\ell} = -(f_\ell^*, f_j') \tag{9.70}$$

vanishes. The value of $\beta_{j\ell}$ in the Minkowski-space scalar product (9.43) is

$$\beta_{j\ell} = -(f_\ell^*, f_j') = -i \int d^3x \Big[f_\ell(x)\, \partial_t f_j'(x) - \big(\partial_t f_\ell(x)\big) f_j'(x) \Big]. \tag{9.71}$$

This integral for $\beta_{j\ell}$ is nonzero, for example, when the functions f_ℓ and f_j' have different frequencies but are not spatially orthogonal.

An example due to Rindler. Let us consider the two metrics

$$ds^2 = dx^2 + dy^2 + dz^2 - dt^2 = dr^2 + dy^2 + dz^2 - r^2 du^2 \tag{9.72}$$

in which y and z are the same, r plays the role of x and u that of time. The first metric has $g_{ik} = \eta_{ik}$ and $g = |\det(\eta)| = 1$. The second metric has

$$g_{ik} = \begin{pmatrix} -r^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad g = r^2. \tag{9.73}$$

The inverse metric is

$$g^{ik} = \begin{pmatrix} -r^{-2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{9.74}$$

The equation of motion is

$$\partial_i \Big[\sqrt{g} g^{ik} \partial_k f \Big] = m^2 \sqrt{g} f. \tag{9.75}$$

So we must solve

$$\partial_u(-r^{-1}\partial_u f) + \partial_r(r\partial_r f) + \partial_y(r\partial_y f) + \partial_z(r\partial_z f) = m^2 r f. \tag{9.76}$$

Now u, r, y, z are independent coordinates, so this equation of motion is

$$-r^{-1}\partial_u^2 f + \partial_r(r\partial_r f) + r\partial_y^2 f + r\partial_z^2 f = m^2 r f \tag{9.77}$$

or

$$-r^{-2}\partial_u^2 f + r^{-1}\partial_r f + \partial_r^2 f + \partial_y^2 f + \partial_z^2 f = m^2 f. \tag{9.78}$$

Let's make f a plane wave in the y and z directions

$$f(u, r, y, z) = f(u, r) e^{iyp_y + izP_z}. \tag{9.79}$$

If we also set

$$M^2 = m^2 + P_y^2 + P_z^2, \tag{9.80}$$

then we must solve

$$-r^{-2}\partial_u^2 f + r^{-1}\partial_r f + \partial_r^2 f = M^2 f \tag{9.81}$$

where $f = f(u, r)$. We now make the Daniel-Middleton transformation, separating the dependence of f upon r and u

$$f(u, r) = a(u)b(r). \tag{9.82}$$

Our differential equation (9.81) reduces to

$$-r^{-2}\ddot{a}b + r^{-1}ab' + ab'' = M^2 ab \tag{9.83}$$

in which dots denote ∂_u and primes ∂_r. Dividing by a, we get

$$-\frac{\ddot{a}}{a}\frac{b}{r^2} + \frac{b'}{r} + b'' - M^2 b = 0. \tag{9.84}$$

As $a(u)$, we choose

$$a(u) = e^{i\omega u} \quad \text{and} \quad \frac{\ddot{a}}{a} = -\omega^2. \tag{9.85}$$

Then our equation (9.84) for $b(r)$ is

$$b'' + \frac{b'}{r} - \Big(M^2 - \frac{\omega^2}{r^2}\Big)b = 0 \tag{9.86}$$

or equivalently

$$r^2 b'' + rb' - \Big(M^2 r^2 - \omega^2\Big)b = 0. \tag{9.87}$$

Yi's solution is a modified Bessel function $I_\nu(z)$ for imaginary ν.

Bogoliubov's β matrix is

$$\beta_{j\ell} = -(f_\ell^*, f_j'). \tag{9.88}$$

The scalar product (9.47) uses the metric of one, not both, of the solutions. If we choose the flat-space metric, then we need to write the solution $f_j'(u, r, y, z)$ in terms of the flat-space coordinates t, x, y, z. The mean occupation number (9.69) is the sum

$$\sum_j {}'\langle 0|a_j^\dagger a_j|0\rangle' = \sum_{ij} |\beta_{ij}|^2 = \sum_{ij} |(f_j^*, f_i')|^2. \tag{9.89}$$

9.4 Accelerated coordinate systems

We recall the Lorentz transformations

$$\begin{aligned} t' &= \gamma(t - vx), \quad x' = \gamma(x - vt), \quad y' = y, \quad \text{and} \quad z' = z \\ t &= \gamma(t' + vx'), \quad x = \gamma(x' + vt'), \quad y = y', \quad \text{and} \quad z = z' \end{aligned} \tag{9.90}$$

in which $\gamma = 1/\sqrt{1 - v^2}$. The velocities (in the x direction) are

$$u = \frac{dx}{dt} \quad \text{and} \quad u' = \frac{dx'}{dt'} = \frac{\gamma(dx - vdt)}{\gamma(dt - vdx)} = \frac{(u - v)}{(1 - vu)}. \tag{9.91}$$

So the accelerations (in the x direction) are

$$\begin{aligned} a &= \frac{du}{dt} \\ a' &= \frac{du'}{dt'} = d\Big[\frac{(u - v)}{(1 - vu)}\Big] \Big/ \gamma(dt - vdx) \\ &= \Big[\frac{du}{(1 - vu)} + \frac{(u - v)vdu}{(1 - vu)^2}\Big] \Big/ \gamma(dt - vdx) \\ &= \Big[\frac{du(1 - vu)}{(1 - vu)^2} + \frac{(u - v)vdu}{(1 - vu)^2}\Big] \Big/ \gamma(dt - vdx) \\ &= \frac{du(1 - v^2)}{\gamma(1 - vu)^2(dt - vdx)} \\ &= \frac{(1 - v^2)a}{\gamma(1 - vu)^2(1 - vu)} \\ &= \frac{(1 - v^2)a}{\gamma(1 - vu)^3} = \frac{a}{\gamma^3(1 - vu)^3}. \end{aligned} \tag{9.92}$$

Now we let the acceleration a' be a constant. That is, the acceleration in the (instantaneous) rest frame of the frame moving instantaneously at $u = v$ in the laboratory frame is a constant, $a' = \alpha$. In this case, since $u = v$, we get an equation (Rindler, 2006, sec. 3.7)

$$\begin{aligned} a' = \alpha &= \frac{a}{\gamma^3(1 - vu)^3} = \frac{a}{\gamma^3(1 - v^2)^3} = \frac{a}{(1 - v^2)^{3/2}} \\ &= \frac{1}{(1 - v^2)^{3/2}}\frac{du}{dt} = \frac{1}{(1 - u^2)^{3/2}}\frac{du}{dt} = \frac{d}{dt}\left(\frac{u}{\sqrt{1 - u^2}}\right) \end{aligned} \tag{9.93}$$

that we can integrate

$$\frac{u}{\sqrt{1 - u^2}} = \alpha\,(t - t_0). \tag{9.94}$$

Squaring and solving for u, we find

$$u = \frac{dx}{dt} = \frac{\alpha(t-t_0)}{\sqrt{1+\alpha^2(t-t_0)^2}} \tag{9.95}$$

which we can integrate to

$$x = \alpha^{-1}\int dt\,\frac{\alpha^2(t-t_0)}{\sqrt{1+\alpha^2(t-t_0)^2}} = \frac{\sqrt{1+\alpha^2(t-t_0)^2}-1}{\alpha} + x_0. \tag{9.96}$$

The proper time of the accelerating frame is $\tau = t'$. An interval dt' of proper time is one at which $dx' = 0$. So $dt' = \sqrt{1-v^2}dt$ or $dt\tau = \sqrt{1-u^2}dt$ where $u(t)$ is the velocity (9.95). Integrating the equation

$$dt' = d\tau = \sqrt{1-u^2}dt = \sqrt{1-\frac{\alpha^2(t-t_0)^2}{1+\alpha^2(t-t_0)^2}}\,dt = \frac{dt}{\sqrt{1+\alpha^2(t-t_0)^2}}, \tag{9.97}$$

we get

$$\alpha(t'-t_0') = \alpha(\tau-\tau_0) = \int \frac{\alpha\,dt}{\sqrt{1+\alpha^2(t-t_0)^2}} = \operatorname{arcsinh}(\alpha(t-t_0)). \tag{9.98}$$

So

$$\alpha(t-t_0) = \sinh(\alpha(\tau-\tau_0)). \tag{9.99}$$

The formula (9.96) for x now gives

$$\alpha(x-x_0) = \cosh(\alpha(\tau-\tau_0)). \tag{9.100}$$

9.5 Scalar field in an accelerating frame

For simplicity, we'll work with a real scalar field (1.54)

$$\phi(x) = \int \frac{d^3p}{\sqrt{(2\pi)^3 2p^0}}\left[a(p)\,e^{ip\cdot x} + a^\dagger(p)\,e^{-ip\cdot x}\right]. \tag{9.101}$$

If the field is quantized in a box of volume V, then the expansion of the field is

$$\phi(x) = \sum_k \frac{1}{\sqrt{2\omega_k V}}\left[a(k)\,e^{ik\cdot x} + a^\dagger(k)\,e^{-ik\cdot x}\right]. \tag{9.102}$$

For the scalar field (9.101), the zero-temperature correlation function is

the mean value in the vacuum

$$\begin{aligned}
\langle 0|\phi(\boldsymbol{x},t)\phi(\boldsymbol{x}',t')|0\rangle &= \langle 0| \int \frac{d^3p}{\sqrt{(2\pi)^3 2p^0}} \left[a(p)\, e^{ip\cdot x} + a^\dagger(p)\, e^{-ip\cdot x}\right] \\
&\quad \times \int \frac{d^3p'}{\sqrt{(2\pi)^3 2p'^0}} \left[a(p')\, e^{ip'\cdot x'} + a^\dagger(p')\, e^{-ip'\cdot x'}\right] |0\rangle \\
&= \langle 0| \int \frac{d^3p d^3p'}{(2\pi)^3 \sqrt{2p^0 2p'^0}}\, a(p) a^\dagger(p')\, e^{ip\cdot x - ip'\cdot x'} |0\rangle \\
&= \int \frac{d^3p d^3p'}{(2\pi)^3 \sqrt{2p^0 2p'^0}}\, \delta^3(\boldsymbol{p}-\boldsymbol{p}') e^{ip\cdot x - ip'\cdot x'} \\
&= \int \frac{d^3p}{(2\pi)^3 2p^0}\, e^{ip\cdot(x-x')}. \qquad (9.103)
\end{aligned}$$

This integral is simpler for massless fields. Setting $p = |\boldsymbol{p}|$ and $r = |\boldsymbol{x}-\boldsymbol{x}'|$, we add a small imaginary part to the exponential and find

$$\begin{aligned}
\langle 0|\phi(\boldsymbol{x},t)\phi(\boldsymbol{x}',t')|0\rangle &= \int \frac{d^3p}{(2\pi)^3 2p}\, e^{ipr\cos\theta - ip(t-t'-i\epsilon)} \\
&= \int \frac{pdp\, d\cos\theta}{(2\pi)^2 2}\, e^{ipr\cos\theta - ip(t-t'-i\epsilon)} \\
&= \int \frac{pdp}{(2\pi)^2 2} \left(\frac{e^{ipr} - e^{-ipr}}{ipr}\right) e^{-ip(t-t'-i\epsilon)} \\
&= \int_0^\infty \frac{dp}{(2\pi)^2 2ir} \left(e^{ipr} - e^{-ipr}\right) e^{-ip(t-t'-i\epsilon)} \\
&= \frac{1}{(2\pi)^2 2ir} \left(-\frac{1}{i(r-(t-t'))} - \frac{1}{i(r+(t-t'))}\right) \\
&= \frac{1}{(2\pi)^2 2r} \left(\frac{1}{r-(t-t')} + \frac{1}{r+(t-t')}\right) \\
&= \frac{1}{(2\pi)^2[(\boldsymbol{x}-\boldsymbol{x}')^2 - (t-t')^2]}. \qquad (9.104)
\end{aligned}$$

This is the zero-temperature correlation function.

In the instantaneous rest frame of an accelerated observer moving in the x direction with coordinates (9.99 & 9.100), this two-point function is

$$\begin{aligned}
\langle 0|\phi(\boldsymbol{x},t)\phi(\boldsymbol{x}',t')|0\rangle &= \frac{\alpha^2/(2\pi)^2}{\left[\cosh(\alpha\tau) - \cosh(\alpha\tau')\right]^2 - \left[\sinh(\alpha\tau) - \sinh(\alpha\tau')\right]^2} \\
&= -\frac{\alpha^2}{(4\pi)^2 \sinh^2[\alpha(\tau-\tau')/2]} \qquad (9.105)
\end{aligned}$$

in which the identities

$$\cosh(\alpha\tau)\cosh(\alpha\tau') - \sinh(\alpha\tau)\sinh(\alpha\tau') = \cosh(\alpha(\tau-\tau')) \tag{9.106}$$

and

$$\cosh(\alpha(\tau-\tau')) - 1 = 2\sinh(\alpha(\tau-\tau')/2) \tag{9.107}$$

as well as $\cosh^2\alpha\tau - \sinh^2\alpha\tau = 1$ were used.

Now let's compute the same correlation function at a finite inverse temperature $\beta = 1/(k_B T)$

$$\langle\phi(\mathbf{0},\tau)\phi(\mathbf{0},\tau')\rangle_\beta = \mathrm{Tr}\big[\phi(\mathbf{0},0)\phi(\mathbf{0},t)e^{-\beta H}\big]\Big/\mathrm{Tr}\big(e^{-\beta H}\big)\ . \tag{9.108}$$

The mean value of the number operator for a given momentum k is

$$\langle a^\dagger(k)\,a(k)\rangle_\beta = n_k = \mathrm{Tr}\big[a^\dagger(k)\,a(k)e^{-\beta H}\big]\Big/\mathrm{Tr}\big(e^{-\beta H}\big) \tag{9.109}$$

in which

$$H_0 = \sum_k \omega_k\big(a^\dagger(k)\,a(k) + \tfrac{1}{2}\big). \tag{9.110}$$

The trace over all momenta with $k' \neq k$ is unity, and we are left with

$$\begin{aligned}\langle a^\dagger\,a\rangle_\beta = n_k &= \mathrm{Tr}\big[a^\dagger\,ae^{-\beta\omega_k a^\dagger a}\big]\Big/\mathrm{Tr}\big(e^{-\beta\omega_k a^\dagger a}\big)\\ &= -\frac{1}{\omega_k\mathrm{Tr}\big(e^{-\beta\omega_k a^\dagger a}\big)}\frac{\partial}{\partial\beta}\mathrm{Tr}\big(e^{-\beta\omega_k a^\dagger a}\big)\end{aligned} \tag{9.111}$$

in which $a^\dagger a \equiv a^\dagger(k)\,a(k)$ and the $1/2$ terms have cancelled. The trace is

$$\mathrm{Tr}\big(e^{-\beta\omega_k a^\dagger a}\big) = \sum_n e^{-\beta n\omega_k} = \frac{1}{1-e^{-\beta\omega_k}}. \tag{9.112}$$

So we have

$$\langle a^\dagger\,a\rangle_\beta = -\frac{\big(1-e^{-\beta\omega_k}\big)}{\omega_k}\frac{(-\omega_k e^{-\beta\omega_k})}{\big(1-e^{-\beta\omega_k}\big)^2} = \frac{1}{e^{\beta\omega_k}-1}. \tag{9.113}$$

In the trace (9.108), only terms that don't change the number of quanta in each mode contribute. So the mean value of the correlation function at inverse temperature $\beta = 1/(k_B T)$ is

$$\begin{aligned}\langle\phi(\mathbf{0},\tau)\phi(\mathbf{0},\tau')\rangle_\beta = \sum_k \frac{1}{2kV}\Big\{&\Big(e^{\hbar\omega_k/k_BT}-1\Big)^{-1}e^{i\omega_k(\tau-\tau')}\\ &+\Big[\Big(e^{\hbar\omega_k/k_BT}-1\Big)^{-1}+1\Big]e^{-i\omega_k(\tau-\tau')}\Big\}.\end{aligned} \tag{9.114}$$

In the continuum limit, this τ, τ' correlation function is two integrals. For massless particles, the simpler one requires some regularization because it involves zero-point energies

$$\begin{aligned} A_\beta &= \sum_k \frac{1}{2kV}\, e^{-ik(\tau-\tau')} = \int \frac{d^3k}{(2\pi)^3 2k}\, e^{-ik(\tau-\tau')} = \int \frac{k^2 dk}{(2\pi)^2 k} e^{-ik(\tau-\tau')} \\ &= \int_0^\infty \frac{kdk}{(2\pi)^2} e^{-ik(\tau-\tau')}. \end{aligned} \tag{9.115}$$

We send $\tau - \tau' \to \tau - \tau' + i\epsilon$ and find

$$\begin{aligned} A_\beta &= \int_0^\infty \frac{kdk}{(2\pi)^2} e^{-ik(\tau-\tau'+i\epsilon)} = i\frac{d}{d\tau}\int_0^\infty \frac{dk}{(2\pi)^2} e^{-ik(\tau-\tau'+i\epsilon)} \\ &= i\frac{d}{d\tau}\int_0^\infty \frac{dk}{(2\pi)^2} e^{-ik(\tau-\tau'+i\epsilon)} = \frac{1}{(2\pi)^2}\frac{d}{d\tau}\frac{1}{(\tau-\tau')} \\ &= -\frac{1}{(2\pi)^2}\frac{1}{(\tau-\tau')^2} \end{aligned} \tag{9.116}$$

as $\epsilon \to 0$.

The second integral is

$$B_\beta = \int_0^\infty \frac{kdk}{(2\pi)^2}\left(e^{\beta k} - 1\right)^{-1} 2\cos(k(\tau-\tau')) \tag{9.117}$$

which Mathematica says is

$$B_\beta = \frac{1}{(2\pi)^2(\tau-\tau')^2} - \frac{\operatorname{csch}^2(\pi(\tau-\tau')/\beta)}{4\beta^2}. \tag{9.118}$$

Thus the finite-temperature correlation function is

$$\langle\phi(\mathbf{0},\tau)\phi(\mathbf{0},\tau')\rangle_\beta = A_\beta + B_\beta = -\frac{1}{4\beta^2\sinh^2(\pi(\tau-\tau')/\beta)}. \tag{9.119}$$

Equating this formula to the zero–temperature correlation function (9.105) in the accelerating frame, we find

$$\frac{1}{4\beta^2\sinh^2(\pi(\tau-\tau')/\beta)} = \frac{\alpha^2}{(4\pi)^2\sinh^2[\alpha(\tau-\tau')/2]} \tag{9.120}$$

which says redundantly

$$k_B T = \frac{\alpha}{2\pi} \quad \text{and} \quad \pi k_B T = \frac{\alpha}{2}. \tag{9.121}$$

So a detector accelerating uniformly with acceleration α in the vacuum feels

a nonzero temperature

$$T = \frac{\hbar\alpha}{2\pi c k_B}. \tag{9.122}$$

This result (Davies, 1975) is equivalent to the finding (Hawking, 1974) that a gravitational field of local acceleration g makes empty space radiate at a temperature

$$T = \frac{\hbar g}{2\pi c k_B}. \tag{9.123}$$

Black holes are not black.

9.6 Maximally symmetric spaces

The spheres S^2 and S^3 and the hyperboloids H^2 and H^3 are maximally symmetric spaces. A transformation $x \to x'$ is an **isometry** if $g'_{ik}(x') = g_{ik}(x')$ in which case the distances $g_{ik}(x)dx^i dx^k = g'_{ik}(x')dx'^i dx'^k = g_{ik}(x')dx'^i dx'^k$ are the same. To see what this symmetry condition means, we consider the infinitesimal transformation $x'^\ell = x^\ell + \epsilon y^\ell(x)$ under which to lowest order $g_{ik}(x') = g_{ik}(x) + g_{ik,\ell}\epsilon y^\ell$ and $dx'^i = dx^i + \epsilon y^i_{,j} dx^j$. The symmetry condition requires

$$g_{ik}(x)dx^i dx^k = (g_{ik}(x) + g_{ik,\ell}\epsilon y^\ell)(dx^i + \epsilon y^i_{,j} dx^j)(dx^k + \epsilon y^k_{,m} dx^m) \tag{9.124}$$

or

$$0 = g_{ik,\ell}\, y^\ell + g_{im}\, y^m_{,k} + g_{jk}\, y^j_{,i}. \tag{9.125}$$

The vector field $y^i(x)$ must satisfy this condition if $x'^i = x^i + \epsilon y^i(x)$ is to be a symmetry of the metric $g_{ik}(x)$. Since the covariant derivative of the metric tensor vanishes, $g_{ik;\ell} = 0$, we may write the condition on the symmetry vector $y^\ell(x)$ as

$$0 = y_{i;k} + y_{k;i}. \tag{9.126}$$

In more detail, we subtract $g_{ik;\ell}y^\ell$, which vanishes, from the condition (9.125) that y be a Killing vector:

$$\begin{aligned}
0 &= g_{ik,\ell}y^\ell + g_{im}y^m_{,k} + g_{jk}y^j_{,i} - g_{ik;\ell}y^\ell \\
&= g_{ik,\ell}y^\ell + g_{im}y^m_{,k} + g_{jk}y^j_{,i} - g_{ik,\ell}y^\ell + g_{im}\Gamma^m{}_{\ell k}y^\ell + g_{kj}\Gamma^j{}_{\ell i}y^\ell \\
&= g_{im}y^m_{,k} + g_{jk}y^j_{,i} + g_{im}\Gamma^m{}_{\ell k}y^\ell + g_{kj}\Gamma^j{}_{\ell i}y^\ell \\
&= g_{im}(y^m_{,k} + \Gamma^m{}_{\ell k}y^\ell) + g_{jk}(y^j_{,i} + \Gamma^j{}_{\ell i}y^\ell) \\
&= g_{im}y^m_{;k} + g_{jk}y^j_{;i} = y_{i;k} + y_{k;i}.
\end{aligned} \tag{9.127}$$

Students leery of the last step may use the vanishing of the covariant derivative of the metric tensor and the derivation property

$$(AB)_{;k} = A_{;k}B + A\,B_{;k} \tag{9.128}$$

of covariant derivatives, to show that

$$0 = (g_{im}y^m)_{;k} + (g_{jk}y^j)_{;i} = y_{i;k} + y_{k;i}. \tag{9.129}$$

The symmetry vector y^ℓ is a **Killing** vector (Wilhelm Killing, 1847–1923). We may use symmetry condition (9.125) or (9.126) either to find the symmetries of a space with a known metric or to find a metric with given symmetries.

Example 9.1 (Killing vectors of the sphere S^2) The first Killing vector is $(y_1^\theta, y_1^\phi) = (0, 1)$. Since the components of y_1 are constants, the symmetry condition (9.125) says $g_{ik,\phi} = 0$ which tells us that the metric is independent of ϕ. The other two Killing vectors are $(y_2^\theta, y_2^\phi) = (\sin\phi, \cot\theta\cos\phi)$ and $(y_3^\theta, y_3^\phi) = (\cos\phi, -\cot\theta\sin\phi)$. The symmetry condition (9.125) for $i = k = \theta$ and Killing vectors y_2 and y_3 tell us that $g_{\theta\phi} = 0$ and that $g_{\theta\theta,\theta} = 0$. So $g_{\theta\theta}$ is a constant, which we set equal to unity. Finally, the symmetry condition (9.125) for $i = k = \phi$ and the Killing vectors y_2 and y_3 tell us that $g_{\phi\phi,\theta} = 2\cot\theta g_{\phi\phi}$ which we integrate to $g_{\phi\phi} = \sin^2\theta$. The 2-dimensional space with Killing vectors y_1, y_2, y_3 therefore has the metric

$$g_{ik} = \begin{pmatrix} g_{\theta\theta} & g_{\theta\phi} \\ g_{\phi\theta} & g_{\phi\phi} \end{pmatrix} = \begin{pmatrix} \boldsymbol{e}_\theta\cdot\boldsymbol{e}_\theta & \boldsymbol{e}_\theta\cdot\boldsymbol{e}_\phi \\ \boldsymbol{e}_\phi\cdot\boldsymbol{e}_\theta & \boldsymbol{e}_\phi\cdot\boldsymbol{e}_\phi \end{pmatrix} = \begin{pmatrix} R^2 & 0 \\ 0 & R^2\sin^2\theta \end{pmatrix} \tag{9.130}$$

of the sphere S^2. □

Example 9.2 (Killing vectors of the hyperboloid H^2) The metric

$$(g_{ij}) = R^2\begin{pmatrix} 1 & 0 \\ 0 & \sinh^2\theta \end{pmatrix}. \tag{9.131}$$

of the hyperboloid H^2 is diagonal with $g_{\theta\theta} = R^2$ and $g_{\phi\phi} = R^2 \sinh^2\theta$. The Killing vector $(y_1^\theta, y_1^\phi) = (0, 1)$ satisfies the symmetry condition (9.125). Since $g_{\theta\theta}$ is independent of θ and ϕ, the $\theta\theta$ component of (9.125) implies that $y^\theta_{,\theta} = 0$. Since $g_{\phi\phi} = R^2 \sinh^2\theta$, the $\phi\phi$ component of (9.125) says that $y^\phi_{,\phi} = -\coth\theta\, y^\theta$. The $\theta\phi$ and $\phi\theta$ components of (9.125) give $y^\theta_{,\phi} = -\sinh^2\theta\, y^\phi_{,\theta}$. The vectors $y_2 = (y_2^\theta, y_2^\phi) = (\sin\phi, \coth\theta\cos\phi)$ and $y_3 = (y_3^\theta, y_3^\phi) = (\cos\phi, -\coth\theta\sin\phi)$ satisfy both of these equations. □

The **Lie derivative** $\mathcal{L}_y$ of a scalar field A is defined in terms of a vector field $y^\ell(x)$ as $\mathcal{L}_y A = y^\ell A_{,\ell}$. The Lie derivative $\mathcal{L}_y$ of a contravariant vector F^i is

$$\mathcal{L}_y F^i = y^\ell F^i_{,\ell} - F^\ell y^i_{,\ell} = y^\ell F^i_{;\ell} - F^\ell y^i_{;\ell} \tag{9.132}$$

in which the second equality follows from $y^\ell \Gamma^i_{\ell k} F^k = F^\ell \Gamma^i_{\ell k} y^k$. The Lie derivative $\mathcal{L}_y$ of a covariant vector V_i is

$$\mathcal{L}_y V_i = y^\ell V_{i,\ell} + V_\ell y^\ell_{,i} = y^\ell V_{i;\ell} + V_\ell y^\ell_{;i}. \tag{9.133}$$

Similarly, the Lie derivative $\mathcal{L}_y$ of a rank-2 covariant tensor T_{ik} is

$$\mathcal{L}_y T_{ik} = y^\ell T_{ik,\ell} + T_{\ell k} y^\ell_{,i} + T_{i\ell} y^\ell_{,k}. \tag{9.134}$$

We see now that the condition (9.125) that a vector field y^ℓ be a symmetry of a metric g_{jm} is that its Lie derivative

$$\mathcal{L}_y g_{ik} = g_{ik,\ell}\, y^\ell + g_{im}\, y^m_{,k} + g_{jk}\, y^j_{,i} = 0 \tag{9.135}$$

must vanish.

A maximally symmetric space (or spacetime) in d dimensions has d translation symmetries and $d(d-1)/2$ rotational symmetries which gives a total of $d(d+1)/2$ symmetries associated with $d(d+1)/2$ Killing vectors. Thus for $d = 2$, there is one rotation and two translations. For $d = 3$, there are three rotations and three translations. For $d = 4$, there are six rotations and four translations.

A maximally symmetric space has a curvature tensor that is simply related to its metric tensor

$$R_{ijk\ell} = c\,(g_{ik} g_{j\ell} - g_{i\ell} g_{jk}) \tag{9.136}$$

where c is a constant (Zee, 2013, IX.6). Since $g^{ki} g_{ik} = g^k_k = d$ is the number of dimensions of the space(time), the Ricci tensor and the curvature scalar of a maximally symmetric space are

$$R_{j\ell} = g^{ki} R_{ijk\ell} = c\,(d-1)\, g_{j\ell} \quad \text{and} \quad R = g^{\ell j} R_{j\ell} = c\, d(d-1). \tag{9.137}$$

9.7 Conformal algebra

An isometry requires that $g'_{ik}(x') = g_{ik}(x')$. The looser condition

$$g'_{ik}(x') = \Omega(x')\, g_{ik}(x') \quad \text{with} \quad \Omega(x') > 0 \tag{9.138}$$

says that two metrics are **conformally** related. An equivalent condition is

$$g'_{ik}(x') = x^{\ell}_{,i'} x^{m}_{,k'} g_{\ell m}(x) = \Omega(x') g_{ik}(x'). \tag{9.139}$$

This condition guarantees that angles do not change:

$$\frac{g'_{ik} u_i v_k}{\sqrt{g'_{ij} u_i u_j}\sqrt{g'_{ij} v_i v_j}} = \frac{g_{ik} u_i v_k}{\sqrt{g_{ij} u_i u_j}\sqrt{g_{ij} v_i v_j}}. \tag{9.140}$$

For the infinitesimal transformation

$$x'^{\ell} = x^{\ell} + \epsilon y^{\ell}, \tag{9.141}$$

we approximate the factor Ω as

$$\Omega^2(x') \approx 1 + \epsilon\omega(x'). \tag{9.142}$$

The conformal condition (9.139) then says that

$$g_{\ell m}(\delta^{\ell}_{i} - \epsilon y^{\ell}_{,i})(\delta^{m}_{k} - \epsilon y^{m}_{,k}) = (1 + \epsilon\omega)(g_{ik} + \epsilon y^{\ell} g_{ik,\ell}) \tag{9.143}$$

or

$$g_{im} y^{m}_{,k} + g_{\ell k} y^{\ell}_{,i} + y^{\ell} g_{ik,\ell} + \omega g_{ik} = 0. \tag{9.144}$$

This is Killing's conformal condition.

Multiplication by g^{ik} and summing over i and k gives

$$\delta^{k}_{m} y^{m}_{,k} + \delta^{i}_{\ell} y^{\ell}_{,i} + g^{ik} y^{\ell} g_{ik,\ell} + 4\omega = 0 \tag{9.145}$$

or

$$2 y^{k}_{,k} + y^{\ell} g^{ik} g_{ik,\ell} + 4\omega = 0. \tag{9.146}$$

The function $\omega(x)$ therefore

$$\omega = -\frac{1}{4}\left(2 y^{i}_{,i} + y^{\ell} g^{ik} g_{ik,\ell}\right). \tag{9.147}$$

Substituting this formula for ω into Killing's conformal condition (9.144) gives a condition on the metric g and on the vector y

$$g_{im} y^{m}_{,k} + g_{\ell k} y^{\ell}_{,i} + y^{\ell} g_{ik,\ell} - \left(\frac{1}{2} y^{j}_{,j} + \frac{1}{4} y^{\ell} g^{jn} g_{jn,\ell}\right) g_{ik} = 0. \tag{9.148}$$

When $\omega = 0$, Killing's conformal condition (9.144)

$$g_{im} y^m_{,k} + g_{\ell k} y^\ell_{,i} + y^\ell g_{ik,\ell} = 0 \tag{9.149}$$

is the same as his isometry condition (9.125) .

Using the vanishing of the covariant derivative of the metric tensor as in (9.127), we can write Killing's conformal condition (9.144) more succinctly as

$$y_{i;k} + y_{k;i} + \omega\, g_{ik} = 0 \tag{9.150}$$

or as

$$\mathcal{L}_y g_{ik} = -\omega\, g_{ik}. \tag{9.151}$$

9.8 Conformal algebra in flat space

To find Killing's conformal vectors y in flat space, we set $g_{ik} = \eta_{ik}$ and use the formula (9.147) for ω to find

$$\omega = -\frac{1}{2} y^i_{,i} \tag{9.152}$$

which in d dimensions of spacetime is $\omega = -2y^i_{,i}/d$. So in flat-space, Killing's conformal condition (9.144) is

$$y_{i,k} + y_{k,i} = \frac{1}{2}\eta_{ik}\, y^\ell_{,\ell} \quad \text{or} \quad y_{i,k} + y_{k,i} = \frac{2}{d}\eta_{ik}\, y^\ell_{,\ell}. \tag{9.153}$$

Infinitesimal transformations

$$x'^i = x^i + \epsilon\, y^i \tag{9.154}$$

that obey this rule generate the conformal algebra of Minkowski space.

We already know some of these vectors. The vector

$$y^\ell = a^\ell + b^\ell{}_k x^k \tag{9.155}$$

with a^i and $b^\ell{}_k$ constant and b antisymmetric, $b_{ik} = -b_{ki}$, has a vanishing divergence $y^\ell_{,\ell} = 0$ and satisfies Killing's conformal condition (9.153)

$$y_{i,k} + y_{k,i} = b_{ik} + b_{ki} = \frac{2}{d}\eta_{ik}\, y^\ell_{,\ell} = 0. \tag{9.156}$$

These y's generate the translations a^i and the Lorentz transformations b_{ik}.

A conformal transformation $x \to x'$ changes the metric by no more than an overall factor. That means that

$$\Omega(x')\, \eta_{ik} dx'^i dx'^k = \eta_{ik} dx^i dx^k. \tag{9.157}$$

So if space is just stretched by a constant factor σ, so that $x'^i = \sigma x^i$, then $dx'^i = \sigma dx^i$, and the stretch condition (9.157) is satisfied with $\Omega(x') = \sigma^{-2}$. This kind of conformal transformation is a **dilation**, which some call a dilatation. The Killing vector y is

$$y^i = \sigma x^i \tag{9.158}$$

with σ a constant. This vector obeys Killing's flat-space conformal condition (9.153) because $y^\ell_{,\ell} = c\,d$ in d dimensions, and so

$$y_{i,k} + y_{k,i} = \eta_{ij}\, c\, \delta^j_k + \eta_{kj}\, c\, \delta^j_i = 2\eta_{ik}\, c = \frac{2}{d}\, \eta_{ik}\, y^\ell_{,\ell}\,. \tag{9.159}$$

The **inversion**

$$x'^i = \frac{r^2\, x^i}{x^2} \tag{9.160}$$

where r is a length and $x^2 = x^k x_k$ also obeys Killing's conformal condition (9.153). The change in x'^i is

$$dx'^i = \frac{r^2}{x^2}\left(\delta^i_j - \frac{2x^i x_j}{x^2}\right) dx^j \tag{9.161}$$

so with $\Omega \equiv \Omega(x')$

$$\begin{aligned}
\Omega\eta_{ik} dx'^i dx'^k &= \Omega\eta_{ik}\left(\frac{r^2}{x^2}\right)^2 \left(\delta^i_j - \frac{2x^i x_j}{x^2}\right) dx^j \left(\delta^k_\ell - \frac{2x^k x_\ell}{x^2}\right) dx^\ell \\
&= \Omega\left(\frac{r^2}{x^2}\right)^2 \left[\eta_{j\ell} - \frac{2\eta_{jk}x^k x_\ell}{x^2} - \frac{2\eta_{i\ell}x^i x_j}{x^2} + \frac{4x^2 x_j x_\ell}{(x^2)^2}\right] dx^j dx^\ell \\
&= \Omega\left(\frac{r^2}{x^2}\right)^2 \eta_{j\ell}\, dx^j dx^\ell.
\end{aligned} \tag{9.162}$$

So the stretch condition (9.157) is satisfied with

$$\Omega(x') = \left(\frac{x^2}{r^2}\right)^2. \tag{9.163}$$

Since both inversions and translations obey Killing's conformal condition (9.153), we may combine them so as to shrink an inversion to its infinitesimal form. We invert, translate by a tiny vector a, and re-invert and so find as

the legitimately infinitesimal form of an inversion the transformation

$$x^i \to \frac{x^i}{x^2} \to \frac{x^i}{x^2} + a^i \to \left(\frac{x^i}{x^2} + a^i\right) \Big/ \eta_{jk} \left(\frac{x^j}{x^2} + a^j\right) \left(\frac{x^k}{x^2} + a^k\right)$$

$$= \left(\frac{x^i}{x^2} + a^i\right) \Big/ \eta_{jk} \left(\frac{1}{x^2} + \frac{2a \cdot x}{x^2} + a^2\right) \tag{9.164}$$

$$= x^2 \left(\frac{x^i}{x^2} + a^i\right) \Big/ \left(1 + 2a \cdot x + a^2 x^2\right) \simeq \left(x^i + a^i x^2\right)(1 - 2a \cdot x)$$

$$= x^i + a_k(\eta^{ik} x^2 - 2x^i x^k) \tag{9.165}$$

which is a **conformal transformation**.

If we use ∂^i to differentiate Killing's conformal condition (9.153), then we get

$$\partial^i y_{i,k} + \partial^i y_{k,i} = \frac{2}{d} \eta_{ik}\, \partial^i\, y^{\ell}_{,\ell} \tag{9.166}$$

or, with $\partial^2 = \partial^i \partial_i$ and $\partial \cdot y = \partial_i y^i$,

$$\left(1 - \frac{2}{d}\right) \partial_k\, \partial \cdot y = -\, \partial^2 y_k \tag{9.167}$$

which says that in two dimensions ($d = 2$) every Killing vector is a solution of Laplace's equation:

$$\partial^2 y_k = 0 \qquad \text{if} \quad d = 2. \tag{9.168}$$

There are infinitely many Killing vectors in two-dimensions, a fact exploited in string theory.

Differentiating the same equation (9.167) again, we get

$$(d-2)\, \partial_i \partial_k\, \partial \cdot y = -\, d\, \partial^2 y_{k,i} \tag{9.169}$$

which says that $\partial^2 y_{k,i} = \partial^2 y_{i,k}$. Applying this symmetry and ∂^2 to Killing's conformal condition (9.153), we get

$$\eta_{ik}\, \partial^2 \partial \cdot y = d\, \partial^2 y_{k,i}. \tag{9.170}$$

Comparing the right-hand sides of the last two equations (9.169 & 9.170), we find

$$\left[(d-2)\, \partial_i \partial_k + \eta_{ik} \partial^2\right]\, \partial \cdot y = 0. \tag{9.171}$$

Differentiating the same equation (9.167) a third time, we have

$$\partial^k\, (d-2)\, \partial_k\, \partial \cdot y = -\, d\, \partial^2 \partial^k y_k \tag{9.172}$$

or

$$(d-1)\,\partial^2\,\partial\cdot y = 0. \tag{9.173}$$

In the case of one dimension ($d = 1$), Killing's conformal condition (9.153) reduces to $2\partial_0 y_0 = 2\partial_0 y_0$ and so places no restriction on Killing's vector, as reflected equations (9.167 & 9.173). Every function $y^0(x^0) \equiv y(t)$ is a conformal Killing vector. The condition (9.157) for a finite one-dimensional conformal transformation is

$$\frac{dx'}{dx} = \Omega^{-1}(x'). \tag{9.174}$$

This condition constrains the transformation $x \to x' = x'(x)$ only by requiring that the derivative (9.174) be positive.

The vanishing (9.168) of the laplacian of the Killing vector in $d = 2$ dimensions means that every Killing vector is a solution of Laplace's equation:

$$\partial^2 y^k = 0 \qquad \text{if} \quad d = 2. \tag{9.175}$$

In two-dimensional euclidian space, we let the $d = 2$ space be the complex plane and let y^0 and y^1 be the real and imaginary parts of an analytic (or antianalytic) function:

$$z = x^0 + ix^1 \quad \text{and} \quad f(z) = y^0 + iy^1. \tag{9.176}$$

Then the Cauchy-Riemann conditions

$$y^0_{,0} = y^1_{,1} \quad \text{and} \quad y^0_{,1} = -\,y^1_{,0} \tag{9.177}$$

imply that the real and imaginary parts of every analytic function $f(z)$ are harmonic

$$y^0_{,00} + y^0_{,11} = 0 \quad \text{and} \quad y^1_{,00} + y^1_{,11} = 0 \tag{9.178}$$

because $y^0_{,00} = y^1_{10} = -y^0_{,11}$ and $y^1_{,00} = -y^0_{,10} = -y^1_{,11}$. The real and imaginary parts of every antianalytic function $f(z^*)$ also are harmonic, and so satisfy Killing's condition (9.175). There are infinitely many solutions for the Killing vectors of an infinitesimal conformal transformation in two-dimensional euclidian space.

What about finite conformal transformations in two-dimensional euclidian space? The definition (9.157) of a conformal transformation in flat space is

$$\eta'_{ik} = \Omega(x')\eta_{ik}. \tag{9.179}$$

But metrics transform like this:

$$\eta'_{ik} = \frac{\partial x^j}{\partial x'^i}\frac{\partial x^\ell}{\partial x'^k}\eta_{j\ell} \quad \text{and} \quad \eta'^{ik} = \frac{\partial x'^i}{\partial x^j}\frac{\partial x'^k}{\partial x^\ell}\eta^{j\ell}. \tag{9.180}$$

In two-dimensional euclidian space, η is the 2×2 matrix η is the 2×2 matrix

$$\eta = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{9.181}$$

so these equations say that

$$\begin{aligned}
\Omega = \Omega\eta^{00} &= \frac{\partial x'^0}{\partial x^0}\frac{\partial x'^0}{\partial x^0}\eta^{00} + \frac{\partial x'^0}{\partial x^1}\frac{\partial x'^0}{\partial x^1}\eta^{11} = \left(\frac{\partial x'^0}{\partial x^0}\right)^2 + \left(\frac{\partial x'^0}{\partial x^1}\right)^2 \\
\Omega = \Omega\eta^{11} &= \frac{\partial x'^1}{\partial x^0}\frac{\partial x'^1}{\partial x^0}\eta^{00} + \frac{\partial x'^1}{\partial x^1}\frac{\partial x'^1}{\partial x^1}\eta^{11} = \left(\frac{\partial x'^1}{\partial x^0}\right)^2 + \left(\frac{\partial x'^1}{\partial x^1}\right)^2 \\
0 = \Omega\eta^{01} &= \frac{\partial x'^0}{\partial x^0}\frac{\partial x'^1}{\partial x^0}\eta^{00} + \frac{\partial x'^0}{\partial x^1}\frac{\partial x'^1}{\partial x^1}\eta^{11} = \frac{\partial x'^0}{\partial x^0}\frac{\partial x'^1}{\partial x^0} + \frac{\partial x'^0}{\partial x^1}\frac{\partial x'^1}{\partial x^1}.
\end{aligned} \tag{9.182}$$

More succinctly, these conditions are

$$\begin{aligned}
\left(\frac{\partial x'^0}{\partial x^0}\right)^2 + \left(\frac{\partial x'^0}{\partial x^1}\right)^2 &= \left(\frac{\partial x'^1}{\partial x^0}\right)^2 + \left(\frac{\partial x'^1}{\partial x^1}\right)^2 \\
\frac{\partial x'^0}{\partial x^0}\frac{\partial x'^1}{\partial x^0} &= -\frac{\partial x'^0}{\partial x^1}\frac{\partial x'^1}{\partial x^1}.
\end{aligned} \tag{9.183}$$

These conditions are equivalent either to

$$\frac{\partial x'^0}{\partial x^0} = \frac{\partial x'^1}{\partial x^1} \quad \text{and} \quad \frac{\partial x'^1}{\partial x^0} = -\frac{\partial x'^0}{\partial x^1} \tag{9.184}$$

which are the Cauchy-Riemann conditions $u_x = v_y$ and $v_x = -u_y$ for $f = u + iv$ to be an analytic function of $z = x + iy$ or to

$$\frac{\partial x'^0}{\partial x^0} = -\frac{\partial x'^1}{\partial x^1} \quad \text{and} \quad \frac{\partial x'^1}{\partial x^0} = \frac{\partial x'^0}{\partial x^1} \tag{9.185}$$

which are the Cauchy-Riemann conditions $u_x = -v_y$ and $v_x = u_y$ for $f(\bar{z}) = u + iv$ to be an analytic function of $\bar{z} = x - iy$.

9.8.1 Angles and analytic functions

An analytic function $f(z)$ maps curves in the z plane into curves in the $f(z)$ plane. In general, this mapping preserves angles. To see why, we consider the angle $d\theta$ between two tiny complex lines $dz = \epsilon\exp(i\theta)$ and $dz' = \epsilon\exp(i\theta')$ that radiate from the same point z. This angle $d\theta = \theta' - \theta$ is the phase of the ratio

$$\frac{dz'}{dz} = \frac{\epsilon e^{i\theta'}}{\epsilon e^{i\theta}} = e^{i(\theta'-\theta)}. \tag{9.186}$$

Let's use $w = \rho e^{i\phi}$ for $f(z)$. Then the analytic function $f(z)$ maps dz into

$$dw = f(z + dz) - f(z) \approx f'(z)\, dz \tag{9.187}$$

and dz' into

$$dw' = f(z + dz') - f(z) \approx f'(z)\, dz'. \tag{9.188}$$

The angle $d\phi = \phi' - \phi$ between dw and dw' is the phase of the ratio

$$\frac{dw'}{dw} = \frac{e^{i\phi'}}{e^{i\phi}} = \frac{f'(z)\, dz'}{f'(z)\, dz} = \frac{dz'}{dz} = \frac{e^{i\theta'}}{e^{i\theta}} = e^{i(\theta' - \theta)}. \tag{9.189}$$

So as long as the derivative $f'(z)$ does not vanish, the angle in the w-plane is the same as the angle in the z-plane

$$d\phi = d\theta. \tag{9.190}$$

Analytic functions preserve angles. They are **conformal** maps.

What if $f'(z) = 0$? In this case, $dw \approx f''(z)\, dz^2/2$ and $dw' \approx f''(z)\, dz'^2/2$, and so the angle $d\phi = d\phi' - d\phi$ between these two tiny complex lines is the phase of the ratio

$$\frac{dw'}{dw} = \frac{e^{i\phi'}}{e^{i\phi}} = \frac{f''(z)\, dz'^2}{f''(z)\, dz^2} = \frac{dz'^2}{dz^2} = e^{2i(\theta' - \theta)}. \tag{9.191}$$

So angles are doubled, $d\phi = 2d\theta$.

In general, if the first nonzero derivative is $f^{(n)}(z)$, then

$$\frac{dw'}{dw} = \frac{e^{i\phi'}}{e^{i\phi}} = \frac{f^{(n)}(z)\, dz'^n}{f^{(n)}(z)\, dz^n} = \frac{dz'^n}{dz^n} = e^{ni(\theta' - \theta)} \tag{9.192}$$

and so $d\phi = nd\theta$. The angles increase by a factor of n.

Example 9.3 (z^n) The function $f(z) = z^n$ has only one nonzero derivative $f^{(k)}(0) = n!\, \delta_{nk}$ at the origin $z = 0$. So at $z = 0$ the map $z \to z^n$ scales angles by n, $d\phi = n\, d\theta$, but at $z \neq 0$ the first derivative $f^{(1)}(z) = nz^{n-1}$ is not equal to zero. So z^n is conformal except at the origin. □

Example 9.4 (Möbius transformation) The function

$$f(z) = \frac{az + b}{cz + d} \tag{9.193}$$

maps (straight) lines into lines and circles and circles into circles and lines, unless $ad = bc$ in which case it is the constant b/d. □

But if we require that $f(z)$ be entire and invertible on the Riemann sphere (the complex plane with the "point" $z = \infty$ mapped onto the north pole) so that the transformations form the **special conformal group**, then the function $f(z)$ cannot vanish at more than one value of z or $f^{-1}(0)$ would not be unique. Similarly, $f(z)$ can't have more than one pole. So the only functions $f(z)$ that are entire and invertible on the Riemann sphere are those of the **projective** transformation

$$f(z) = \frac{az+b}{cz+d} \quad \text{with} \quad ad - bc = 1 \tag{9.194}$$

and a, b, c, and d complex, which is a special case of the fractional linear or Möbius transformation (9.193). These matrices form the group $SL(2, \mathbb{C})$ which is isomorphic to the Lorentz group $SO(3,1)$. So the requirement that the finite transformations form a group has reduced the symmetry of the conformal group to that of the Lorentz group.

In two-dimensional Minkowski space, we must solve Killing's equation for the vector of an infinitesimal transformation

$$y^k_{,11} = y^k_{,00}. \tag{9.195}$$

So we use Dirac's light-cone coordinates. We set

$$x^\pm = t \pm x. \tag{9.196}$$

Then

$$\begin{aligned} ds^2 = & -dt^2 + dx^2 = -(dt+dx)(dt-dx) = \eta_{ik} dx^i dx^k \\ & = \eta_{--} dx^- dx^- + \eta_{++} dx^+ dx^+ + \eta_{-+} dx^+ dx^- + \eta_{+-} dx^- dx^+ \end{aligned} \tag{9.197}$$

with $\eta_{-+} = \eta_{+-} = -\frac{1}{2}$ and $\eta_{--} = \eta_{++} = 0$. So the inverse metric has $\eta^{-+} = \eta^{+-} = -2$. The light-cone derivatives are

$$\partial_\pm = \frac{1}{2}\left(\partial_t \pm \partial_x\right). \tag{9.198}$$

They act like this:

$$\begin{aligned} \partial_+ x^+ = 1 \quad & \text{and} \quad \partial_- x^- = 1 \\ \partial_+ x^- = 0 \quad & \text{and} \quad \partial_- x^+ = 0. \end{aligned} \tag{9.199}$$

In light-cone coordinates, the equation (9.195) for the $d = 2$ Killing vector is

$$\partial_- \partial_+ y^k = \partial_+ \partial_- y^k = 0. \tag{9.200}$$

Any vector $y^k(x^+)$ satisfies this equation because

$$\partial_- y^k(x^+) = y'^k \partial_- x^+ = 0. \tag{9.201}$$

Similarly, any function $y^k(x^-)$ obeys the same equation. So there are two sets of infinitely many solutions $y^k(x^+)$ and $y^k(x^-)$ of Killing's equation (9.200). We let

$$y^k(x^+, x^-) = y^k_+(x^+) + y^k_-(x^-). \tag{9.202}$$

To understand finite conformal transformations in 2-d Minkowski space, we return to the definition (9.179)

$$\eta'_{ik} = \Omega(x')\eta_{ik}. \tag{9.203}$$

of a conformal transformation in flat space and recall how (9.204) metrics transform

$$\eta'_{ik} = \frac{\partial x^j}{\partial x'^i}\frac{\partial x^\ell}{\partial x'^k}\eta_{j\ell} \quad \text{and} \quad \eta'^{ik} = \frac{\partial x'^i}{\partial x^j}\frac{\partial x'^k}{\partial x^\ell}\eta^{j\ell}. \tag{9.204}$$

In two-dimensional Minkowski space, η is the 2×2 matrix

$$\eta = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{9.205}$$

so these equations say that

$$\begin{aligned}
-\Omega = \Omega\eta^{00} &= \frac{\partial x'^0}{\partial x^0}\frac{\partial x'^0}{\partial x^0}\eta^{00} + \frac{\partial x'^0}{\partial x^1}\frac{\partial x'^0}{\partial x^1}\eta^{11} = -\left(\frac{\partial x'^0}{\partial x^0}\right)^2 + \left(\frac{\partial x'^0}{\partial x^1}\right)^2 \\
\Omega = \Omega\eta^{11} &= \frac{\partial x'^1}{\partial x^0}\frac{\partial x'^1}{\partial x^0}\eta^{00} + \frac{\partial x'^1}{\partial x^1}\frac{\partial x'^1}{\partial x^1}\eta^{11} = -\left(\frac{\partial x'^1}{\partial x^0}\right)^2 + \left(\frac{\partial x'^1}{\partial x^1}\right)^2 \\
0 = \Omega\eta^{01} &= \frac{\partial x'^0}{\partial x^0}\frac{\partial x'^1}{\partial x^0}\eta^{00} + \frac{\partial x'^0}{\partial x^1}\frac{\partial x'^1}{\partial x^1}\eta^{11} = -\frac{\partial x'^0}{\partial x^0}\frac{\partial x'^1}{\partial x^0} + \frac{\partial x'^0}{\partial x^1}\frac{\partial x'^1}{\partial x^1}.
\end{aligned} \tag{9.206}$$

More succinctly, these conditions are

$$\begin{aligned}
\left(\frac{\partial x'^0}{\partial x^0}\right)^2 - \left(\frac{\partial x'^0}{\partial x^1}\right)^2 &= -\left(\frac{\partial x'^1}{\partial x^0}\right)^2 + \left(\frac{\partial x'^1}{\partial x^1}\right)^2 \\
\frac{\partial x'^0}{\partial x^0}\frac{\partial x'^1}{\partial x^0} &= \frac{\partial x'^0}{\partial x^1}\frac{\partial x'^1}{\partial x^1}.
\end{aligned} \tag{9.207}$$

In terms of light-cone coordinates, these conditions are

$$\begin{aligned}
\partial_+ x'^0\, \partial_- x'^0 &= -\,\partial_+ x'^1\, \partial_- x'^1 \\
\partial_+ x'^0\, \partial_- x'^1 &= -\,\partial_- x'^0\, \partial_+ x'^1.
\end{aligned} \tag{9.208}$$

We can satisfy them by having x'^k be functions either of x^+ or of x^-:

$$x'^k(t,x) = x^k(x^+) \quad \text{or} \quad x'^k(t,x) = x^k(x^-) \tag{9.209}$$

but not both as would be okay (9.202) for infinitesimal transformations.

Example 9.5 (Liouville's equation) Liouville's equation is

$$\phi_{tt} - \phi_{xx} + 2e^{\phi} = 0 \tag{9.210}$$

or

$$\partial_+ \partial_- \phi = -\tfrac{1}{2}\, e^{\phi}. \tag{9.211}$$

It has as a solution

$$e^{\phi} = \frac{4 f'(x^+) g'(x^-)}{(f(x^+) - g(x^-))^2} \tag{9.212}$$

in which $f'\, g' > 0$. □

In more than two dimensions ($d \geq 3$), the vanishing (9.173) of the laplacian of the divergence $\partial \cdot y$ and the third-order equation (9.171) imply that the all the second derivatives of the divergence vanish

$$\partial_i\, \partial_k\, (\partial \cdot y) = 0. \tag{9.213}$$

So the divergence of Killing's vector must be at most linear in the coordinates

$$\partial \cdot y = a + b_i x^i \quad \text{for} \quad d \geq 3. \tag{9.214}$$

To see what this implies, we differentiate Killing's conformal condition (9.153)

$$y_{i,k\ell} + y_{k,i\ell} = \frac{2}{d} \eta_{ik} (\partial \cdot y)_{,\ell}, \tag{9.215}$$

add to it the same equation with k and ℓ interchanged

$$y_{i,\ell k} + y_{\ell,ik} = \frac{2}{d} \eta_{i\ell} (\partial \cdot y)_{,k}, \tag{9.216}$$

and then subtract the same equation with i, k, ℓ permuted

$$y_{k,\ell i} + y_{\ell,ki} = \frac{2}{d} \eta_{k\ell} (\partial \cdot y)_{,i} \tag{9.217}$$

so as to get

$$2 y_{i,k\ell} = \frac{2}{d} \left[\eta_{ik} (\partial \cdot y)_{,\ell} + \eta_{i\ell} (\partial \cdot y)_{,k} - \eta_{k\ell} (\partial \cdot y)_{,i} \right]. \tag{9.218}$$

So if we substitute the at-most-linear condition (9.214) into this equation (9.218), then we find

$$2 y_{i,k\ell} = \frac{2}{d} \left[\eta_{ik} b_\ell + \eta_{i\ell} b_k - \eta_{k\ell} b_i \right] \tag{9.219}$$

which says that y^i can be at most quadratic in the coordinates.

$$y_i = \alpha_i + \beta_{ij}x^j + \gamma_{ijk}x^j x^k \tag{9.220}$$

in which $\gamma_{ijk} = \gamma_{ikj}$.

Killing's conformal condition (9.153) must hold for all x's and is linear in the vector y, so we can impose it successively on a_i, b_{ij}, and c_{ijk} by differentiating with respect to the variables x^ℓ and setting them equal to zero. The vector α^i is an arbitrary translation. The condition on β_{ij} is

$$\beta_{ik} + \beta_{ki} = \frac{2}{d}\eta_{ik}\,\beta^\ell{}_\ell \tag{9.221}$$

which says that β_{ij} is the sum

$$\beta_{ij} = b_{ij} + c\,\eta_{ij} \tag{9.222}$$

of an antisymmetric part $b_{ij} = -b_{ji}$ that generates a Lorentz transformation and a term proportional to η_{ij} that generates a scale transformation. The conformal condition (9.153) makes the quadratic term satisfy the relations

$$\gamma_{ik\ell} + \gamma_{ki\ell} = 2\eta_{ik}d_\ell \tag{9.223}$$

$$\gamma_{i\ell k} + \gamma_{\ell ik} = 2\eta_{i\ell}d_k \tag{9.224}$$

$$\gamma_{k\ell i} + \gamma_{\ell ki} = 2\eta_{k\ell}d_i \tag{9.225}$$

in which

$$d_\ell = \frac{1}{d}\gamma^r{}_{r\ell}. \tag{9.226}$$

Subtracting the last equation (9.225) from the sum of the first two (9.223 & 9.224), we find

$$\gamma_{ik\ell} = \eta_{ik}d_\ell + \eta_{i\ell}d_k - \eta_{k\ell}d_i. \tag{9.227}$$

The corresponding infinitesimal transformation is

$$x'^i = x^i + 2(x\cdot d)x^i - d^i x^2 \tag{9.228}$$

which is called a **special conformal transformation** whose exponential or finite form is

$$x'^i = \frac{x^i - d^i x^2}{1 - 2d\cdot x + d^2x^2}. \tag{9.229}$$

The general Killing vector and its infinitesimal transformation are

$$y^i = a^i + b^i{}_k x^k + cx^i + d_k(\eta^{ik}x^2 - 2x^i x^k) \tag{9.230}$$

and

$$x'^i = x^i + a^i + b^i{}_k x^k + cx^i + d_k(\eta^{ik}x^2 - 2x^i x^k) \tag{9.231}$$

in which we see a translation, a Lorentz transformation, a dilation, and a special conformal transformation.

The differential forms of the Poincaré transformations and the dilation D and the inversion k are

$$\begin{aligned} &P_i = \partial_i, \quad J_{ik} = (x_i\,\partial_k - x_k\,\partial_i), \\ &D = x^i\,\partial_i, \quad \text{and} \quad K^i = (\eta^{ik}\,x^j\,x_j - 2x^i\,x^k)\,\partial_k. \end{aligned} \tag{9.232}$$

The commutators of the conformal Lie algebra are

$$\begin{aligned} &[P^i, P^j] = 0, \quad [K^i, K^j] = 0, \\ &[D, P^i] = -P^i, \quad [D, J_{ij}] = 0, \quad [D, K^i] = K^i, \\ &[J^{ij}, P^\ell] = -\eta^{i\ell}P^j + \eta^{j\ell}P^i, \quad [J^{ij}, K^\ell] = -\eta^{i\ell}K^j + \eta^{j\ell}K^i \\ &[J^{ij}, J^{\ell m}] = -\eta^{i\ell}J^{jm} - \eta^{jm}J^{i\ell} + \eta^{j\ell}J^{im} + \eta^{im}J^{j\ell}, \\ &[K^i, P^j] = 2J^{ij} + 2\eta^{ij}D. \end{aligned} \tag{9.233}$$

These commutation relations say that D is a Lorentz scalar, and that K^i transforms as a vector.

There are $d\,P$'s, $d\,K$'s, $d(d-1)/2\,J$'s, and one D. That's $(d+2)(d+1)/2$ generators, which is the same number of generators as $SO(d+2)$. In fact, Minkowski space in d dimensions has $SO(d-1,1)$ as its Lorentz group and $SO(d,2)$ as its conformal algebra. The "Lorentz group" of a spacetime with d spatial dimensions and 2 time dimensions is also $SO(d,2)$.

9.9 Maxwell's action is conformally invariant for $d = 4$

Under a change of coordinates from x to x', a covariant vector field $A_i(x)$ changes this way

$$A'_i(x') = x^k_{,i'}\,A_k(x) = \frac{\partial x^k}{\partial x'^i}\,A_k(x). \tag{9.234}$$

So under the tiny change (with a convenient minus sign)

$$x' = x - y \quad \text{or} \quad x = x' + y \tag{9.235}$$

the change in A_i is

$$\begin{aligned} \delta A_i(x) &\equiv A'_i(x) - A_i(x) \\ &= A'_i(x) - A'_i(x') + A'_i(x') - A_i(x) \\ &= y^j\,A'_{i,j}(x') + \left(\delta^j_i + y^j_{,i'}\right)\,A_j(x) - A_i(x) \\ &= y^j\,A'_{i,j}(x') + y^j_{,i'}\,A_j(x). \end{aligned} \tag{9.236}$$

To lowest order in y, this is

$$\delta A_i(x) = y^j A_{i,j}(x) + y^j_{,i} A_j(x). \tag{9.237}$$

So the change in the Maxwell-Faraday tensor is

$$\begin{aligned}
\delta F_{ik} &= \delta\left(\partial_i A_k - \partial_k A_i\right) = \partial_i\left(\delta A_k\right) - \partial_k\left(\delta A_i\right) \\
&= \partial_i\left(y^j A_{k,j}(x) + y^j_{,k} A_j(x)\right) - \partial_k\left(y^j A_{i,j}(x) + y^j_{,i} A_j(x)\right) \\
&= y^j \partial_j F_{ik} + y^j_{,i} A_{k,j} - y^j_{,k} A_{i,j} + y^j_{,ki} A_j - y^j_{,ik} A_j + y^j_{,k} \partial_i A_j - y^j_{,i} \partial_k A_j \\
&= y^j \partial_j F_{ik} + y^j_{,i} F_{jk} + y^j_{,k} F_{ij}.
\end{aligned} \tag{9.238}$$

Thus the change in Maxwell's action density is

$$\begin{aligned}
\delta\left(F^{ik} F_{ik}\right) &= 2\, F^{ik}\, \delta F_{ik} = 2\, F^{ik}\left(y^j \partial_j F_{ik} + y^j_{,i} F_{jk} + y^j_{,k} F_{ij}\right) \\
&= \partial_j\left(y^j F^{ik} F_{ik}\right) - (\partial \cdot y) F^{ik} F_{ik} + 2 y^j_{,i} F^{ik} F_{jk} + 2 y^j_{,k} F^{ik} F_{ij} \\
&= \partial_j\left(y^j F^{ik} F_{ik}\right) - (\partial \cdot y) F^{ik} F_{ik} + 2 y_{j,i} F^{ik} F^j{}_k + 2 y_{j,k} F^{ik} F_i{}^j \\
&= \partial_j\left(y^j F^{ik} F_{ik}\right) - (\partial \cdot y) F^{ik} F_{ik} + 4 y_{j,i} F^{ik} F^j{}_k \\
&= \partial_j\left(y^j F^{ik} F_{ik}\right) - (\partial \cdot y) F^{ik} F_{ik} + 4 y_{k,i} F^{ij} F^k{}_j.
\end{aligned} \tag{9.239}$$

We can rewrite this as

$$\delta\left(F^{ik} F_{ik}\right) = \partial_j\left(y^j F^{ik} F_{ik}\right) + 2 F^{ij} F^k{}_j \left(y_{k,i} + y_{i,k} - \tfrac{1}{2}\eta_{ik} \partial \cdot y\right). \tag{9.240}$$

But if y is a conformal Killing vector obeying (9.153), then this change in the Maxwell action density is

$$\delta\left(F^{ik} F_{ik}\right) = \partial_j\left(y^j F^{ik} F_{ik}\right) + 2 F^{ij} F^k{}_j \left(\frac{2}{d}\eta_{ik} y^\ell_{,\ell} - \frac{1}{2}\eta_{ik} \partial \cdot y\right). \tag{9.241}$$

Iff $d = 4$, this change is a total derivative

$$\delta\left(F^{ik} F_{ik}\right) = \partial_j\left(y^j F^{ik} F_{ik}\right) \iff d = 4. \tag{9.242}$$

The action density of pure (only gauge fields), classical Yang-Mills theory also is conformally invariant in four dimensions ($d = 4$). Quantum effects introduce a mass scale, however, and break conformal invariance.

9.10 Massless scalar field theory is conformally invariant if $d = 2$

Under a tiny change of coordinates $x = x' + y$, the change in a scalar field ϕ is

$$\delta\phi = y^j \phi_{,j} = y_j \phi^{,j}. \tag{9.243}$$

So the change in the action density is

$$\begin{aligned}\delta\left(\frac{1}{2}\phi^{,i}\phi_{,i}\right) &= \phi^{,i}\,\delta\phi_{,i} = \phi^{,i}\,\partial_i\delta\phi = \phi^{,i}\,\partial_i\left(y_j\phi^{,j}\right)\\ &= \phi^{,i}\phi^{,j}y_{j,i} + \phi^{,i}y_j\phi^{,j}_{,i}\\ &= \frac{1}{2}\phi^{,i}\phi^{,j}\left(y_{j,i}+y_{i,j}\right) + \phi^{,i}\phi^{,j}_{,i}\,y_j.\end{aligned} \tag{9.244}$$

If y obeys the condition (9.153) for a conformal Killing vector, then the change (9.244) in the action density is a total divergence in two dimensions

$$\begin{aligned}\delta\left(\frac{1}{2}\phi^{,i}\phi_{,i}\right) &= \frac{1}{2}\phi^{,i}\phi^{,j}\frac{2}{d}\eta_{ij}y^{\ell}_{,\ell} + \phi^{,i}\phi^{,j}_{,i}\,y_j\\ &= \frac{1}{d}\phi^{,i}\phi_{,i}\,y^{\ell}_{,\ell} + \phi^{,i}\phi_{,i\ell}\,y^{\ell} = \frac{1}{d}\phi^{,i}\phi_{,i}\,y^{\ell}_{,\ell} + \frac{1}{2}\left(\phi^{,i}\phi_{,i}\right)_{,\ell}\,y^{\ell}\\ &= \partial_\ell\left(\frac{1}{2}\phi^{,i}\phi_{,i}\,y^{\ell}\right) \iff d=2.\end{aligned} \tag{9.245}$$

Thus the classical theory of a free massless scalar field is conformally invariant in two-dimensional spacetimes.

The rest of this chapter is at best a first draft, not ready for human consumption.

9.11 Christoffel symbols as nonabelian gauge fields

A contravariant vector V^i transforms like $dx'^i = x'^i_{,k}dx^k$ as

$$V'^i(x') = \frac{\partial x'^i}{\partial x^k}\,V^k(x) = x'^i_{,k}\,V^k(x) = E^i{}_k(x)\,V^k(x). \tag{9.246}$$

The 4×4 matrix $E^i{}_k(x) = x'^i_{,k}$ depends upon the spacetime point x and is a member of the huge noncompact group $GL(4,\mathbb{R})$.

The insight of Yang and Mills (section 5.1) lets us define a **covariant derivative** $D_\ell = \partial_\ell + A_\ell$ of a contravariant vector V^i

$$(D_\ell V)^k = (\partial_\ell\delta^k_j + A^k_{\ell j})V^j \tag{9.247}$$

that transforms as

$$\left[(D_\ell V)^k\right]' = \frac{\partial x^i}{\partial x'^\ell}\frac{\partial x'^k}{\partial x^m}(D_i V)^m. \tag{9.248}$$

We need

$$\begin{aligned}\left[(\partial_\ell \delta^k_j + A^k_{\ell j})V^j\right]' &= (\partial'_\ell \delta^k_j + A'^k_{\ell j})V'^j = V'^k_{,\ell'} + A'^k_{\ell j}V'^j \\ &= x^i_{,\ell'} x'^k_{,m}(\partial_i \delta^m_n + A^m_{in})V^n = x^i_{,\ell'} x'^k_{,m}(V^m_{,i} + A^m_{in}V^n)\end{aligned} \tag{9.249}$$

or

$$V'^k_{,\ell'} + A'^k_{\ell j}V'^j = x^i_{,\ell'} x'^k_{,m}(V^m_{,i} + A^m_{in}V^n). \tag{9.250}$$

Decoding the left-hand side, we get

$$\partial_{\ell'}(x'^k_{,m}V^m) + A'^k_{\ell j}V'^j = x^i_{,\ell'} x'^k_{,m}(V^m_{,i} + A^m_{in}V^n) \tag{9.251}$$

or

$$x^i_{,\ell'}\partial_i(x'^k_{,m}V^m) + A'^k_{\ell j}V'^j = x^i_{,\ell'} x'^k_{,m}(V^m_{,i} + A^m_{in}V^n) \tag{9.252}$$

or

$$x^i_{,\ell'} x'^k_{,m}V^m_{,i} + x^i_{,\ell'} x'^k_{,mi}V^m + A'^k_{\ell j}V'^j = x^i_{,\ell'} x'^k_{,m}(V^m_{,i} + A^m_{in}V^n). \tag{9.253}$$

After a cancelation, we get

$$x^n_{,\ell'} x'^k_{,mn}V^m + A'^k_{\ell j}V'^j = x^i_{,\ell'} x'^k_{,m}A^m_{in}V^n \tag{9.254}$$

or

$$A'^k_{\ell j}V'^j = x^p_{,\ell'} x'^k_{,m}A^m_{pn}V^n - x^n_{,\ell'} x'^k_{,mn}V^m \tag{9.255}$$

which is

$$A'^k_{\ell j}x'^j_{,m}V^m = x^p_{,\ell'} x'^k_{,m}A^m_{pn}V^n - x^n_{,\ell'} x'^k_{,mn}V^m \tag{9.256}$$

or after the interchange of m and n in the second term

$$A'^k_{\ell j}x'^j_{,m}V^m = x^p_{,\ell'} x'^k_{,n}A^n_{pm}V^m - x^n_{,\ell'} x'^k_{,mn}V^m. \tag{9.257}$$

Since the vector V^m is arbitrary, we can promote this to an equation with three free indexes

$$A'^k_{\ell j}x'^j_{,m} = x^p_{,\ell'} x'^k_{,n}A^n_{pm} - x^n_{,\ell'} x'^k_{,mn} \tag{9.258}$$

and then multiply by the inverse of $x'^j_{,m}$

$$A'^k_{\ell j}x'^j_{,m}x^m_{,i'} = x^p_{,\ell'} x'^k_{,n}A^n_{pm}x^m_{,i'} - x^n_{,\ell'} x'^k_{,mn}x^m_{,i'} \tag{9.259}$$

so as to get

$$A'^k_{\ell i} = x^m_{,\ell'} x'^k_{,p}A^p_{nm}x^n_{,i'} - x^p_{,\ell'} x'^k_{,np}x^n_{,i'}. \tag{9.260}$$

Partial derivatives of the same kind commute, so $x'^k_{,np} = x'^k_{,pn}$. It follows

that the inhomogeneous term in the transformation law (9.260) for A is symmetric in ℓ and i

$$-x^p_{,\ell'}x'^k_{,np}x^n_{,i'} = -x^n_{,i'}x'^k_{,np}x^p_{,\ell'} = -x^p_{,i'}x'^k_{,pn}x^n_{,\ell'} = -x^p_{,i'}x'^k_{,np}x^n_{,\ell'}. \quad (9.261)$$

The homogeneous term $x^m_{,\ell'}x'^k_{,p}A^p_{nm}x^n_{,i'}$ in the transformation law (9.260) for A is symmetric in ℓ and i if A^p_{nm} is symmetric in n and m, $A^p_{nm} = A^p_{mn}$. Thus we can use gauge fields that are symmetric in their lower indexes, $A^p_{nm} = A^p_{mn}$.

The transformations are

$$V'^k = x'^k_{,j}V^j \equiv E^k{}_jV^j \quad \text{and} \quad V'_\ell = x^i_{,\ell'}V_i \equiv E^{-1\mathsf{T}}{}_\ell{}^iV_i = E^{-1i}{}_\ell V_i \quad (9.262)$$

and

$$\begin{aligned} A'^k_{\ell i} &= x^m_{,\ell'}x'^k_{,p}A^p_{nm}x^n_{,i'} - x^p_{,\ell'}x'^k_{,np}x^n_{,i'} \\ &= E^{-1m}{}_\ell E^k{}_pA^p_{nm}E^{-1n}{}_i - x^p_{,\ell'}x'^k_{,np}x^n_{,i'}. \end{aligned}$$

We decode the inhomogeneous term

$$\begin{aligned} -x^p_{,\ell'}x'^k_{,np}x^n_{,i'} &= -(\partial_{\ell'}x'^k_{,n})x^n_{,i'} = -(\partial_{\ell'}E^k{}_n)x^n_{,i'} = -(\partial_{\ell'}E^k{}_n)E^{-1n}{}_i \\ &= -E^{-1n}{}_i(\partial_{\ell'}E^k{}_n) = E^k{}_n(\partial_{\ell'}E^{-1n}{}_i) = x'^k_{,n}(\partial_{\ell'}x^n_{,i'}) \end{aligned} \quad (9.263)$$

So on a contravariant vector $V' = E^{-1}V$, A' is

$$A'_\ell = E^{-1m}{}_\ell EA_mE^{-1} + E\partial_{\ell'}E^{-1} \quad (9.264)$$

or

$$A'^k_{\ell i} = E^{-1m}{}_\ell E^k_pA^p_{mn}E^{-1n}_i + E^k_n\partial_{\ell'}E^{-1n}_i. \quad (9.265)$$

This gauge field is the affine connection, aka, Christoffel symbol (of the second kind) $A^k{}_{\ell i} = \Gamma^k{}_{\ell i}$. The usual formula for the covariant derivative of a contravariant vector is

$$(D_\ell V)^k = (\partial_\ell\delta^k_j + A^k_{\ell j})V^j = (\partial_\ell\delta^k_j + \Gamma^k{}_{\ell j})V^j = \partial_\ell V^k + \Gamma^k{}_{\ell j}V^j. \quad (9.266)$$

We can write the chain rule identities

$$\frac{\partial x^i}{\partial x'^k}\frac{\partial x'^k}{\partial x^\ell} = \delta^i_\ell = \frac{\partial x'^i}{\partial x^k}\frac{\partial x^k}{\partial x'^\ell} \quad (9.267)$$

in terms of the matrix $E^k{}_\ell = x'^k_{,\ell}$ as

$$\frac{\partial x^i}{\partial x'^k}E^k{}_\ell = \delta^i_\ell = E^i{}_k\frac{\partial x^k}{\partial x'^\ell} \quad (9.268)$$

which show that the inverse of the matrices $E^k{}_\ell$ and $E^i{}_k$ are

$$E^{-1i}{}_k = \frac{\partial x^i}{\partial x'^k} = x^i_{,k'} \quad \text{and} \quad E^{-1k}{}_\ell = \frac{\partial x^k}{\partial x'^\ell} = x^k_{,\ell'}. \tag{9.269}$$

The very general identity

$$0 = \partial_i I = \partial_i (C\, C^{-1}) = C_{,i}\, C^{-1} + C\, C^{-1}_{,i} \tag{9.270}$$

will soon be useful. So we can write

$$\begin{aligned} E^m{}_n\, E^{-1n}{}_{k,\ell'} &= E^m{}_n\, \partial_{\ell'} \frac{\partial x^n}{\partial x'^k} = E^m{}_n\, \partial_{k'} \frac{\partial x^n}{\partial x'^\ell} \\ &= E^m{}_n\, E^{-1n}{}_{\ell,k'} = -E^m{}_{n,k'}\, E^{-1n}{}_\ell. \end{aligned} \tag{9.271}$$

A covariant vector V_j transforms like a derivative ∂_j

$$V'_j(x') = \frac{\partial x^k}{\partial x'^j}\, V_k(x) = x^k_{,j'}\, V_k(x) \equiv E^{-1k}{}_j\, V_k. \tag{9.272}$$

The Yang-Mills trick now requires that we find a gauge field B_ℓ so that the covariant derivative of a covariant vector

$$(D_\ell V)_k = (\partial_\ell \delta^j_k + B^j_{\ell k}) V_j \tag{9.273}$$

transforms as

$$\big[(D_\ell V)_k\big]' = \frac{\partial x^i}{\partial x'^\ell} \frac{\partial x^n}{\partial x'^k} (D_i V)_n. \tag{9.274}$$

We need

$$\begin{aligned} \big[(\partial_\ell \delta^j_k + B^j_{\ell k}) V_j\big]' &= (\partial'_\ell \delta^j_k + B'^j_{\ell k}) V'_j = V'_{k,\ell'} + B'^j_{\ell k} V'_j \\ &= x^i_{,\ell'} x^n_{,k'} (\partial_i \delta^p_n + B^p_{in}) V_p = x^i_{,\ell'} x^n_{,k'} (V_{n,i} + B^p_{in} V_p) \end{aligned} \tag{9.275}$$

or

$$V'_{k,\ell'} + B'^j_{\ell k} V'_j = x^i_{,\ell'} x^n_{,k'} (V_{n,i} + B^p_{in} V_p). \tag{9.276}$$

Decoding the left-hand side, we get

$$\begin{aligned} (x^n_{,k'}\, V_n)_{,\ell'} + B'^j_{\ell k}\, x^n_{,j'}\, V_n &= x^n_{,k'\ell'}\, V_n + x^n_{,k'}\, V_{n,\ell'} + B'^j_{\ell k}\, x^n_{,j'}\, V_n \\ &= x^n_{,k'\ell'}\, V_n + x^n_{,k'}\, V_{n,i}\, x^i_{,\ell'} + B'^j_{\ell k}\, x^n_{,j'}\, V_n. \end{aligned} \tag{9.277}$$

So combining the last two equations, we find

$$x^n_{,k'\ell'}\, V_n + x^n_{,k'}\, V_{n,i}\, x^i_{,\ell'} + B'^j_{\ell k}\, x^n_{,j'}\, V_n = x^i_{,\ell'} x^n_{,k'}\, V_{n,i} + x^i_{,\ell'} x^n_{,k'}\, B^p_{in} V_p \tag{9.278}$$

or

$$x^n_{,k'\ell'}\, V_n + B'^j_{\ell k}\, x^n_{,j'}\, V_n = x^i_{,\ell'} x^n_{,k'}\, B^p_{in} V_p. \tag{9.279}$$

Interchanging p and n in the third term

$$x^n_{,k'\ell'}\, V_n + B'^j_{\ell k}\, x^n_{,j'}\, V_n = x^i_{,\ell'} x^p_{,k'}\, B^n_{ip}\, V_n \tag{9.280}$$

and then using the arbitrariness of V_n, we get

$$x^n_{,k'\ell'} + B'^j_{\ell k}\, x^n_{,j'} = x^i_{,\ell'} x^p_{,k'}\, B^n_{ip}. \tag{9.281}$$

Multiplying by $x'^m_{,n}$ gives

$$x^n_{,k'\ell'}\, x'^m_{,n} + B'^j_{\ell k}\, x^n_{,j'}\, x'^m_{,n} = x^n_{,k'\ell'}\, x'^m_{,n} + B'^m_{\ell k} = x^i_{,\ell'} x^p_{,k'}\, B^n_{ip}\, x'^m_{,n} \tag{9.282}$$

or

$$B'^m_{\ell k} = x^i_{,\ell'} x^p_{,k'}\, B^n_{ip}\, x'^m_{,n} - x^n_{,k'\ell'}\, x'^m_{,n}. \tag{9.283}$$

The inhomogeneous term $-x^n_{,k'\ell'}\, x'^m_{,n}$ is symmetric in k and ℓ. The homogeneous term $x^i_{,\ell'} x^p_{,k'}\, B^n_{ip}\, x'^m_{,n}$ also is symmetric in k and ℓ. Thus we can use gauge fields that are symmetric in their lower indexes, $B^n_{ip} = B^n_{pi}$.

In terms of the matrices

$$E^k{}_\ell = x'^k_{,\ell} \quad \text{and} \quad E^{-1k}{}_\ell = \frac{\partial x^k}{\partial x'^\ell} = x^k_{,\ell'} \tag{9.284}$$

and in view of the identities (9.270) and (9.271), the transformation rule (9.283) for B has the equivalent form

$$B'^m_{\ell k} = (E^{-1\mathsf{T}})_\ell{}^i E^m{}_n B^n_{ip} E^{-1p}{}_k + E^m{}_{n,\ell'} E^{-1n}{}_k. \tag{9.285}$$

The transformation law for A_ℓ is

$$A'^k_{\ell i} = E^{-1m}{}_\ell E^{-1n}{}_i A^p_{nm} E^k{}_p - E^{-1n}{}_i \partial_{\ell'} E^k{}_n. \tag{9.286}$$

We write the formula (9.285) for B' as

$$\begin{aligned} -B'^m_{\ell k} &= -(E^{-1\mathsf{T}})_\ell{}^i E^m{}_n B^n_{ip} E^{-1p}{}_k - E^{-1n}{}_k\, E^m{}_{n,\ell'} \\ &= -(E^{-1\mathsf{T}})_\ell{}^i E^m{}_n B^n_{ip} E^{-1p}{}_k - E^{-1n}{}_\ell\, E^m{}_{n,k'} \end{aligned} \tag{9.287}$$

and do the substitutions $k \to i$, $i \to m$, and $m \to k$

$$-B'^k_{\ell i} = -E^{-1m}{}_\ell E^k{}_n B^n_{mp} E^{-1p}{}_i - E^{-1n}{}_i\, \partial_{\ell'} E^k{}_n. \tag{9.288}$$

The inhomogeneous terms in the equations for A' and B' are the same. We interchange p and n in the formula for B'

$$\begin{aligned} -B'^k_{\ell i} &= -E^{-1m}{}_\ell E^k{}_p B^p_{mn} E^{-1n}{}_i - E^{-1n}{}_i\, \partial_{\ell'} E^k{}_n \\ &= -E^{-1m}{}_\ell E^{-1n}{}_i B^p_{mn} E^k{}_p - E^{-1n}{}_i\, \partial_{\ell'} E^k{}_n. \end{aligned} \tag{9.289}$$

Since the gauge fields are symmetric in their lower indexes, we can write this as

$$-B'^{k}_{\ell i} = -E^{-1m}{}_{\ell} E^{-1n}{}_{i} B^{p}_{nm} E^{k}{}_{p} - E^{-1n}{}_{i}\, \partial_{\ell'} E^{k}{}_{n}. \tag{9.290}$$

Since the transformation laws are the same, we can identify the two gauge fields apart from a minus sign

$$A^{k}_{\ell i} = -B^{k}_{\ell i} = \Gamma^{k}{}_{i\ell}. \tag{9.291}$$

The gauge fields A and B correspond to the Christoffel symbols, the Γ's of the usual notation,

$$\begin{aligned} A^{i}_{\ell k} \quad &: \quad D_\ell V^i = V^i_{;\ell} = V^i_{,\ell} + \Gamma^{i}{}_{k\ell} V^k = V^i_{,\ell} + e^i \cdot e_{k,\ell} V^k \\ B^{k}_{\ell i} \quad &: \quad D_\ell V_i = V_{i;\ell} = V^i_{,\ell} - \Gamma^{k}{}_{i\ell}\, V_k = V_{i,\ell} - e^k \cdot e_{i,\ell}\, V_k \end{aligned} \tag{9.292}$$

9.12 Spin connection

The Lorentz index a on a tetrad c^a_i is subject to local Lorentz transformations. A spin-one-half field ψ_b also has an index b that is subject to local Lorentz transformations. We can use the insight of Yang and Mills to introduce a gauge field, called the **spin connection** ω^{ab}_i which multiplies the generators J_{ab} of the Lorentz group

$$\omega_i = \omega^{ab}_i J_{ab}. \tag{9.293}$$

Since the generators J_{ab} are antisymmetric $J_{ab} = -J_{ba}$, the spin connection is also antisymmetric

$$\omega^{ab}_i = -\omega^{ba}_i. \tag{9.294}$$

So for each spacetime index i, there are six independent ω^{ab}_i's.

The spin connection is

$$\omega^{a}{}_{b\,\ell} = -c^k_b \big(c^a_{k,\ell} - \Gamma^j_{k\ell}\, c^a_j\big) = c^a_j\, c^k_b\, \Gamma^j{}_{k\ell} - c^a_{k,\ell}\, c^k_b. \tag{9.295}$$

Under a general coordinate transformation and a local Lorentz transformation, the spin connection (9.295) transforms as

$$\omega'^{a}{}_{b\,\ell} = \frac{\partial x^i}{\partial x'^\ell} \Big[L^a{}_d\, \omega^d{}_{e\,i} - (\partial_i L^a{}_e) \Big] L^{-1e}{}_b. \tag{9.296}$$

This example is a work in progress....

Example 9.6 (Relativistic particle) The lagrangian of a relativistic particle of mass m is $L = -m\sqrt{1-\dot{q}^2}$ in units with $c = 1$. The hamiltonian is

$H = \sqrt{p^2 + m^2}$. The path integral for the first step (7.63) toward $Z(\beta)$ is

$$\langle q_1 | e^{-\epsilon H} | q_a \rangle = \int_{-\infty}^{\infty} e^{-\epsilon\sqrt{p'^2+m^2}}\; e^{ip'(q_1-q_a)/\hbar}\, \frac{dp'}{2\pi\hbar} \tag{9.297}$$

which is a nontrivial integral. In the $m \to 0$ limit, it is

$$\begin{aligned}
\langle q_1 | e^{-\epsilon H} | q_a \rangle &= \int_{-\infty}^{\infty} e^{-\epsilon|p'|}\; e^{ip'(q_1-q_a)/\hbar}\, \frac{dp'}{2\pi\hbar} \\
&= 2\int_0^{\infty} e^{-\epsilon p'} \cos\Big(\frac{p'(q_1-q_a)}{\hbar}\Big)\, \frac{dp'}{2\pi\hbar} \\
&= \int_0^{\infty} e^{-\epsilon p' + ip'(q_1-q_a)/\hbar} + e^{-\epsilon p - ip'(q_1-q_a)/\hbar}\, \frac{dp'}{2\pi\hbar} \\
&= \Big(\frac{1}{\epsilon - i(q_1-q_a)/\hbar} + \frac{1}{\epsilon + i(q_1-q_a)/\hbar}\Big)\frac{1}{2\pi\hbar} \\
&= \Big(\frac{2\epsilon}{\epsilon^2 + (q_1-q_a)^2/\hbar^2}\Big)\frac{1}{2\pi\hbar} \\
&= \frac{1}{\pi\hbar\epsilon}\Big(\frac{1}{1+(q_1-q_a)^2/(\hbar\epsilon)^2}\Big).
\end{aligned} \tag{9.298}$$

So the partition function is $n \to \infty$ limit of the product of the n integrals

$$Z(\beta) = \left(\frac{1}{\pi\hbar\epsilon}\right)^n \int_{-\infty}^{\infty} \frac{dq_n}{1+(q_n-q_{n-1})^2/(\hbar\epsilon)^2} \cdots \frac{dq_1}{1+(q_1-q_n)^2/(\hbar\epsilon)^2}$$

in which $q_n = q_a$.

$$Z(\beta) = \left(\frac{1}{\pi\hbar\epsilon}\right)^n (\pi\hbar\epsilon)^n. \tag{9.299}$$

In one space dimension, quantum mechanics gives the result

$$Z(\beta) = \frac{L}{\pi\hbar\beta c} \tag{9.300}$$

in which c has reappeared. and we must take the $L \to \infty$ limit. In two space dimensions, it gives

$$\begin{aligned}
Z(\beta) &= \frac{L^2}{(2\pi\hbar)^2}\int_0^{\infty} e^{-\beta c\sqrt{m^2c^2+p^2}} 2\pi p dp = \frac{L^2}{4\pi\hbar^2}\int_0^{\infty} e^{-\beta c\sqrt{m^2c^2+u}} du \\
&= \frac{(Lmc)^2}{4\pi\hbar^2}\int_0^{\infty} e^{-\beta mc^2\sqrt{1+v}} dv = \frac{(Lmc)^2}{4\pi\hbar^2}\frac{2(1+\beta mc^2)}{(\beta mc^2)^2} e^{-\beta mc^2} \\
&= \frac{L^2}{2\pi\hbar^2}\frac{(1+\beta mc^2)}{(\beta c)^2} e^{-\beta mc^2}.
\end{aligned} \tag{9.301}$$

□

10

Field Theory on a Lattice

10.1 Scalar Fields

To represent a hermitian field $\varphi(x)$, we put a real number $\varphi(i,j,k,\ell)$ at each vertex of the lattice and label a lattice of spacetime points x by a 4-vector of integers as

$$x = a(i,j,k,\ell) = as \tag{10.1}$$

where a is the lattice spacing and

$$s = (i,j,k,\ell) \tag{10.2}$$

is a 4-vector of lattice sites. The derivative $\partial_i\varphi(x)$ is approximated as

$$\partial_i\varphi(x) \approx \frac{\varphi(x+\hat{i}) - \varphi(x)}{a} \tag{10.3}$$

in which x is the discrete position 4-vector and $\hat{i}$ is a unit 4-vector pointing in the i direction. So the euclidian action is the sum over all lattice sites of

$$\begin{aligned} S_e &= \sum_x \frac{1}{2}\left(\partial_i\varphi(x)\right)^2\, a^4 + \frac{1}{2}m^2\,\varphi^2(x)\,a^4 \\ &= \sum_x \frac{1}{2}\Big(\varphi(x+\hat{i}) - \varphi(x)\Big)^2\, a^2 + \frac{1}{2}m^2\,\varphi^2(x)\,a^4 \end{aligned} \tag{10.4}$$

We switch to dimensionless variables

$$\phi = a\,\varphi \quad \text{and} \quad \mu = a\,m. \tag{10.5}$$

In these terms, the action density is

$$S_e = -\frac{1}{2}\sum_{xi}\phi(x+\hat{i})\phi(x) + \frac{1}{2}\left(8+\mu^2\right)\,\phi^2(x) \tag{10.6}$$

or if the self interaction happens to be quartic

$$\begin{aligned} S_e &= \frac{1}{2}\left(\partial_i\phi(x)\right)^2\, a^4 + \frac{1}{2}m^2\,\phi^2(x)\,a^4 + \frac{\lambda}{4}\,\phi^4(x)\,a^4 \\ &= -\frac{1}{2}\sum_{xi}\phi(x+\hat{i})\phi(x) + \frac{1}{2}\left(8+\mu^2\right)\,\phi^2(x) + \frac{\lambda}{4}\,\phi^4(x). \end{aligned} \tag{10.7}$$

10.2 Finite-temperature field theory

Since the Boltzmann operator $e^{-\beta H} = e^{-H/(kT)}$ is the time evolution operator $e^{-itH/\hbar}$ at the imaginary time $t = -i\hbar\beta = -i\hbar/(kT)$, the formulas of finite-temperature field theory are those of quantum field theory with t replaced by $-iu = -i\hbar\beta = -i\hbar/(kT)$.

Our hamiltonian is $H = K + V$ where K and V are sums over the vertices of the spatial lattice $v = a(i,j,k) = (ai, aj, ak)$ of the cubes of volume a^3 times the squared momentum and the potential energy

$$\begin{aligned} aH = aK + aV &= \frac{a^4}{2}\sum_v \varpi_v^2 + \frac{a^4}{2}\sum_v (\nabla\varphi_v)^2 + m^2\varphi_v^2 + P(\varphi_v) \\ &= \frac{1}{2}\sum_v \pi_v^2 + \frac{1}{2}\sum_v (\nabla\phi_v)^2 + \mu^2\phi_v^2 + P(\phi_v). \end{aligned} \tag{10.8}$$

We are assuming that the action if quadratic in the time derivatives of the fields. A matrix element of the first term of the Trotter product formula (7.7)

$$e^{-\beta(K+V)} = \lim_{n\to\infty}\left(e^{-\beta K/n}\,e^{-\beta V/n}\right)^n \tag{10.9}$$

is the imaginary-time version of (7.133) with $\epsilon = \hbar\beta/n$

$$\langle\phi_1|e^{-\epsilon K}\,e^{-\epsilon V}|\phi_a\rangle = e^{-\epsilon V(\phi_a)}\prod_v\left[\int\frac{a^3 d\pi'_v}{2\pi}e^{a^3[-\epsilon\pi_v^2/2 + i(\phi_{1v}-\phi_{av})\pi'_v]}\right]. \tag{10.10}$$

Setting $\dot{\phi}_{av} = (\phi_{1v} - \phi_{av})/\epsilon$, we find, instead of (7.134)

$$\langle\phi_1|e^{-\epsilon K}\,e^{-\epsilon V}|\phi_a\rangle = \prod_v\left[\left(\frac{a^3}{2\pi\epsilon}\right)^{1/2} e^{-\epsilon a^3[\dot{\phi}_{av}^2 + (\nabla\phi_{av})^2 + m^2\phi_{av}^2 + P(\phi_v)]/2}\right]. \tag{10.11}$$

The product of $n = \hbar\beta/\epsilon$ such inverse-temperature intervals is

$$\langle\phi_b|e^{-\beta H}|\phi_a\rangle = \prod_v\left[\left(\frac{a^3 n}{2\pi\beta}\right)^{n/2}\int e^{-S_{ev}}D\phi_v\right] \tag{10.12}$$

in which the euclidian action (10.4 or 10.7) is a sum over all the vertices $x = a(i, j, k, \ell)$ of the spacetime lattice

$$S_{ev} = \frac{\beta}{n}\frac{a^3}{2}\sum_{j=0}^{n-1}\left[\dot{\phi}_{jv}^2 + (\nabla\phi_{jv})^2 + m^2\phi_{jv}^2 + P(\phi_v)\right] \tag{10.13}$$

where $\dot{\phi}_{jv} = n(\phi_{j+1,v} - \phi_{j,v})/\beta$ and $D\phi_v = d\phi_{n-1,v}\cdots d\phi_{1,v}$.

The amplitude $\langle\phi_b|e^{-(\beta_b-\beta_a)H}|\phi_a\rangle$ is the integral over all fields that go from $\phi_a(\boldsymbol{x})$ at β_a to $\phi_b(\boldsymbol{x})$ at β_b each weighted by an exponential

$$\langle\phi_b|e^{-(\beta_b-\beta_a)H}|\phi_a\rangle = \int e^{-S_e[\phi]}\,D\phi \tag{10.14}$$

of its euclidian action

$$S_e[\phi] = \int_{\beta_a}^{\beta_b} du \int d^3x\,\frac{1}{2}\left[\dot{\phi}^2 + (\nabla\phi)^2 + m^2\phi^2 + P(\phi)\right] \tag{10.15}$$

in which $D\phi$ is the $n \to \infty$ limit of the product over all spatial vertices v

$$D\phi = \prod_v\left[\left(\frac{a^3 n}{2\pi(\beta_b - \beta_a)}\right)^{n/2} d\phi_{n-1,v}\cdots d\phi_{1,v}\right]. \tag{10.16}$$

Equivalently, the Boltzmann operator is

$$e^{-(\beta_b-\beta_a)H} = \int |\phi_b\rangle e^{-S_e[\phi]}\,\langle\phi_a|\,D\phi\,D\phi_a D\phi_b \tag{10.17}$$

in which $D\phi_a D\phi_b = \prod_v d\phi_{a,v}d\phi_{b,v}$ is an integral over the initial and final states.

The trace of the Boltzmann operator is the partition function

$$Z(\beta) = \mathrm{Tr}(e^{-\beta H}) = \int e^{-S_e[\phi]}\,\langle\phi_a|\phi_b\rangle\,D\phi\,D\phi_a D\phi_b = \int e^{-S_e[\phi]}\,D\phi\,D\phi_a \tag{10.18}$$

which is an integral over all fields that go back to themselves in euclidian time β. So we must require that the field φ is periodic in euclidian time. If na is the extend of the lattice in euclidian time, then

$$\varphi(i, j, k, \ell + n) = \varphi(i, j, k, \ell). \tag{10.19}$$

As a convenience, in scalar field theoeries one usually also imposes periodic boundary conditions in the spatial directions. One then has full periodicity: for any four multiples of the lengths L_i of the lattice in the four directions $n_i = nL_i$, one has

$$\varphi(i + n_1, j + n_2, k + n_3, \ell + n_4) = \varphi(i, j, k, \ell). \tag{10.20}$$

Like a position operator (7.107), a field at an imaginary time $t = -iu = -i\hbar\beta$ is defined as

$$\phi_e(\boldsymbol{x}, u) = \phi_e(\boldsymbol{x}, \hbar\beta) = e^{uH/\hbar}\phi(\boldsymbol{x}, 0)\, e^{-uH/\hbar} \tag{10.21}$$

in which $\phi(\boldsymbol{x}) = \phi(\boldsymbol{x}, 0) = \phi_e(\boldsymbol{x}, 0)$ is the field at time zero, which obeys the commutation relations (7.122). The euclidian-time-ordered product of several fields is their product with newer (higher $u = \hbar\beta$) fields standing to the left of older (lower $u = \hbar\beta$) fields as in the definition (7.109).

We set $\hbar = 1$ to avoid clutter and irrelevant thinking. The euclidian path integrals for the mean values of euclidian-time-ordered-products of fields are similar to those (7.161 & 7.148) for ordinary time-ordered-products. The euclidian-time-ordered-product of the fields $\phi(x_j) = \phi(\boldsymbol{x}_j, u_j)$ is the path integral

$$\langle n|\mathcal{T}[\phi_e(x_1)\phi_e(x_2)]|n\rangle = \frac{\displaystyle\int \langle n|\phi_b\rangle\phi(x_1)\phi(x_2)e^{-S_e[\phi]}\langle\phi_a|n\rangle\, D\phi}{\displaystyle\int \langle n|\phi_b\rangle e^{-S_e[\phi]}\langle\phi_a|n\rangle\, D\phi} \tag{10.22}$$

in which the integrations are over all paths that go from before u_1 and u_2 to after both euclidian times. The analogous result for several fields is

$$\langle n|\mathcal{T}[\phi_e(x_1)\cdots\phi_e(x_k)]|n\rangle = \frac{\displaystyle\int \langle n|\phi_b\rangle\phi(x_1)\cdots\phi(x_k)e^{-S_e[\phi]/\hbar}\langle\phi_a|n\rangle\, D\phi}{\displaystyle\int \langle n|\phi_b\rangle e^{-S_e[\phi]/\hbar}\langle\phi_a|n\rangle\, D\phi} \tag{10.23}$$

in which the integrations are over all paths that go from before the times $u_1, \ldots, u_k$ to after them.

In the low-temperature $\beta = 1/(kT) \to \infty$ limit, the Boltzmann operator is proportional to the outer product $|0\rangle\langle 0|$ of the ground-state kets, $e^{-\beta H} \to e^{-\beta E_0}|0\rangle\langle 0|$. In this limit, the integrations are over all fields that run from $u = -\infty$ to $u = \infty$, and the only energy eigenstate $|n\rangle$ that contributes is the ground state $|0\rangle$ of the theory

$$\langle 0|\mathcal{T}[\phi_e(x_1)\cdots\phi_e(x_k)]|0\rangle = \frac{\displaystyle\int \langle 0|\phi_b\rangle\phi(x_1)\cdots\phi(x_k)e^{-S_e[q]/\hbar}\langle\phi_a|0\rangle\, D\phi}{\displaystyle\int \langle 0|\phi_b\rangle e^{-S_e[q]/\hbar}\langle\phi_a|0\rangle\, D\phi}. \tag{10.24}$$

Formulas like this one are used in lattice gauge theory. For a free scalar field

theory, they take the form of a multiple integral over lattice sites s

$$Z_0(\beta) = \mathrm{Tr}(e^{-\beta H_0}) = \prod_s d\phi_s e^{-S_{0e}} \tag{10.25}$$

in which the free euclidian action (10.6) is a sum over the spacetime lattice with nearest-neighbor coulings

$$\begin{aligned} S_{0e} = & -\frac{1}{2}\sum_{xi} \phi(x+\hat{i})\phi(x) + \frac{1}{2}\left(8+\mu^2\right)\phi^2(x) \\ = & \frac{1}{2}\sum_{n,m} \phi_n K_{nm} \phi_m. \end{aligned} \tag{10.26}$$

The matrix K is $n^4 \times n^4$. Its elements are

$$K_{su} = -\sum_{i>0} \left[\delta_{s+i,u} + \delta_{s-i,u} - 2\delta_{su}\right] + \mu^2 \delta_{su} \tag{10.27}$$

in which $i = (1,0,0,0)$ etc. The integral is a gaussian. If the lattice is four dimensional with n^4 sites, then $Z(\beta)$ is

$$\begin{aligned} Z_0(\beta) = & \prod_s d\phi_s \exp\left(-\frac{1}{2}\sum_{s,s'} \phi_s K_{ss'} \phi_{s'}\right) \\ = & \sqrt{\left(\frac{(2\pi)^{n^4}}{\det K}\right)} \end{aligned} \tag{10.28}$$

in which $\beta = na$. In the presence of a classical current J, the otherwise free partition function becomes

$$Z_0(\beta, J) = \prod_s d\phi_s \exp\left(-\frac{1}{2}\sum_{s,s'} \phi_s K_{ss'} \phi_{s'} + \sum_s J_s \phi_s\right) \tag{10.29}$$

We shift the field to

$$\phi' = \phi + K^{-1} J \tag{10.30}$$

so that

$$\begin{aligned} -\frac{1}{2}\phi' K \phi' + J\cdot\phi' = & -\frac{1}{2}\left(\phi + JK^{-1}\right) K \left(\phi + K^{-1}J\right) + J\cdot\left(\phi + K^{-1}J\right) \\ = & -\frac{1}{2}\phi K \phi + \frac{1}{2} J\cdot K^{-1} J. \end{aligned} \tag{10.31}$$

and

$$
\begin{aligned}
Z_0(\beta, J) &= \prod_s d\phi_s \exp\left(-\frac{1}{2}\sum_{s,s'} \phi_s K_{ss'} \phi_{s'} + \sum_{s,s'} J_s K^{-1}_{s,s'} J_{s'}.\right) \\
&= \sqrt{\left(\frac{(2\pi)^{n^4}}{\det K}\right)} \exp\left(\frac{1}{2}\sum_{s,s'} J_s K^{-1}_{s,s'} J_{s'}\right).
\end{aligned}
\tag{10.32}
$$

The formula (10.29) now gives us the propagator as

$$
\langle \phi_s \phi_{s'} \rangle = Z_0(\beta, J)^{-1} \frac{\partial^2}{\partial J_s \partial J_{s'}} Z_0(\beta, J)\bigg|_{J=0} = K^{-1}_{ss'}. \tag{10.33}
$$

To find $K^{-1}_{ss'}$, we use

$$
\sum_t K_{st} K^{-1}_{tu} = \delta_{su} = \int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4}\, e^{ik\cdot(s-u)}. \tag{10.34}
$$

We also use this formula to write K as

$$
K_{su} = \int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4}\, K(k)\, e^{ik\cdot(s-u)} \tag{10.35}
$$

where

$$
K(k) = 4\sum_{j=1}^{4} \sin^2(k_j/2) + \mu. \tag{10.36}
$$

10.3 The Propagator

10.4 Pure Gauge Theory

The gauge-covariant derivative is defined in terms of the generators t_a of a compact Lie algebra

$$
[t_a, t_b] = i f_{abc} t_c \tag{10.37}
$$

and a gauge-field matrix $A_i = igA_i^b\, t_b$ as

$$
D_i = \partial_i - A_i = \partial_i - igA_i^b\, t_b \tag{10.38}
$$

summed over all the generators, and g is a coupling constant. Since the group is compact, we may raise and lower group indexes without worrying about factors or minus signs.

The Faraday matrix is

$$
F_{ij} = [D_i, D_j] = [I\partial_i - A_i(x), I\partial_j - A_j(x)] = -\partial_i A_j + \partial_j A_i + [A_i, A_j] \tag{10.39}
$$

in matrix notation. With more indices exposed, it is

$$\begin{aligned}(F_{ij})_{cd} &= (-\partial_i A_j + \partial_j A_i + [A_i, A_j])_{cd} \\ &= -igt^b_{cd}\left(\partial_i A^b_j - \partial_j A^b_i\right) + \left([igt^b A^b_i, igt^e A^e_j]\right)_{cd}.\end{aligned} \tag{10.40}$$

Summing over repeated indices, we get

$$\begin{aligned}(F_{ij})_{cd} &= -igt^b_{cd}\left(\partial_i A^b_j - \partial_j A^b_i\right) - g^2 A^b_i A^e_j \left([t^b, t^e]\right)_{cd} \\ &= -igt^b_{cd}\left(\partial_i A^b_j - \partial_j A^b_i\right) - g^2 A^b_i A^e_j i f_{bef} t^f_{cd} \\ &= -igt^b_{cd}\left(\partial_i A^b_j - \partial_j A^b_i\right) - ig^2 A^b_i A^e_j f_{bef} t^f_{cd} \\ &= -igt^b_{cd}\left(\partial_i A^b_j - \partial_j A^b_i\right) - ig^2 A^f_i A^e_j f_{feb} t^b_{cd} \\ &= -igt^b_{cd}\left(\partial_i A^b_j - \partial_j A^b_i + gA^f_i A^e_j f_{feb}\right) \\ &= -igt^b_{cd}\left(\partial_i A^b_j - \partial_j A^b_i + gf_{bfe}\, A^f_i A^e_j\right) = -igt^b_{cd}\, F^b_{ij}\end{aligned} \tag{10.41}$$

where

$$F^b_{ij} = \partial_i A^b_j - \partial_j A^b_i + gf_{bfe}\, A^f_i A^e_j \tag{10.42}$$

is the Faraday tensor.

The action density of this tensor is

$$L_F = -\frac{1}{4} F^b_{ij} F^{ij}_b. \tag{10.43}$$

The trace of the square of the Faraday matrix is

$$\begin{aligned}\mathrm{Tr}\left[F_{ij}F^{ij}\right] &= \mathrm{Tr}\left[igt^b\, F^b_{ij}\, igt^c\, F^{ij}_c\right] \\ &= -g^2 F^b_{ij}\, F^{ij}_c\, \mathrm{Tr}(t^b t^c) = -g^2 F^b_{ij}\, F^{ij}_c\, k\delta_{bc} \\ &= -kg^2\, F^b_{ij}\, F^{ij}_b.\end{aligned} \tag{10.44}$$

So the Faraday action density is

$$L_F = -\frac{1}{4} F^b_{ij} F^{ij}_b = \frac{1}{4kg^2}\,\mathrm{Tr}\left[F_{ij}F^{ij}\right] = \frac{1}{2g^2}\,\mathrm{Tr}\left[F_{ij}F^{ij}\right]. \tag{10.45}$$

The theory described by this action density, without scalar or spinor fields, is called **pure** gauge theory.

The purpose of a gauge field is to make the gauge-invariant theories. So if a field ψ_a is transformed by the group element $g(x)$

$$\psi'_a(x) = g(x)_{ab}\psi_b(x), \tag{10.46}$$

then we want the covariant derivative of the field to go as

$$\begin{aligned}(D_i\psi(x))' &= \left(\partial_i - A_i'(x)\right) g(x)\psi(x) \\ &= g(x)D_i\psi(x) = g(x)\left[\left(\partial_i - A_i(x)\right)\psi(x)\right].\end{aligned} \tag{10.47}$$

So the gauge field must go as

$$\partial_i g - A_i' g = -gA_i \tag{10.48}$$

or as

$$A_i'(x) = g(x)A_i(x)g^{-1}(x) + (\partial_i g(x))g^{-1}(x). \tag{10.49}$$

So if $g(x) = \exp(-i\theta^a(x)t^a)$, then a gauge transforms as

$$A_i'(x) = e^{-i\theta^a(x)t^a}\, A_i(x)\, e^{i\theta^a(x)t^a} + (\partial_i e^{-i\theta^a(x)t^a})\, e^{i\theta^a(x)t^a}. \tag{10.50}$$

How does an exponential of a path-ordered, very short line integral of gauge fields go? We will evaluate how the path-ordered exponential in which g_0 is a coupling constant

$$\begin{aligned}P\exp\left(g_0A_i(x)dx^i\right)' &= P\exp\left(\left[g(x)g_0A_i(x)g^{-1}(x) + (\partial_i g(x))g^{-1}(x)\right]dx^i\right) \\ &= P\exp\left(\left[g(x)g_0A_i(x)g^{-1}(x) + \partial_i \log g(x)\right]dx^i\right)\end{aligned} \tag{10.51}$$

changes under the gauge transformation (10.50) in the limitt $dx^i \to 0$. We find

$$\begin{aligned}P\exp\left(g_0A_i(x)dx^i\right)' &= P\Big[g(x)e^{g_0A_i(x)dx^i}g^{-1}(x) \\ &\quad \times e^{\log g(x+dx^i/2) - \log g(x) + \log g(x) - \log g(x-dx^i/2)}\Big] \\ &= P\Big[g(x)e^{g_0A_i(x)dx^i}g^{-1}(x) \\ &\quad \times g(x+dx^i/2)g^{-1}(x)g(x)g^{-1}(x-dx^i/2)\Big] \\ &= g(x+dx^i/2)e^{g_0A_i(x)dx^i}g^{-1}(x-dx^i/2).\end{aligned} \tag{10.52}$$

Putting together a chain of such infinitesimal links, we get

$$P\exp\left(\int_y^x g_0A_i(x')dx'^i\right)' = g(x)P\exp\left(\int_y^x g_0A_i(x')dx'^i\right)g^{-1}(y). \tag{10.53}$$

In particular, this means that the trace of a closed loop is gauge invariant

$$\begin{aligned}\left[\mathrm{Tr}P\exp\left(\oint g_0 A_i(x)dx^i\right)\right]' &= \mathrm{Tr}g(x)P\exp\left(\oint g_0 A_i(x)dx^i\right)g^{-1}(x)\\ &= \mathrm{Tr}P\exp\left(\oint g_0 A_i(x)dx^i\right).\end{aligned} \tag{10.54}$$

10.5 Pure Gauge Theory on a Lattice

Wilson's lattice gauge theory is inspired by these last two equations. Another source of inspiration is the approximation for a loop of tiny area $dx \wedge dy$ which in the joint limit $dx \to 0$ and $dy \to 0$ is

$$\begin{aligned}W = P\exp\left(\oint g_0 A_i(x)dx^i\right) &= \exp\Big[g_0\Big(A_{y,x} - A_{x,y} + g_0[A_x, A_y]\Big)dxdy\Big]\\ &= \exp\Big(-g_0 F_{xy}\,dxdy\Big).\end{aligned} \tag{10.55}$$

To derive this formula, we will ignore the bare coupling constant g_0 for the moment and apply the Baker-Campbell-Hausdorf identity

$$e^A e^B = \exp\left(A + B + \frac{1}{2}[A,B] + \frac{1}{12}[A,[A,B]] + \frac{1}{12}[B,[B,A]] + \dots\right). \tag{10.56}$$

to the product

$$\begin{aligned}W = &\exp\Big(A_x\,dx\Big)\exp\Big(A_y\,dy + A_{y,x}\,dxdy\Big)\\ &\times\exp\Big(-A_x\,dx - A_{x,y}\,dxdy\Big)\exp\Big(-A_y\,dy\Big).\end{aligned} \tag{10.57}$$

We get

$$\begin{aligned}W = &\exp\Big(A_x\,dx + A_y\,dy + A_{y,x}\,dxdy + \frac{1}{2}[A_x\,dx, (A_y + A_{y,x}\,dx)dy]\Big)\\ &\times\exp\Big(-A_x\,dx - A_y\,dy - A_{x,y}\,dxdy + \frac{1}{2}[(A_x + A_{x,y}\,dy)dx, A_y\,dy]\Big).\end{aligned} \tag{10.58}$$

Applying again the BCH identity, we get

$$W = \exp\Big[\Big(A_{y,x} - A_{x,y} + [A_x, A_y]\Big)dxdy\Big] = \exp\Big(F_{xy}\,dxdy\Big), \tag{10.59}$$

an identity that is the basis of lattice gauge theory.

Restoring g_0, we divide the trace of W by the dimension n of the matrices t_a and subtract from unity

$$\begin{aligned} 1 - \frac{1}{n}\mathrm{Tr}\left[P\exp\left(\oint g_0 A_i(x)dx^i\right)\right] &= 1 - \frac{1}{n}\mathrm{Tr}\Big[\exp\Big(g_0 F_{xy}\,dxdy\Big)\Big] \\ &= -\frac{1}{n}\mathrm{Tr}\Big[g_0 F_{xy}\,dxdy + \frac{1}{2}\Big(g_0 F_{xy}\,dxdy\Big)^2\Big]. \end{aligned} \tag{10.60}$$

The generators of $SU(n)$, $SO(n)$, and $Sp(2n)$ are traceless, so the first term vanishes, and we get

$$1 - \frac{1}{n}\mathrm{Tr}\left[P\exp\left(\oint g_0 A_i(x)dx^i\right)\right] = -\frac{1}{2n}\mathrm{Tr}\Big(g_0 F_{xy}\,dxdy\Big)^2. \tag{10.61}$$

Recalling the more explicit form (10.41) of the Faraday matrix, we have

$$1 - \frac{1}{n}\mathrm{Tr}(W) = \frac{g_0^2}{2n}\mathrm{Tr}\Big[t_a F^a_{xy}\, t_b F^b_{xy}\,\big(dxdy\big)^2\Big] = \frac{kg_0^2}{2n}\big(F^a_{xy}dxdy\big)^2 \tag{10.62}$$

in which k is the constant of the normalization $\mathrm{Tr}(t_a t_b) = k\delta_{ab}$. For $SU(2)$ with $t_a = \sigma_a/2$ and for $SU(3)$ with $t_a = \lambda_a/2$, this constant is $k = 1/2$.

The Wilson action is a sum over all the smallest squares of the lattice, called the plaquettes, of the quantity

$$S_\square = \frac{n}{2kg_0^2}\left[1 - \frac{1}{n}\mathrm{Tr}(W)\right] = \frac{1}{4}\left(F^a_{ij}\right)^2 a^4 \tag{10.63}$$

in which a is the lattice spacing. The full Wilson action is the sum of this quantity over all the elementary squares of the lattice and over $i, j = 1, 2, 3, 4$.

References

Amdahl, David, and Cahill, Kevin. 2016. Path integrals for awkward actions. *arXiv*, 1611.06685.

Davies, P. C. W. 1975. Scalar particle production in Schwarzschild and Rindler metrics. *J. Phys.*, **A8**, 609–616.

DeWitt, Bryce S. 1967. Quantum Theory of Gravity. II. The Manifestly Covariant Theory. *Phys. Rev.*, **162(5)**, 1195–1239.

Faddeev, L. D., and Popov, V. N. 1967. Feynman diagrams for the Yang-Mills field. *Phys. Lett. B*, **25(1)**, 29–30.

Hawking, S. W. 1974. Black hole explosions. *Nature*, **248**, 30–31.

Kato, T. 1978. *Topics in Functional Analysis.* Academic Press. Pages 185–195.

Rindler, Wolfgang. 2006. *Relativity: Special, General, and Cosmological.* 2d edn. Oxford University Press.

Srednicki, Mark. 2007. *Quantum Field Theory.* Cambridge University Press.

Trotter, H. F. 1959. On the product of semi-groups of operators. *Proc. Ann. Math.*, **10**, 545–551.

Weinberg, Steven. 1995. *The Quantum Theory of Fields.* Vol. I Foundations. Cambridge University Press.

Weinberg, Steven. 1996. *The Quantum Theory of Fields.* Vol. II Modern Applications. Cambridge, UK: Cambridge University Press.

Zee, A. 2013. *Einstein Gravity in a Nutshell.* New Jersey: Princeton University Press.

Zee, Anthony. 2010. *Quantum Field Theory in a Nutshell.* 2d edn. Princeton University Press.

Index