Physics 523: Quantum Field Theory I

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Quantum fields and special relativity

1. States
A Lorentz transformation $\Lambda$ is implemented by a unitary operator $U(\Lambda)$ which replaces the state $|p, \sigma\rangle$ of a massive particle of momentum $p$ and spin $\sigma$ along the $z$-axis by the state

$$U(\Lambda)|p, \sigma\rangle = \sqrt{(\Lambda p)^0} \sum_{s'} D_{s's}(W(\Lambda, p)) |\Lambda p, s'\rangle$$  \hspace{1cm} (1.1)$$

where $W(\Lambda, p)$ is a Wigner rotation

$$W(\Lambda, p) = L^{-1}(\Lambda p)\Lambda L(p)$$  \hspace{1cm} (1.2)$$

and $L(p)$ is a standard Lorentz transformation that takes $(m, \vec{0})$ to $p$.

1.2 Creation operators
The vacuum is invariant under Lorentz transformations and translations

$$U(\Lambda, a)|0\rangle = |0\rangle.$$  \hspace{1cm} (1.3)$$

A creation operator $a^\dagger(p, \sigma)$ makes the state $|p, \sigma\rangle$ from the vacuum state $|0\rangle$

$$|p\sigma\rangle = a^\dagger(p, \sigma)|0\rangle.$$  \hspace{1cm} (1.4)$$

The creation and annihilation operators obey either the commutation relation

$$[a(p, s), a^\dagger(p', s')] = a(p, s) a^\dagger(p', s') - a^\dagger(p', s') a(p, s) = \delta_{ss'} \delta^{(3)}(p - p')$$  \hspace{1cm} (1.5)$$
Quantum fields and special relativity

or the anticommutation relation

\[ [a(p, s), a^\dagger(p', s')] = a(p, s) a^\dagger(p', s') + a^\dagger(p', s') a(p, s) = \delta_{ss'} \delta^{(3)}(p - p'). \]

(1.6)

The two kinds of relations are written together as

\[ [a(p, s), a^\dagger(p', s')] = a(p, s) a^\dagger(p', s') + a^\dagger(p', s') a(p, s) = \delta_{ss'} \delta^{(3)}(p - p'). \]

(1.7)

A bracket \([A, B]\) with no signed subscript is interpreted as a commutator.

Equations (1.1 & 1.4) give

\[ U(\Lambda) a^\dagger(p, \sigma)|0\rangle = \sqrt{\frac{(\Lambda p)^0}{p^0}} \sum_{s'} D^{(j)}_{s' \sigma}(W(\Lambda, p)) a^\dagger(\Lambda p, s')|0\rangle. \]

(1.8)

And (1.3) gives

\[ U(\Lambda) a^\dagger(p, \sigma) U^{-1}(\Lambda)|0\rangle = \sqrt{\frac{(\Lambda p)^0}{p^0}} \sum_{s'} D^{(j)}_{s' \sigma}(W(\Lambda, p)) a^\dagger(\Lambda p, s')|0\rangle. \]

(1.9)

SW in chapter 4 concludes that

\[ U(\Lambda) a^\dagger(p, \sigma) U^{-1}(\Lambda) = \sqrt{\frac{(\Lambda p)^0}{p^0}} \sum_{s'} D^{(j)}_{s' \sigma}(W(\Lambda, p)) a^\dagger(\Lambda p, s'). \]

(1.10)

If \(U(\Lambda, b)\) follows \(\Lambda\) by a translation by \(b\), then

\[ U(\Lambda, b) a^\dagger(p, \sigma) U^{-1}(\Lambda, b) = e^{-i(\Lambda p)\cdot a} \sqrt{\frac{(\Lambda p)^0}{p^0}} \sum_{s'} D^{(j)}_{s' \sigma}(W(\Lambda, p)) a^\dagger(\Lambda p, s') \]

(1.11)
1.3 How fields transform

The adjoint of this equation is

\[ U(\Lambda, b) a(p, \sigma) U^{-1}(\Lambda, b) = e^{i(\Lambda p) \cdot a} \sqrt{\frac{(\Lambda p)^0}{p^0}} \sum_{s'} D_{s'\sigma}^{(j)}(W(\Lambda, p)) a(\Lambda p, s') \]

\[ = e^{i(\Lambda p) \cdot a} \sqrt{\frac{(\Lambda p)^0}{p^0}} \sum_{s'} D_{s'\sigma'}^{(j)}(W(\Lambda, p)) a(\Lambda p, s') \]

\[ = e^{i(\Lambda p) \cdot a} \sqrt{\frac{(\Lambda p)^0}{p^0}} \sum_{s'} D_{s\sigma'}^{(j)}(W^{-1}(\Lambda, p)) a(\Lambda p, s'). \] \hspace{1cm} (1.12)

These equations (1.11 & 1.12) are (5.1.11 & 5.1.12) of SW.

1.3 How fields transform

The “positive frequency” part of a field is a linear combination of annihilation operators

\[ \psi_+^\ell(x) = \sum_{\sigma} \int d^3 p \ u_\ell(x; p, \sigma) a(p, \sigma). \] \hspace{1cm} (1.13)

The “negative frequency” part of a field is a linear combination of creation operators of the antiparticles

\[ \psi_-^\ell(x) = \sum_{\sigma} \int d^3 p \ v_\ell(x; p, \sigma) b^\dagger(p, \sigma). \] \hspace{1cm} (1.14)

To have the fields (1.13 & 1.14) transform properly under Poincaré transformations

\[ U(\Lambda, a) \psi_+^\ell(x) U^{-1}(\Lambda, a) = \sum_\ell D_{\ell\ell}(\Lambda^{-1}) \psi_+^\ell(\Lambda x + a) \]

\[ = \sum_\ell D_{\ell\ell}(\Lambda^{-1}) \sum_\sigma \int d^3 p \ u_\ell(\Lambda x + a; p, \sigma) a(p, \sigma) \]

\[ U(\Lambda, a) \psi_-^\ell(x) U^{-1}(\Lambda, a) = \sum_\ell D_{\ell\ell}(\Lambda^{-1}) \psi_-^\ell(\Lambda x + a) \]

\[ = \sum_\ell D_{\ell\ell}(\Lambda^{-1}) \sum_\sigma \int d^3 p \ v_\ell(\Lambda x + a; p, \sigma) b^\dagger(p, \sigma) \] \hspace{1cm} (1.15)

the spinors \( u_\ell(x; p, \sigma) \) and \( v_\ell(x; p, \sigma) \) must obey certain rules which we’ll now determine.
Quantum fields and special relativity

First (1.12 & 1.13) give

\[ U(\Lambda, a) \psi_\ell^+(x) U^{-1}(\Lambda, a) = U(\Lambda, a) \sum_\sigma \int d^3p \, u_\ell(x; p, \sigma) a(p, \sigma) U^{-1}(\Lambda, a) \]

\[ = \sum_\sigma \int d^3p \, u_\ell(x; p, \sigma) U(\Lambda, a) a(p, \sigma) U^{-1}(\Lambda, a) \]

\[ = \sum_\sigma \int d^3p \, u_\ell(x; p, \sigma) e^{i(\Lambda p) \cdot a} \sqrt{\frac{(\Lambda p)^0}{p^0}} \sum_{s'} D^{(j)}_{\sigma s'}(W^{-1}(\Lambda, p)) a(\Lambda p, s'). \quad (1.16) \]

Now we use the identity

\[ \frac{d^3p}{p^0} = \frac{d^3(\Lambda p)}{(\Lambda p)^0} \]

(1.17)

to turn (1.16) into

\[ U(\Lambda, a) \psi_\ell^+(x) U^{-1}(\Lambda, a) = \sum_\sigma \int d^3(\Lambda p) \, u_\ell(x; p, \sigma) e^{i(\Lambda p) \cdot a} \]

\[ \times \sqrt{\frac{p^0}{(\Lambda p)^0}} \sum_{s'} D^{(j)}_{\sigma s'}(W^{-1}(\Lambda, p)) a(\Lambda p, s'). \quad (1.18) \]

Similarly (1.11, 1.14, & 1.17) give

\[ U(\Lambda, a) \psi_\ell^-(x) U^{-1}(\Lambda, a) = U(\Lambda, a) \sum_\sigma \int d^3p \, v_\ell(x; p, \sigma) b^\dagger(p, \sigma) U^{-1}(\Lambda, a) \]

\[ = \sum_\sigma \int d^3p \, v_\ell(x; p, \sigma) U(\Lambda, a) b^\dagger(p, \sigma) U^{-1}(\Lambda, a) \]

\[ = \sum_\sigma \int d^3p \, v_\ell(x; p, \sigma) e^{-i(\Lambda p) \cdot a} \sqrt{\frac{(\Lambda p)^0}{p^0}} \sum_{s'} D^{*}_{\sigma s'}(W^{-1}(\Lambda, p)) b^\dagger(\Lambda p, s') \]

\[ = \sum_\sigma \int d^3(\Lambda p) \, v_\ell(x; p, \sigma) e^{-i(\Lambda p) \cdot a} \sqrt{\frac{p^0}{(\Lambda p)^0}} \sum_{s'} D^{*}_{\sigma s'}(W^{-1}(\Lambda, p)) b^\dagger(\Lambda p, s'). \quad (1.19) \]

So to get the fields to transform as in (1.15), equations (1.18 & 1.19) say...
1.3 How fields transform

that we need

\[ \sum_{\ell} D_{\ell\ell}(\Lambda^{-1}) \psi^{+}_{\ell}(\Lambda x + a) = \sum_{\ell} D_{\ell\ell}(\Lambda^{-1}) \sum_{\sigma} \int d^{3}p \, u_{\ell}(\Lambda x + a; p, \sigma) a(p, \sigma) \]

\[ = \sum_{\ell} D_{\ell\ell}(\Lambda^{-1}) \sum_{\sigma} \int d^{3}(\Lambda p) \, u_{\ell}(\Lambda x + a; \Lambda p, \sigma) a(\Lambda p, \sigma) \]

\[ = \sum_{\sigma} \int d^{3}(\Lambda p) \, u_{\ell}(x; p, \sigma) e^{i(\Lambda p) \cdot a} \]  

(1.20)

\[ \times \sqrt{\frac{p^{0}}{(\Lambda p)^{0}}} \sum_{s'} D_{s'\sigma}(^{(j)}(W^{-1}(\Lambda, p)) a(\Lambda p, s') \]

\[ = \sum_{s'} \int d^{3}(\Lambda p) \, u_{\ell}(x; p, s') e^{i(\Lambda p) \cdot a} \]

\[ \times \sqrt{\frac{p^{0}}{(\Lambda p)^{0}}} \sum_{\sigma} D_{s'\sigma}(^{(j)}(W^{-1}(\Lambda, p)) a(\Lambda p, \sigma) \]

and

\[ \sum_{\ell} D_{\ell\ell}(\Lambda^{-1}) \psi^{-}_{\ell}(\Lambda x + a) = \sum_{\ell} D_{\ell\ell}(\Lambda^{-1}) \sum_{\sigma} \int d^{3}p \, v_{\ell}(\Lambda x + a; p, \sigma) b^{\dagger}(p, \sigma) \]

\[ = \sum_{\ell} D_{\ell\ell}(\Lambda^{-1}) \sum_{\sigma} \int d^{3}(\Lambda p) \, v_{\ell}(\Lambda x + a; \Lambda p, \sigma) b^{\dagger}(\Lambda p, \sigma) \]

\[ = \sum_{\sigma} \int d^{3}(\Lambda p) \, v_{\ell}(x; p, \sigma) e^{-i(\Lambda p) \cdot a} \]  

(1.21)

\[ \times \sqrt{\frac{p^{0}}{(\Lambda p)^{0}}} \sum_{s'} D_{s'\sigma}(^{*}(j)(W^{-1}(\Lambda, p)) b^{\dagger}(\Lambda p, s') \]

\[ = \sum_{s'} \int d^{3}(\Lambda p) \, v_{\ell}(x; p, s') e^{-i(\Lambda p) \cdot a} \]

\[ \times \sqrt{\frac{p^{0}}{(\Lambda p)^{0}}} \sum_{\sigma} D_{s'\sigma}(^{*}(j)(W^{-1}(\Lambda, p)) b^{\dagger}(\Lambda p, \sigma) \]

Equating coefficients of the red annihilation and blue creation operators, we find that the fields will transform properly if the spinors \( u \) and \( v \) satisfy the
Quantum fields and special relativity

\[
\sum \ell D_{\ell \ell} (\Lambda^{-1}) u_{\ell} (\Lambda x + a; \Lambda p, \sigma) = \sqrt{\frac{p^0}{(\Lambda p)^0}} \sum_{s'} D_{s' \sigma}^{(j)} (W^{-1}(\Lambda, p)) u_{s'} (x; p, s') e^{i(\Lambda p) \cdot a} 
\]

(1.22)

\[
\sum \ell D_{\ell \ell} (\Lambda^{-1}) v_{\ell} (\Lambda x + a; \Lambda p, \sigma) = \sqrt{\frac{p^0}{(\Lambda p)^0}} \sum_{s'} D_{s' \sigma}^{(j)} (W^{-1}(\Lambda, p)) v_{s'} (x; p, s') e^{-i(\Lambda p) \cdot a} 
\]

(1.23)

which differ from SW’s by an interchange of the subscripts \(\sigma, s'\) on the rotation matrices \(D^{(j)}\). (I think SW has a typo there.) If we multiply both sides of these equations (1.22 & 1.23) by the two kinds of \(D\) matrices, then we get first

\[
\sum \ell, \ell' D_{\ell \ell'} (\Lambda) D_{\ell' \ell} (\Lambda^{-1}) u_{\ell'} (\Lambda x + a; \Lambda p, \sigma) = u_{\ell'} (\Lambda x + a; \Lambda p, \sigma) 
\]

\[
= \sqrt{\frac{p^0}{(\Lambda p)^0}} \sum_{s, \ell} D_{s \sigma}^{(j)} (W^{-1}(\Lambda, p)) D_{\ell' \ell} (\Lambda) u_{\ell} (x; p, s') e^{i(\Lambda p) \cdot a} 
\]

(1.24)

\[
\sum \ell, \ell' D_{\ell \ell'} (\Lambda) D_{\ell' \ell} (\Lambda^{-1}) v_{\ell'} (\Lambda x + a; \Lambda p, \sigma) = v_{\ell'} (\Lambda x + a; \Lambda p, \sigma) 
\]

\[
= \sqrt{\frac{p^0}{(\Lambda p)^0}} \sum_{s, \ell} D_{s' \sigma}^{(j)} (W^{-1}(\Lambda, p)) D_{\ell' \ell} (\Lambda) v_{\ell} (x; p, s') e^{-i(\Lambda p) \cdot a} 
\]

(1.25)
and then with $W \equiv W(\Lambda, p)$

$$
\sum_{\sigma} D^{(j)}_{\sigma s}(W) u_{\ell'}(\Lambda x + a; \Lambda p, \sigma) = \sum_{\sigma} D^{(j)}_{\sigma s}(W^{-1}) D^{(j)}_{\sigma s}(W) D_{\ell' \ell}(\Lambda) u_{\ell}(x; p, s') e^{i(\Lambda p) \cdot a}
$$

$$
\sum_{\sigma} D^{(j)}_{\sigma s}(W) v_{\ell'}(\Lambda x + a; \Lambda p, \sigma) = \sum_{\sigma} D^{(j)}_{\sigma s}(W^{-1}) D^{(j)}_{\sigma s}(W) D_{\ell' \ell}(\Lambda) v_{\ell}(x; p, s') e^{-i(\Lambda p) \cdot a}
$$

which are equations (5.1.13 & 5.1.14) of SW:

$$
\sum_{s'} u_{\ell}(\Lambda x + a; \Lambda p, s) D^{(j)}_{s' s}(W(\Lambda, p)) = \sqrt{\frac{p^0}{(\Lambda p)^0}} \sum_{\ell} D_{\ell' \ell}(\Lambda) u_{\ell}(x; p, \sigma) e^{i(\Lambda p) \cdot a}
$$

$$
\sum_{s'} v_{\ell}(\Lambda x + a; \Lambda p, s) D^{(j)}_{s' s}(W(\Lambda, p)) = \sqrt{\frac{p^0}{(\Lambda p)^0}} \sum_{\ell} D_{\ell' \ell}(\Lambda) v_{\ell}(x; p, \sigma) e^{-i(\Lambda p) \cdot a}
$$

These are the equations that determine the spinors $u$ and $v$ up to a few arbitrary phases.

### 1.4 Translations

When $\Lambda = I$, the $D$ matrices are equal to unity, and these last equations (1.28) say that for $x = 0$

$$
u_{\ell}(a; p, \sigma) = u_{\ell}(0; p, \sigma) e^{i p \cdot a}
$$

$$
v_{\ell}(a; p, \sigma) = v_{\ell}(0; p, \sigma) e^{-i p \cdot a}.
$$

Thus the spinors $u$ and $v$ depend upon spacetime by the usual phase $e^{\pm i p \cdot x}$

$$
u_{\ell}(x; p, \sigma) = (2\pi)^{-3/2} u_{\ell}(p, \sigma) e^{i p \cdot x}
$$

$$
v_{\ell}(x; p, \sigma) = (2\pi)^{-3/2} v_{\ell}(p, \sigma) e^{-i p \cdot x}.
$$
Quantum fields and special relativity

in which the $2\pi$'s are conventional. The fields therefore are Fourier transforms:

$$\psi^+ (x) = (2\pi)^{-3/2} \sum_\sigma \int d^3 p \, e^{ip \cdot x} u_\ell (p, \sigma) a(p, \sigma)$$

and every field of mass $m$ obeys the Klein-Gordon equation

$$(\nabla^2 - \partial_0^2 - m^2) \psi_\ell (x) = (\Box - m^2) \psi_\ell (x) = 0.$$ (1.32)

Since $\exp[i(\Lambda \cdot (\Lambda x + a))] = \exp(ip \cdot x + i\Lambda p \cdot a)$, the conditions simplify to

$$\sum_\sigma u_\ell (\Lambda p, \sigma) D^{(j)}_{sa} (W(\Lambda, p)) = \sqrt{p^0_{(\Lambda p)^0}} \sum_\ell D_{\ell\ell} (\Lambda) u_\ell (p, \sigma)$$

and

$$\sum_\sigma v_\ell (\Lambda p, \sigma) D^{(j)}_{sa} (W(\Lambda, p)) = \sqrt{p^0_{(\Lambda p)^0}} \sum_\ell D_{\ell\ell} (\Lambda) v_\ell (p, \sigma)$$ (1.33)

for all Lorentz transformations $\Lambda$.

1.5 Boosts

Set $p = k = (m, \vec{0})$ and $\Lambda = L(q)$ where $L(q)k = q$. So $L(p) = 1$ and

$$W(\Lambda, p) \equiv L^{-1}(\Lambda p) \Lambda L(p) = L^{-1}(q)L(q) = 1.$$ (1.34)

Then the equations (1.33) are

$$u_\ell (q, \sigma) = \sqrt{q^0_{(\Lambda p)^0}} \sum_\ell D_{\ell\ell} (L(q)) u_\ell (\sigma)$$

and

$$v_\ell (q, \sigma) = \sqrt{q^0_{(\Lambda p)^0}} \sum_\ell D_{\ell\ell} (L(q)) v_\ell (\sigma).$$ (1.35)

Thus a spinor at finite momentum is given by a representation $D(\Lambda)$ of the Lorentz group (see the online notes of chapter 10 of my book for its finite-dimensional nonunitary representations) acting on the spinor at zero 3-momentum $p = k = (m, \vec{0})$. We need to find what these spinors are.
1.6 Rotations

Now set \( p = k = (m, \vec{0}) \) and \( \Lambda = R \) a rotation so that \( W = R \). For rotations, the spinor conditions (1.33) are

\[
\sum_{s} u_{\ell}(\vec{0}, s) D^{(j)}_{s\sigma}(R) = \sum_{\ell} D_{\ell\ell}(R) u_{\ell}(\vec{0}, \sigma) \\
\sum_{s} v_{\ell}(\vec{0}, s) D^{\dagger(j)}_{s\sigma}(R) = \sum_{\ell} D_{\ell\ell}(R) v_{\ell}(\vec{0}, \sigma).
\]

(1.36)

The representations \( D^{(j)}_{s\sigma}(R) \) of the rotation group are \((2j + 1) \times (2j + 1)\)-dimensional unitary matrices. For a rotation of angle \( \theta \) about the \( \vec{\theta} = \theta \) axis, they are the ones taught in courses on quantum mechanics (and discussed in the notes of chapter 10)

\[
D^{(j)}_{s\sigma}(\theta) = \left[ e^{-i\theta J^{(j)}} \right]_{s\sigma}
\]

(1.37)

where \([J_{a}, J_{b}] = i\epsilon_{abc} J_{c}\). The representations \( D_{\ell\ell}(R) \) of the rotation group are finite-dimensional unitary matrices. For a rotation of angle \( \theta \) about the \( \vec{\theta} = \theta \) axis, they are

\[
D_{\ell\ell}(\theta) = \left[ e^{-i\theta J} \right]_{\ell\ell}
\]

(1.38)

in which \([J_{a}, J_{b}] = i\epsilon_{abc} J\). For tiny rotations, the conditions (1.36) require (because of the complex conjugation of the antiparticle condition) that the spinors obey the rules

\[
\sum_{s} u_{\ell}(\vec{0}, s)(J_{a}^{(j)})_{s\sigma} = \sum_{\ell} (J_{a})_{\ell\ell} u_{\ell}(\vec{0}, \sigma) \\
\sum_{s} v_{\ell}(\vec{0}, s)(-J_{a})_{s\sigma}^{(j)} = \sum_{\ell} (J_{a})_{\ell\ell} v_{\ell}(\vec{0}, \sigma)
\]

(1.39)

for \( a = 1, 2, 3 \).

1.7 Spin-zero fields

Spin-zero fields have no spin or Lorentz indexes. So the boost conditions (1.210) merely require that \( u(q) = \sqrt{m/q^0} u(0) \) and \( v(q) = \sqrt{m/q^0} v(0) \).

The conventional normalization is \( u(0) = 1/\sqrt{2m} \) and \( v(0) = 1/\sqrt{2m} \). The spin-zero spinors then are

\[
u(p) = (2p^0)^{-1/2} \quad \text{and} \quad v(p) = (2p^0)^{-1/2}.
\]

(1.40)

For simplicity, let’s first consider a neutral scalar field so that \( b(p, s) = \)
Quantum fields and special relativity

The definitions \((1.13)\) and \((1.14)\) of the positive-frequency and negative-frequency fields and their behavior \((1.30)\) under translations then give us

\[
\begin{align*}
\phi^+(x) &= \int \frac{d^3p}{\sqrt{(2\pi)^32p^0}} a(p) e^{ip\cdot x} \\
\phi^-(x) &= \int \frac{d^3p}{\sqrt{(2\pi)^32p^0}} a^\dagger(p) e^{-ip\cdot x}.
\end{align*}
\] (1.41)

Note that

\[
[\phi^\pm(x)]^\dagger = \phi^{\mp}(x).
\] (1.42)

Since \([a(p), a(p')])_\pm = 0\), it follows that

\[
[\phi^+(x), \phi^+(y)]_\mp = 0 \quad \text{and} \quad [\phi^-(x), \phi^-(y)]_\mp = 0
\] (1.43)

whatever the values of \(x\) and \(y\) as long as we use commutators for bosons and anticommutators for fermions.

But the commutation relation

\[
[a(p, s), a^\dagger(q, t)]_\mp = \delta_{st} \delta_3(p - q)
\] (1.44)

makes the commutator

\[
[\phi^+(x), \phi^-(y)]_\mp = \int \frac{d^3p d^3p'}{(2\pi)^3\sqrt{2p^02p'^0}} e^{ip\cdot x} e^{-ip'\cdot y} \delta_3(p - p')
\]

\[
= \int \frac{d^3p}{(2\pi)^32p^0} e^{ip(x-y)} = \Delta_+(x - y)
\] (1.45)

nonzero even for \((x - y)^2 > 0\) as we’ll now verify.

For space-like \(x\), the Lorentz-invariant function \(\Delta_+(x)\) can only depend upon \(x^2 > 0\) since the time \(x^0\) and its sign are not Lorentz invariant. So we choose a Lorentz frame with \(x^0 = 0\) and \(|x| = \sqrt{x^2}\). In this frame,

\[
\Delta_+(x) = \int \frac{d^3p}{(2\pi)^3\sqrt{2p^0}} e^{ip\cdot x}
\]

\[
= \int \frac{p^2 dp \ d\cos\theta}{(2\pi)^3\sqrt{2p^0}} e^{ipx\cos\theta}
\] (1.46)

where \(p = |p|\) and \(x = |x|\). Now

\[
\int d\cos\theta \ e^{ipx\cos\theta} \to (e^{ipx} - e^{-ipx})/(ipx) = 2\sin(px)/(px),
\] (1.47)

so the integral \((1.46)\) is

\[
\Delta_+(x) = \frac{1}{4\pi^2 x} \int_0^\infty \frac{\sin(px) p dp}{\sqrt{p^2 + m^2}}
\] (1.48)
1.7 Spin-zero fields

with \( u \equiv p/m \)

\[
\Delta_+(x) = \frac{m}{4\pi^2x} \int_0^\infty \frac{\sin(mxu)\, u\, du}{\sqrt{u^2+1}} = \frac{m}{4\pi^2x} K_1(mx^2) \tag{1.49}
\]

a Hankel function.

To get a Lorentz-invariant, causal theory, we use the arbitrary parameters \( \kappa \) and \( \lambda \) setting

\[
\phi(x) = \kappa \phi^+(x) + \lambda \phi^-(x) \tag{1.50}
\]

Now the adjoint rule \([1.42]\) and the commutation relations \([1.45\text{ and }1.45]\)
give

\[
[\phi(x), \phi^+(y)] = [\kappa \phi^+(x) + \lambda \phi^-(x), \kappa^* \phi^-(y) + \lambda \phi^+(y)] = [\kappa^2 \phi^+(x), \phi^-(y)] + [\lambda^2 \phi^-(x), \phi^+(y)] = [\kappa^2 \Delta_+(x-y) \mp |\lambda|^2 \Delta_+(y-x)] \tag{1.51}
\]

But when \((x-y)^2 > 0\), \( \Delta_+(x-y) = \Delta_+(y-x) \). Thus these conditions are

\[
[\phi(x), \phi^+(y)] = (|\kappa|^2 \mp |\lambda|^2) \Delta_+(x-y)
\]

\[
[\phi(x), \phi^+(y)] = \kappa \lambda \left( [\phi^+(x), \phi^-(y)] + [\phi^-(x), \phi^+(y)] \right) = \kappa \lambda (\Delta_+(x-y) \mp \Delta_+(y-x)).
\]

The first of these equations implies that we choose the minus sign and so that we use commutation relations and not anticommutation relations for spin-zero fields. This is the spin-statistics theorem for spin-zero fields. SW proves the theorem for arbitrary massive fields in section 5.7.

We also must set

\[
|\kappa| = |\lambda|. \tag{1.53}
\]

The second equation then is automatically satisfied. The common magnitude and the phases of \( \kappa \) and \( \lambda \) are arbitrary, so we choose \( \kappa = \lambda = 1 \). We then have

\[
\phi(x) = \phi^+(x) + \phi^-(x) = \phi^+(x) + \phi^+(x) = \phi^+(x). \tag{1.54}
\]

Now the interaction density \( \mathcal{H}(x) \) will commute with \( \mathcal{H}(y) \) for \((x-y)^2 > 0\), and we have a chance of having a Lorentz-invariant, causal theory.

The field \([1.54]\)

\[
\phi^+(x) = \int \frac{d^3p}{\sqrt{(2\pi)^3 2p^0}} \left[ a(p) e^{ip\cdot x} + a^\dagger(p) e^{-ip\cdot x} \right] \tag{1.55}
\]
obeys the **Klein-Gordon equation**

\[
(\nabla^2 - \partial_0^2 - m^2) \phi(x) \equiv (\Box - m^2) \phi(x) = 0. \tag{1.56}
\]

### 1.8 Conserved charges

If the field \( \phi \) adds and deletes charged particles, an interaction \( \mathcal{H}(x) \) that is a polynomial in \( \phi \) will not commute with the charge operator \( Q \) because \( \phi^+ \) will lower the charge and \( \phi^- \) will raise it. The standard way to solve this problem is to start with two hermitian fields \( \phi_1 \) and \( \phi_2 \) of the same mass.

One defines a complex scalar field as a complex linear combination of the two fields

\[
\phi(x) = \frac{1}{\sqrt{2}} (\phi_1(x) + i\phi_2(x))
\]

\[
= \int \frac{d^3p}{\sqrt{(2\pi)^32p^0}} \left[ \frac{1}{\sqrt{2}} (a_1(p) + ia_2(p)) e^{ip \cdot x} + \frac{1}{\sqrt{2}} (a_1^\dagger(p) + ia_2^\dagger(p)) e^{-ip \cdot x} \right].
\tag{1.57}
\]

Setting

\[
a(p) = \frac{1}{\sqrt{2}} (a_1(p) + ia_2(p)) \quad \text{and} \quad b^\dagger(p) = \frac{1}{\sqrt{2}} (a_1^\dagger(p) + ia_2^\dagger(p)) \tag{1.58}
\]

so that

\[
b(p) = \frac{1}{\sqrt{2}} (a_1(p) - ia_2(p)) \quad \text{and} \quad a(p) = \frac{1}{\sqrt{2}} (a_1^\dagger(p) - ia_2^\dagger(p)) \tag{1.59}
\]

we have

\[
\phi(x) = \int \frac{d^3p}{\sqrt{(2\pi)^32p^0}} \left[ a(p) e^{ip \cdot x} + b^\dagger(p) e^{-ip \cdot x} \right]. \tag{1.60}
\]

and

\[
\phi^\dagger(x) = \int \frac{d^3p}{\sqrt{(2\pi)^32p^0}} \left[ b(p) e^{ip \cdot x} + a^\dagger(p) e^{-ip \cdot x} \right]. \tag{1.61}
\]

Since the commutation relations of the real creation and annihilation operators are for \( i, j = 1, 2 \)

\[
[a_i(p), a_j^\dagger(p')] = \delta_{ij} \delta^3(p - p') \quad \text{and} \quad [a_i(p), a_j(p')] = 0 = [a_i^\dagger(p), a_j^\dagger(p')]
\tag{1.62}
\]

the commutation relations of the complex creation and annihilation operators are

\[
[a(p), a^\dagger(p')] = \delta^3(p - p') \quad \text{and} \quad [b(p), b^\dagger(p')] = \delta^3(p - p') \tag{1.63}
\]
1.9 Parity, charge conjugation, and time reversal

with all other commutators vanishing.

Now $\phi(x)$ lowers the charge of a state by $q$ if $a^\dagger$ adds a particle of charge $q$ and if $b^\dagger$ adds a particle of charge $-q$. Similarly, $\phi^\dagger(x)$ raises the charge of a state by $q$

$$[Q, \phi(x)] = -q\phi(x) \quad \text{and} \quad [Q, \phi^\dagger(x)] = q\phi^\dagger(x). \quad (1.64)$$

So an interaction with as many $\phi(x)$’s as $\phi^\dagger(x)$’s conserves charge.

1.9 Parity, charge conjugation, and time reversal

If the unitary operator $P$ represents parity on the creation operators

$$Pa_1^\dagger(p)P^{-1} = \eta a_1^\dagger(-p) \quad \text{and} \quad Pa_2^\dagger(p)P^{-1} = \eta a_2^\dagger(-p) \quad (1.65)$$

with the same phase $\eta$. Then

$$Pa_1(p)P^{-1} = \eta^* a_1(-p) \quad \text{and} \quad Pa_2(p)P^{-1} = \eta^* a_2(-p) \quad (1.66)$$

and so both

$$Pa_1^\dagger(p)P^{-1} = \eta a_1(-p) \quad \text{and} \quad Pa(p)P^{-1} = \eta^* a(-p) \quad (1.67)$$

and

$$Pb_1^\dagger(p)P^{-1} = \eta b_1(-p) \quad \text{and} \quad Pb(p)P^{-1} = \eta^* b(-p). \quad (1.68)$$

Thus if the field

$$\phi_1(x) = \int \frac{d^3p}{\sqrt{(2\pi)^32p^0}} \left[ a_1(p)e^{ip\cdot x} + a_1^\dagger(p)e^{-ip\cdot x} \right] \quad (1.69)$$

or $\phi_2(x)$, or the complex field (1.60) is to go into a multiple of itself under parity, then we need $\eta = \eta^*$ so that $\eta$ is real. Then the fields transform under parity as

$$P\phi_1(x)P^{-1} = \eta^* \phi_1(x^0, -x) = \eta\phi_1(x^0, -x) \quad (1.70)$$

Since $P^2 = I$, we must have $\eta = \pm 1$. SW allows for a more general phase by having parity act with the same phase on $a$ and $b^\dagger$. Both schemes imply that the parity of a hermitian field is $\pm 1$ and that the state

$$|ab \rangle = \int d^3p f(p^2) a_1^\dagger(p) b_1^\dagger(-p) |0 \rangle \quad (1.71)$$

has even or positive parity, $P|ab \rangle = |ab \rangle$. 

So an interaction with as many $\phi(x)$’s as $\phi^\dagger(x)$’s conserves charge.
Charge conjugation works similarly. If the unitary operator $C$ represents charge conjugation on the creation operators

$$Ca_1^\dagger(p)C^{-1} = \xi a_1^\dagger(p) \quad \text{and} \quad Ca_2^\dagger(p)C^{-1} = -\xi a_2^\dagger(p) \quad (1.72)$$

with the same phase $\xi$. Then

$$Ca_1(p)C^{-1} = \xi^* a_1(p) \quad \text{and} \quad Ca_2(p)C^{-1} = -\xi^* a_2(p) \quad (1.73)$$

and so since $a = (a_1 + ia_2)/\sqrt{2}$ and $b = (a_1 - ia_2)/\sqrt{2}$

$$Ca(p)C^{-1} = \xi^* b(p) \quad \text{and} \quad Cb(p)C^{-1} = \xi^* a(p) \quad (1.74)$$

and since $a^\dagger = (a_1^\dagger - ia_2^\dagger)/\sqrt{2}$ and $b^\dagger = (a_1^\dagger + ia_2^\dagger)/\sqrt{2}$

$$Ca^\dagger(p)C^{-1} = \xi b^\dagger(p) \quad \text{and} \quad Cb^\dagger(p)C^{-1} = \xi a^\dagger(p). \quad (1.75)$$

Thus under charge conjugation, the field (1.66) becomes

$$C\phi(x)C^{-1} = \int \frac{d^3p}{\sqrt{(2\pi)^32^3}} \left[ \xi^* b(p)e^{ipx} + \xi a^\dagger(p)e^{-ipx} \right] \quad (1.76)$$

and so if it is to go into a multiple of itself or of its adjoint under charge conjugation then we need $\xi = \xi^*$ so that $\xi$ is real. We then get

$$C\phi(x)C^{-1} = \xi^* \phi^\dagger(x) = \xi \phi^\dagger(x). \quad (1.77)$$

Since $C^2 = I$, we must have $\xi = \pm 1$. SW allows for a more general phase by having charge conjugation act with the same phase on $a$ and $b^\dagger$. Both schemes imply that the charge-conjugation parity of a hermitian field is $\pm 1$ and that the state

$$|ab\rangle = \int d^3p f(p^2) a^\dagger(p) b^\dagger(p) |0\rangle \quad (1.78)$$

has even or positive charge-conjugation parity, $\zeta|ab\rangle = |ab\rangle$.

The time-reversal operator $T$ is antilinear and antunitary. So if

$$Ta_1(p)T^{-1} = \zeta a_1(-p) \quad \text{and} \quad Ta_2(p)T^{-1} = -\zeta a_2(-p) \quad (1.79)$$

then

$$Ta(p)T^{-1} = T \frac{1}{\sqrt{2}}(a_1(p) + ia_2(p))T^{-1} = \frac{1}{\sqrt{2}}(Ta_1(p)T^{-1} - iTa_2(p)T^{-1})$$

$$= \zeta \frac{1}{\sqrt{2}}[a_1(-p) + ia_2(-p)] = \zeta^* a(-p) \quad (1.80)$$
1.10 Vector fields

Then one has

$$T \phi(x) T^{-1} = T \int d^3 p \sqrt{(2\pi)^3 2p^0} \left[ a(p) e^{ip \cdot x} + b^\dagger(p) e^{-ip \cdot x} \right] T^{-1}$$

$$= \int d^3 p \sqrt{(2\pi)^3 2p^0} \left[ Ta(p) T^{-1} e^{-ip \cdot x} + T b^\dagger(p) T^{-1} e^{ip \cdot x} \right]$$

$$= \int d^3 p \sqrt{(2\pi)^3 2p^0} \left[ \zeta^* a(-p) e^{-ip \cdot x} + \zeta b^\dagger(-p) e^{ip \cdot x} \right].$$

(1.82)

So if $\zeta$ is real, then after replacing $-p$ by $p$, we get

$$T \phi(x) T^{-1} = \zeta^* \int d^3 p \sqrt{(2\pi)^3 2p^0} \left[ a(p) e^{i(p^0 \cdot x^0 + ip \cdot x)} + b^\dagger(p) e^{-i(p^0 \cdot x^0 + ip \cdot x)} \right]$$

$$= \zeta^* \phi(-x^0, x) = \zeta \phi(-x^0, x).$$

(1.83)

Since $T^2 = I$, the phase $\zeta = \pm 1$. SW lets $\zeta$ be complex but defined only for complex scalar fields and not for their real and imaginary parts.

1.10 Vector fields

Vector fields transform like the 4-vector $x^i$ of spacetime. So

$$D_{\bar{\ell}\ell}(\Lambda) = \Lambda_{\bar{\ell}}^\ell$$

(1.84)

for $\bar{\ell}, \ell = 0, 1, 2, 3$. Again we start with a hermitian field labelled by $i = 0, 1, 2, 3$

$$\phi^+(x) = (2\pi)^{-3/2} \sum_s \int d^3 p e^{ip \cdot x^0} u^i(p, s) a(p, s)$$

$$\phi^-(x) = (2\pi)^{-3/2} \sum_s \int d^3 p e^{-ip \cdot x^0} u^i(p, s) a^\dagger(p, s).$$

(1.85)
The boost conditions \((1.210)\) say that

\[
u^i(p, s) = \sqrt{\frac{m}{p^0}} \sum_k L(p)^i_k u^k(\vec{0}, s)
\]

\[
v^i(p, s) = \sqrt{\frac{m}{p^0}} \sum_k L(p)^i_k v^k(\vec{0}, s).
\]

(1.86)

The rotation conditions \((1.39)\) give

\[
\sum_{\bar{s'}} u^i(\vec{0}, \bar{s})(J_a^{(j)})_{\bar{s}s} = \sum_k (J_a^{(j)})^i_k u^k(\vec{0}, s)
\]

\[
- \sum_{\bar{s'}} v^i(\vec{0}, \bar{s})(J^{s(j)}_a)_{\bar{s}s} = \sum_k (J^i_a) v^k(\vec{0}, s).
\]

(1.87)

The \((2j + 1) \times (2j + 1)\) matrices \((J_a^{(j)})_{\bar{s}s}\) are the generators of the \((2j + 1) \times (2j + 1)\) representation of the rotation group. (See my online notes on group theory.) You learned that

\[
\sum_{a=1}^3 \sum_{s=-j}^j (J_a^{(j)})^2 = \sum_{a=1}^3 \sum_{s=-j}^j (J_a^{(j)})_{\bar{s}s} (J_a^{(j)})_{s's'} = j(j + 1)\delta_{ss'} (1.88)
\]

and that

\[
J_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad J_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}
\]

(1.89)

in courses on quantum mechanics.

For \(k = 1, 2, 3\), the three \(4 \times 4\) matrices \((J_k)^i_j\) are the generators of rotations in the vector representation of the Lorentz group. Their nonzero components are

\[
(J_k)^i_j = -i\epsilon_{ijk}
\]

(1.90)

for \(i, j, k = 1, 2, 3\), while \((J_k)^0_0 = 0\), \((J_k)^0_j = 0\), and \((J_k)^i_0 = 0\) for \(i, j, k = 1, 2, 3\). So

\[
(J^2)^i_j = 2\delta^i_j
\]

(1.91)

with \((J^2)^0_0 = 0\), \((J^2)^0_j = 0\), and \((J^2)^i_0 = 0\) for \(i, j = 1, 2, 3\). Apart from a factor of \(i\), the \(J_k\)'s are the \(4 \times 4\) matrices \(J_a = iR_a\) of my online notes on
the Lorentz group
\[
\mathcal{J}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, \quad \mathcal{J}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \quad \mathcal{J}_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
\] (1.92)

Since \((\mathcal{J}_a)^0_k = 0\) for \(a, k = 1, 2, 3\), the spin conditions \((1.87)\) give for \(i = 0\)
\[
\sum_s u^0(\vec{0}, \vec{s})(\mathcal{J}_a^{(j)})_{\vec{s} \vec{s}} = 0 \quad \text{and} \quad -\sum_s v^0(\vec{0}, \vec{s})(\mathcal{J}_a^{s(j)})_{\vec{s} \vec{s}} = 0.
\] (1.93)

Multiplying these equations from the right by \((\mathcal{J}_a^{(j)})_{\vec{s} \vec{s'}}\) while summing over \(a = 1, 2, 3\) and using the formula \((1.88)\) \([\mathcal{J}^{(j)}]_{ss'}^2 = j(j + 1) \delta_{ss'}\), we find
\[
j(j + 1) u^0(\vec{0}, s) = 0 \quad \text{and} \quad j(j + 1) v^0(\vec{0}, s) = 0.
\] (1.94)

Thus \(u^0(\vec{0}, \sigma)\) and \(v^0(\vec{0}, \sigma)\) can be anything if the field represents particles of spin \(j = 0\), but \(u^0(\vec{0}, \sigma)\) and \(v^0(\vec{0}, \sigma)\) must both vanish if the field represents particles of spin \(j > 0\).

Now we set \(i = 1, 2, 3\) in the spin conditions \((1.87)\) and again multiply from the right by \((\mathcal{J}_a^{(j)})_{ss'}\) while summing over \(a = 1, 2, 3\) and using the formula \((1.88)\) \((\mathcal{J}^{(j)})^2 = j(j + 1)\). The Lorentz rotation matrices generate a \(j = 1\) representation of the group of rotations.
\[
\sum_{k=1}^{3} (\mathcal{J}_a)^i_k (\mathcal{J}_a)^k_l = j(j + 1) \delta^i_l = 2 \delta^i_l.
\] (1.95)

So the remaining conditions on the fields are
\[
j(j + 1) u^i(\vec{0}, s') = \sum_{ss'a} u^i(\vec{0}, s)(\mathcal{J}_a^{(j)})_{\vec{s} \vec{s}} (\mathcal{J}_a^{(j)})_{\vec{s} \vec{s'}} = \sum_{kss'a} (\mathcal{J}_a)^i_k u^k(\vec{0}, s) (\mathcal{J}_a^{(j)})_{\vec{s} \vec{s'}}
\]
\[
= \sum_{kss'a} (\mathcal{J}_a)^i_k (\mathcal{J}_a)^k_\ell u^\ell(\vec{0}, s') = 2 \delta^i_\ell u^\ell(\vec{0}, s')
\]
\[
j(j + 1) v^i(\vec{0}, s') = \sum_{ss'a} v^i(\vec{0}, s)(\mathcal{J}_a^{s(j)})_{\vec{s} \vec{s}} (\mathcal{J}_a^{s(j)})_{\vec{s} \vec{s'}} = \sum_{kss'a} (\mathcal{J}_a)^i_k v^k(\vec{0}, s) (\mathcal{J}_a^{s(j)})_{\vec{s} \vec{s'}}
\]
\[
= \sum_{kss'a} (\mathcal{J}_a)^i_k (\mathcal{J}_a)^k_\ell v^\ell(\vec{0}, s') = 2 \delta^i_\ell v^\ell(\vec{0}, s').
\] (1.96)

Thus if \(j = 0\), then for \(i = 1, 2, 3\) both \(u^i(\vec{0}, s)\) and \(v^i(\vec{0}, s)\) must vanish, while if \(j > 0\), then since \(j(j + 1) = 2\), the spin \(j\) must be unity, \(j = 1\).
111 Vector field for spin-zero particles

The only nonvanishing components are constants taken conventionally as

\[ u^0(\vec{0}) = i\sqrt{m/2} \quad \text{and} \quad v^0(\vec{0}) = -i\sqrt{m/2}. \] (1.97)

At finite momentum the boost conditions (1.210) give them as

\[ u^\mu(p) = ip^\mu/\sqrt{2p^0} \quad \text{and} \quad v^\mu(p) = -ip^\mu/\sqrt{2p^0}. \] (1.98)

The vector field \( \phi^\mu(x) \) of a spin-zero particle is then the derivative of a scalar field \( \phi(x) \)

\[ \phi^\mu(x) = \partial^\mu \phi(x) = \int \frac{d^3p}{\sqrt{(2\pi)^3(2p^0)}} \left[ ip^\mu a(p) e^{ip\cdot x} - ip^\mu b^\dagger(p) e^{-ip\cdot x} \right] \] (1.99)

112 Vector field for spin-one particles

We start with the \( s = 0 \) spinors \( u^i(\vec{0}, 0) \) and \( v^i(\vec{0}, 0) \) and note that since

\[ (J_3^i)_{s0} = 0, \] the \( a = 3 \) rotation conditions (1.87) imply that

\[ (J_3^i)_{k} u^k(\vec{0}, 0) = iR_3 u^i(\vec{0}, 0) = 0 \quad \text{and} \quad (J_3^i)_{k} v^k(\vec{0}, 0) = iR_3 v^i(\vec{0}, 0) = 0. \] (1.100)

Referring back to the explicit formulas for the generators of rotations and setting \( u, v = (0, x, y, z) \) we see that

\[ J_3 u = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ -iy \\ x \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \] (1.101)

and

\[ J_3 v = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ -iy \\ x \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \] (1.102)

Thus only the 3-component \( z \) can be nonzero. The conventional choice is

\[ u^\mu(\vec{0}, 0) = v^\mu(\vec{0}, 0) = \frac{1}{\sqrt{2m}} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \] (1.103)

We now form the linear combinations of the rotation conditions (1.87)
that correspond to the raising and lowering matrices $J_\pm^{(1)} = J_1^{(1)} \pm i J_2^{(1)}$

$$J_+^{(1)} = \sqrt{2} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad J_-^{(1)} = \sqrt{2} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \quad (1.104)$$

Their Lorentz counterparts are

$$J_\pm^{(1)} = J_1^{(1)} \pm i J_2^{(1)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \mp 1 \\ 0 & 0 & -i \end{pmatrix}. \quad (1.105)$$

In these terms, the rotation conditions (1.87) for the $j = 1$ spinors $u^i(\vec{0}, s)$ are

$$\sum_s u^i(\vec{0}, \bar{s})(J_\pm^{(1)})_{ss} = \sum_k (J_\pm)^i_k u^k(\vec{0}, s). \quad (1.106)$$

But

$$J_1^{(1)} \pm i J_2^{(1)} = J_+^{(1)} \mp i J_-^{(1)} = J_. \quad (1.107)$$

So the rotation conditions (1.87) for the $j = 1$ spinors $v^i(\vec{0}, s)$ are

$$-\sum_s v^i(\vec{0}, \bar{s})(J_+^{(1)})_{ss} = \sum_k (J_\pm)^i_k v^k(\vec{0}, s). \quad (1.108)$$

So for the plus sign and the choice $s = 0$, the condition (1.106) gives $u^i(\vec{0}, 1)$ as

$$\sum_s u^i(\vec{0}, \bar{s}) J_+^{(1)} = \sqrt{2} u^i(\vec{0}, 1) = (J_+)^i_k u^k(\vec{0}, 0) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -i \end{pmatrix} \frac{1}{\sqrt{2m}} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad (1.109)$$

or

$$u^i(\vec{0}, 1) = \frac{1}{2\sqrt{m}} \begin{pmatrix} 0 \\ 1 \\ -i \\ 0 \end{pmatrix}. \quad (1.110)$$
Similarly, the minus sign and the choice $s = 0$ give for $u^i(\tilde{0}, -1)$

$$
\sum_s u^i(\tilde{0}, s) J^{(1)}_{-s0} = \sqrt{2} u^i(\tilde{0}, -1) = (J_-)^i_k u^k(\tilde{0}, 0) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & -1 & i & 0 \end{pmatrix} \frac{1}{\sqrt{2m}} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}
$$

or

$$
u^i(\tilde{0}, -1) = \frac{1}{2\sqrt{m}} \begin{pmatrix} 0 \\ 1 \\ -i \\ 0 \end{pmatrix}.
$$

The rotation condition (1.108) for the $j = 1$ spinors $\nu^i(\tilde{0}, s)$ with the minus sign and the choice $s = 0$ gives

$$
-\sum_s \nu^i(\tilde{0}, s) J^{(1)}_{-s0} = -\sqrt{2} \nu^i(\tilde{0}, -1) = (J_+)^i_k \nu^k(\tilde{0}, 0) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 1 & i & 0 \end{pmatrix} \frac{1}{\sqrt{2m}} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}
$$

or

$$
\nu^i(\tilde{0}, -1) = \frac{1}{2\sqrt{m}} \begin{pmatrix} 0 \\ 1 \\ i \\ 0 \end{pmatrix}.
$$

Similarly, the plus sign and the choice $s = 0$ give

$$
-\sum_s \nu^i(\tilde{0}, s) J^{(1)}_{+s0} = -\sqrt{2} \nu^i(\tilde{0}, 1) = (J_-)^i_k \nu^k(\tilde{0}, 0) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & -1 & i & 0 \end{pmatrix} \frac{1}{\sqrt{2m}} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}
$$

or

$$
\nu^i(\tilde{0}, 1) = \frac{1}{2\sqrt{m}} \begin{pmatrix} 0 \\ -1 \\ i \\ 0 \end{pmatrix}.
$$

The boost conditions (1.210) now give for $i, k = 0, 1, 2, 3$

$$
u^i(\vec{p}, s) = \nu'^i(\vec{p}, s) = \sqrt{m/\vec{p}\cdot} L^i_k(\vec{p}) \nu^k(\tilde{0}, s) = e^i(\vec{p}, s)/\sqrt{2\vec{p}\cdot}
$$

(1.117)
where
\[ e^i(\vec{p}, s) = L_k^i(\vec{p}) e^k(\vec{0}, s) \] (1.118)

and
\[ e(\vec{0}, 0) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad e(\vec{0}, 1) = -\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \\ 0 \\ 0 \end{pmatrix}, \quad \text{and} \quad e(\vec{0}, -1) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -i \\ 0 \end{pmatrix}. \] (1.119)

A single massive vector field is then
\[ \phi_i(x) = \phi^+_i(x) + \phi^-_i(x) = \sum_{s=\pm 1} \int \frac{d^3p}{(2\pi)^3 2p^0} e^i(\vec{p}, s) a(\vec{p}, s) e^{ip \cdot x} + e^i(\vec{p}, s) a^\dagger(\vec{p}, s) e^{-ip \cdot x}. \] (1.120)

The commutator/anticommutator of the positive and negative frequency parts of the field is
\[ [\phi^+_i(x), \phi^-_k(y)]_\mp = \int \frac{d^3p}{(2\pi)^3 2p^0} e^{ip \cdot (x-y)} \Pi_{ik}(\vec{p}) \] (1.121)

where \( \Pi \) is a sum of outer products of 4-vectors
\[ \Pi_{ik}(\vec{p}) = \sum_{s=\pm 1} e^i(\vec{p}, s)e^* k(\vec{p}, s). \] (1.122)

At \( \vec{p} = 0 \), the matrix \( \Pi \) is the unit matrix on the spatial coordinates
\[ \Pi(\vec{0}) = \sum_{s=\pm 1} e^i(\vec{0}, s)e^* k(\vec{0}, s) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \] (1.123)

So \( \Pi(\vec{p}) \) is
\[ \Pi(\vec{p}) = L \Pi(0) L^T = L \eta L^T + L \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} L^T \] (1.124)

or
\[ \Pi(\vec{p})_{ik} = \eta_{ik} + p_i p_k / m^2. \] (1.125)
This equation lets us write the commutator (1.126) in terms of the Lorentz-invariant function \( \Delta_+ (x-y) \) as

\[
[\phi^+ i(x), \phi^+ k(y)] \equiv (\eta^{ik} - \partial^i \partial^k / m^2) \int \frac{d^3 p}{\sqrt{(2\pi)^3}} 2^p(x-y) \nonumber
\]

\[
= (\eta^{ik} - \partial^i \partial^k / m^2) \Delta_+ (x-y).
\]

(1.126)

As for a scalar field, we set

\[
v^i(x) = \kappa \phi^+ i(x) + \lambda \phi^+ i(x) \nonumber
\]

and find for \((x-y)^2 > 0\) since \(\Delta_+ (x-y) = \Delta_+ (y-x)\) for \(x, y\) spacelike

\[
[v(x), v^i(y)] \equiv (|\kappa|^2 + |\lambda|^2) (\eta^{ik} - \partial^i \partial^k / m^2) \Delta_+ (x-y)
\]

\[
[v(x), v(y)] \equiv (1 + 1) \kappa \lambda (\eta^{ik} - \partial^i \partial^k / m^2) \Delta_+ (x-y).
\]

(1.128)

So we must choose the minus sign and set \(|\kappa| = |\lambda|\). So then

\[
v^i(x) = v^+ i(x) + v^- i(x) = v^+ i(x) + v^+ i(x) \nonumber
\]

is real. This is a second example of the spin-statistics theorem.

If two such fields have the same mass, then we can combine them as we combined scalar fields

\[
v^i(x) = v^+ i(x) + i v^+_2 i(x).
\]

(1.130)

These fields obey the Klein-Gordon equation

\[
(\Box - m^2)v^i(x) = 0.
\]

(1.131)

And since both

\[
p^i = L^i \cdot k \quad \text{and} \quad \epsilon^k (\vec{p}) = L^k \epsilon^i (0)
\]

(1.132)

it follows that

\[
p \cdot \epsilon (\vec{p}) = k \cdot \epsilon (0) = 0.
\]

(1.133)

So the field \(v^i\) also obeys the rule

\[
\partial_i v^i(x) = 0.
\]

(1.134)

These equations (1.133) and (1.134) are like those of the electromagnetic field in Lorentz gauge. But one can’t get quantum electrodynamics as the \(m \to 0\) limit of just any such theory. For the interaction \(\mathcal{H} = J_i v^i\) would lead to a rate for \(v\)-boson production like

\[
J_i J_k \Pi^{ik} (\vec{p})
\]

(1.135)

which diverges as \(m \to 0\) because of the \(p^i p^k / m^2\) term in \(\Pi^{ik} (\vec{p})\). One can
1.13 Lorentz group

avoid this divergence by requiring that $\partial_i J^i = 0$ which is current conservation.

Under parity, charge conjugation, and time reversal, a vector field transforms as

$$P v^\alpha(x) P^{-1} = - \eta^* P^a b v^b (P x)$$

$$C v^\alpha(x) C^{-1} = \xi^* v^\alpha(x)$$

$$T v^\alpha(x) T^{-1} = \zeta^* P^a b v^b (-P x).$$

(1.136)

1.13 Lorentz group

The Lorentz group $O(3,1)$ is the set of all linear transformations $L$ that leave invariant the Minkowski inner product

$$xy \equiv x \cdot y - x^0 y^0 = x^T \eta y$$

in which $\eta$ is the diagonal matrix

$$\eta = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.$$  

(1.137)

(1.138)

So $L$ is in $O(3,1)$ if for all 4-vectors $x$ and $y$

$$(Lx)^T \eta L y = x^T L^T \eta Ly = x^T \eta y.$$  

(1.139)

Since $x$ and $y$ are arbitrary, this condition amounts to

$$L^T \eta L = \eta.$$  

(1.140)

Taking the determinant of both sides and recalling that $\det A^T = \det A$ and that $\det(AB) = \det A \det B$, we have

$$(\det L)^2 = 1.$$  

(1.141)

So $\det L = \pm 1$, and every Lorentz transformation $L$ has an inverse. Multiplying \((\ref{1.140})\) by $\eta$, we get

$$\eta L^T \eta L = \eta^2 = I$$  

(1.142)

which identifies $L^{-1}$ as

$$L^{-1} = \eta L^T \eta.$$  

(1.143)
The subgroup of $O(3, 1)$ with $\det L = 1$ is the proper Lorentz group $SO(3, 1)$. The subgroup of $SO(3, 1)$ that leaves invariant the sign of the time component of timelike vectors is the **proper orthochronous Lorentz group** $SO^+(3, 1)$.

To find the Lie algebra of $SO^+(3, 1)$, we take a Lorentz matrix $L = I + \omega$ that differs from the identity matrix $I$ by a tiny matrix $\omega$ and require $L$ to obey the condition (1.140) for membership in the Lorentz group

\[
(I + \omega^T) \eta (I + \omega) = \eta + \omega^T \eta + \eta \omega + \omega^T \omega = \eta.
\]  

(1.144)

Neglecting $\omega^T \omega$, we have $\omega^T \eta = -\eta \omega$ or since $\eta^2 = I$

\[
\omega^T = -\eta \omega \eta.
\]  

(1.145)

This equation implies that the matrix $\omega_{ab}$ is antisymmetric when both indexes are down

\[
\omega_{ab} = -\omega_{ba}.
\]  

(1.146)

To see why, we write it (1.145) as $\omega^e_a = -\eta_{ab} \omega^b_c \eta^{ce}$ and the multiply both sides by $\eta_{de}$ so as to get $\omega_{da} = \eta_{de} \omega^e_a = -\eta_{ab} \omega^b_c \eta^{ce} \eta_{de} = -\omega_{ac} \delta^c_d = -\omega_{ad}$.

The key equation (1.145) also tells us that under transposition the time-time and space-space elements of $\omega$ change sign, while the time-space and spacetime elements do not. That is, the tiny matrix $\omega$ is for infinitesimal $\theta$ and $\lambda$ a linear combination

\[
\omega = \theta \cdot R + \lambda \cdot B
\]  

(1.147)

of three antisymmetric space-space matrices

\[
R_1 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix} \quad R_2 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0
\end{pmatrix} \quad R_3 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]  

(1.148)

and of three symmetric time-space matrices

\[
B_1 = \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} \quad B_2 = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} \quad B_3 = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}
\]  

(1.149)

all of which satisfy condition (1.145). The three $R_\ell$ are $4 \times 4$ versions of the familiar rotation generators; the three $B_\ell$ generate Lorentz boosts.
If we write \( L = I + \omega \) as
\[
L = I - i\theta_\ell R_\ell + i\lambda_\ell iB_\ell \equiv I - i\theta_\ell J_\ell + i\lambda_\ell K_\ell
\]
then the three matrices \( J_\ell = iR_\ell \) are imaginary and antisymmetric, and therefore hermitian. But the three matrices \( K_\ell = iB_\ell \) are imaginary and symmetric, and so are antihermitian. The \( 4 \times 4 \) matrix \( L = \exp(i\theta_\ell J_\ell - i\lambda_\ell K_\ell) \) is not unitary because the Lorentz group is not compact.

### 1.14 Gamma matrices and Clifford algebras

In component notation, \( L = I + \omega \) is
\[
L^a_b = \delta^a_b + \omega^a_b,
\]
the matrix \( \eta \) is \( \eta_{cd} = \eta^{cd} \), and \( \omega^T = -\eta \omega \eta \) is
\[
\omega^a_b = (\omega^T)^a_b = -\eta \omega \eta^a_b = -\eta_{bc} \omega^c_d \eta^{da} = -\omega_{bd} \eta^{da} = -\omega^a_b.
\]
Lowering index \( a \) we get
\[
\omega_{eb} = \eta_{ea} \omega^a_b = -\omega_{bd} \eta^{da} \eta_{ea} = -\omega_{bd} \delta^d_e = -\omega_{be}
\]
That is, \( \omega_{ab} \) is antisymmetric
\[
\omega_{ab} = -\omega_{ba}.
\]

A representation of the Lorentz group is generated by matrices \( D(L) \) that represent matrices \( L \) close to the identity matrix by sums over \( a, b = 0, 1, 2, 3 \)
\[
D(L) = 1 + \frac{i}{2} \omega_{ab} \mathcal{J}^{ab}.
\]
The generators \( \mathcal{J}^{ab} \) must obey the commutation relations
\[
i[\mathcal{J}^{ab}, \mathcal{J}^{cd}] = \eta^{bc} \mathcal{J}^{ad} - \eta^{ca} \mathcal{J}^{bd} - \eta^{da} \mathcal{J}^{cb} + \eta^{db} \mathcal{J}^{ca}.
\]
A remarkable representation of these commutation relations is provided by matrices \( \gamma^a \) that obey the anticommutation relations
\[
\{\gamma^a, \gamma^b\} = 2\eta^{ab}.
\]
One sets
\[
\mathcal{J}^{ab} = -\frac{i}{4} [\gamma^a, \gamma^b]
\]
where \( \eta \) is the usual flat-space metric (1.138). Any four \( 4 \times 4 \) matrices that
satisfy these anticommutation relations form a set of Dirac gamma matrices. They are not unique. Is $S$ is any nonsingular $4 \times 4$ matrix, then the matrices

$$
\gamma^a = S \gamma^a S^{-1} \tag{1.159}
$$

also are a set of Dirac’s gamma matrices.

Any set of matrices obeying the anticommutation relations (1.157) for any $n \times n$ diagonal matrix $\eta$ with entries that are $\pm 1$ is called a Clifford algebra.

As a homework problem, show that

$$
[J^{ab}, \gamma^c] = -i \gamma^a \eta^{bc} + i \gamma^b \eta^{ac}. \tag{1.160}
$$

One can use these commutation relations to derive the commutation relations (1.156) of the Lorentz group.

The gamma matrices are a vectors in the sense that for $L$ near the identity

$$
D(L) \gamma^c D^{-1}(L) \approx (I + i \frac{1}{2} \omega_{ab} \mathcal{J}^{ab}) \gamma^c (I - i \frac{1}{2} \omega_{ab} \mathcal{J}^{ab})
$$

in which we used (1.152) to write $-\omega^c_a = \omega_a^c$. The finite $\omega$ form is

$$
D(L) \gamma^a D^{-1}(L) = L^a_c \gamma^c. \tag{1.162}
$$

The unit matrix is a scalar

$$
D(L) I D^{-1}(L) = I. \tag{1.163}
$$

The generators of the Lorentz group form an antisymmetric tensor

$$
D(L) \mathcal{J}^{ab} D^{-1}(L) = L^a_c L^b_d \mathcal{J}^{cd}. \tag{1.164}
$$
1.15 Dirac’s gamma matrices

Out of four gamma matrices, one can also make totally antisymmetric tensors of rank-3 and rank-4

\[ A_{abc} \equiv \gamma^a \gamma^b \gamma^c \quad \text{and} \quad B_{abcd} \equiv \gamma^a \gamma^b \gamma^c \gamma^d \]  

(1.165)

where the brackets mean that one inserts appropriate minus signs so as to achieve total antisymmetry. Since there are only four \( \gamma \) matrices in four spacetime dimensions, any rank-5 totally antisymmetric tensor made from them must vanish, \( C_{abcde} = 0 \).

Notation: The parity transformation is

\[ \beta = i \gamma^0 \]  

(1.166)

It flips the spatial gamma matrices but not the temporal one

\[ \beta \gamma^i \beta^{-1} = - \gamma^i \quad \text{and} \quad \beta \gamma^0 \beta^{-1} = \gamma^0 \]  

(1.167)

It flips the generators of boosts but not those of rotations

\[ \beta \mathcal{J}^{0i} \beta^{-1} = - \mathcal{J}^{0i} \quad \text{and} \quad \beta \mathcal{J}^{ik} \beta^{-1} = \mathcal{J}^{ik} \]  

(1.168)

1.15 Dirac’s gamma matrices

Weinberg’s chosen set of Dirac matrices is

\[ \gamma^0 = -i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -\gamma^{0\dagger} \quad \text{and} \quad \gamma^i = -i \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} = \gamma^{i\dagger} \]  

(1.169)

in which the \( \sigma \)'s are Pauli’s 2 \( \times \) 2 hermitian matrices

\[ \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \text{and} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \]  

(1.170)

which are the gamma matrices of 3-dimensional spacetime. With this choice of \( \gamma \)'s, the matrix \( \beta \) is

\[ \beta = i \gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \beta^{\dagger}. \]  

(1.171)

In spacetimes of five dimensions, the fifth gamma matrix \( \gamma^4 \) which traditionally is called \( \gamma^5 = \gamma_5 \) is

\[ \gamma^5 = \gamma_5 = -i\gamma^0 \gamma^1 \gamma^2 \gamma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]  

(1.172)

It anticommutes with all four Dirac gammas and its square is unity, as it must if it is to be the fifth gamma in 5-space:

\[ \{ \gamma^a, \gamma^b \} = 2\eta^{ab} \]  

(1.173)
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for $a, b = 0, 1, 2, 3, 4$ with $\eta^{44} = 1$ and $\eta^{40} = \eta^{04} = 0$.

With Weinberg’s choice of $\gamma$’s, the Lorentz boosts are

$$
\mathcal{J}^{0a} = -\frac{i}{4} [\gamma^i, \gamma^0] = -\frac{i}{4} \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad \mathcal{J}^{10} = -i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathcal{J}^{20} = -i \begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}
$$

(1.174)

The Lorentz rotation matrices are

$$
\mathcal{J}^{ik} = \frac{i}{4} [\gamma^i, \gamma^k] = \frac{i}{4} \begin{pmatrix} -\sigma^i \sigma^k & 0 \\ 0 & -\sigma^i \sigma^k \end{pmatrix}, \quad \mathcal{J}^{jk} = \frac{i}{4} \begin{pmatrix} \sigma^j \sigma^k & 0 \\ 0 & \sigma^j \sigma^k \end{pmatrix}
$$

(1.175)

The Dirac representation of the Lorentz group is reducible, as SW’s choice of gamma matrices makes apparent. The Dirac rotation matrices are

$$
\mathcal{J}_i = \frac{1}{2} \begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}
$$

(1.176)

Some useful relations are

$$
\beta \gamma^a \beta = -\gamma^a, \quad \beta \mathcal{J}^{ab} \beta = \mathcal{J}^{ab} \quad \text{and} \quad \beta D(L)^\dagger \beta = D(L)^{-1}
$$

(1.177)

as well as

$$
\beta \gamma_5 \beta = -\gamma_5 \quad \text{and} \quad \beta (\gamma_5 \gamma^a)^\dagger \beta = -\gamma_5 \gamma^a.
$$

(1.178)

1.16 Dirac fields

The positive- and negative-frequency parts of a Dirac field are

$$
\psi^+_{\ell}(x) = (2\pi)^{-3/2} \sum_s \int d^3p \ u_\ell(\vec{p}, s) e^{ip \cdot x} a(\vec{p}, s)
$$

$$
\psi^-_{\ell}(x) = (2\pi)^{-3/2} \sum_s \int d^3p \ v_\ell(\vec{p}, s) e^{-ip \cdot x} b^{\dagger}(\vec{p}, s).
$$

(1.179)
1.16 Dirac fields

The rotation conditions (1.39) are

\[
\sum_{\ell} u_{\ell}(\bar{0}, \bar{s})(J_{\ell}^{(j)})_{\bar{s}s} = \sum_{\ell} (\mathcal{J}_{\ell})_{\ell\ell} u_{\ell}(\bar{0}, s)
\]

\[
\sum_{\ell} v_{\ell}(\bar{0}, \bar{s})(-J_{\ell}^{(j)})_{\bar{s}s} = \sum_{\ell} (\mathcal{J}_{\ell})_{\ell\ell} v_{\ell}(\bar{0}, s).
\]  

(1.180)

The Dirac rotation matrices (1.176) are

\[
\mathcal{J}_{\ell} = \frac{1}{2} \begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}
\]  

(1.181)

so we set the four values \(\ell, \bar{\ell} = 1, 2, 3, 4\) to \(\ell = (m, \pm)\) with \(m = \pm \frac{1}{2}\). And we consider \(u_{\ell}(s)\) to be \(u_{\ell}^+(s)\) stacked upon \(u_{\ell}^-(s)\) and similarly take \(v_{\ell}(s)\) to be \(v_{m}^+(s)\) above \(v_{m}^-(s)\) where \(u_{\ell}^\pm(s)\) and \(v_{m}^\pm(s)\) are, a priori, \(2 \times (2j + 1)\)-dimensional matrices with indexes \(m = \pm 1/2\) and \(s = -j, \ldots, j\). That is,

\[
\begin{pmatrix} u_1(s) \\ u_2(s) \\ u_3(s) \\ u_4(s) \end{pmatrix} = \begin{pmatrix} u_{+1/2}(s) \\ u_{-1/2}(s) \\ u_{+1/2}(s) \\ u_{-1/2}(s) \end{pmatrix}
\]

and

\[
\begin{pmatrix} v_1(s) \\ v_2(s) \\ v_3(s) \\ v_4(s) \end{pmatrix} = \begin{pmatrix} v_{+1/2}(s) \\ v_{-1/2}(s) \\ v_{+1/2}(s) \\ v_{-1/2}(s) \end{pmatrix}.
\]  

(1.182)

We then have four equations

\[
\sum_{\ell} u_{\ell}^+(\bar{0}, \bar{s})(J_{\ell}^{(j)})_{\bar{s}s} = \sum_{m} \frac{1}{2} \sigma^i_{m\bar{m}} u_{m}^+(\bar{0}, s)
\]

\[
\sum_{\ell} u_{\ell}^-(\bar{0}, \bar{s})(J_{\ell}^{(j)})_{\bar{s}s} = \sum_{m} \frac{1}{2} \sigma^i_{m\bar{m}} u_{m}^-(\bar{0}, s)
\]

\[
\sum_{\ell} v_{\ell}^+(\bar{0}, \bar{s})(-J_{\ell}^{(j)})_{\bar{s}s} = \sum_{m} \frac{1}{2} \sigma^i_{m\bar{m}} v_{m}^+(\bar{0}, s)
\]

\[
\sum_{\ell} v_{\ell}^-(\bar{0}, \bar{s})(-J_{\ell}^{(j)})_{\bar{s}s} = \sum_{m} \frac{1}{2} \sigma^i_{m\bar{m}} v_{m}^-(\bar{0}, s).
\]  

(1.183)

SW defines the four \(2 \times (2j + 1)\) matrices

\[
U_{ms}^+ = u_{m}^+(\bar{0}, s) \quad \text{and} \quad U_{ms}^- = u_{m}^-(\bar{0}, s)
\]

\[
V_{ms}^+ = v_{m}^+(\bar{0}, s) \quad \text{and} \quad V_{ms}^- = v_{m}^-(\bar{0}, s).
\]  

(1.184)

in terms of which the four Dirac rotation conditions (1.183) are

\[
U^+ J_{i}^{(j)} = \frac{1}{2} \sigma_i U^+ \quad \text{and} \quad U^- J_{i}^{(j)} = \frac{1}{2} \sigma_i U^-
\]

\[
V^+ (-J_{i}^{(j)}) = \frac{1}{2} \sigma_i V^+ \quad \text{and} \quad V^- (-J_{i}^{(j)}) = \frac{1}{2} \sigma_i V^-.
\]  

(1.185)
Taking the complex conjugate of the second of these equations, we get
\[ -J_i^{(j)} = V^{*+1}(\frac{1}{2}\sigma^*\sigma)V^{*+} = V^{*+1}(-\frac{1}{2}\sigma_2 \sigma^i \sigma_2)V^{*+} \]
\[ -J_i^{(j)} = V^{-*1}(\frac{1}{2}\sigma^*\sigma)V^{-*} = V^{-*1}(-\frac{1}{2}\sigma_2 \sigma^i \sigma_2)V^{-*} \]
(1.186)
or more simply
\[ J_i^{(j)} = (\sigma_2 V^{*+})^{-1} \frac{1}{2}\sigma^i (\sigma_2 V^{*+}) \]
\[ J_i^{(j)} = (\sigma_2 V^{-*})^{-1} \frac{1}{2}\sigma^i (\sigma_2 V^{-*}) . \]
(1.187)
The $2 \times 2$ Pauli matrices $\vec{\sigma}$ and the $(2j + 1) \times (2j + 1)$ matrices $\vec{J}^{(j)}$ both generate irreducible representations of the rotation group. So by writing
\[ U^+ J_k^{(j)} = \frac{1}{2} \sigma_i U^+ J_k^{(j)} = = \frac{1}{2} \sigma_i \frac{1}{2} \sigma_k U^+ \]
(1.188)
and similar equations for $U^-, V^+, V^-$, we see that
\[ U^+ D^{(j)}(\vec{0}) = U^+ e^{-i\vec{0} \cdot \vec{J}^{(j)}} = e^{-i\vec{0} \cdot \vec{J}} U^+ = D^{(1/2)}(\vec{0}) U^+ \]
\[ U^- D^{(j)}(\vec{0}) = U^- e^{-i\vec{0} \cdot \vec{J}^{(j)}} = e^{-i\vec{0} \cdot \vec{J}} U^- = D^{(1/2)}(\vec{0}) U^- \]
(1.189)
and similar equations for $V^\pm$.
\[ \sigma_2 V^{*+} D^{(j)}(\vec{0}) = \sigma_2 V^{*+} e^{-i\vec{0} \cdot \vec{J}^{(j)}} = e^{-i\vec{0} \cdot \vec{J}} \sigma_2 V^{*+} = D^{(1/2)}(\vec{0}) \sigma_2 V^{*+} \]
\[ \sigma_2 V^{-*} D^{(j)}(\vec{0}) = \sigma_2 V^{-*} e^{-i\vec{0} \cdot \vec{J}^{(j)}} = e^{-i\vec{0} \cdot \vec{J}} \sigma_2 V^{-*} = D^{(1/2)}(\vec{0}) \sigma_2 V^{-*} . \]
(1.190)

Now recall Schur’s lemma (section 10.7 of PM):

Part 1: If $D_1$ and $D_2$ are inequivalent, irreducible representations of a group $G$, and if $D_1(g)A = AD_2(g)$ for some matrix $A$ and for all $g \in G$, then the matrix $A$ must vanish, $A = 0$.

Part 2: If for a finite-dimensional, irreducible representation $D(g)$ of a group $G$, we have $D(g)A = AD(g)$ for some matrix $A$ and for all $g \in G$, then $A = cI$. That is, any matrix that commutes with every element of a finite-dimensional, irreducible representation must be a multiple of the identity matrix.

Part 1 tells us that $D^{(j)}(\vec{0})$ and $D^{(1/2)}(\vec{0})$ must be equivalent. So $j = 1/2$ and $2j + 1 = 2$. A Dirac field must represent particles of spin 1/2.

Part 2 then says that the matrices $U^\pm$ must be multiples of the $2 \times 2$ identity matrix
\[ U^+ = c_+ I \quad \text{and} \quad U^- = c_- I \]
(1.191)
and that the matrices $\sigma_2 V^{*+}$ must be multiples of the $2 \times 2$ identity matrix
\[ \sigma_2 V^{*+} = d_+ I \quad \text{and} \quad \sigma_2 V^{-*} = d_- I \]
(1.192)
or more simply

\[ V^+ = -i d_+ \sigma_2 \quad \text{and} \quad V^- = -i d_- \sigma_2. \quad (1.193) \]

That is,

\[ v_m^+(\bar{0}, s) = d_+ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad v_m^-(\bar{0}, s) = d_- \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \quad (1.194) \]

Going back to \( \ell = (m, \pm) \) by using the index code \( \ref{1.182} \), we have for the \( u \)'s

\[
\begin{align*}
    u_1(1/2) &= c_+ \quad \text{and} \quad u_{1/2}^-(1/2) = 0 \\
    v_1(1/2) &= v_2(1/2) = 0 \\
    u_{1/2}^+(1/2) &= c_+ \quad \text{and} \quad u_{1/2}^-(1/2) = 0 \\
    v_{1/2}^+(1/2) &= v_{1/2}^-(1/2) = 0 \\
    u_{1/2}^+(1/2) &= d_+ \quad \text{and} \quad v_{1/2}^+(1/2) = d_- \\
    v_{1/2}^+(1/2) &= d_+ \quad \text{and} \quad v_{1/2}^-(1/2) = d_- \\
\end{align*}
\]

So

\[
\begin{align*}
    u(\bar{0}, m = \frac{1}{2}) &= \begin{bmatrix} u_{1/2}^+(1/2) \\ u_{1/2}^+(1/2) \\ u_{1/2}^+(1/2) \\ u_{1/2}^-(1/2) \end{bmatrix} = \begin{bmatrix} c_+ \\ 0 \\ c_- \\ 0 \end{bmatrix} \quad \text{and} \quad u(\bar{0}, m = -\frac{1}{2}) = \begin{bmatrix} v_{1/2}^+(1/2) \\ v_{1/2}^+(1/2) \\ v_{1/2}^+(1/2) \\ v_{1/2}^-(1/2) \end{bmatrix} = \begin{bmatrix} 0 \\ c_+ \\ 0 \\ c_- \end{bmatrix}, \\
    v(\bar{0}, m = \frac{1}{2}) &= \begin{bmatrix} v_{1/2}^+(1/2) \\ v_{1/2}^+(1/2) \\ v_{1/2}^+(1/2) \\ v_{1/2}^-(1/2) \end{bmatrix} = \begin{bmatrix} 0 \\ d_+ \\ 0 \\ d_- \end{bmatrix} \quad \text{and} \quad v(\bar{0}, m = -\frac{1}{2}) = \begin{bmatrix} v_{1/2}^+(1/2) \\ v_{1/2}^+(1/2) \\ v_{1/2}^+(1/2) \\ v_{1/2}^-(1/2) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ d_+ \\ 0 \end{bmatrix}.
\end{align*}
\]
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To put more constraints on $c_\pm$ and $d_\pm$, we recall that under parity

$$P a(\vec{p}, s) P^{-1} = \eta^*_a a(-\vec{p}, s) \quad \text{and} \quad P b^\dagger(\vec{p}, s) P^{-1} = \eta_b b^\dagger(-\vec{p}, s) \quad (1.205)$$

and so

$$P \psi^+ (x) P^{-1} = (2\pi)^{-3/2} \sum_s \int d^3 p \ u_\ell(\vec{p}, s) \ e^{ip \cdot x} \eta^*_a a(-\vec{p}, s)$$

$$= (2\pi)^{-3/2} \sum_s \int d^3 p \ u_\ell(-\vec{p}, s) \ e^{ip \cdot \vec{p}} \eta^*_a a(\vec{p}, s) \quad (1.206)$$

$$P \psi^- (x) P^{-1} = (2\pi)^{-3/2} \sum_s \int d^3 p \ v_\ell(\vec{p}, s) \ e^{-ip \cdot x} \eta_b b^\dagger(-\vec{p}, s)$$

$$= (2\pi)^{-3/2} \sum_s \int d^3 p \ v_\ell(-\vec{p}, s) \ e^{-ip \cdot \vec{p}} \eta_b b^\dagger(\vec{p}, s) \ .$$

We recall the relations (1.177)

$$\beta \gamma^0 \beta = -\gamma^0, \quad \beta J^{ab\dagger} \beta = J^{ab}, \quad \text{and} \quad \beta D(L) \beta = D(L)^{-1} \quad (1.207)$$

and in particular, since $J^{0i\dagger} = -J^{0i}$, the rule

$$\beta J^{0i} \beta = J^{0i\dagger} = -J^{0i}. \quad (1.208)$$

We also have the pseudounitarity relation

$$\beta D^\dagger(L) \beta = D^{-1}(L). \quad (1.209)$$

In general spinors at finite momentum are related to those at zero momentum by

$$u_\ell(q, s) = \sqrt{m/q^0} \sum_\ell D_{\ell\ell}(L(q)) u_\ell(0, s)$$

$$v_\ell(q, s) = \sqrt{m/q^0} \sum_\ell D_{\ell\ell}(L(q)) v_\ell(0, s) \quad (1.210)$$

which for Dirac spinors is

$$u(p, s) = \sqrt{m/p^0} D(L(p)) u(0, s)$$

$$v(p, s) = \sqrt{m/p^0} D(L(p)) v(0, s) \quad (1.211)$$

So now by using the boost rule (1.208) we have

$$u_\ell(-\vec{p}, s) = \sqrt{m/p^0} D(L(-\vec{p})) u(0, s) = \sqrt{m/p^0} D(L(\vec{p}))-1 u(0, s) \quad (1.212)$$

$$= \sqrt{m/p^0} \beta D(L(\vec{p})) \beta u(0, s) \quad (1.213)$$
and

\[ v_\ell(-\vec{p}, s) = \sqrt{m/p^0} D(L(-\vec{p})) v(0, s) = \sqrt{m/p^0} D(L(\vec{p}))^{-1} v(0, s) \]  
\[ = \sqrt{m/p^0} \beta D(L(\vec{p})) \beta v(0, s). \] (1.214)

So under parity

\[ P \psi^+(x) P^{-1} = (2\pi)^{-3/2} \sum_s \int d^3 p \sqrt{m/p^0} \beta D(L(\vec{p})) \beta u(0, s) e^{ip \cdot \vec{x}} \eta_s^* a(\vec{p}, s) \]
\[ P \psi^-(x) P^{-1} = (2\pi)^{-3/2} \sum_s \int d^3 p \sqrt{m/p^0} \beta D(L(\vec{p})) \beta v(0, s) e^{-ip \cdot \vec{x}} \eta_b^\dagger a(\vec{p}, s). \] (1.215)

So to have \( P \psi^\pm_\ell (x) P^{-1} \propto \psi^\pm_\ell (x) \), we need

\[ \beta u(0, s) = b_u u(0, s) \quad \text{and} \quad \beta v(0, s) = b_v v(0, s). \] (1.216)

We then get

\[ P \psi^+_\ell (t, \vec{x}) P^{-1} = b_u \beta \eta_u^* \psi^+_\ell (t, -\vec{x}) \quad \text{and} \quad P \psi^-_\ell (t, \vec{x}) P^{-1} = b_v \beta \eta_b \psi^-_\ell (t, -\vec{x}). \] (1.217)

Here since \( P^2 = 1 \), these factors are just signs, \( b_u^2 = b_v^2 = 1 \). The eigenvalue equations \( \text{(1.217)} \) tell us that \( c_- = b_u c_+ \) and that \( d_- = b_v d_+ \). So rescaling the fields we get

\[ u(0, m = \frac{1}{\sqrt{2}}) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ b_u \\ 0 \end{bmatrix} \quad \text{and} \quad u(0, m = -\frac{1}{\sqrt{2}}) = -\frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 0 \\ b_u \end{bmatrix}, \]
\[ v(0, m = \frac{1}{\sqrt{2}}) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ b_v \end{bmatrix} \quad \text{and} \quad v(0, m = -\frac{1}{\sqrt{2}}) = -\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ b_v \end{bmatrix}. \] (1.218)

If the annihilation and creation operators \( a(p, s) \) and \( a^\dagger(p, s) \) obey the rule

\[ [a(p, s), a^\dagger(p', s')]_{\pm} = \delta_{ss'} \delta^3(\vec{p} - \vec{p}') \] (1.219)

and if the field is the sum of the positive- and negative-frequency parts \( 2.55 \)

\[ \psi^+_\ell (x) = (2\pi)^{-3/2} \sum_s \int d^3 p \ u_\ell(\vec{p}, s) e^{ip \cdot \vec{x}} a(\vec{p}, s) \]
\[ \psi^-_\ell (x) = (2\pi)^{-3/2} \sum_s \int d^3 p \ v_\ell(\vec{p}, s) e^{-ip \cdot \vec{x}} b^\dagger(\vec{p}, s). \] (1.220)
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with arbitrary coefficients $\kappa$ and $\lambda$

$$\psi(x) = \kappa \psi^\dagger(x) + \lambda \psi^-(x)$$  \hspace{1cm} (1.222)

then

$$[\psi(x), \psi^\dagger(y)]_\pm = [\kappa \psi^\dagger(x) + \lambda \psi^-(x), \kappa^* \psi^\dagger(y) + \lambda^* \psi^-(y)]$$

$$= \int \frac{d^3p}{(2\pi)^3} \sum_s \left[ |\kappa|^2 u_\ell(p, s) u^*_\ell(p, s) e^{ip(x-y)} \mp |\lambda|^2 v_\ell(p, s) v^*_\ell(p, s) e^{-ip(x-y)} \right]$$

$$= \int \frac{d^3p}{(2\pi)^3} \sum_s \left[ |\kappa|^2 N_{\ell\ell'}(p) e^{ip(x-y)} \mp |\lambda|^2 N_{\ell\ell'}(p) e^{-ip(x-y)} \right]$$  \hspace{1cm} (1.223)

where

$$N_{\ell\ell'}(p) = \sum_s u_\ell(p, s) u^*_\ell(p, s)$$

$$M_{\ell\ell'}(p) = \sum_s v_\ell(p, s) v^*_\ell(p, s).$$  \hspace{1cm} (1.224)

When $\vec{p} = 0$, these matrices are

$$N_{\ell\ell'}(0) = \sum_s u_\ell(0, s) u^*_\ell(0, s)$$

$$N(0) = \begin{bmatrix} 1 & 0 \\ b_u & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & b_u \\ 0 & 0 & 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0 & 1 & b_u \\ b_u & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & b_u \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 1 & 0 & b_u \\ 0 & 0 & 0 \\ b_u & 0 & 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & b_u \\ 0 & 0 & 0 \end{bmatrix} = \frac{1 + b_u \beta}{2}$$

(1.225)
and
\[ M_{\ell'\ell}(0) = \sum_s v_\ell(\vec{0}, s) v_\ell^*(\vec{0}, s) \]
\[ M(0) = \frac{1}{2} \begin{pmatrix} 0 & 1 & 0 \cr 0 & 1 & 0 \cr b_v & b_v & 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 & 0 & b_v \cr 0 & 0 & 0 \cr 0 & b_v & 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & b_v \\ 0 & 0 & 0 \end{pmatrix} \]
\[ = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & b_v \\ 0 & 0 & 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 & 0 & b_v & 0 \\ 0 & 0 & 0 & 0 \\ 0 & b_v & 0 & 1 \end{pmatrix} = \frac{1 + b_v \beta}{2}. \] (1.226)

So then using the boost relations (1.211) we find
\[ N(\vec{p}) = \sum_s u_\ell(\vec{p}, s) u^*_\ell(\vec{p}, s) = \frac{m}{\vec{p}^0} D(L(p)) \sum_s u_\ell(\vec{0}, s) u^*_\ell(\vec{0}, s) D^\dagger(L(p)) \]
\[ = \frac{m}{2\vec{p}^0} D(L(p)) (1 + b_m \beta) D^\dagger(L(p)) \]
\[ M(\vec{p}) = \sum_s v_\ell(\vec{p}, s) v^*_\ell(\vec{p}, s) = \frac{m}{\vec{p}^0} D(L(p)) \sum_s v_\ell(\vec{0}, s) v^*_\ell(\vec{0}, s) D^\dagger(L(p)) \]
\[ = \frac{m}{2\vec{p}^0} D(L(p)) (1 + b_v \beta) D^\dagger(L(p)). \] (1.227)

The pseudounitarity relation (1.209)
\[ \beta D^\dagger(L) \beta = D^{-1}(L). \] (1.228)
gives
\[ \beta D^\dagger(L) = D^{-1}(L) \beta \] (1.229)
which implies that
\[ D(L) \beta D^\dagger(L) = \beta. \] (1.230)
The pseudounitarity relation also says that
\[ D^\dagger(L) = \beta D^{-1}(L) \beta \] (1.231)
so that
\[ D(L) D^\dagger(L) = D(L) \beta D^{-1}(L) \beta. \] (1.232)
Also since the gammas form a 4-vector (1.162)
\[ D(L) \gamma^a D^{-1}(L) = L_c^a \gamma^c \] (1.233)
and since $\beta = i\gamma^0$, we have
\[
D(L(p)) \beta D^{-1}(L(p)) = D(L(p)) i\gamma^0 D^{-1}(L(p)) = iL_c^0(p) \gamma^c = -iL^c_0 \gamma_c.
\]
(1.234)

Now
\[
p^a = L^a_b(p) k^b = L^a_0(p) m
\]
(1.235)
so
\[
D(L(p)) \beta D^{-1}(L(p)) = -i p^c \gamma_c / m
\]
(1.236)
which implies that
\[
D(L) D^\dagger(L) = -i (p^c \gamma_c / m) \beta.
\]
(1.237)

Thus
\[
N(\vec{p}) = \frac{m}{2p^0} D(L(p)) \left( 1 + b_u \beta \right) D^\dagger(L(p)) = \frac{m}{2p^0} \left[(-i p^c \gamma_c / m) \beta + b_u \beta \right]
\]
\[
= \frac{1}{2p^0} \left[ -i p^c \gamma_c + b_u m \right] \beta
\]
(1.238)
and
\[
M(\vec{p}) = \frac{m}{2p^0} D(L(p)) \left( 1 + b_v \beta \right) D^\dagger(L(p)) = \frac{m}{2p^0} \left[(-i p^c \gamma_c / m) \beta + b_v \beta \right]
\]
\[
= \frac{1}{2p^0} \left[ -i p^c \gamma_c + b_v m \right] \beta.
\]
(1.239)

We now put the spin sums (1.238) and (1.239) in the (anti)commutator (1.223) and get
\[
[\psi_\ell(x), \psi^\dagger_{\ell'}(y)]_\mp = \int \frac{d^3p}{(2\pi)^3 2p^0} \left[ \kappa^2 \left[(-i p^c \gamma_c + b_u m) \beta \right]_{\ell \ell'} e^{ip(x-y)} \right.
\]
\[
\quad + |\lambda|^2 \left[(-i p^c \gamma_c + b_v m) \beta \right]_{\ell' \ell} e^{-ip(x-y)} \left. \right]\left[(-i p^c \gamma_c + b_u m) \beta \right]_{\ell' \ell} e^{ip(x-y)} \right]
\]
\[
= \kappa^2 \left[(-i \partial_c \gamma^c + b_u m) \beta \right]_{\ell \ell'} \int \frac{d^3p}{(2\pi)^3 2p^0} e^{ip(x-y)}
\]
\[
\quad + |\lambda|^2 \left[(-\partial_c \gamma^c + b_v m) \beta \right]_{\ell' \ell} \int \frac{d^3p}{(2\pi)^3 2p^0} e^{-ip(x-y)}
\]
\[
= \kappa^2 \left[(-i \partial_c \gamma^c + b_u m) \beta \right]_{\ell \ell'} \Delta_+(x-y)
\]
\[
\quad + |\lambda|^2 \left[(-\partial_c \gamma^c + b_v m) \beta \right]_{\ell' \ell} \Delta_+(y-x).
\]
(1.240)

Recall that for $(x-y) > 0$, i.e. spacelike, $\Delta_+(x-y) = \Delta_+(y-x)$. So its first
derivatives are odd. So for $x - y$ spacelike
\[ [\psi(x), \psi(y)]_\pm = |\kappa|^2 \left[ (\gamma^e - b \gamma^c + a) \gamma^c \right]_{\ell \ell'} \Delta_+ (x - y) \]
\[ \mp |\lambda|^2 \left[ (\gamma^e + b \gamma^c - a) \gamma^c \right]_{\ell \ell'} \Delta_+ (x - y) \]
\[ = (|\kappa|^2 \pm |\lambda|^2) \left[ (\gamma^e - b \gamma^c) \gamma^c \right]_{\ell \ell'} \Delta_+ (x - y) \]
\[ \pm (|\kappa|^2 b + |\lambda|^2 b) \gamma^c \gamma^c \Delta_+ (x - y). \] (1.241)

To get the first term to vanish, we need to choose the lower sign (that is, use anticommutators) and set $|\kappa| = |\lambda|$. To get the second term to be zero, we must set $b_u = -b_v$. We may adjust $\kappa$ and $b_u$ so that
\[ \kappa = \lambda \quad \text{and} \quad b_u = -b_v = 1. \] (1.242)

In particular, a spin-one-half field must obey anticommutation relations
\[ [\psi(x), \psi(y)]_+ \equiv \{\psi(x), \psi(y)\} = 0 \quad \text{for} \quad (x - y)^2 > 0. \] (1.243)

Finally then, the Dirac field is
\[ \psi(x) = \sum_s \int \frac{d^3 p}{(2\pi)^{3/2}} \left[ u_\ell(\vec{p}, s) e^{i p \cdot x} a(\vec{p}, s) + v_\ell(\vec{p}, s) e^{-i p \cdot x} b^\dagger(\vec{p}, s) \right]. \] (1.244)

The zero-momentum spinors are
\[ u(0, m = \frac{1}{2}) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad u(0, m = -\frac{1}{2}) = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \]
\[ v(0, m = \frac{1}{2}) = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \quad \text{and} \quad v(0, m = -\frac{1}{2}) = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}. \] (1.245)

The spin sums are
\[ [N(\vec{p})]_{\ell m} = \sum_s u_\ell(\vec{p}, s) u^*_m(\vec{p}, s) = \left[ \frac{1}{2 \vec{p} \cdot \hat{p}} (p^c \gamma^c + m) \beta \right]_{\ell m}, \]
\[ [M(\vec{p})]_{\ell m} = \sum_s v_\ell(\vec{p}, s) v^*_m(\vec{p}, s) = \left[ \frac{1}{2 \vec{p} \cdot \hat{p}} (p^c \gamma^c - m) \beta \right]_{\ell m}. \] (1.246)

The Dirac anticommutator is
\[ [\psi(x), \psi(y)]_+ \equiv \{\psi(x), \psi(y)\} = [(\gamma^e - b \gamma^c + a) \gamma^c \Delta_+ (x - y). \] (1.247)
Two standard abbreviations are
\[ \beta \equiv i\gamma^0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad \bar{\psi} \equiv \psi^\dagger \beta = \psi^\dagger \gamma^0 = \begin{bmatrix} \psi_x^* & \psi_x \\ \psi_x & \psi_x^* \end{bmatrix}. \] (1.248)

A Majorana fermion is represented by a field like
\[ \psi_L(x) = \sum_s \int \frac{d^3p}{(2\pi)^{3/2}} \left[ u_L(p, s) e^{ip \cdot x} a(p, s) + v_L(p, s) e^{-ip \cdot x} a^\dagger(p, s) \right]. \] (1.249)

Since \( C = \gamma_2 \beta \) it follows that \( C^{-1} = \beta \gamma_2 \) and so that \( J^{*ab} = -\beta C J^{ab} C^{-1} \beta = -\gamma_2 \beta J^{ab} \beta \gamma_2 \beta \). But \( \beta \gamma_2 \beta = -\beta \beta \gamma_2 = -i^2 \gamma_0^2 \gamma_2 = -\gamma_2 \). So \( J^{*ab} = -\gamma_2 J^{ab} \). Thus
\[ D^*(L) = e^{-i\omega^{ab} J^{*ab}} = e^{-i\omega^{ab} (-\gamma_2 J^{ab} \gamma_2)} = \gamma_2 e^{i\omega^{ab} J^{ab} \gamma_2} \gamma_2 = \gamma_2 D(L) \gamma_2. \] (1.250)

Now with SW’s \( \gamma \)'s,
\[ \gamma_2 u(\vec{0}, \pm \frac{1}{2}) = v(\vec{0}, \pm \frac{1}{2}) \quad \text{and} \quad \gamma_2 v(\vec{0}, \pm \frac{1}{2}) = u(\vec{0}, \pm \frac{1}{2}). \] (1.251)

Thus the hermitian conjugate of a Majorana field is
\[
\psi^\dagger(x) = \sum_s \int \frac{d^3p}{(2\pi)^{3/2}} \left[ u^*(\vec{p}, s) e^{-ip \cdot x} a^\dagger(\vec{p}, s) + v^*(\vec{p}, s) e^{ip \cdot x} a(\vec{p}, s) \right] \\
= \sum_s \int \frac{d^3p}{(2\pi)^{3/2}} \left[ v^*(\vec{p}, s) e^{ip \cdot x} a(\vec{p}, s) + u^*(\vec{p}, s) e^{-ip \cdot x} a^\dagger(\vec{p}, s) \right] \\
= \sum_s \int \frac{d^3p}{(2\pi)^{3/2}} \left[ D^*(L(p)) v^*(\vec{0}, s) e^{ip \cdot x} a(\vec{p}, s) + D^*(L(p)) u^*(\vec{0}, s) e^{-ip \cdot x} a^\dagger(\vec{p}, s) \right] \\
= \gamma_2 \sum_s \int \frac{d^3p}{(2\pi)^{3/2}} \left[ D(L(p)) \gamma_2 u(\vec{0}, s) e^{ip \cdot x} a(\vec{p}, s) + D(L(p)) \gamma_2 v(\vec{0}, s) e^{-ip \cdot x} a^\dagger(\vec{p}, s) \right] \\
= \gamma_2 \sum_s \int \frac{d^3p}{(2\pi)^{3/2}} \left[ D(L(p)) u(\vec{0}, s) e^{ip \cdot x} a(\vec{p}, s) + D(L(p)) v(\vec{0}, s) e^{-ip \cdot x} a^\dagger(\vec{p}, s) \right] \\
= \gamma_2 \sum_s \int \frac{d^3p}{(2\pi)^{3/2}} \left[ u(\vec{p}, s) e^{ip \cdot x} a^\dagger(\vec{p}, s) + v(\vec{p}, s) e^{-ip \cdot x} a(\vec{p}, s) \right] = \gamma_2 \psi(x). \] (1.252)

The parity rules \( \{1.253\} \) now are
\[ P\psi^+_L(t, \vec{x}) P^{-1} = \beta \eta_\alpha^L \psi^+_L(t, -\vec{x}) \quad \text{and} \quad P\psi^-_L(t, \vec{x}) P^{-1} = -\beta \eta_\alpha^L \psi^-_L(t, -\vec{x}). \] (1.253)

So to have a Dirac field survive a parity transformation, we need the phase
of the particle to be minus the complex conjugate of the phase of the antiparticle

$$\eta_a^* = -\eta_b \quad \text{or} \quad \eta_b = -\eta_a^*. \quad (1.254)$$

So the intrinsic parity of a particle-antiparticle state is odd. So negative-parity bosons like $\pi^0, \rho_0, J/\psi$ can be interpreted as s-wave bound states of quark-antiquark pairs. Under parity a Dirac field goes as

$$P\psi(t, \vec{x})P^{-1} = \eta^* \beta \psi(t, -\vec{x}). \quad (1.255)$$

If a Dirac particle is the same as its antiparticle, then its intrinsic parity must be odd under complex conjugation, $\eta = -\eta^*$. So the intrinsic parity of a Majorana fermion must be imaginary

$$\eta = \pm i. \quad (1.256)$$

But this means that if we express a Dirac field $\psi$ as a complex linear combination

$$\psi = \frac{1}{\sqrt{2}}(\chi_1 + i\chi_2) \quad (1.257)$$
of two Majorana fields with intrinsic parities $\eta_1^* = \pm i$ and $\eta_2^* = \pm i$, then under parity

$$P\psi(t, \vec{x})P^{-1} = \frac{1}{\sqrt{2}} \left( \eta_1^* \beta \phi_1(t, -\vec{x}) + i\eta_2^* \beta \phi_2(t, -\vec{x}) \right) \quad (1.258)$$
so we need $\eta_1^* = \eta_2^*$ to have

$$P\psi(t, \vec{x})P^{-1} = \frac{1}{\sqrt{2}} \left( \eta_1^* \beta \phi_1(t, -\vec{x}) + i\eta_2^* \beta \phi_2(t, -\vec{x}) \right) = \eta_1^* \beta \psi(t, -\vec{x}). \quad (1.259)$$

But in that case the Dirac field has intrinsic parity $\eta = \pm i$.

The equation (1.236) that shows how beta goes under $D(L(p))$

$$D(L(p)) \beta D^{-1}(L(p)) = -i p^\nu \gamma_\nu / m \quad (1.260)$$
tells us that the spinors (1.211)

$$u(p, s) = \sqrt{m / p^0} D(L(p)) u(\vec{0}, s) \quad \text{and} \quad v(p, s) = \sqrt{m / p^0} D(L(p)) v(\vec{0}, s) \quad (1.261)$$
are eigenstates of \(-i p^\gamma c/m\) with eigenvalues \(\pm 1\)

\[
\begin{align*}
( - i p^\gamma c/m ) u(p, s) &= D(L(p)) \beta D^{-1}(L(p)) \sqrt{\frac{m}{p^0}} D(L(p)) u(\vec{0}, s) \\
&= \sqrt{\frac{m}{p^0}} D(L(p)) \beta u(\vec{0}, s) = \sqrt{\frac{m}{p^0}} D(L(p)) u(\vec{0}, s) = u(p, s) \\
( - i p^\gamma c/m ) v(p, s) &= D(L(p)) \beta D^{-1}(L(p)) \sqrt{\frac{m}{p^0}} D(L(p)) v(\vec{0}, s) \\
&= \sqrt{\frac{m}{p^0}} D(L(p)) \beta v(\vec{0}, s) = -\sqrt{\frac{m}{p^0}} D(L(p)) v(\vec{0}, s) = -v(p, s).
\end{align*}
\]

So

\[
(i p^\gamma c + m) u(p, s) = 0 \quad \text{and} \quad (-i p^\gamma c + m) v(p, s) = 0
\]

which implies that a Dirac field obeys Dirac’s equation

\[
(\gamma^a \partial_a + m) \psi(x) = (\gamma^a \partial_a + m) \sum_s \int \frac{d^3p}{(2\pi)^3/2} \left[ u(\vec{p}, s) e^{ipx} a(\vec{p}, s) + v(\vec{p}, s) e^{-ipx} b^\dagger(\vec{p}, s) \right]
\]

\[
= \sum_s \int \frac{d^3p}{(2\pi)^3/2} \left[ (\gamma^a \partial_a + m) u(\vec{p}, s) e^{ipx} a(\vec{p}, s) + (\gamma^a \partial_a + m) v(\vec{p}, s) e^{-ipx} b^\dagger(\vec{p}, s) \right]
\]

\[
= \sum_s \int \frac{d^3p}{(2\pi)^3/2} \left[ (i\gamma^a p_a + m) u(\vec{p}, s) e^{ipx} a(\vec{p}, s) + (-i\gamma^a p_a + m) v(\vec{p}, s) e^{-ipx} b^\dagger(\vec{p}, s) \right] = 0.
\]

SW shows that under complex conjugation

\[
u^*(p, s) = -\beta C v(p, s) \quad \text{and} \quad u^*(p, s) = -\beta C u(p, s).
\]

So for a Dirac field to survive charge conjugation, the particle-antiparticle phases must be related

\[
\xi_b = \xi_a^*.
\]

Then under charge conjugation a Dirac field goes as

\[
C \psi(x) C^{-1} = -\xi^* \beta C \psi^*(x).
\]

If a Dirac particle is the same as its antiparticle, then \(\xi\) must be real (and
1.16 Dirac fields

η imaginary), ξ = ±1, and must satisfy the reality condition

$$\psi(x) = -\beta C \psi^*(x). \quad (1.268)$$

Suppose a particle and its antiparticle form a bound state

$$|\Phi\rangle = \sum_{ss'} \int d^3p d^3p' \chi(p, s; p', s') a^\dagger(p, s) b^\dagger(p', s') |0\rangle. \quad (1.269)$$

Under charge conjugation

$$C |\Phi\rangle = \xi_a \xi_b \sum_{ss'} \int d^3p d^3p' \chi(p, s; p', s') b^\dagger(p, s) a^\dagger(p', s') |0\rangle$$

$$= - \xi_a \xi_b \sum_{ss'} \int d^3p d^3p' \chi(p, s; p', s') a^\dagger(p', s') b^\dagger(p, s) |0\rangle \quad (1.270)$$

$$= - \xi_a \xi_b \sum_{ss'} \int d^3p d^3p' \chi(p', s'; p, s) a^\dagger(p, s) b^\dagger(p', s') |0\rangle$$

$$= - \xi_a \xi_b |\Phi\rangle = -\xi_a \xi_{a^*} |\Phi\rangle = -|\Phi\rangle.$$

The intrinsic charge-conjugation parity of a bound state of a particle and its antiparticle is odd.
2

Feynman diagrams

2.1 Time-dependent perturbation theory

Most physics problems are insoluble, and we must approximate the unknown solution by numerical or analytic methods. The most common analytic method is called perturbation theory and is based on an assumption that something is small compared to something simple that we can analyze.

In time-dependent perturbation theory, we write the hamiltonian $H$ as the sum $H = H_0 + V$ of a simple hamiltonian $H_0$ and a small, complicated part $V$ which we give the time dependence induced by $H_0$

$$V(t) = e^{iH_0t/\hbar} Ve^{-iH_0t/\hbar}. \quad (2.1)$$

We alter the time dependence of states by the same simple exponential (and set $\hbar = 1$)

$$|\psi, t\rangle = e^{iH_0t} e^{-iHt} |\psi\rangle \quad (2.2)$$

and find as its time derivative

$$i\frac{d}{dt} |\psi, t\rangle = e^{iH_0t} (H - H_0) e^{-iHt} |\psi\rangle = e^{iH_0t} V e^{-iH_0t} e^{iH_0t} e^{-iHt} |\psi\rangle$$

$$= V(t) |\psi, t\rangle. \quad (2.3)$$

We can formally solve this differential equation by time-ordering factors of $V(t)$ in expansion of the exponential

$$|\psi, t\rangle = T[e^{-i\int_0^t V(t') dt'}] |\psi\rangle$$

$$= \left\{ 1 - i \int_0^t V(t') dt' - \frac{1}{2!} \int T[V(t')V(t'')] dt'dt'' + \ldots \right\} |\psi\rangle \quad (2.4)$$

so that $V$’s of later times occur to the left of $V$’s of earlier times, that is, if $t > t'$ then $T[V(t)V(t')] = V(t)V(t')$ and so forth.
2.2 Dyson’s expansion of the S matrix

The time-evolution operator in the interaction picture is the time-ordered exponential of the integral over time of the interaction hamiltonian

$$T[e^{-i \int V(t) dt}].$$  \hfill (2.5)

Time ordering has $V(t_>)$ to the left of $V(t_\leq)$ if the time $t_>$ is later than the time $V(t_\leq)$. The interaction hamiltonian is

$$V(t) = \int H(t, x) \, d^3 x.$$  \hfill (2.6)

The density of the interaction hamiltonian is (or is taken to be) a sum of terms

$$H(t, x) = \sum_i g_i H_i(t, x)$$  \hfill (2.7)

each of which is a monomial in the fields $\psi_\ell(x)$ and their adjoints $\psi_\ell^\dagger(x)$. A generic field is an integral over momentum

$$\psi_\ell(x) = \sum_s \int \frac{d^3 p}{(2\pi)^{3/2}} \left[ u_\ell(p, s, n) a(p, s, n) e^{ip \cdot x} + v_\ell(p, s, n) b^\dagger(p, s, n) e^{-ip \cdot x} \right]$$  \hfill (2.8)
in which $n$ labels the kind of field.

The elements of the S matrix are amplitudes for an initial state $|p_1, s_1, n_1; \ldots; p_k, s_k, n_k\rangle$ to evolve into a final state $|p'_1, s'_1, n'_1; \ldots; p'_{k'}, s'_{k'}, n'_{k'}\rangle$. The case considered most often is when $k = 2$ incoming particles and $k' = 2$ or $3$ outgoing particles, but in high-energy collisions, $k'$ can be much larger than 2 or 3. An S-matrix amplitude is

$$S_{p'_1, s'_1; n'_1; \ldots; p_1, s_1, n_1; \ldots; p_k, s_k, n_k} = \langle p'_1, s'_1, n'_1; \ldots; p'_{k'}, s'_{k'}, n'_{k'}|T[e^{-i \int H(x) \, d^4 x}]| p_1, s_1, n_1; \ldots; p_k, s_k, n_k\rangle = \langle 0|a(p_{k'}, s_{k'}, n_{k'}; \ldots; a(p_1, s_1, n_1)$$

$$\times \sum_{N=0}^{\infty} \frac{(-i)^n}{N!} \int d^4 x_1 \ldots d^4 x_N T[H(x_1) \ldots H(x_N)]$$

$$\times a^\dagger(p_1, s_1, n_1; \ldots; a^\dagger(p_k, s_k, n_k)|0\rangle$$  \hfill (2.9)
in which $|0\rangle$ is the vacuum state. It is the mean value in the vacuum state of an infinite polynomial in creation and annihilation operators.

To make sense of it, the first step is to normally order the time-evolution operator by moving all annihilation operators to the right of all creation operators. We assume for now that $H(x)$ itself is already normally ordered.
To do that we use these commutation relations with plus signs for bosons and minus signs for fermions:

\[ a(p, s, n) a^\dagger(p', s', n') = \pm a(p', s', n') a(p, s, n) + \delta^3(\vec{p} - \vec{p}') \delta_{ss'} \delta_{nn'} \]

\[ a(p, s, n) a(p', s', n') = \pm a(p', s', n') a(p, s, n) \]

(2.10)

\[ a^\dagger(p, s, n) a^\dagger(p', s', n') = \pm a^\dagger(p', s', n') a^\dagger(p, s, n). \]

Since

\[ a(p, s, n) |0\rangle = 0 \quad \text{and} \quad \langle 0 | a^\dagger(p, s, n) = 0, \]

(2.11)

the S-matrix amplitude is just the left-over pieces proportional to products of delta functions multiplied by \( u \)'s and \( v \)'s and Fourier factors all divided by \( 2\pi \)'s.

Each such piece arises from the pairing of one annihilation operator with one creation operator. Such pairs arise in six ways:

1. A particle \( p', s', n' \) in the final state can pair with a field adjoint \( \psi^\dagger_\ell(x) \) in \( H(x) \) and yield the factor

\[ [a(p', s', n'), \psi^\dagger_\ell(x)]_\mp = u_\ell(p', s', n') e^{-ip' \cdot x}(2\pi)^{-3/2}. \]

(2.12)

2. An antiparticle \( p', s', n' \) in the final state can pair with a field \( \psi_\ell(x) \) in \( H(x) \) and yield the factor

\[ [b(p', s', n'), \psi_\ell(x)]_\mp = v_\ell(p', s', n') e^{-ip' \cdot x}(2\pi)^{-3/2}. \]

(2.13)

3. A particle \( p, s, n \) in the initial state can pair with a field \( \psi_\ell(x) \) in \( H(x) \) and yield the factor

\[ [\psi_\ell(x), a(p', s, n)]_\mp = u_\ell(p, s, n) e^{ip \cdot x}(2\pi)^{-3/2}. \]

(2.14)

4. An antiparticle \( p, s, n \) in the initial state can pair with a field adjoint \( \psi^\dagger_\ell(x) \) in \( H(x) \) and yield the factor

\[ [\psi^\dagger_\ell(x), b(p, s, n)]_\mp = v^*_\ell(p, s, n) e^{ip \cdot x}(2\pi)^{-3/2}. \]

(2.15)

5. A particle (or antiparticle) \( p', s', n' \) in the final state can pair with a particle (or antiparticle) \( p, s, n \) in the initial state and yield

\[ [a(p', s', n'), a^\dagger(p, s, n)]_\mp = \delta^3(\vec{p} - \vec{p}') \delta_{ss'} \delta_{nn'}. \]

(2.16)

6. A field \( \psi_\ell(x) = \psi^+_\ell(x) + \psi^-_\ell(x) \) in \( H(x) \) can pair with a field adjoint in \( H'(y) \), \( \psi^\dagger_\ell(y) = \psi^+_\ell(y) + \psi^-_\ell(y) \). But for \( \psi^+_\ell(x) \) to cross \( \psi^\dagger_\ell(y) \) the time \( x^0 \) must be later than \( y^0 \), and for \( \psi^-_\ell(y) \) to cross \( \psi^\dagger_\ell(x) \), the time \( y^0 \) must
be later than \( x^0 \). If for instance \( H(x) = \psi(x)\phi(x) \), then in Dyson’s expansion these terms would appear

\[
\theta(x^0 - y^0) \psi(x)\phi(x) \psi(y)\phi(y) + \theta(y^0 - x^0) \psi(x)\phi(y) \psi(y)\phi(x) .
\]

(2.17)

The resulting pairings are the propagator

\[
\theta(x^0 - y^0) [\psi(x), \psi(y)] \pm \theta(y^0 - x^0) [\psi(y), \psi(x)] \equiv -i\Delta_{\ell m}(x, y)
\]

(2.18)

in which the \( \pm \) signs will be explained later.

One then integrates over \( N \) spacetimes and the implicit momenta. The result is defined by a set of rules and Feynman diagrams. In general, the \( 1/N! \) in Dyson’s expansion is cancelled by the \( N! \) ways of labelling the \( x_i \)’s. For example, in the term

\[
\frac{1}{2!} \int d^4x_1 d^4x_2 T[H(x_1)H(x_2)]
\]

(2.19)

one can have \( H(x_1) \) absorb an incoming electron of momentum \( p \) and have \( H(x_2) \) absorb an incoming electron of momentum \( p' \) or the reverse.

But some processes require special combinatorics. So people often write

\[
H(x) = \frac{g}{3!}\phi^3(x) \quad \text{or} \quad H(x) = \frac{g}{4!}\phi^4(x)
\]

(2.20)

to compensate for multiple possible pairings. But these factorials don’t always cancel. Fermions introduce minus signs. The surest way to check the signs and factorials in each process until one has gained sufficient experience.

### 2.3 The Feynman propagator for scalar fields

Adding \( \pm i\epsilon \) to the denominator of a pole term of an integral formula for a function \( f(x) \) can slightly shift the pole into the upper or lower half plane, causing the pole to contribute if a ghost contour goes around the upper half-plane or the lower half-plane. Such an \( i\epsilon \) can impose a boundary condition on a Green’s function.

The Feynman propagator \( \Delta_F(x) \) is a Green’s function for the Klein-Gordon differential operator \( \Box \) pp. 274–280

\[
(m^2 - \Box)\Delta_F(x) = \delta^4(x)
\]

(2.21)

in which \( x = (x^0, \mathbf{x}) \) and

\[
\Box = \Delta - \frac{\partial^2}{\partial x^2} = \Delta - \frac{\partial^2}{\partial (x^0)^2}
\]

(2.22)
is the four-dimensional version of the laplacian $\triangle \equiv \nabla \cdot \nabla$. Here $\delta^4(x)$ is the four-dimensional Dirac delta function

$$
\delta^4(x) = \int \frac{d^4q}{(2\pi)^4} \exp[i(q \cdot x - q^0x^0)] = \int \frac{d^4q}{(2\pi)^4} e^{iqx} \tag{2.23}
$$

in which $qx = q \cdot x - q^0x^0$ is the Lorentz-invariant inner product of the 4-vectors $q$ and $x$. There are many Green’s functions that satisfy Eq. (2.21). Feynman’s propagator $\Delta_F(x)$

$$
\Delta_F(x) = \int \frac{d^4q}{(2\pi)^4} \frac{\exp(iqx)}{q^2 + m^2 - i\epsilon} = \int \frac{d^4q}{(2\pi)^4} \int_{-\infty}^{\infty} dq^0 \frac{e^{iqx - iq^0x^0}}{2\pi q^2 + m^2 - i\epsilon}. \tag{2.24}
$$

is the one that satisfies boundary conditions that will become evident when we analyze the effect of its $i\epsilon$. The quantity $E_q = \sqrt{q^2 + m^2}$ is the energy of a particle of mass $m$ and momentum $q$ in natural units with the speed of light $c = 1$. Using this abbreviation and setting $\epsilon' = \epsilon / 2E_q$, we may write the denominator as

$$
q^2 + m^2 - i\epsilon = q \cdot q - (q^0)^2 + m^2 - i\epsilon = (E_q - i\epsilon' - q^0)(E_q - i\epsilon' + q^0) + \epsilon'^2 \tag{2.25}
$$

in which $\epsilon'^2$ is negligible. Dropping the prime on $\epsilon$, we do the $q^0$ integral

$$
I(q) = -\int_{-\infty}^{\infty} \frac{dq^0}{2\pi} e^{-iq^0x^0} \frac{1}{[q^0 - (E_q - i\epsilon)][q^0 - (E_q + i\epsilon)]}. \tag{2.26}
$$

As shown in Fig. 2.1, the integrand

$$
e^{-iq^0x^0} \frac{1}{[q^0 - (E_q - i\epsilon)][q^0 - (E_q + i\epsilon)]} \tag{2.27}
$$

has poles at $E_q - i\epsilon$ and at $-E_q + i\epsilon$. When $x^0 > 0$, we can add a ghost contour that goes clockwise around the lower half-plane and get

$$
I(q) = ie^{-iE_qx^0} \frac{1}{2E_q} x^0 > 0. \tag{2.28}
$$

When $x^0 < 0$, our ghost contour goes counterclockwise around the upper half-plane, and we get

$$
I(q) = ie^{iE_qx^0} \frac{1}{2E_q} x^0 < 0. \tag{2.29}
$$

Using the step function $\theta(x) = (x + |x|)/2$, we combine (2.28) and (2.29) to get

$$
-\epsilon I(q) = \frac{1}{2E_q} \left[ \theta(x^0) e^{-iE_qx^0} + \theta(-x^0) e^{iE_qx^0} \right]. \tag{2.30}
$$
2.3 The Feynman propagator for scalar fields

Ghost Contours and the Feynman Propagator

Figure 2.1 In equation (2.27), the function $f(q^0)$ has poles at $\pm(E_q - i\epsilon)$, and the function $\exp(-iq^0x^0)$ is exponentially suppressed in the lower half plane if $x^0 > 0$ and in the upper half plane if $x^0 < 0$. So we can add a ghost contour ($\ldots$) in the LHP if $x^0 > 0$ and in the UHP if $x^0 < 0$.

In terms of the Lorentz-invariant function

$$\Delta_+(x) = \frac{1}{(2\pi)^3} \int \frac{d^3q}{2E_q} \exp[i(q \cdot x - E_q x^0)]$$

(2.31)

and with a factor of $-i$, Feynman’s propagator (2.24) is

$$-i\Delta_F(x) = \theta(x^0) \Delta_+(x) + \theta(-x^0) \Delta_+(x,-x^0).$$

(2.32)
The integral (2.31) defining $\Delta_+(x)$ is insensitive to the sign of $q$, and so

$$\Delta_+(x) = 1 = \left(\frac{2\pi}{E_q}\right)^3 \int \frac{d^3q}{2E_q} \exp[i(q \cdot x + E_q x^0)] = \Delta_+(x, -x^0).$$

Thus we arrive at the Standard form of the Feynman propagator

$$-i\Delta_F(x) = \theta(x^0) \Delta_+(x) + \theta(-x^0) \Delta_+(x). \quad (2.34)$$

The annihilation operators $a(q)$ and the creation operators $a^\dagger(p)$ of a scalar field $\phi(x)$ satisfy the commutation relations

$$[a(q), a^\dagger(p)] = \delta^3(q - p) \quad \text{and} \quad [a(q), a(p)] = [a^\dagger(q), a^\dagger(p)] = 0. \quad (2.35)$$

Thus the commutator of the positive-frequency part

$$\phi^+(x) = \int \frac{d^3p}{\sqrt{(2\pi)^32p^0}} \exp[i(p \cdot x - p^0 x^0)] a(p) \quad (2.36)$$

of a scalar field $\phi = \phi^+ + \phi^-$ with its negative-frequency part

$$\phi^-(y) = \int \frac{d^3q}{\sqrt{(2\pi)^32q^0}} \exp[-i(q \cdot y - q^0 y^0)] a^\dagger(q) \quad (2.37)$$

is the Lorentz-invariant function $\Delta_+(x - y)$

$$[\phi^+(x), \phi^-(y)] = \int \frac{d^3p d^3q}{(2\pi)^32\sqrt{q^0 p^0}} e^{ipx - iqy} [a(p), a^\dagger(q)]$$

$$= \int \frac{d^3p}{(2\pi)^32p^0} e^{ip(x - y)} = \Delta_+(x - y) \quad (2.38)$$

in which $p(x - y) = p \cdot (x - y) - p^0(x^0 - y^0)$.

At points $x$ that are space-like, that is, for which $x^2 = x^2 - (x^0)^2 \equiv r^2 > 0$, the Lorentz-invariant function $\Delta_+(x)$ depends only upon $r = +\sqrt{x^2}$ and has the value [11] p. 202

$$\Delta_+(x) = \frac{m}{4\pi^2r} K_1(mr) \quad (2.39)$$

in which the Hankel function $K_1$ is

$$K_1(z) = -\frac{\pi}{2} [J_1(iz) + iN_1(iz)] = \frac{1}{z} - \frac{z}{2} \left[ \ln \left(\frac{z}{2}\right) + \gamma - \frac{1}{2} \right] + \ldots \quad (2.40)$$

where $J_1$ is the first Bessel function, $N_1$ is the first Neumann function, and $\gamma = 0.57721 \ldots$ is the Euler-Mascheroni constant.
2.4 Application to a cubic scalar field theory

The Feynman propagator arises most simply as the mean value in the vacuum of the time-ordered product of the fields \( \phi(x) \) and \( \phi(y) \)

\[
\mathcal{T} \{ \phi(x)\phi(y) \} \equiv \theta(x^0 - y^0)\phi(x)\phi(y) + \theta(y^0 - x^0)\phi(y)\phi(x).
\]

(2.41)

The operators \( a(p) \) and \( a^\dagger(p) \) respectively annihilate the vacuum ket \( a(p)|0\rangle = 0 \) and bra \( \langle 0|a^\dagger(p) = 0 \), and so by (2.36 & 2.37) do the positive- and negative-frequency parts of the field \( \phi^+(z)|0\rangle = 0 \) and \( \langle 0|\phi^-(z) = 0 \). Thus the mean value in the vacuum of the time-ordered product is

\[
\langle 0|\mathcal{T} \{ \phi(x)\phi(y) \} |0\rangle = \langle 0|\theta(x^0 - y^0)\phi(x)\phi(y) + \theta(y^0 - x^0)\phi(y)\phi(x)|0\rangle \\
= \langle 0|\theta(x^0 - y^0)\phi^+(x)\phi^-(y) + \theta(y^0 - x^0)\phi^-(y)\phi^+(x)|0\rangle \\
= \langle 0|\theta(x^0 - y^0)[\phi^+(x), \phi^-(y)] \\
+ \theta(y^0 - x^0)[\phi^+(y), \phi^-(x)]|0\rangle.
\]

(2.42)

But by (2.38), these commutators are \( \Delta_+(x - y) \) and \( \Delta_+(y - x) \). Thus the mean value in the vacuum of the time-ordered product of two real scalar fields

\[
\langle 0|\mathcal{T} \{ \phi(x)\phi(y) \} |0\rangle = \theta(x^0 - y^0)\Delta_+(x - y) + \theta(y^0 - x^0)\Delta_+(y - x) \\
= -i\Delta_F(x - y) = \int \frac{d^4q}{(2\pi)^4} \frac{e^{iq\cdot x}}{q^2 + m^2 - i\epsilon}
\]

(2.43)

is the Feynman propagator \( 2.32 \) multiplied by \(-i\).

2.4 Application to a cubic scalar field theory

The action density

\[
\mathcal{L} = -\frac{1}{2} \partial_\nu \phi \partial^\nu \phi - \frac{1}{2} m^2 \phi^2 - \frac{g}{3!} \phi^3,
\]

(2.44)

describes a scalar field with cubic interactions. We let

\[
V(t) = \frac{g}{3!} \int \phi(x)^3 \, d^3x
\]

(2.45)

in which the field \( \phi \) has the free-field time dependence

\[
\phi(x) = \int \frac{d^3p}{\sqrt{(2\pi)^32p^0}} \left[ a(p)e^{ip\cdot x} + a^\dagger(p)e^{-ip\cdot x} \right]
\]

(2.46)
in which \( p \cdot x = p^\mu x^\mu \) and \( p^0 = \sqrt{p^2 + m^2} \). The amplitude for the scattering of bosons with momenta \( p \) and \( k \) to momenta \( p' \) and \( k' \) is to
lowest order in the coupling constant $g$

$$A = \langle p', k' | - \frac{1}{2!} \int T[V(t')V(t'')] \, dt' \, dt'' \, | p, k)$$

$$= - \frac{g^2}{2!(3!)^2} \langle p', k' | \int T[\phi^3(x)\phi^3(y)] \, d^4xdy \, | p, k).$$

(2.47)

This process happens in three ways, so $A = A_1 + A_2 + A_3$.

The first way is for the incoming particles to be absorbed at the same vertex $x$ or $y$ and for the outgoing particles to be emitted at the other vertex $y$ or $x$. We cancel the $2!$ by choosing the initial particles to be absorbed at $y$ and the final particles to be emitted at $x$. Then using the expansion (2.46) of the field, we find

$$A_1 = - \frac{g^2}{4} \langle 0 | a(k')a(p') \int \frac{d^3p''d^3k''d^3p''d^3k'''}{(2\pi)^6 \sqrt{2p'^0k''^02p''^0k'''^0}} \, a^\dagger(p''')a^\dagger(k''')$$

$$\times e^{-i(p''''+k''')}xT[\phi(x)\phi(y)]a(p'')a(k''')e^{i(p'''+k'')}y \, d^4xd'y \, a^\dagger(p)a^\dagger(k) | 0 \rangle$$

(2.48)

after canceling two factors of 3 because each of the 3 fields $\phi^3(x)$ and each of the 3 fields $\phi^3(y)$ could be the one to remain in the time-ordered product $T[\phi(x)\phi(y)]$. The commutation relations $[a(p), a^\dagger(k)] = \delta^3(p-k)$ now give

$$A_1 = - g^2 \int \frac{d^4xd^4y}{(2\pi)^6 \sqrt{2p'^0k''^02p''^0k'''^0}} e^{-i(p+k)-y-i(p'+k')} \langle 0 | T[\phi(x)\phi(y)] | 0 \rangle$$

(2.49)

in which the mean value in the vacuum of the time-ordered product

$$\langle 0 | T[\phi(x)\phi(y)] | 0 \rangle = - i \Delta_F(x-y) = - i \int \frac{d^4q}{(2\pi)^4} \frac{\exp(\imath qx)}{q^2 + m^2 - \imath \epsilon}$$

(2.50)

is Feynman’s propagator (2.24) & (2.43). Thus the amplitude $A_1$ is

$$A_1 = ig^2 \int \frac{d^4xd^4yd^4q}{(2\pi)^{10} \sqrt{2p'^0k''^02p''^0k'''^0}} \frac{e^{i(p+k)-y-i(p'+k')x+\imath qx}}{q^2 + m^2 - \imath \epsilon}$$

(2.51)

Thanks to Dirac’s delta function, the integrals over $x$, $y$, and $q$ are easy, and in the smooth $\epsilon \to 0$ limit $A_1$ is

$$A_1 = \frac{g^2}{(2\pi)^2 \sqrt{2p'^0k''^02p''^0k'''^0}} \frac{\delta^4(p+k-q)\delta^4(q-p'-k')}{q^2 + m^2 - \imath \epsilon}$$

$$= \frac{\delta^4(p+k-p'-k')}{(2\pi)^2 \sqrt{2p'^0k''^02p''^0k'''^0}} \frac{ig^2}{(p+k)^2 + m^2},$$

(2.52)
The sum of the three amplitudes is

\[
A = \frac{\delta^4(p + k - p' - k')}{{16\pi}^2\sqrt{p^0k^0p'^0k'^0}} \left[ \frac{ig^2}{(p + k)^2 + m^2} + \frac{ig^2}{(p - k')^2 + m^2} + \frac{ig^2}{(p - p')^2 + m^2} \right]
\]

in which the delta function conserves energy and momentum.

2.5 Feynman’s propagator for fields with spin

The time-ordered product for fields with spin is defined so as to compensate for the minus sign that arises when a Fermi field is moved past a Fermi field. Thus the mean value in the vacuum of the time-ordered product \( T\{\psi(x)\psi^\dagger_m(y)\} \) is

\[
\langle 0 | T\{\psi(x)\psi^\dagger_m(y)\} | 0 \rangle = \langle 0 | \theta(x^0 - y^0)\psi(x)\psi^\dagger_m(y) \pm \theta(y^0 - x^0)\psi^\dagger_m(y)\psi(x) | 0 \rangle
\]

\[
= \langle 0 | \theta(x^0 - y^0)\psi^+(x)\psi^\dagger_m^-(y) \pm \theta(y^0 - x^0)\psi^\dagger_m^+(y)\psi^-(x) | 0 \rangle
\]

\[
= \langle 0 | \theta(x^0 - y^0)[\psi^+_m(x), \psi^\dagger_m^-(y)] \pm \theta(y^0 - x^0)[\psi^\dagger_m^+(y), \psi^-_m(x)] | 0 \rangle. \tag{2.54}
\]

in which the upper signs are used for bosons and the lower ones for fermions.

The expansions

\[
\psi^+_\ell(x) = (2\pi)^{-3/2} \sum_s \int d^3p \ u_\ell(\vec{p}, s) e^{ip \cdot x} a(\vec{p}, s)
\]

\[
\psi^-_\ell(x) = (2\pi)^{-3/2} \sum_s \int d^3p \ v_\ell(\vec{p}, s) e^{-ip \cdot x} b^\dagger(\vec{p}, s) \tag{2.55}
\]

give for the (anti)commutators

\[
[\psi^+_\ell(x), \psi^\dagger_m(y)] = \left[ (2\pi)^{-3/2} \sum_s \int d^3p \ u_\ell(\vec{p}, s) e^{ip \cdot x} a(\vec{p}, s), \right.
\]

\[
\left. (2\pi)^{-3/2} \sum_r \int d^3k \ u_m^\ast(\vec{k}, r) e^{-ik \cdot y} a^\dagger(\vec{k}, r) \right] \pm \tag{2.56}
\]

\[
= \sum_s \int \frac{d^3p}{(2\pi)^3} u_\ell(\vec{p}, s) u_m^\ast(\vec{p}, s) e^{ip \cdot (x-y)}
\]
Feynman diagrams

\[ [\psi^+_m(y), \psi^-_\ell(x)]_\mp = \left[ (2\pi)^{-3/2} \sum_s \int d^3p \, v_m^*(\vec{p}, s) e^{ip \cdot y} b(\vec{p}, s), \right. \]
\[ \left. (2\pi)^{-3/2} \sum_r \int d^3k \, v_\ell(k, r) e^{-ik \cdot x} b^\dagger(k, r) \right]_\mp (2.57) \]

\[
= \sum_s \int \frac{d^3p}{(2\pi)^3} v_\ell(\vec{p}, s) v_m^*(\vec{p}, s) e^{ip \cdot (y-x)}. \]

Putting these expansions into the formula (2.54) for the time-ordered product, we get for its mean value in the vacuum

\[
\langle 0 | T \{ \psi_\ell(x) \psi^+_m(y) \} | 0 \rangle = \theta(x^0 - y^0) \sum_s \int \frac{d^3p}{(2\pi)^3} u_\ell(\vec{p}, s) u_m^*(\vec{p}, s) e^{ip \cdot (x-y)}
\]
\[ \pm \theta(y^0 - x^0) \sum_s \int \frac{d^3p}{(2\pi)^3} v_\ell(\vec{p}, s) v_m^*(\vec{p}, s) e^{ip \cdot (y-x)}. \]

(2.58)

2.6 Feynman’s propagator for spin-one-half fields

For fields of spin one half, the spin sums are

\[
[N(\vec{p})]_\ell m = \sum_s u_\ell(\vec{p}, s) u_m^*(\vec{p}, s) = \left[ \frac{1}{2p^0} (-i p^c \gamma^c + m) \beta \right]_\ell m \]
\[
[M(\vec{p})]_\ell m = \sum_s v_\ell(\vec{p}, s) v_m^*(\vec{p}, s) = \left[ \frac{1}{2p^0} (-i p^c \gamma^c - m) \beta \right]_\ell m. \]

(2.59)

so for spin-one-half fields Feynman’s propagator is

\[
\langle 0 | T \{ \psi_\ell(x) \psi^+_m(y) \} | 0 \rangle = \theta(x^0 - y^0) \int \frac{d^3p}{(2\pi)^3} \left[ \frac{1}{2p^0} (-i p^c \gamma^c + m) \beta \right]_\ell m e^{ip \cdot (x-y)}
\]
\[ - \theta(y^0 - x^0) \int \frac{d^3p}{(2\pi)^3} \left[ \frac{1}{2p^0} (-i p^c \gamma^c - m) \beta \right]_\ell m e^{ip \cdot (y-x)}. \]

(2.60)
Using the derivative term \(-\partial_c \gamma^c\) to generate \(-ip_c \gamma^c\), we get

\[
\langle 0 | T \{ \psi_\ell(x) \bar{\psi}_m(y) \} | 0 \rangle = \theta(x^0 - y^0) \left[ (-\partial_c \gamma^c + m) \beta \right]_{\ell m} \int \frac{d^3p}{(2\pi)^32p^0} e^{i\mathbf{p} \cdot (x-y)} - \theta(y^0 - x^0) \left[ (\partial_c \gamma^c - m) \beta \right]_{\ell m} \int \frac{d^3p}{(2\pi)^32p^0} e^{i\mathbf{p} \cdot (y-x)}
\]

\[
= \theta(x^0 - y^0) \left[ (-\partial_c \gamma^c + m) \beta \right]_{\ell m} \Delta_+(x-y) + \theta(y^0 - x^0) \left[ (\partial_c \gamma^c - m) \beta \right]_{\ell m} \Delta_+(y-x).
\]

(2.61)

Since the derivative of the step function is a delta function

\[
\partial_0 \theta(x^0 - y^0) = \delta(x^0 - y^0),
\]

(2.62)

we can write Feynman’s propagator as

\[
\langle 0 | T \{ \psi_\ell(x) \bar{\psi}_m(y) \} | 0 \rangle = \left[ (-\partial_c \gamma^c + m) \beta \right]_{\ell m} \times \left( \theta(x^0 - y^0) \Delta_+(x-y) + \theta(y^0 - x^0) \Delta_+(y-x) \right)
\]

\[
+ \partial_0 \gamma^0 \beta \theta(x^0 - y^0) \Delta_+(x-y)
\]

\[
- \partial_0 \gamma^0 \beta \theta(y^0 - x^0) \Delta_+(y-x)
\]

(2.63)

and so also as

\[
\langle 0 | T \{ \psi_\ell(x) \bar{\psi}_m(y) \} | 0 \rangle = \left[ (-\partial_c \gamma^c + m) \beta \right]_{\ell m} \times \left( \theta(x^0 - y^0) \Delta_+(x-y) + \theta(y^0 - x^0) \Delta_+(y-x) \right)
\]

\[
+ \gamma^0 \beta \delta(x^0 - y^0) \Delta_+(x-y) - \gamma^0 \beta \delta(y^0 - x^0) \Delta_+(y-x)
\]

\[
= \left[ (-\partial_c \gamma^c + m) \beta \right]_{\ell m} \times \left( \theta(x^0 - y^0) \Delta_+(x-y) - \theta(y^0 - x^0) \Delta_+(y-x) \right)
\]

\[
+ \gamma^0 \beta \delta(x^0 - y^0) \left( \Delta_+(x-y) - \Delta_+(y-x) \right)
\]

(2.64)

But at equal times \(\Delta_+(x-y) = \Delta_+(y-x)\). So the ugly final term vanishes, and Feynman’s propagator for spin-one-half fields is

\[
\langle 0 | T \{ \psi_\ell(x) \bar{\psi}_m(y) \} | 0 \rangle = \left[ (-\partial_c \gamma^c + m) \beta \right]_{\ell m} \times \left( \theta(x^0 - y^0) \Delta_+(x-y) + \theta(y^0 - x^0) \Delta_+(y-x) \right).
\]

(2.65)
But this last piece is Feynman’s propagator for real scalar fields

\[ \langle 0 | T \{ \phi(x) \phi(y) \} | 0 \rangle = \theta(x^0 - y^0) \Delta_+(x-y) + \theta(y^0 - x^0) \Delta_+(y-x) = -i \Delta_F(x-y) = \int \frac{d^4q}{(2\pi)^4} \frac{-i}{q^2 + m^2 - i\epsilon} e^{iq \cdot (x-y)}. \]

so Feynman’s propagator for spin-one-half fields is

\[ \langle 0 | T \{ \psi(x) \psi^\dagger_m(y) \} | 0 \rangle \equiv -i \Delta_{\ell m}(x-y) \]

\[ = \left[ ( - \partial_c \gamma^c + m ) \beta \right]_{\ell m} \int \frac{d^4q}{(2\pi)^4} \frac{-i}{q^2 + m^2 - i\epsilon} e^{iq \cdot (x-y)}. \] (2.66)

Letting the derivatives act on the exponential, we get

\[ \langle 0 | T \{ \psi(x) \psi^\dagger_m(y) \} | 0 \rangle \equiv -i \Delta_{\ell m}(x-y) \]

\[ = -i \int \frac{d^4q}{(2\pi)^4} \left[ ( - i q_a \gamma^a + m ) \beta \right]_{\ell m} e^{iq \cdot (x-y)}. \] (2.67)

Since

\[ ( - i q_a \gamma^a + m ) ( i q_a \gamma^a + m ) = q^2 + m^2, \] (2.68)

people often write this as

\[ \langle 0 | T \{ \psi(x) \psi^\dagger_m(y) \} | 0 \rangle \equiv -i \Delta_{\ell m}(x-y) \]

\[ = -i \int \frac{d^4q}{(2\pi)^4} \left[ \frac{1}{i q_a \gamma^a + m - i\epsilon} \beta \right]_{\ell m} e^{iq \cdot (x-y)}. \] (2.69)

Feynman was a master of notation (and of everything else). He set \( \not{p} = p_a \gamma^a \) and wrote

\[ \langle 0 | T \{ \psi(x) \psi^\dagger_m(y) \} | 0 \rangle \equiv -i \Delta_{\ell m}(x-y) \]

\[ = -i \int \frac{d^4q}{(2\pi)^4} \left[ \frac{1}{i \not{q} + m - i\epsilon} \beta \right]_{\ell m} e^{iq \cdot (x-y)}. \] (2.70)
In the common notation $\overline{\psi} = \psi^\dagger \beta$, his propagator is

$$\langle 0 | T \{ \psi(x) \overline{\psi}_m(y) \} | 0 \rangle \equiv -i \Delta_{\ell m} (x - y) \beta$$

$$= -i \int \frac{d^4q}{(2\pi)^4} \frac{\left[ \left( \frac{-i\mathbf{q}}{\mathbf{q}^2 + m^2 - i\epsilon} \right)_{\ell m} \right]}{\mathbf{q}^2 + m^2 - i\epsilon} e^{i\mathbf{q}(x-y)}$$

$$= -i \int \frac{d^4q}{(2\pi)^4} \left( \frac{1}{i\mathbf{q} + m - i\epsilon} \right)_{\ell m} e^{i\mathbf{q}(x-y)}. \quad (2.71)$$

2.7 Application to a theory of fermions and bosons

Let us consider first the theory

$$H(x) = g_{\ell m} \psi_{\ell}^\dagger(x) \overline{\psi}_m(x) \phi(x) \quad (2.72)$$

where $g_{\ell m}$ is a coupling constant, $\psi(x)$ is the field of a fermion $f$, and $\phi(x) = \phi^\dagger(x)$ is a real boson $b$. Let’s compute the amplitude $A$ for $f + b \rightarrow f' + b'$.

The lowest-order term is

$$A = -\frac{1}{2!} \int d^4x d^4y \langle 0 | a(p', s') b(k') T \left[ H(x) H(y) \right] b^\dagger(k) a^\dagger(p, s) | 0 \rangle \quad (2.73)$$

$$= -\frac{g_{\ell m} g_{\ell' m'}}{2!} \int d^4x d^4y \langle 0 | a(p', s') b(k') T \left[ \psi_{\ell}^\dagger(x) \overline{\psi}_m(x) \phi(x) \psi_{\ell'}^\dagger(y) \overline{\psi}_{m'}(y) \phi(y) \right]$$

Here the operators $b(k')$ and $b^\dagger(k)$ are the boson deletion and addition operators. Either the boson field $\phi(y)$ deletes the boson from the initial state and the boson field $\phi(x)$ deletes the boson from the final state or the boson field $\phi(x)$ deletes the boson from the initial state and the boson field $\phi(y)$ deletes the boson from the final state. These give the same result. So we cancel the $2!$ and let $\phi(y)$ delete the boson from the initial state and have
\( \phi(x) \) add the boson of the final state. We then get

\[
A = - g_{\ell m} g_{\ell m'} \int d^4 x d^4 y \langle 0 | a(p', s') b(k') T \left[ \psi^\dagger_{\ell'}(x) \psi_m(x) \phi^-(x) \psi_{\ell'}(y) \psi_{m'}(y) \phi^+(y) \right] \\
\times b^\dagger(k) a^\dagger(p, s) | 0 \rangle

= - g_{\ell m} g_{\ell m'} \int d^4 x d^4 y \langle 0 | a(p', s') b(k') T \left[ \psi^\dagger_{\ell'}(x) \psi_m(x) \right] \\
\times \int \frac{d^3 k''}{\sqrt{(2\pi)^3 2k''}} b^\dagger(k'') e^{-ik'' \cdot x} \psi^\dagger_{\ell'}(y) \psi_{m'}(y) \phi^+(y) \right] b^\dagger(k) a^\dagger(p, s) | 0 \rangle

= - g_{\ell m} g_{\ell m'} \int d^4 x d^4 y \langle 0 | a(p', s') T \left[ \psi^\dagger_{\ell'}(x) \psi_m(x) \right] \\
\times \frac{1}{\sqrt{(2\pi)^3 2k''}} e^{-ik'' \cdot x} \psi^\dagger_{\ell'}(y) \psi_{m'}(y) \phi^+(y) \] b^\dagger(k) a^\dagger(p, s) | 0 \rangle.

We now let \( \phi^+(y) \) delete the boson from the initial state.

\[
A = - g^2 \int d^4 x d^4 y \langle 0 | a(p', s') T \left[ \psi^\dagger_{\ell'}(x) \psi_m(x) \right] \\
\times \psi^\dagger_{\ell'}(y) \psi_{m'}(y) \int \frac{d^3 k''}{\sqrt{(2\pi)^3 2k''}} b^\dagger(k'') e^{ik'' \cdot y} \right] b^\dagger(k) a^\dagger(p, s) | 0 \rangle

= - g_{\ell m} g_{\ell m'} \int d^4 x d^4 y \langle 0 | a(p', s') T \left[ \psi^\dagger_{\ell'}(x) \psi_m(x) \right] \\
\times \psi^\dagger_{\ell'}(y) \psi_{m'}(y) \int \frac{d^3 k''}{\sqrt{(2\pi)^3 2k''}} \delta^3(k'' - k) e^{ik'' \cdot y} \right] a^\dagger(p, s) | 0 \rangle

= - g_{\ell m} g_{\ell m'} \int d^4 x d^4 y \langle 0 | a(p', s') T \left[ \psi^\dagger_{\ell'}(x) \psi_m(x) \right] \\
\times \frac{1}{\sqrt{(2\pi)^3 2k''}} e^{-ik'' \cdot x} \right] a^\dagger(p, s) | 0 \rangle

= - g_{\ell m} g_{\ell m'} \int d^4 x d^4 y \langle 0 | a(p', s') T \left[ \psi^\dagger_{\ell'}(x) \psi_m(x) \psi^\dagger_{\ell'}(y) \psi_{m'}(y) \right] a^\dagger(p, s) | 0 \rangle.
Now there are two terms. In one the initial fermion is deleted at \(y\) and the final fermion is added at \(x\)

\[
A_1 = -g_{\ell m} g_{\ell' m'} \int \frac{dx \, dy}{(2\pi)^6} e^{ik_{y'-ik'}x} \langle 0 | a(p', s') T \left[ \psi_{\ell}^{+\dagger}(x) \psi_{m}(x) \psi_{\ell'}^{\dagger}(y) \psi_{m'}^{+}(y) \right] a^{\dagger}(p, s) \rangle 0
\]

\[
= -g_{\ell m} g_{\ell' m'} \int \frac{dx \, dy}{(2\pi)^6} e^{ik_{y'-ik'}x} \langle 0 | a(p', s') T \left[ \int \frac{d^3 p''}{(2\pi)^3} e^{-ip'' \cdot x} a^{\dagger}(p'', s') u_{\ell}^{+}(p'', s'') \right] \psi_{m}(x) \psi_{\ell'}^{\dagger}(y) \langle 0 | a^{\dagger}(p, s) \rangle 0
\]

\[
\times \psi_{m}(x) \psi_{\ell'}^{\dagger}(y) \int \frac{d^3 p''}{(2\pi)^3} e^{ip'' \cdot y} u_{m'}(p'', s'') \langle 0 | a^{\dagger}(p, s) \rangle 0
\]

\[
= -g_{\ell m} g_{\ell' m'} u_{\ell}^{+}(p', s') u_{m'}(p, s) \int \frac{dx \, dy}{(2\pi)^6} e^{ik_{y'-ik'}x} e^{-ip' \cdot y} \langle 0 | T \left[ \psi_{m}(x) \psi_{\ell'}^{\dagger}(y) \right] 0 \rangle
\]

(2.76)

In terms of SW’s definition (6.2.31) of the fermion propagator, \(A_1\) is

\[
A_1 = -g_{\ell m} g_{\ell' m'} u_{\ell}^{+}(p', s') u_{m'}(p, s) \int \frac{dx \, dy}{(2\pi)^6} e^{ik_{y'-ik'}x} e^{-ip' \cdot y} (\Delta_{ml'}(x, y))
\]

(2.77)

In the other term, the initial fermion is deleted at \(x\) and the final fermion is added at \(y\)

\[
A_2 = -g_{\ell m} g_{\ell' m'} \int \frac{dx \, dy}{(2\pi)^6} e^{ik_{y'-ik'}x} \langle 0 | a(p', s') T \left[ \psi_{\ell}^{+\dagger}(x) \psi_{m}(x) \psi_{\ell'}^{+\dagger}(y) \psi_{m'}^{+}(y) \right] a^{\dagger}(p, s) \rangle 0
\]

\[
= -g_{\ell m} g_{\ell' m'} \int \frac{dx \, dy}{(2\pi)^6} e^{ik_{y'-ik'}x} \langle 0 | a(p', s') T \left[ \psi_{\ell}^{+\dagger}(x) \psi_{\ell'}^{+\dagger}(y) \psi_{m'}^{+}(y) \psi_{m}(x) \right] a^{\dagger}(p, s) \rangle 0
\]

(2.78)

In which the minus sign arises from the transposition \(\psi_{\ell}^{+\dagger}(x) \psi_{\ell'}^{+\dagger}(y) \rightarrow \psi_{\ell'}^{+\dagger}(y) \psi_{\ell}^{+\dagger}(x)\). The earlier transposition \(\psi_{m}^{+}(x) \psi_{m'}^{+}(y) \rightarrow \psi_{m'}^{+}(y) \psi_{m}^{+}(x)\) produced two minus signs or one plus sign. Inserting the expansions of \(\psi_{\ell'}^{+\dagger}(y)\)
and $\psi^+(x)$, we have

$$A_2 = g_{\ell m} g_{\ell' m'} \int \frac{d^4x d^4y}{(2\pi)^{2k^0/2k^3}} e^{ik' - ik \cdot x} \langle 0 | a(p', s') T \left[ \int \frac{d^3p''}{\sqrt{(2\pi)^3}} u^*_\ell(p'', s') a^+(p'', s'') e^{-ip'' \cdot y} \right] \psi^\dagger_\ell(x) \psi_{m'}(y) | 0 \rangle.$$

	

$$= g_{\ell m} g_{\ell' m'} u^*_\ell(p', s') u_{m}(p, s) \int \frac{d^4x d^4y}{(2\pi)^{12k^0/2k^3}} e^{i(k' - p) \cdot y - i(k' - p') \cdot x} T \left[ \psi^\dagger_\ell(x) \psi_{m'}(y) \right] | 0 \rangle.$$


$$A_2 = -g_{\ell m} g_{\ell' m'} \int \frac{d^4x d^4y}{(2\pi)^{12k^0/2k^3}} e^{i(k' - p) \cdot y - i(k' - p') \cdot x} (0 | T \left[ \psi_{m'}(y) \psi^\dagger_\ell(x) \right] | 0 \rangle.$$ (2.79)

In terms of SW’s definition (6.2.31) of the fermion propagator, $A_2$ is

$$A_2 = -g_{\ell m} g_{\ell' m'} u^*_\ell(p', s') u_{m}(p, s) \int \frac{d^4x d^4y}{(2\pi)^{12k^0/2k^3}} e^{i(k' - p) \cdot y - i(k' - p') \cdot x} (0 | T \left[ \psi_{m'}(y) \psi^\dagger_\ell(x) \right] | 0 \rangle.$$ (2.80)

The full amplitude for $f + b \rightarrow f' + b'$ is the sum $A = A_1 + A_2$ of the two amplitudes

$$A = -g_{\ell m} g_{\ell' m'} u^*_\ell(p', s') u_{m'}(p, s) \int \frac{d^4x d^4y}{(2\pi)^{12k^0/2k^3}} e^{i(k + p) \cdot y - i(k' + p') \cdot x} ( - i\Delta_{m'}(x, y) )$$

$$- g_{\ell m} g_{\ell' m'} u^*_\ell(p', s') u_{m}(p, s) \int \frac{d^4x d^4y}{(2\pi)^{12k^0/2k^3}} e^{i(k' - p) \cdot y - i(k' - p') \cdot x} ( - i\Delta_{m'}(y, x) ).$$ (2.81)

Interchanging $x$ and $y$, $m$ and $m'$, and $\ell$ and $\ell'$ in $A_1$, we get

$$A = -g_{\ell' m} g_{\ell m} u^*_\ell(p', s') u_{m'}(p, s) \int \frac{d^4x d^4y}{(2\pi)^{12k^0/2k^3}} e^{i(k + p) \cdot x - i(k' + p') \cdot y} ( - i\Delta_{m'}(y, x) )$$

$$- g_{\ell' m} g_{\ell m} u^*_\ell(p', s') u_{m}(p, s) \int \frac{d^4x d^4y}{(2\pi)^{12k^0/2k^3}} e^{i(k' - p) \cdot y - i(k' - p') \cdot x} ( - i\Delta_{m'}(y, x) ).$$ (2.82)

Combining terms, we get

$$A = -g_{\ell' m} g_{\ell m} u^*_\ell(p', s') u_{m}(p, s) \int \frac{d^4x d^4y}{(2\pi)^{12k^0/2k^3}} e^{ip \cdot y - i(p' + p') \cdot y} ( - i\Delta_{m'}(y, x) )$$

$$\times \left( e^{i(k' - p) \cdot y} + e^{i(k - p') \cdot y} \right).$$ (2.83)

which agrees with SW’s (6.1.27) when his boson is restricted to a single scalar field.
We now replace mean value in the vacuum of the time-ordered product by its value \([2.70]\)

\[
\langle 0\vert \mathcal{T}\left\{ \psi_t(x)\psi^\dagger_{m}(y) \right\} \vert 0 \rangle \equiv -i\Delta_{\ell m}(x-y) = -i \int \frac{d^4q}{(2\pi)^4} \frac{\left( (-iq\gamma^\mu + m)\beta \right)_{\ell m}}{q^2 + m^2 - i\epsilon} e^{iq\cdot x}.
\]

Replacing \(\ell, m, x, y\) by \(m', \ell, y, x\), we get

\[
\langle 0\vert \mathcal{T}\left\{ \psi_{m'}(y)\psi^\dagger_{\ell}(x) \right\} \vert 0 \rangle \equiv -i\Delta_{m'\ell}(y-x) = -i \int \frac{d^4q}{(2\pi)^4} \frac{\left( (-iq\gamma^\mu + m)\beta \right)_{m'\ell}}{q^2 + m^2 - i\epsilon} e^{iq\cdot (y-x)}.
\]

We then find

\[
A = \frac{\langle 0\vert \mathcal{T}\left\{ \psi_{m'}(y)\psi^\dagger_{\ell}(x) \right\} \vert 0 \rangle}{\langle 0\vert \mathcal{T}\left\{ \psi_{m'}(y)\psi^\dagger_{\ell}(x) \right\} \vert 0 \rangle} = -\int \frac{d^4x d^4y}{(2\pi)^6 \sqrt{2k^0 k^0}} \frac{\left( e^{ip\cdot x - ip'\cdot y} + e^{ik'\cdot y - ik\cdot x} \right)}{\left( -i\beta \right)_{m'\ell}^\dagger e^{iq\cdot (y-x)}}.
\]

In matrix notation, this is

\[
A = -\int \frac{d^4x d^4y}{(2\pi)^6 \sqrt{2k^0 k^0}} \left( e^{ip\cdot x - ip'\cdot y} + e^{ik'\cdot y - ik\cdot x} \right)\beta_{m'\ell}^\dagger e^{iq\cdot (y-x)}.
\]

The \(d^4x\) and \(d^4y\) integrations give

\[
A = \frac{i\int d^4q}{(2\pi)^2 \sqrt{2k^0 k^0}} \left( \delta(q - k' - p')\delta(p + k - q) + \delta(q + k - p')\delta(p - k' - q) \right)\beta_{m'\ell}^\dagger e^{iq\cdot (y-x)}.
\]

which is

\[
A = \frac{i\delta(p' + k' - p - k)}{8\pi^2 \sqrt{k^0 k^0}} \left( \beta_{m'\ell}^\dagger \right) e^{iq\cdot (y-x)}.
\]
2.8 Feynman propagator for spin-one fields

The general form of the propagator is

\[
\langle 0 | T [\psi_a(x)\psi_b^\dagger(y)] | 0 \rangle = \theta(x^0 - y^0) \sum_s \int \frac{d^3p}{(2\pi)^3} u_a(\vec{p}, s) u_b^\dagger(\vec{p}, s) e^{ip(\vec{x} - \vec{y})} \\
\pm \theta(y^0 - x^0) \sum_s \int \frac{d^3p}{(2\pi)^3} v_a(\vec{p}, s) v_b^\dagger(\vec{p}, s) e^{ip(\vec{y} - \vec{x})}.
\]

(2.90)

We use the upper (+) sign for spin-one fields because they are bosons. For single real massive vector field

\[
\psi^a(x) = \sum_{s=\pm 1} \int \frac{d^3p}{\sqrt{(2\pi)^32p^0}} \left[ e^a(\vec{p}, s)a(\vec{p}, s)e^{ip\vec{x}} + e^{*a}(\vec{p}, s)a^\dagger(\vec{p}, s)e^{-ip\vec{x}} \right],
\]

(2.91)

the mean value in the vacuum of its time-ordered product is

\[
\langle 0 | T [\psi_a(x)\psi_b^\dagger(y)] | 0 \rangle = \theta(x^0 - y^0) \sum_s \int \frac{d^3p}{(2\pi)^32p^0} c_a(\vec{p}, s) c_b^\dagger(\vec{p}, s) e^{ip(\vec{x} - \vec{y})} \\
+ \theta(y^0 - x^0) \sum_s \int \frac{d^3p}{(2\pi)^32p^0} c_a(\vec{p}, s) c_b^\dagger(\vec{p}, s) e^{ip(\vec{y} - \vec{x})}.
\]

(2.92)

The spin-one spin sum is

\[
\sum_s c_a(\vec{p}, s) c_b^\dagger(\vec{p}, s) = \eta_{ab} + \frac{p_a p_b}{m^2}.
\]

(2.93)

So the mean value (2.58) is

\[
\langle 0 | T [\psi_a(x)\psi_b^\dagger(y)] | 0 \rangle = \theta(x^0 - y^0) \int \frac{d^3p}{(2\pi)^32p^0} \left( \eta_{ab} + \frac{p_a p_b}{m^2} \right) e^{ip(\vec{x} - \vec{y})} \\
+ \theta(y^0 - x^0) \int \frac{d^3p}{(2\pi)^32p^0} \left( \eta_{ab} + \frac{p_a p_b}{m^2} \right) e^{ip(\vec{y} - \vec{x})}.
\]

(2.94)
in which the momentum \( p_a \) is physical (aka, on the mass shell) in that
\( p^0 = \sqrt{\mathbf{p}^2 + m^2} \) and \( p_0^0 = \mathbf{p}^2 + m^2 \). In terms of derivatives, we have

\[
\langle 0 | \mathcal{T} [\psi_a(x) \psi_b^\dagger(y)] | 0 \rangle = \theta(x^0 - y^0) \left( \eta_{ab} - \frac{\partial_a \partial_b}{m^2} \right) \int \frac{d^3p}{(2\pi)^32p^0} e^{ip(x-y)} + \theta(y^0 - x^0) \left( \eta_{ab} - \frac{\partial_a \partial_b}{m^2} \right) \int \frac{d^3p}{(2\pi)^32p^0} e^{ip(y-x)}. \tag{2.95}
\]

In this formula, we may interpret the double time derivative as \( \partial_0^2 = \nabla^2 - m^2 \).

For spatial values of \( a \) and \( b \), we can move the derivatives to the left of the step functions. And we can move the product \( \partial_0 \partial_1 \) of one spatial and one time derivative by the argument we used for spin one-half. We find

\[
\langle 0 | \mathcal{T} [\psi_a(x) \psi_b^\dagger(y)] | 0 \rangle = \left( \eta_{ab} - \frac{\partial_a \partial_b}{m^2} \right) \left[ \theta(x^0 - y^0) \int \frac{d^3p}{(2\pi)^32p^0} e^{ip(x-y)} + \theta(y^0 - x^0) \int \frac{d^3p}{(2\pi)^32p^0} e^{ip(y-x)} \right] = \left( \eta_{ab} - \frac{\partial_a \partial_b}{m^2} \right) \left[ \theta(x^0 - y^0) \Delta_+(x-y) + \theta(y^0 - x^0) \Delta_+(y-x) \right] = -i \left( \eta_{ab} - \frac{\partial_a \partial_b}{m^2} \right) \int \frac{d^4q}{(2\pi)^4} \frac{e^{iq(x-y)}}{q^2 + m^2 - i\epsilon} \tag{2.96}
\]

in which \( \partial_0^2 = \nabla^2 - m^2 \).

We can relax the rule \( \partial_0^2 = \nabla^2 - m^2 \) if we add an extra term to the propagator. The extra term is

\[
\frac{i}{m^2} (\nabla^2 - m^2 - \partial_0^2) \int \frac{d^4q}{(2\pi)^4} \frac{e^{iq(x-y)}}{q^2 + m^2 - i\epsilon} = \frac{i}{m^2} \int \frac{d^4q}{(2\pi)^4} \frac{-q^2 - m^2 + p_0^2}{q^2 + m^2 - i\epsilon} e^{iq(x-y)} = -\frac{i}{m^2} \delta^4(x-y). \tag{2.97}
\]

So the Feynman propagator for spin-one fields is

\[
\langle 0 | \mathcal{T} [\psi_a(x) \psi_b^\dagger(y)] | 0 \rangle = -i \int \frac{d^4q}{(2\pi)^4} \left( \eta_{ab} + \frac{p_a p_b}{m^2} \right) \frac{e^{iq(x-y)}}{q^2 + m^2 - i\epsilon} - \frac{i}{m^2} \delta_0^a \delta_0^b \delta^4(x-y). \tag{2.98}
\]

Suppose the vector field \( \psi_a(x) \) interacts with a current \( j^a(x) \) thru a term \( \psi_a(x) j^a(x) \) in \( \mathcal{H}(x) \). Then in Dyson’s expansion, the awkward second term

\[
\int \frac{d^4q}{(2\pi)^4} \frac{\eta_{ab} + \frac{p_a p_b}{m^2}}{q^2 + m^2 - i\epsilon} e^{iq(x-y)} - \frac{i}{m^2} \delta_0^a \delta_0^b \delta^4(x-y).
\]
in the Feynman propagator (2.98) contributes the term
\[-i\mathcal{H}_2(x) = \frac{1}{2} \int [-ij^a(x)][-ij^b(y)] \left[ -\frac{i}{m^2} \delta_a^0 \delta_b^0 \delta^4(x-y) \right] d^4y = \frac{i[j^0(x)]^2}{2m^2}.\] (2.99)

So if \(\mathcal{H}(x)\) also contains the term
\[\mathcal{H}_C(x) = \frac{[j^0(x)]^2}{2m^2},\] (2.100)
then the effect of the awkward second term in the Feynman propagator is cancelled. This actually happens in a natural way as SW explains in chapter 7.

### 2.9 The Feynman rules

1) Draw all diagrams of the order you are working at. Label each internal line with its (unphysical) 4-momentum considered to flow in the direction of the arrow or in either direction if the particle is uncharged.

2) For each vertex of type \(i\), include the factor
\[-i(2\pi)^4 g_i \delta^4 \left( \sum p + q - p' - q' \right)\] (2.101)
which makes the sum of the 4-momenta \(p + q\) entering each vertex add up to sum of the 4-momenta \(p' + q'\) leaving each vertex.

For each outgoing line include the factor
\[\frac{u_\hat{\mathcal{L}}(\mathbf{p}', s, n')}{(2\pi)^{3/2}} \quad \text{or} \quad \frac{v_\mathcal{L}(\mathbf{p}', s, n')}{(2\pi)^{3/2}}\] (2.102)
for arrows pointing out \((u_\hat{\mathcal{L}}(\mathbf{p}', s, n'))\) or pointing in \((v_\mathcal{L}(\mathbf{p}', s, n'))\).

For each incoming line include the factor
\[\frac{u_\mathcal{L}(\mathbf{p}, s, n)}{(2\pi)^{3/2}} \quad \text{or} \quad \frac{v_\hat{\mathcal{L}}(\mathbf{p}, s, n)}{(2\pi)^{3/2}}\] (2.103)
for arrows pointing in \((u_\mathcal{L}(\mathbf{p}, s, n))\) or pointing out \((v_\hat{\mathcal{L}}(\mathbf{p}, s, n'))\).

For each internal line of a spin-zero particle carrying momentum \(q^a\) include the factor
\[-i \frac{1}{(2\pi)^4 q^2 + m^2 - i\epsilon}.\] (2.104)

For each internal line of a spin-one-half particle with ends labelled by \(\ell\) and \(m\) and carrying momentum \(q^a\) include the factor
\[-i \frac{[( - i q + m)\beta]_{\ell m}}{(2\pi)^4 q^2 + m^2 - i\epsilon}.\] (2.105)
For each internal line of a spin-one particle with ends labelled by $\ell$ and $m$ and carrying momentum $q^a$ include the factor
\[
\frac{-i}{(2\pi)^4} \frac{\eta_{\ell m} + q\epsilon_{\ell m}/m^2}{q^2 + m^2 - i\epsilon}
\]
and keep in mind the delta-function term in (2.98).

3. Integrate the product of all these factors over all the internal momenta and sum over $\ell$ and $m$, etc.

4) Add the results of all the Feynman diagrams.

2.10 Fermion-antifermion scattering

We can watch Feynman’s rules emerge in fermion-antifermion scattering. We consider a fermion interacting with a neutral scalar boson (2.72)
\[
H(x) = g_{\ell m} \psi_\ell^\dagger(x) \psi_m(x) \phi(x).
\]

The initial state is $|p, s; q, t\rangle = a^\dagger(p, s) b^\dagger(q, t)|0\rangle$ and the final state is $|p', s'; q', t'\rangle = a^\dagger(p', s') b^\dagger(q', t')|0\rangle$. The lowest-order term is
\[
A = -\frac{1}{2!} \int d^4x d^4y \langle 0| b(q', t') a(p', s') T[H(x) H(y)] a^\dagger(p, s) b^\dagger(q, t)|0\rangle
\]
\[
= -\frac{g_{\ell m} g_{\ell' m'}}{2!} \int d^4x d^4y \langle 0| b(q', t') a(p', s') T[\psi^\dagger_\ell(x) \psi_m(x) \phi(x) \psi^\dagger_{\ell'}(y) \psi_{m'}(y) \phi(y)]
\times a^\dagger(p, s) b^\dagger(q, t)|0\rangle
\]

in which the operators $a$ and $b$ delete fermions and antifermions. There is only one Feynman diagram

\[
\begin{aligned}
q(t) & \quad q'(t') \\
\quad \quad \quad \text{---} \\
p(s) & \quad p'(s')
\end{aligned}
\]

which appears in the conventional way with incoming particles on the left and outgoing particles on the right. TikZ-Feynman does not easily use SW’s vertical flow.

We cancel the 2! by choosing to absorb the incoming fermion-antifermion
pair at vertex $y$ and to add the outgoing fermion-antifermion pair at vertex $x$. We are left with

$$A = - \frac{g_{\ell m} g_{\ell' m'}}{(2\pi)^6} \int d^4x \int d^4y \langle 0| u^*_\ell(p',s') v_m(q',t') e^{-i x \cdot (p' + q')} T \left[ \phi(x) \phi(y) \right]$$

$$\times u_{m'}(p,s) v^*_{\ell'}(q,t) e^{i y \cdot (p+q)} |0\rangle.$$  \hspace{1cm} (2.109)

Adding in the scalar propagator (2.50), we get

$$A = - \frac{g_{\ell m} g_{\ell' m'}}{(2\pi)^6} \int d^4x \int d^4y u^*_\ell(p',s') v^*_m(q',t') e^{-i x \cdot (p' + q')}$$

$$\times u_{m'}(p,s) v^*_{\ell'}(q,t) e^{i y \cdot (p+q)} \int \frac{d^4k}{(2\pi)^4} \frac{-i e^{ik \cdot (x-y)}}{k^2 + m^2 - i\epsilon}.$$ \hspace{1cm} (2.110)

The integration over $y$ conserves 4-momentum at vertex $y$, and the integration over $x$ conserves 4-momentum at vertex $x$. We then have

$$A = i \frac{g_{\ell m} g_{\ell' m'}}{(2\pi)^2} \int d^4k \ u^*_\ell(p',s') v_m(q',t') \delta(k - p' - q') \delta(p + q - k)$$

$$\times u_{m'}(p,s) v^*_{\ell'}(q,t) \frac{1}{k^2 + m^2 - i\epsilon}.$$ \hspace{1cm} (2.111)

or

$$A = i \delta(p + q - k' - q') \frac{g_{\ell m} g_{\ell' m'}}{(2\pi)^2} \ u^*_\ell(p',s') v_m(q',t') \ u_{m'}(p,s) v^*_{\ell'}(q,t) \frac{1}{(p + k)^2 + m^2}.$$ \hspace{1cm} (2.112)
3

Action

3.1 Lagrangians and Hamiltonians

A transformation is a symmetry of a theory if the action is invariant or changes by a surface term. So we choose to work with actions that are symmetrical. The action is normally an integral over spacetime of an action density often called a lagrangian. Often the action density itself is invariant under the transformation of the symmetry.

There are procedures, sometimes clumsy procedures, for computing the hamiltonian from the lagrangian. The hamiltonian often is not invariant under the transformation of the symmetry. So it’s very hard to find a suitably symmetrical theory by starting with a hamiltonian. But once one has a hamiltonian, one can compute scattering amplitudes energies, and states with these energies.

3.2 Canonical variables

In quantum mechanics, we use the equal-time commutation relations

\[ [q_i, p_k] = i\delta_{jk}, \quad [q_i, q_k] = 0, \quad \text{and} \quad [p_i, p_k] = 0. \] (3.1)

In general, the operators \( q_i \) and \( q_i(t) = e^{iHt}q_ie^{-iHt} \) do not commute. Quantum field theory promotes these equal-time commutation relations to ones in which the indexes \( i \) and \( k \) denote different points of space

\[ [q^m(\vec{x}, t), p_n(\vec{y}, t)]_\mp = i\delta(\vec{x} - \vec{y})\delta^n_m, \]
\[ [q^m(\vec{x}, t), q^m(\vec{y}, t)]_\mp = 0, \quad \text{and} \quad [p_n(\vec{x}, t), p_m(\vec{y}, t)]_\mp = 0. \] (3.2)

The commutator of a real scalar field

\[ \phi(x) = \int \frac{d^3p}{\sqrt{(2\pi)^32p^0}} \left[ a(\vec{p}) e^{ip \cdot x} + a^\dagger(\vec{p}) e^{-ip \cdot x} \right] \] (3.3)
Action is

\[ \phi(x, y) = \Delta(x - y) = \int \frac{d^3p}{(2\pi)^3 2p^0} \left( e^{ip(x-y)} - e^{-ip(x-y)} \right). \] (3.4)

At equal times, one has

\[ \Delta(x - y, 0) = 0, \quad \frac{\partial}{\partial x^0} \Delta(x - y)|_{x^0=y^0} = -i \delta^3(x - y), \quad \text{and} \quad \frac{\partial^2}{\partial x^0 \partial y^0} \Delta(x - y)|_{x^0=y^0} = 0. \] (3.5)

So a real field \( \phi \) and its time derivative \( \dot{\phi} \) satisfy the equal-time commutation relations (3.2)

\[ [\phi(x), \phi(y)] = i \delta^3(x - y), \quad [\phi(x), \dot{\phi}(y)] = 0, \quad \text{and} \quad [\dot{\phi}(x), \phi(y)] = 0. \] (3.6)

A complex scalar field

\[ \phi(x) = \frac{1}{\sqrt{2}} \left[ \phi_1(x) + i \phi_2(x) \right] \] (3.7)

obeys the commutation relations

\[ [\phi(x), \phi^\dagger(y)] = \frac{1}{2} \left[ \phi_1(x) + i \phi_2(x), \phi_1(y) - i \phi_2(y) \right] \]
\[ = \frac{1}{2} \left( [\phi_1(x), \phi_1(x)] + [\phi_2(x), \phi_2(x)] \right) = \Delta(x - y) \] (3.8)

and

\[ [\phi(x), \phi(y)] = \frac{1}{2} \left[ \phi_1(x) + i \phi_2(x), \phi_1(y) + i \phi_2(y) \right] \]
\[ = \frac{1}{2} \left( [\phi_1(x), \phi_1(x)] - [\phi_2(x), \phi_2(x)] \right) = 0. \] (3.9)

So the complex scalar fields \( \phi(x) \) and \( \dot{\phi}(x) \) obey the equal-time commutation relations (3.6)

\[ [\phi(x, t), \dot{\phi}^\dagger(y, t)] = i \delta^3(x - y) \quad \text{and} \quad [\phi(x, t), \phi^\dagger(y, t)] = 0 \] (3.10)
\[ [\dot{\phi}(x, t), \dot{\phi}^\dagger(y, t)] = 0 \quad \text{and} \quad [\dot{\phi}^\dagger(x, t), \dot{\phi}^\dagger(y, t)] = 0 \] (3.11)
\[ [\phi(x, t), \phi(y, t)] = 0 \quad \text{and} \quad [\phi^\dagger(x, t), \phi^\dagger(y, t)] = 0. \] (3.12)
3.3 Principle of stationary action in field theory

If $\phi(x)$ is a scalar field, and $L(\phi)$ is its action density, then its action $S[\phi]$ is the integral over all of spacetime

$$S[\phi] = \int L(\phi(x)) \, d^4x. \quad (3.13)$$

The principle of least (or stationary) action says that the field $\phi(x)$ that satisfies the classical equation of motion is the one for which the first-order change in the action due to any tiny variation $\delta \phi(x)$ in the field vanishes,

$$\delta S[\phi] = 0.$$ And to keep things simple, we’ll assume that the action (or Lagrange) density $L(\phi)$ is a function only of the field $\phi$ and its first derivatives $\partial_a \phi = \partial \phi/\partial x^a$. The first-order change in the action then is

$$\delta S[\phi] = \int \left[ \frac{\partial L}{\partial \phi} \delta \phi + \frac{\partial L}{\partial (\partial_a \phi)} \delta (\partial_a \phi) \right] d^4x \quad (3.14)$$

in which we sum over the repeated index $a$ from 0 to 3. Now $\delta (\partial_a \phi) = \partial_a (\phi + \delta \phi) - \partial_a \phi = \partial_a \delta \phi$. So we may integrate by parts and drop the surface terms because we set $\delta \phi = 0$ on the surface at infinity

$$\delta S[\phi] = \int \left[ \frac{\partial L}{\partial \phi} \delta \phi + \frac{\partial L}{\partial (\partial_a \phi)} \partial_a (\delta \phi) \right] d^4x = \int \left[ \frac{\partial L}{\partial \phi} - \partial_a \frac{\partial L}{\partial (\partial_a \phi)} \right] \delta \phi \, d^4x. \quad (3.15)$$

This first-order variation is zero, for arbitrary $\delta \phi$ only if the field $\phi(x)$ satisfies Lagrange’s equation

$$\partial_a \left( \frac{\partial L}{\partial (\partial_a \phi)} \right) \equiv \frac{\partial}{\partial x^a} \left[ \frac{\partial L}{\partial (\partial_a \phi/\partial x^a)} \right] = \frac{\partial L}{\partial \phi} \quad (3.16)$$

which is the classical equation of motion.

**Example 3.1** (Theory of a scalar field) The action density of a scalar field $\phi$ of mass $m$ is

$$L = \frac{1}{2} (\dot{\phi})^2 - \frac{1}{2} (\nabla \phi)^2 - \frac{1}{2} m^2 \phi^2 = -\frac{1}{2} \partial_a \phi \partial^a \phi - \frac{1}{2} m^2 \phi^2. \quad (3.17)$$

Lagrange’s equation (3.16) for this action density is

$$\nabla^2 \phi - \ddot{\phi} = \partial_a \partial^a \phi = m^2 \phi \quad (3.18)$$

which is the Klein-Gordon equation (1.56).
In a theory of several fields \( \phi_1, \ldots, \phi_n \) with action density \( L(\phi_k, \partial_a \phi_k) \), the fields obey \( n \) copies of Lagrange’s equation

\[
\frac{\partial}{\partial x^a} \left( \frac{\partial L}{\partial (\partial_a \phi_k)} \right) = \frac{\partial L}{\partial \phi_k}
\]

one for each \( k \).

### 3.4 Symmetries and conserved quantities in field theory

An action density \( L(\phi_i, \partial_a \phi_i) \) that is invariant under a transformation of the coordinates \( x^a \) or of the fields \( \phi_i \) and their derivatives \( \partial_a \phi_i \) is a symmetry of the action density. Such a symmetry implies that something is conserved or time independent.

Suppose that an action density \( L(\phi_i, \partial_a \phi_i) \) is unchanged when the fields \( \phi_i \) and their derivatives \( \partial_a \phi_i \) change by \( \delta \phi_i \) and by \( \delta(\partial_a \phi_i) \)

\[
0 = \delta L = \sum_i \frac{\partial L}{\partial \phi_i} \delta \phi_i + \frac{\partial L}{\partial \partial_a \phi_i} \delta (\partial_a \phi_i).
\]

For many transformations (but not for Lorentz transformations), we can set \( \delta(\partial_a \phi_i) = \partial_a (\delta \phi_i) \). In such cases by using Lagrange’s equations (3.19) to rewrite \( \partial L/\partial \phi_i \), we find

\[
0 = \sum_i \left( \partial_a \frac{\partial L}{\partial \partial_a \phi_i} \right) \delta \phi_i + \frac{\partial L}{\partial \partial_a \phi_i} \partial_a \delta \phi_i = \partial_a \sum_i \frac{\partial L}{\partial \partial_a \phi_i} \delta \phi_i
\]

which says that the current

\[
J^a = \sum_i \frac{\partial L}{\partial \partial_a \phi_i} \delta \phi_i
\]

has zero divergence

\[
\partial_a J^a = 0.
\]

Thus the time derivative of the volume integral of the charge density \( J^0 \)

\[
Q_V = \int_V J^0 \, d^3x
\]

is the flux of current \( \vec{J} \) entering through the boundary \( S \) of the volume \( V \)

\[
\dot{Q}_V = \int_V \partial_0 J^0 \, d^3x = -\int_V \partial_k J^k \, d^3x = -\int_S J^k \, d^2S_k.
\]
If no current enters \( V \), then the charge \( Q \) inside \( V \) is conserved. When the volume \( V \) is the whole universe, the charge is the integral over all of space
\[
Q = \int J^0 \, d^3x = \int \sum_i \frac{\partial L}{\partial \dot{\phi}_i} \delta \phi_i \, d^3x = \int \sum_i \pi_i \, \delta \phi_i \, d^3x
\tag{3.26}
\]
in which \( \pi_i \) is the momentum conjugate to the field \( \phi_i \)
\[
\pi_i = \frac{\partial L}{\partial \dot{\phi}_i}.
\tag{3.27}
\]

**Example 3.2** \((O(n)\) symmetry and its charge) Suppose the action density \( L \) is the sum of \( n \) copies of the quadratic action density (3.17)
\[
L = \sum_{i=1}^{n} \left( \frac{1}{2} \phi_i^2 - \frac{1}{2} (\nabla \phi_i)^2 - \frac{1}{2} m^2 \phi_i^2 \right) = -\frac{1}{2} \frac{\partial}{\partial a} \phi_i \partial^a \phi - \frac{1}{2} m^2 \phi_i^2
\tag{3.28}
\]
and \( A_{ij} \) is any constant antisymmetric matrix, \( A_{ij} = -A_{ji} \). Then if the fields change by \( \delta \phi_i = \epsilon \sum_j A_{ij} \phi_j \), the change (3.20) in the action density
\[
\delta L = -\epsilon \sum_{i,j=1}^{n} \left[ m^2 \phi_i A_{ij} \phi_j + \partial^a \phi_i A_{ij} \partial_a \phi_j \right] = 0
\tag{3.29}
\]
vanishes. Thus the charge (3.26) associated with the matrix \( A \)
\[
Q_A = \int \sum_i \pi_i \delta \phi_i \, d^3x = \epsilon \int \sum_i \pi_i A_{ij} \phi_j \, d^3x
\tag{3.30}
\]
is conserved. There are \( n(n-1)/2 \) antisymmetric \( n \times n \) imaginary matrices; they generate the group \( O(n) \) of \( n \times n \) orthogonal matrices.

An action density \( L(\phi_i, \partial_a \phi_i) \) that is invariant under a spacetime translation, \( x'^a = x^a + \delta x^a \), depends upon \( x^a \) only through the fields \( \phi_i \) and their derivatives \( \partial_a \phi_i \)
\[
\frac{\partial L}{\partial x^a} = \sum_i \left( \frac{\partial L}{\partial \phi_i} \frac{\partial \phi_i}{\partial x^a} + \frac{\partial L}{\partial \partial_a \phi_i} \frac{\partial^2 \phi_i}{\partial x^a \partial x^a} \right).
\tag{3.31}
\]
Using Lagrange’s equations (3.19) to rewrite \( \partial L/\partial \phi_i \), we find
\[
0 = \sum_i \partial_b \left( \frac{\partial L}{\partial \partial_b \phi_i} \right) \partial_a \phi_i + \frac{\partial L}{\partial \partial_b \phi_i} \frac{\partial^2 \phi_i}{\partial x^b \partial x^a} - \frac{\partial L}{\partial x^a}
\]
\[
0 = \partial_b \left[ \sum_i \left( \frac{\partial L}{\partial \partial_b \phi_i} \frac{\partial \phi_i}{\partial x^a} - \delta^a_b L \right) \right]
\tag{3.32}
\]
that the energy-momentum tensor
\[ T^b_a = \sum_i \frac{\partial L}{\partial \dot{\phi}_i} \frac{\partial \phi_i}{\partial x^a} - \delta_a^b L \] (3.33)
has zero divergence, \( \partial_b T^b_a = 0 \).

Thus the time derivative of the 4-momentum \( P_a V \) inside a volume \( V \)
\[ P_a V = \int_V \left( \sum_i \frac{\partial L}{\partial \dot{\phi}_i} \frac{\partial \phi_i}{\partial x^a} - \delta_a^0 L \right) d^3x = \int_V T^0_a d^3x \] (3.34)
is equal to the flux entering through \( V \)'s boundary \( S \)
\[ \partial_0 P_a V = \int_V \partial_0 T^0_a d^3x = - \int_V \partial_k T^k_a d^3x = - \int_S T^k_a d^2k. \] (3.35)
The invariance of the action density \( L \) under spacetime translations implies the conservation of energy \( P_0 \) and momentum \( \vec{P} \).

The momentum \( \pi_i(x) \) that is canonically conjugate to the field \( \phi_i(x) \) is the derivative of the action density \( L \) with respect to the time derivative of the field
\[ \pi_i = \frac{\partial L}{\partial \dot{\phi}_i}. \] (3.36)
If one can express the time derivatives \( \dot{\phi}_i \) of the fields in terms of the fields \( \phi_i \) and their momenta \( \pi_i \), then hamiltonian of the theory is the spatial integral of
\[ H = P_0 = T^0_0 = \left( \sum_{i=1}^n \pi_i \dot{\phi}_i \right) - L \] (3.37)
in which \( \dot{\phi}_i = \dot{\phi}_i(\phi, \pi) \).

**Example 3.3** (Hamiltonian of a scalar field) The hamiltonian density (3.37) of the theory (3.17) is
\[ H = \frac{1}{2} \pi^2 + \frac{1}{2} (\nabla \phi)^2 + \frac{1}{2} m^2 \phi^2. \] (3.38)

A Lorentz transformation (1.15) of a field is like this
\[ U(\Lambda, a) \psi(x) U^{-1}(\Lambda, a) = \sum_{\ell} D_{\ell} \Lambda^{-1} \psi(\Lambda x + a). \] (3.39)
The action is invariant under Lorentz transformations. The action density
is constructed so as to be invariant when the fields transform this way:

$$\psi'_\ell(x) = \sum_\ell D_{\ell\ell}(A^{-1})\psi_\ell(x) \tag{3.40}$$

The action density is constructed so as to be invariant when the fields and their derivatives of the fields transform under infinitesimal Lorentz transformations as

$$\delta\psi_\ell(x) = \frac{i}{2} \omega^{ab}(J_{ab})^\ell_m \psi^m(x) \tag{3.41}$$

$$\delta(\partial_k \psi_\ell)(x) = \frac{i}{2} \omega^{ab}(J_{ab})^\ell_m \partial_k \psi^m(x) + \omega_j \partial_j \psi_\ell(x).$$

The invariance of the action density says that

$$0 = \frac{\partial L}{\partial \psi_\ell} \delta \psi_\ell + \frac{\partial L}{\partial \partial_k \psi_\ell} \delta(\partial_k \psi_\ell) \tag{3.42}$$

$$= \frac{\partial L}{\partial \psi_\ell} \frac{i}{2} \omega^{ab}(J_{ab})^\ell_m \psi^m + \frac{\partial L}{\partial \partial_k \psi_\ell} \left[ \frac{i}{2} \omega^{ab}(J_{ab})^\ell_m \partial_k \psi^m + \omega_k \partial_k \psi_\ell \right].$$

Remembering that $\omega_{ab} = -\omega_{ba}$ and $\omega^{ab} = -\omega^{ba}$, we rewrite the last bit as

$$\omega^b_k \partial_b \psi_\ell = \eta_{ka} \omega^{ab} \partial_b \psi_\ell = \frac{1}{2} (\eta_{ka} \omega^{ab} \partial_b - \eta_{kb} \omega^{ab} \partial_a) \psi_\ell. \tag{3.43}$$

We then have

$$0 = \frac{\partial L}{\partial \psi_\ell} \frac{i}{2} \omega^{ab}(J_{ab})^\ell_m \psi^m \tag{3.44}$$

or

$$0 = \frac{\partial L}{\partial \psi_\ell} \frac{i}{2} (J_{ab})^\ell_m \psi^m \tag{3.45}$$

The equations of motion now give

$$0 = \left( \partial_k \frac{\partial L}{\partial \partial_k \psi_\ell} \right) \frac{i}{2} (J_{ab})^\ell_m \psi^m \tag{3.46}$$

or

$$0 = \partial_k \left( \frac{\partial L}{\partial \partial_k \psi_\ell} \frac{i}{2} (J_{ab})^\ell_m \psi^m \right) + \frac{\partial L}{\partial \partial_k \psi_\ell} \frac{1}{2} (\eta_{ka} \partial_b - \eta_{kb} \partial_a) \psi_\ell$$
or
\[ 0 = \partial_k \left( \frac{\partial L}{\partial \partial_k \psi^\ell} \frac{i}{2} (J_{ab})^\ell_m \psi^m \right) + \frac{1}{2} \left( \frac{\partial L}{\partial \partial^a \psi^\ell} \partial_b - \frac{\partial L}{\partial \partial^b \psi^\ell} \partial_a \psi^\ell \right). \]

We recall (3.33) the energy-momentum tensor
\[ T^b_a = \sum_i \frac{\partial L}{\partial \partial^a \phi_i} \frac{\partial \phi_i}{\partial x^b} - \delta^b_a L \] (3.47)
or
\[ T_{ba} = \sum_i \frac{\partial L}{\partial \partial^b \psi^\ell} \frac{\partial \psi^\ell}{\partial x^a} - \eta_{ba} L \] (3.48)
which has zero divergence, \( \partial_b T^b_a = 0 \). In terms of it, we have
\[ 0 = \partial_k \left( \frac{\partial L}{\partial \partial_k \psi^\ell} \frac{i}{2} (J_{ab})^\ell_m \psi^m \right) - \frac{1}{2} (T_{ab} - T_{ba}). \]

So Belinfante defined the symmetric energy-momentum tensor as
\[ \Theta^{ab} = T^{ab} - \frac{i}{2} \partial_k \left[ \frac{\partial L}{\partial \partial^a \psi^\ell} (J^{ab})^\ell_m \psi^m \right. \]
\[ \left. - \frac{\partial L}{\partial \partial^b \psi^\ell} (J^{kb})^\ell_m \psi^m - \frac{\partial L}{\partial \partial^b \psi^\ell} (J^{ka})^\ell_m \psi^m \right]. \] (3.49)
The quantity in the square brackets is antisymmetric in \( a, k \). So
\[ \partial_a \Theta^{ab} = \partial_a T^{ab} - \frac{i}{2} \partial_a \partial_k \left[ \frac{\partial L}{\partial \partial^a \psi^\ell} (J^{ab})^\ell_m \psi^m \right. \]
\[ \left. - \frac{\partial L}{\partial \partial^b \psi^\ell} (J^{kb})^\ell_m \psi^m - \frac{\partial L}{\partial \partial^b \psi^\ell} (J^{ka})^\ell_m \psi^m \right] \]
\[ = \partial_a T^{ab} = 0. \] (3.50)
The Belinfante energy-momentum tensor is symmetric
\[ \Theta^{ab} = \Theta^{ba} \] (3.51)
and so it is the one to use when gravity is involved.

The quantity
\[ M^{abc} = x^b \Theta^{ac} - x^c \Theta^{ab} \] (3.52)
is a conserved current
\[ \partial_a M^{abc} = \partial_a \left( x^b \Theta^{ac} - x^c \Theta^{ab} \right) = \Theta^{bc} - \Theta^{cb} = 0. \] (3.53)
So the angular-momentum operators
\[ J^{bc} = \int M^{0bc} d^3 x = \int \left( x^b \Theta^{0c} - x^c \Theta^{0b} \right) d^3 x \] (3.54)
are conserved.
4
Quantum electrodynamics

4.1 Global $U(1)$ symmetry

In most theories of charged fields, the action density is invariant when the charged fields change by a phase transformation

$$\psi'_\ell(x) = e^{i\theta_\ell} \psi_\ell(x).$$  \hspace{1cm} \text{(4.1)}

If the phase $\theta$ is independent of $x$, then the symmetry is called a \textbf{global $U(1)$ symmetry}.

Section \ref{sec:3.4} describes Noether’s theorem according to which a current

$$J^a = \sum_\ell \frac{\partial L}{\partial \partial^a \psi_\ell} \delta \psi_\ell$$  \hspace{1cm} \text{(4.2)}

has zero divergence

$$\partial_a J^a = 0.$$  \hspace{1cm} \text{(4.3)}

The charge

$$Q = \int J^0 \, d^3x$$  \hspace{1cm} \text{(4.4)}

is conserved due to the global $U(1)$ symmetry. Here for tiny $\theta_\ell$, $\delta \psi_\ell = i\theta_\ell \psi_\ell$, and the charge density $J^0$ is

$$J^0 = \sum_\ell \frac{\partial L}{\partial \partial^0 \psi_\ell} \delta \psi_\ell = \sum_\ell \frac{\partial L}{\partial \partial^0 \psi_\ell} i\theta_\ell \psi_\ell = \sum_\ell \pi_\ell i\theta_\ell \psi_\ell.$$  \hspace{1cm} \text{(4.5)}

One imagines that the angle is proportional to the charge of the field $\psi_\ell$,

$$\theta_\ell = q_\ell \theta.$$
4.2 Abelian gauge invariance

Quantum electrodynamics is a theory in which the action density is invariant when the charged fields change by a phase transformation that varies with the spacetime point $x$

$$\psi'_\ell(x) = e^{i\theta_\ell(x)} \psi_\ell(x).$$ (4.6)

Such a symmetry is called a **local** $U(1)$ symmetry. A theory with a local $U(1)$ symmetry also has a global $U(1)$ symmetry and so it conserves charge.

Quantities like $\psi'_\ell(x) \psi_\ell(x)$ are intrinsically invariant under local $U(1)$ symmetries because

$$(\psi'_\ell(x))' \psi'_\ell(x) = \psi'_\ell(x) e^{-i\theta_\ell(x)} e^{i\theta_\ell(x)} \psi_\ell(x) = \psi'_\ell(x) \psi_\ell(x).$$ (4.7)

Derivatives of fields present problems, however, because

$$\partial_a (e^{i\theta_\ell(x)} \psi_\ell(x)) = e^{i\theta_\ell(x)} [\partial_a (\psi_\ell(x)) + i (\partial_a \theta_\ell(x)) \psi_\ell(x)] \neq e^{i\theta_\ell(x)} \partial_a (\psi_\ell(x))$$ (4.8)

so things like $\psi'_\ell(x) \partial_a \psi_\ell(x)$ and like $(\partial_a \psi'_\ell(x)) \partial_a \psi_\ell(x)$ are not invariant under local phase transformations.

The trick is to introduce a field $A_a(x)$ that transforms so as to cancel the awkward term $e^{i\theta_\ell} i (\partial_a \theta) \psi_\ell$ in the derivative (4.8). We want a new derivative $D_a \psi_\ell$ that transforms like $\psi_\ell$.

$$(D_a \psi_\ell)' = e^{i\theta_\ell} D_a \psi_\ell.$$ (4.9)

So we set

$$D_a = \partial_a + i A_a$$ (4.10)

and require that

$$(D_a \psi_\ell)' = (\partial_a + i A'_a) \psi'_\ell = e^{i\theta_\ell} D_a \psi_\ell = e^{i\theta_\ell} (\partial_a + i A_a) \psi_\ell.$$ (4.11)

That is, we insist that

$$(\partial_a + i A'_a) \psi'_\ell = (\partial_a + i A'_a) e^{i\theta_\ell} \psi_\ell = e^{i\theta_\ell} (\partial_a + i A_a) \psi_\ell.$$ (4.12)

So we need

$$i \partial_a \theta_\ell + i A'_a = i A_a$$ (4.13)

or

$$A'_a(x) = A_a(x) - i \partial_a \theta_\ell.$$ (4.14)

This is what we need except that the field $A_a$ does not carry the index $\ell$. 
The solution to this problem is to define the symmetry transformation (4.6) so that the angle \( \theta_{\ell} \) is proportional to the charge of the field \( \psi_{\ell} \)

\[
\psi'_{\ell}(x) = e^{i q_{\ell} \theta(x)} \psi_{\ell}(x).
\]

(4.15)

This definition has the advantage that the charge density (4.5) becomes

\[
J^0 = i \theta \sum_{\ell} q_{\ell} \pi_{\ell} \psi_{\ell}
\]

(4.16)

which makes more sense than the old formula (4.5). More importantly, the definition (4.15) means that equations (4.9–4.22) change to

\[
(D_a \psi_{\ell})' = e^{i q_{\ell} \theta} D_a \psi_{\ell}.
\]

(4.17)

So we make the **covariant derivative**

\[
D_a \psi_{\ell} = (\partial_a + i q_{\ell} A_a) \psi_{\ell}
\]

(4.18)

depend upon the field \( \psi_{\ell} \) and require that

\[
(D_a \psi_{\ell})' = (\partial_a + i q_{\ell} A'_a) \psi'_{\ell} = e^{i q_{\ell} \theta} D_a \psi_{\ell} = e^{i q_{\ell} \theta} (\partial_a + i q_{\ell} A_a) \psi_{\ell}.
\]

(4.19)

That is, we insist that

\[
(\partial_a + i q_{\ell} A'_a) \psi'_{\ell} = (\partial_a + i q_{\ell} A_a) e^{i q_{\ell} \theta} \psi_{\ell} = e^{i q_{\ell} \theta} (\partial_a + i q_{\ell} A_a) \psi_{\ell}.
\]

(4.20)

So we need

\[
i q_{\ell} \partial_a \theta + i q_{\ell} A'_a = i q_{\ell} A_a
\]

(4.21)

or

\[
A'_a(x) = A_a(x) - i \partial_a \theta(x)
\]

(4.22)

which is much better than the old rule (4.14). The twin rules

\[
\psi'_{\ell}(x) = e^{i q_{\ell} \theta(x)} \psi_{\ell}(x)
\]

\[
A'_a(x) = A_a(x) - i \partial_a \theta(x)
\]

(4.23)

constitute an **abelian gauge transformation**.

Note that a field \( \psi_{\ell} \) couples to the electromagnetic field \( A_a \) in a way \( q_{\ell} A_a \psi_{\ell} \) that is proportional to its charge \( q_{\ell} \).

To insure that the right charge \( q_{\ell} \) appears in the right place, we can introduce a charge operator \( q \) such that \( q \psi_{\ell} = q_{\ell} \psi_{\ell} \) and redefine the abelian gauge transformation (4.23) as

\[
\psi'_{\ell}(x) = e^{i q \theta(x)} \psi_{\ell}(x) = e^{i q_{\ell} \theta(x)} \psi_{\ell}(x)
\]

\[
A'_a(x) = A_a(x) - i \partial_a \theta(x).
\]

(4.24)
So
\[ \partial_b A'_a = \partial_b A_a - i \partial_b \partial_a \theta \] \hspace{1cm} (4.25)
is not invariant, but the antisymmetric combination
\[ \partial_b A'_a - \partial_a A'_b = \partial_b A_a - i \partial_b \partial_a \theta - \partial_a A_b + i \partial_a \partial_b \theta = \partial_b A_a - \partial_a A_b \] \hspace{1cm} (4.26)
is invariant. Maxwell introduced this combination
\[ F_{ba} = \partial_b A_a - \partial_a A_b. \] \hspace{1cm} (4.27)
Thus
\[ L = -\frac{1}{4} F_{ba} F^{ba} \] \hspace{1cm} (4.28)
is a Lorentz-invariant, gauge-invariant action density for the electromagnetic field.

### 4.3 Coulomb-gauge quantization

The first step in the canonical quantization of a gauge theory is to pick a gauge. The most physical gauge for electrodynamics is the Coulomb gauge defined by the gauge condition
\[ 0 = \nabla \cdot \vec{A}. \] \hspace{1cm} (4.29)
If the action density (4.28) is modified by an interaction with a current \( J^a \)
\[ L = -\frac{1}{4} F_{ba} F^{ba} + A_a J^a \] \hspace{1cm} (4.30)
then the equation of motion is
\[ \partial_b F^{ba} = -J^a \] \hspace{1cm} (4.31)
while the homogeneous equations
\[ 0 = \partial_a F_{bc} + \partial_b F_{ca} + \partial_c F_{ab} \] \hspace{1cm} (4.32)
follow from the antisymmetry of \( F_{ab} \).

Because of the antisymmetry of \( F_{ab} \), the derivative \( \dot{A}_0 \) does not appear in the action density (4.30). So for \( a = 0 \), the equation of motion (4.31) actually is a **constraint**
\[ \partial_i F^{i0} = - \partial_i E^i = -J^0 \] \hspace{1cm} (4.33)
known as Gauss’s law
\[ \nabla \cdot \vec{E} = \rho = J^0. \] \hspace{1cm} (4.34)
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This constraint together with the Coulomb gauge condition \(4.29\) lets us express \(A^0\) in terms of the charge density \(\rho = J^0\). We find

\[
\nabla^2 A_0 - \partial_0 \nabla \cdot \vec{A} = \nabla \cdot E = J^0
\]

or

\[
\nabla^2 A_0 = J^0.
\]

The solution is

\[
A^0(\vec{x}, t) = \int \frac{J^0(\vec{y}, t)}{4\pi|\vec{x} - \vec{y}|} d^3y.
\]

The quantum fields in this gauge are the transverse parts of \(\vec{A}\) and their conjugate momentum \(\vec{E}\).

### 4.4 QED in the interaction picture

The Coulomb-gauge hamiltonian in the interaction picture is

\[
H = H_0 + V
\]

\[
H_0 = \frac{1}{2} \int \vec{E}^2 + \vec{B}^2 \, d^3x + H_{\psi, 0}
\]

\[
V = -\int \vec{J} \cdot \vec{A} \, d^3x + V_C + V_\psi
\]

in which \(B = \nabla \times \vec{A}\) and both \(\vec{E}\) and \(\vec{A}\) are transverse, that is

\[
\nabla \cdot \vec{E} = 0 \quad \text{and} \quad \nabla \cdot \vec{A} = 0
\]

and the Coulomb potential energy is

\[
V_C = \frac{1}{2} \int \frac{J^0(\vec{x}, 0)J^0(\vec{y}, 0)}{4\pi|\vec{x} - \vec{y}|} \, d^3x d^3y.
\]

The electromagnetic field is

\[
A_b(\vec{x}) = \int \frac{d^3p}{\sqrt{(2\pi)^32p^0}} \sum_s \left[ e^{i\vec{p} \cdot \vec{x}} \epsilon^b(p, s)a(p, s) + e^{-i\vec{p} \cdot \vec{x}} \epsilon^b(p, s)a^\dagger(p, s) \right].
\]

The polarization vectors may be chosen to be

\[
e^b(p, \pm 1) = R(\hat{p}) \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ \pm i \\ 0 \end{pmatrix}
\]
where \( R(\hat{p}) \) rotates \( \hat{z} \) into \( \hat{p} \). Thus
\[
\vec{p} \cdot \vec{e}(p, s) = 0
\]
and
\[
e^0(p, s) = 0
\]
because \( A^0 \) is a dependent variable \([4.37]\). The commutation relations are
\[
[a(p, s), a^\dagger(p', s')] = \delta_{ss'} \delta^3(\vec{p} - \vec{p}') \quad \text{and} \quad [a(p, s), a(p', s')] = 0. \quad (4.45)
\]
The term \( H_0 \) in the hamiltonian \((4.38)\) is
\[
H_0 = \int \sum_s \frac{1}{2} p^0 [a(p, s), a^\dagger(p, s)]_+ d^3p
\]
\[
= \int \sum_s p^0 \left( a^\dagger(p, s)a(p, s) + \frac{1}{2} \delta^3(\vec{p} - \vec{p}) \right) d^3p. \quad (4.46)
\]
The interaction is
\[
V(t) = e^{iH_0 t} \left[ -\int \vec{J}(\vec{x}, 0) \cdot \vec{A}(\vec{x}, 0) d^3x + \int \frac{J^0(\vec{x}, 0)J^0(\vec{y}, 0)}{8\pi|\vec{x} - \vec{y}|} d^3x d^3y + V_m(0) \right] e^{-iH_0 t}
\]
in which \( V_m(0) \) is the non-electromagnetic part of the matter interaction. Since \( A^0 = 0, \vec{J} \cdot \vec{A} = J \cdot A \).

### 4.5 Photon propagator

The photon propagator is
\[
- i \Delta_{ab}(x - y) = \langle 0 | T[A_a(x), A_b(y)] | 0 \rangle. \quad (4.48)
\]
Inserting the formula for the electromagnetic field \([4.41]\), we get
\[
- i \Delta_{ij}(x - y) = \int \frac{d^3p}{(2\pi)^3 2p^0} \left( \delta_{ij} - \frac{p_i p_j}{p^2} \right)
\]
\[
\times \left[ e^{ip \cdot (x - y)} \theta(x^0 - y^0) + e^{-ip \cdot (x - y)} \theta(y^0 - x^0) \right]
\]
\[
= \int \frac{d^4q}{(2\pi)^4} \frac{-i}{q^2 - i\epsilon} \left( \delta_{ij} - \frac{q_i q_j}{q^2} \right) e^{iq \cdot (x - y)} \quad (4.49)
\]
with the understanding that this propagator vanishes when \( a \) or \( b \) is zero. If \( n^c = (1, 0, 0, 0) \) has only a time component, one can write the \( \delta_{ij} - q_i q_j/q^2 \) as
\[
\left( \delta_{ab} - \frac{q_a q_b}{q^2} \right) \theta(ab) = \eta_{ab} + q^0 (q_a n_b + q_b n_a) - q_a q_b + q^2 n_a n_b \quad (4.50)
\]
in which \( q^0 \) is arbitrary and may be chosen to be determined by conservation of energy. The terms \( q_a n_b, q_b n_a, \) and \( q_a q_b \) act like \( \partial_a n_b, \partial_b n_a, \) and \( \partial_a \partial_b \) so they appear in the S-matrix as \( \partial_a J^a n_b, \) and \( \partial_b J^b n_a, \) etc., which vanish because of current conservation. The remaining term
\[
\frac{q^2 n_a n_b}{q^2} \frac{-i}{q^2 - i \epsilon} = -im_a n_b
\] (4.51)
gives in the S-matrix a term
\[
T = \frac{1}{2} \int d^4 x d^4 y \left[ -i J^0(x) \right] \left[ -i J^0(y) \right] \frac{-i}{(2\pi)^4} \int \frac{d^4 q}{q^2} e^{iq(x-y)}
\]
\[
= \frac{1}{2} \int d^4 x d^4 y J^0(x) J^0(y) \frac{d^3 q}{(2\pi)^3} \frac{\delta(x^0 - y^0)}{q^2} e^{iq(x-y)}
\] (4.52)
\[
= \frac{i}{2} \int d^3 x d^3 y dt \frac{J^0(\vec{x}, t) J^0(\vec{y}, t)}{4\pi |\vec{x} - \vec{y}|}
\] (4.53)
which cancels the Coulomb term
\[
T_C = -i \frac{1}{2} \int d^3 x d^3 y dt \frac{J^0(\vec{x}, t) J^0(\vec{y}, t)}{4\pi |\vec{x} - \vec{y}|}.
\] (4.54)
The bottom line is that the effective photon propagator is
\[
- i \Delta_{ab}(x-y) = \int \frac{d^4 q}{(2\pi)^4} \frac{-im_{ab}}{q^2 - i \epsilon} e^{iq(x-y)} = -i \Delta_{ab}(y - x).
\] (4.55)
Dirac’s action density is
\[
L_\psi = - \bar{\psi} \left( \gamma^a \partial_a + m \right) \psi
\] (4.56)
with \( \bar{\psi} = i \psi^\dagger \gamma^0. \) The conjugate momentum is
\[
\pi = \frac{\partial L}{\partial \dot{\psi}} = -\bar{\psi} \gamma^0 = -i \psi^\dagger \gamma^0 \gamma^0 = i \psi^\dagger.
\] (4.57)
The hamiltonian is
\[
H = \int (\pi \dot{\psi} - L) d^3 x = \int \left[ i \psi^\dagger \dot{\psi} + \bar{\psi} (\gamma^a \partial_a + m) \psi \right] d^3 x
\]
\[
= \int \bar{\psi} (\vec{\gamma} \cdot \vec{\nabla} + m) \psi \ d^3 x.
\] (4.58)
The free Dirac field aka the Dirac field in the interaction picture is
\[
\psi(x) = \int \frac{d^3 p}{(2\pi)^{3/2}} \sum_s \left[ u(p, s) e^{ipx} a(p, s) + v(p, s) e^{-ipx} b^\dagger(p, s) \right]
\] (4.59)
The addition and deletion operators obey the anticommutation relations
\[
[a(p, s), a^\dagger(p', s')]_+ = [b(p, s), b^\dagger(p', s')]_+ = \sigma_{ss'} \delta^3(\vec{p} - \vec{p}') \quad (4.60)
\]
with the other anticommutators equal to zero. Putting \( \psi(x) \) into \( H \) gives
\[
H = \int \sum_s p^0 \left[ a^\dagger(p, s)a(p, s) + b^\dagger(p, s)b(p, s) - \delta^3(\vec{p} - \vec{p}') \right] d^3p. \quad (4.61)
\]

4.6 Feynman’s rules for QED

The action density is
\[
L = -\frac{1}{4} F_{ab} F^{ab} - \bar{\psi}[\gamma^a(\partial_a + ieA_a) + m] \psi. \quad (4.62)
\]
The electric current is
\[
J^a = \frac{\partial L}{\partial A^a} = -ie\bar{\psi}\gamma^a \psi. \quad (4.63)
\]
The interaction is
\[
V(t) = ie \int \bar{\psi}(\vec{x}, t) \gamma^a \psi(\vec{x}, t) A_a(\vec{x}, t) d^3x + V_C(t) \quad (4.64)
\]
Draw all appropriate diagrams.
Label each vertex with \( \alpha = 1, 2, 3, \) or 4 on an electron line of momentum \( p \) entering the vertex and \( \beta = 1, \ldots, 4 \) on an electron line of momentum \( p' \) leaving the vertex, and an index \( a \) on the photon line of momentum \( q \). The vertex carries the factor
\[
(2\pi)^4 e\gamma^a_{\beta\alpha} \delta^4(p - p' + q). \quad (4.65)
\]
An outgoing electron line gives
\[
\frac{\bar{u}_\beta(p, s)}{(2\pi)^{3/2}}. \quad (4.66)
\]
An outgoing positron line gives
\[
\frac{v_\alpha(p, s)}{(2\pi)^{3/2}}. \quad (4.67)
\]
An incoming electron line gives
\[
\frac{u_\alpha(p, s)}{(2\pi)^{3/2}}. \quad (4.68)
\]
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\[ p + k \gamma e - e + e - e - e - \]

Figure 4.1 Direct or s-channel diagram for electron-positron scattering.

An incoming positron line gives
\[
\frac{\bar{v}_\beta(p,s)}{(2\pi)^{3/2}}.
\]  \hspace{1cm} (4.69)

An outgoing photon gives
\[
\frac{e^a(p,s)}{\sqrt{(2\pi)^3 2\rho^0}}.
\]  \hspace{1cm} (4.70)

An incoming photon gives
\[
\frac{e_a(p,s)}{\sqrt{(2\pi)^3 2\rho^0}}.
\]  \hspace{1cm} (4.71)

An internal electron line of momentum \( p \) from vertex \( \beta \) to vertex \( \alpha \) gives
\[
\frac{-i}{(2\pi)^4} \frac{(-i\vec{p} + m)_{\alpha\beta}}{p^2 + m^2 - i\epsilon}.
\]  \hspace{1cm} (4.72)

An internal photon line of momentum \( q \) linking vertexes \( a \) and \( b \) gives
\[
\frac{-i}{(2\pi)^4} \frac{\eta_{ab}}{q^2 - i\epsilon}.
\]  \hspace{1cm} (4.73)

Integrate over all momenta, sum over all indexes.
Add up all terms and get the combinatorics and minus signs right.

4.7 Electron-positron scattering

There are two Feynman diagrams for electron-positron scattering at order \( e^2 \) where \( \alpha = e^2/(4\pi\epsilon_0\hbar c) = 0.00729735256 \approx 1/137 \) is the fine-structure constant. The annihilation diagram is represented in Fig. 4.2.
Feynman’s rules give for the annihilation diagram

\[
A_s = \int \frac{d^4 q}{(2\pi)^4} e^{2\pi \delta^4(p + k - q)} (2\pi)^4 e^{2\pi \delta^4(-p' - k' + q)} \times \bar{u}_\gamma(p', s') v_\delta(k', t') u_\alpha(p, s) \bar{v}_\beta(k, t) - \frac{i}{(2\pi)^3} \frac{\eta_{ab}}{q^2 - i\epsilon} \\
= (2\pi)^4 e^{2\pi \delta^4(p + k - p' - k')} \times \bar{u}_\gamma(p', s') v_\delta(k', t') u_\alpha(p, s) \bar{v}_\beta(k, t) - \frac{i}{(2\pi)^3} \frac{\eta_{ab}}{(p + k)^2} \\
= e^{2\pi \delta^4(p + k - p' - k')} \times \bar{u}_\gamma(p', s') v_\delta(k', t') u_\alpha(p, s) \bar{v}_\beta(k, t) - \frac{i}{(2\pi)^3} \frac{\eta_{ab}}{(p + k)^2} \\
= -i \frac{e^2}{(2\pi)^3} \frac{\delta^4(p + k - p' - k')}{(p + k)^2} \bar{v}_\beta(k, t) \gamma_{\beta\alpha} u_\alpha(p, s) \bar{u}_\gamma(p', s') \gamma_{\gamma\delta} v_\delta(k', t').
\]

Suppressing indexes, we get

\[
A_s = -i \frac{e^2}{(2\pi)^3} \delta^4(p + k - p' - k') \frac{\bar{v} \gamma^a u \bar{u}^\prime \gamma^a v'}{(p + k)^2}. \tag{4.75}
\]

The exchange diagram is

\[
A_t = \frac{e^2}{(2\pi)^3} \delta^4(p + k - p' - k') \frac{\bar{u}^\prime \gamma^a u \bar{v} \gamma^a v'}{(p - p')^2}, \tag{4.76}
\]

and we must check our minus signs.

The initial state is \( |p, s; k, t\rangle = a^\dagger(p, s)b^\dagger(k, t)|0\rangle \) and the final state is
The trace of a single gamma matrix is zero. Since $\gamma$ is a fourth spatial gamma matrix (1.173), its square is unity, $\gamma^5 \gamma^5 = 1$, and it anticommutes with the ordinary gammas. So since the trace is cyclic,

$$\text{Tr}(\gamma^a) = \text{Tr}(\gamma^5 \gamma^5 \gamma^a) = -\text{Tr}(\gamma^5 \gamma^a \gamma^5) = -\text{Tr}(\gamma^5 \gamma^5 \gamma^a) = -\text{Tr}(\gamma^a)$$ (4.80)

the trace of a single gamma matrix is zero.

The trace of two gammas

$$\text{Tr}(\gamma^a \gamma^b) = \text{Tr}(2\eta^{ab} - \gamma^b \gamma^a)$$ (4.81)

is

$$\text{Tr}(\gamma^a \gamma^b) = \eta^{ab} \text{Tr}(1) = 4\eta^{ab}.$$ (4.82)
The trace of an odd number of gammas vanishes because
\[ \text{Tr}(\gamma^{a_1} \cdots \gamma^{a_{2n+1}}) = - \text{Tr}(\gamma^{a_5} \gamma^{a_1} \cdots \gamma^{a_{2n+1}}) = \text{Tr}(\gamma^{a_5} \gamma^{a_1} \cdots \gamma^{a_{2n+1}}) \]

which implies that
\[ \text{Tr}(\gamma^{a_1} \cdots \gamma^{a_{2n+1}}) = 0. \] (4.84)

To find the trace of four gammas, we use repeatedly the fundamental rule
\[ \gamma^a \gamma^b = 2 \eta^{ab} - \gamma^b \gamma^a. \] (4.85)

We thus find
\[ \text{Tr}(\gamma^a \gamma^b \gamma^c \gamma^d) = \text{Tr}[2\eta^{ab} \gamma^c \gamma^d - \eta^{ac} \gamma^b \gamma^d + \eta^{ad} \gamma^b \gamma^c] \]

or
\[ \text{Tr}(\gamma^a \gamma^b \gamma^c \gamma^d) = 4(\eta^{ab} \eta^{cd} - \eta^{ac} \eta^{bd} + \eta^{ad} \eta^{bc}). \] (4.87)

Reversing completely the order of the gammas changes nothing:
\[ \text{Tr}(\gamma^a \gamma^b \cdots \gamma^y) = \text{Tr}(\gamma^z \gamma^y \cdots \gamma^b \gamma^a). \] (4.88)

### 4.9 Electron-positron to muon-antimuon

To avoid the extra complications that the sum of two amplitudes \( A_s + A_t \) entail, let’s consider the process \( e^- (p) + e^+ (p') \rightarrow \mu^- (k) + \mu^+ (k') \) which requires only the direct amplitude
\[
A = - i \frac{e^2}{(2\pi)^2} \delta^4 (p + p' - k - k') \frac{\bar{v}(p', s') \gamma^a u(p, s) \bar{u}(k, t') \gamma_a v(k', t')}{(p + p')^2} \]

\[ = - 2\pi i \delta^4 (p + p' - k - k') \mathcal{M} \] (4.89)

where
\[
\mathcal{M} = \frac{e^2}{(2\pi)^2} \frac{\bar{v}(p', s') \gamma^a u(p, s) \bar{u}(k, t) \gamma_a v(k', t')}{(p + p')^2} \] (4.90)

in which I have relabeled the momenta so that \( k \) & \( k' \) belong to the muons.
The probability $|A|^2$ includes the term

$$\bar{v} \gamma^a u \bar{u}_\mu \gamma_\mu v_\mu \left( \bar{v} \gamma^b u \bar{u}_\mu \gamma_\mu v_\mu \right)^*$$

(4.91)

in which $\mu$'s label muons. Since $\bar{v} = iv^\dagger \gamma^0 = v^\dagger \beta$ as well as $\beta^2 = (i \gamma^0)^2 = 1$, and since $[1.177] \beta \gamma^a \beta = -\gamma^a$, we get

$$(\bar{v} \gamma^a u)^* = (v^\dagger \beta \gamma^a u)^* = u^\dagger \gamma^a \beta v = u^\dagger \beta \beta \gamma^a \beta v = u^\dagger \beta \gamma^a v = \bar{v} \gamma^a v.$$  (4.92)

So the electron part of the term (4.91) is

$$\bar{v} \gamma^a u \bar{v} \gamma^b u = \bar{v} \gamma^a u \bar{v} \gamma^b u.$$  (4.93)

The spin sums are

$$\sum_s u_\ell(p, \bar{s}) u_m^*(p, s) = \left[ \frac{1}{2\rho^0} \left( -i p^\dagger \gamma^c + m \right) \beta \right]_{\ell m}$$

$$\sum_s v_\ell(p, \bar{s}) v_m^*(p, s) = \left[ \frac{1}{2\rho^0} \left( -i p^\dagger \gamma^c - m \right) \beta \right]_{\ell m}.$$  (4.94)

Equivalently and more simply,

$$\sum_s u_\ell(p, \bar{s}) \bar{u}_m(p, s) = \left[ \frac{1}{2\rho^0} \left( -i p^\dagger \gamma^c + m \right) \right]_{\ell m}$$

$$\sum_s v_\ell(p, \bar{s}) \bar{v}_m(p, s) = \left[ \frac{1}{2\rho^0} \left( -i p^\dagger \gamma^c - m \right) \right]_{\ell m}.$$  (4.95)

So

$$\sum_{s,s'} \bar{v} \gamma^a u \bar{v} \gamma^b v = \sum_{s',j,\ell,m,n} \bar{v}_{j}(p', s') \gamma^a_{j\ell} u_\ell(p, s) \bar{u}_m(p, s) \gamma^b_{mn} v_n(p', s')$$

$$= \frac{1}{2\rho^0} \sum_{s',j,\ell,m,n} \bar{v}_{j}(p', s') \gamma^a_{j\ell} (-i p^\dagger \gamma^c + m)_{\ell m} \gamma^b_{mn} v_n(p', s')$$

(4.96)

$$= \frac{1}{2\rho^0} \sum_{s',j,\ell,m,n} v_n(p', s') \bar{v}_{j}(p', s') \gamma^a_{j\ell} (-i p^\dagger \gamma^c + m)_{\ell m} \gamma^b_{mn}$$

$$= \frac{1}{2\rho^0} \sum_{s',j,\ell,m,n} (-i p^\dagger - m)_{nj} \gamma^a_{j\ell} (-i p^\dagger + m)_{\ell m} \gamma^b_{mn}$$

$$= \frac{1}{2\rho^0} \rho^0 \text{Tr} \left[ (-i p^\dagger - m_e) \gamma^a (-i p^\dagger + m_e) \gamma^b \right].$$

The muon part of the term (4.91) is

$$\bar{u} \gamma^a u \bar{u} \gamma^b v = \bar{u} \gamma^a u \bar{u} \gamma^b v = \bar{u} \gamma^a u \bar{u} \gamma^a v.$$  (4.97)
4.9 Electron-positron to muon-antimuon

So the sum over all spins gives for the muons

$$\sum_{\alpha,\alpha'} \bar{u} \gamma_\alpha v \gamma_{\alpha'} u = \frac{1}{2k_0^2k_0^0} \operatorname{Tr}[-ik^\mu - m_\mu(\gamma_\beta - ik + m_\mu)\gamma_\alpha].$$ \hspace{1cm} (4.98)

Using the formula (4.86) for the trace of 4 gammas, we evaluate the trace for the electrons in stages:

$$\operatorname{Tr}[-i\gamma^\nu - m_\nu(\gamma_\beta - ik + m_\nu)\gamma^\mu] = - \operatorname{Tr}(\gamma^\mu \gamma^\nu \gamma_\beta - m_\nu^2 \operatorname{Tr}(\gamma^\mu \gamma_\beta)).$$ \hspace{1cm} (4.99)

The 4-gamma term is

$$\operatorname{Tr}[-i\gamma^\nu - m_\nu(\gamma_\beta - ik + m_\nu)\gamma^\mu] = - \operatorname{Tr}[\gamma^\mu \gamma_\beta - m_\nu^2 \operatorname{Tr}(\gamma^\mu \gamma_\beta)].$$

$$= - p'^\nu p_\mu \operatorname{Tr}[\gamma^\mu \gamma_\beta - m_\nu^2 \operatorname{Tr}(\gamma^\mu \gamma_\beta)]$$

$$= - 4 p'^\nu p_\mu (\eta^{\mu\beta} - \eta^{\nu\nu} + \eta^{\nu\mu} \eta^{\beta\beta})$$

$$= - 4(p'^\nu p_\beta - p'^\nu p_\beta + p'^\nu p_\beta).$$ \hspace{1cm} (4.100)

The 2-gamma term is

$$- m_\nu^2 \operatorname{Tr}(\gamma^\mu \gamma_\beta) = - 4m_\nu^2 \eta^{\mu\beta}.$$ \hspace{1cm} (4.101)

So their sum is

$$\operatorname{Tr}[-i\gamma^\nu - m_\nu(\gamma_\beta - ik + m_\nu)\gamma^\mu] = - 4[p'^\nu p_\beta + p'^\nu p_\beta + \eta^{\mu\beta}(m_\nu^2 - p'^\nu p_\beta)].$$ \hspace{1cm} (4.102)

The similar term for muons is

$$\operatorname{Tr}[-i\gamma^\nu - m_\nu(\gamma_\beta - ik + m_\nu)\gamma^\mu] = - 4[k'^\nu k_\beta + k'^\nu k_\beta + \eta^{\mu\beta}(m_\nu^2 - k'^\nu k_\beta)].$$ \hspace{1cm} (4.103)

Their product is

$$\operatorname{Tr}[-i\gamma^\nu - m_\nu(\gamma_\beta - ik + m_\nu)\gamma^\mu] \operatorname{Tr}[-i\gamma^\nu - m_\nu(\gamma_\beta - ik + m_\nu)\gamma^\mu]$$

$$= 16[p'^\nu p_\beta + p'^\nu p_\beta + \eta^{\mu\beta}(m_\nu^2 - p'^\nu p_\beta)] [k'^\nu k_\beta + k'^\nu k_\beta + \eta^{\mu\beta}(m_\nu^2 - k'^\nu k_\beta)]$$

$$= 32[(p'k')(pk') + (p'k')(pk') + m_\nu^2(pp') + m_\nu^2(k'k)].$$

So apart from the delta function and the \((2\pi)\)'s, the squared amplitude summed over final spins and averaged over initial spins is

$$\frac{1}{4} \sum_{s,s'} |\mathcal{M}|^2 = \frac{e^4}{2(2\pi)^6 p^0 p^0 k^0 k^0} \frac{(p'k')(pk') + (p'k')(pk') + m_\nu^2(pp') + m_\nu^2(k'k)}{q^4}. $$ \hspace{1cm} (4.105)

In the rest frame of a collider, \(q^2 = (p + p')^2 = 4E^2 = 4p^0^2\). The cosine of the scattering angle \(\theta\) is \(\cos \theta = \vec{p} \cdot \vec{k}/(|\vec{p}| |\vec{k}|)\).

Since \(m_\nu \approx m_\mu/200\), I will neglect \(m_\nu\) in what follows. One then gets
Quantum electrodynamics

\[ p \cdot p' = -2E^2, \quad p \cdot k = E(|k| \cos \theta - E), \quad \text{and} \quad p \cdot k' = p' \cdot k = -E(|k| \cos \theta + E). \]

Thus

\[
\frac{1}{4} \sum_{s,s'} |M|^2 = \frac{e^4 \left[ E^2(E - k \cos \theta)^2 + E^2(E + k \cos \theta)^2 + 2m_p^2E^2 \right]}{2(2\pi)^6E^416E^4} = \frac{e^4 \left[ (E - k \cos \theta)^2 + (E + k \cos \theta)^2 + 2m_p^2 \right]}{32(2\pi)^6E^6} = \frac{e^4 \left[ 2E^2 + 2k^2 \cos^2 \theta + 2m_p^2 \right]}{32(2\pi)^6E^6} = \frac{e^4 \left[ E^2 + k^2 \cos^2 \theta + m_p^2 \right]}{16(2\pi)^6E^6}
\]

(4.106)

The squared amplitude is

\[
|A|^2 = (2\pi)^3|\delta^4(p + p' - k - k')|^2 \frac{1}{4} \sum_{ss'} |M|^2
\]

(4.107)

\[
= \frac{VT}{(2\pi)^2} \delta^4(p + p' - k - k') \frac{1}{4} \sum_{ss'} |M|^2.
\]

(4.108)

since

\[
\delta^4(0) = \int \frac{d^4x}{(2\pi)^4} = \frac{VT}{(2\pi)^4}
\]

(4.109)

in which \( V \) is the volume of the universe and \( T \) is its infinite time. We also need to switch from delta-function normalization of states to unit normalization. The relation is that between a continuum delta function and a Kronecker delta

\[
\delta^3(\vec{p}' - \vec{p}) = \frac{V}{(2\pi)^3} \delta_{pp'}.
\]

(4.110)

Since there are two particles in the initial and final states, the probability is

\[
P = \left[ \frac{(2\pi)^3}{V} \right]^4 |A|^2 N_f
\]

(4.111)

where \( N_f \) is the number of final states. For a range of momenta \( d^3kd^3k' \), the number of final states is

\[
N_f = \left[ \frac{V}{(2\pi)^3} \right]^2 d^3kd^3k'.
\]

(4.112)
So the rate \( P/T \) is

\[
R = \left[ \frac{(2\pi)^3}{V} \right]^4 |A|^2 \frac{1}{T} N_f = \left[ \frac{(2\pi)^3}{V} \right]^{4} |A|^2 \left[ \frac{V}{(2\pi)^3} \right]^2 d^3 k d^3 k' \\
= \left[ \frac{(2\pi)^3}{V} \right]^{4} |A|^2 \frac{1}{T} N_f = \left[ \frac{(2\pi)^3}{V} \right]^{2} |A|^2 d^3 k d^3 k' \\
= \frac{(2\pi)^4}{V} \delta^4(p + p' - k - k') \frac{1}{4} \sum_{ss'} |M|^2 d^3 k d^3 k' \quad (4.113)
\]

The flux of incoming particles is \( u/V \) where \( u = |\vec{v}_1 - \vec{v}_2| \) is the relative velocity, which with \( c = 1 \) for massless electrons is \( u = 2 \). So the differential cross-section is

\[
d\sigma = \frac{1}{2} (2\pi)^4 \delta^4(p + p' - k - k') \frac{1}{4} \sum_{ss'} |M|^2 d^3 k d^3 k'. \quad (4.114)
\]

Integrating over \( d^3 k' \) sets \( \vec{k}' = -\vec{k} \), and so

\[
d\sigma = \frac{1}{2} (2\pi)^4 \delta^4(p^0 + p'^0 - 2k^0) \frac{1}{4} \sum_{ss'} |M|^2 d^3 k \\
= \frac{1}{2} (2\pi)^4 \delta \left( p^0 + p'^0 - 2\sqrt{k^2 + m^2} \right) \frac{1}{4} \sum_{ss'} |M|^2 k^2 dk d\Omega \quad (4.115)
\]

where \( k = |\vec{k}| \). The derivative of the delta function is \( k^2 p^0/k^0 \), so

\[
d\sigma = \frac{1}{2} (2\pi)^4 \frac{1}{4} \sum_{ss'} |M|^2 k^2 d\Omega \frac{k^0}{2k^0 p^0} = 2\pi^4 \frac{1}{4} \sum_{ss'} |M|^2 \frac{k^0}{p^0} d\Omega \\
= 4\pi^4 \frac{1}{4} \sum_{ss'} |M|^2 \frac{k^0}{p^0} d\Omega. \quad (4.116)
\]
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So then

\[ d\sigma = 4\pi^4 e^4 \left[ E^2 + k^2 \cos^2 \theta + m_\mu^2 \right] \frac{k k^{02}}{4(2\pi)^6 E^5} \frac{1}{p^\mu} \ d\Omega \]

\[ = \frac{e^4}{4(2\pi)^2 E^3} \left[ 1 + \frac{k^2}{E^2} \cos^2 \theta + \frac{m_\mu^2}{E^2} \right] k \ d\Omega \]

\[ = \frac{e^4}{4(2\pi)^2 E^3} \left[ 1 + \frac{E^2 - m_\mu^2}{E^2} \cos^2 \theta + \frac{m_\mu^2}{E^2} \right] k \ d\Omega \]  \hspace{1cm} (4.117)

\[ = \frac{e^4}{4(2\pi)^2 E^2} \sqrt{1 - \frac{m_\mu^2}{E^2}} \left[ 1 + \frac{m_\mu^2}{E^2} \right] \left[ 1 + \frac{m_\mu^2}{E^2} \right] \left[ 1 - \frac{m_\mu^2}{E^2} \right] \cos^2 \theta \ d\Omega \]

\[ = \frac{\alpha^2}{16E^2} \sqrt{1 - \frac{m_\mu^2}{E^2}} \left[ 1 + \frac{m_\mu^2}{E^2} \right] \left[ 1 + \frac{m_\mu^2}{E^2} \right] \left[ 1 - \frac{m_\mu^2}{E^2} \right] \left( 1 + \frac{1}{2} \frac{m_\mu^2}{E^2} \right) \] \hspace{1cm} (4.118)

The total cross-section is the integral over solid angle

\[ \sigma = \frac{\alpha^2}{16E^2} \sqrt{1 - \frac{m_\mu^2}{E^2}} \left[ \left( 1 + \frac{m_\mu^2}{E^2} \right) 4\pi + \left( 1 - \frac{m_\mu^2}{E^2} \right) \int \cos^2 \theta \ d\Omega \right] \]

\[ = \frac{\alpha^2}{16E^2} \sqrt{1 - \frac{m_\mu^2}{E^2}} \left[ \left( 1 + \frac{m_\mu^2}{E^2} \right) 4\pi + \left( 1 - \frac{m_\mu^2}{E^2} \right) \int \cos^2 \theta 2\pi d\cos \theta \right] \]

\[ = \frac{\pi\alpha^2}{4E^2} \sqrt{1 - \frac{m_\mu^2}{E^2}} \left[ \left( 1 + \frac{m_\mu^2}{E^2} \right) + 1 \right] \left( 1 - \frac{m_\mu^2}{E^2} \right) \frac{1}{3} \]

\[ = \frac{\pi\alpha^2}{3E^2} \sqrt{1 - \frac{m_\mu^2}{E^2}} \left( 1 + \frac{1}{2} \frac{m_\mu^2}{E^2} \right) \] \hspace{1cm} (4.118)

where \( E \) is the energy of each electron in the lab frame of the collider. The energy of the collider is \( E_{cm} = 2E \). At very high energies, the x-section is \( \sigma = \pi\alpha^2/(3E^2) \).
4.10 Electron-muon scattering

For the process \( e^- (p_1) + \mu^- (p_2) \rightarrow e^- (p'_1) + \mu^- (p'_2) \), the Feynman rules give

\[
A = -i \frac{e^2}{(2\pi)^2} \delta^4(p_1 + p_2 - p'_1 - p'_2) \frac{\bar{u}(p'_1, s') \gamma^a u(p_1, s) \bar{u}(p'_2, t') \gamma_a u(p_2, t')}{(p_1 - p'_1)^2}
\]

\[
= -2\pi i \delta^4(p_1 + p_2 - p'_1 - p'_2) \mathcal{M}
\]

(4.119)

The product of the traces over \( q^2 = (p_1 - p'_1)^2 \) is

\[
\frac{\text{Tr}[-i\gamma^a(\not{p}_1 + m_e)\gamma^b] \text{Tr}[-i\gamma^a(\not{p}_2 + m_\mu)\gamma^b]}{(p_1 - p'_1)^2}
\]

(4.120)

For \( e^+ - e^- \rightarrow \mu^+ + \mu^- \) we got

\[
\frac{\text{Tr}[-i\gamma^a(\not{p}_1 + m_e)\gamma^b] \text{Tr}[-i\gamma^a(\not{k} + m_\mu)\gamma^b]}{(p + k)^2}
\]

(4.121)
Nonabelian gauge theory

5.1 Yang and Mills invent local nonabelian symmetry

The action of a Yang-Mills theory is unchanged when a spacetime-dependent unitary matrix $U(x) = \exp(-it_a \theta^a(x))$ maps a vector $\psi(x)$ of matter fields to $\psi'(x) = U(x) \psi(x)$. The symmetry $\psi^\dagger U U^\dagger \psi = \psi^\dagger \psi$ is obvious, but how can kinetic terms like $\partial_i \psi^\dagger \partial^i \psi$ be made invariant? Yang and Mills introduced matrices $A_i = t_a A^a_i$ of gauge fields, replaced ordinary derivatives $\partial_i$ by covariant derivatives $D_i \equiv \partial_i + A_i$, and required that covariant derivatives of fields transform like fields

$$(\partial_i + A'_i) U \psi = (\partial_i U + U \partial_i + A'_i U) \psi = U (\partial_i + A_i) \psi.$$  \hspace{1cm} (5.1)$$

Their nonabelian gauge transformation is

$$A'_i(x) = U(x) A_i(x) U^\dagger(x) - (\partial_i U(x)) U^\dagger(x).$$ \hspace{1cm} (5.2)$$

Their Faraday tensor $F_{ik} = [D_i, D_k] = \partial_i A_k - \partial_k A_i + [A_i, A_k]$ transforms as

$$F'_{ik}(x) = U(x) F_{ik} U^{-1}(x) = U(x) [D_i, D_k] U^{-1}(x).$$ \hspace{1cm} (5.3)$$
6

Gravity

This chapter is at best a first draft, not ready for human consumption.

6.1 Christoffel symbols as nonabelian gauge fields

The insight of Yang and Mills (section 5.1) lets us define a covariant derivative $D_\ell = \partial_\ell + A_\ell$ of a contravariant vector $V^i$

$$(D_\ell V)^k = (\partial_\ell \delta^k_j + A^k_{\ell j})V^j$$

(6.1)

that transforms as

$$[(D_\ell V)^k]' = \frac{\partial x^i}{\partial x'^m}(D_\ell V)^m.$$  

(6.2)

So we need

$$[(\partial_\ell \delta^k_j + A^k_{\ell j})V^j]' = (\partial_\ell \delta^k_j + A^k_{\ell j})V'^j + A^k_{\ell j}V'^j$$

$$= x^k_{\ell'}x^m_{\ell}(\partial_\ell \delta^m_i + A^m_{\ell i})V^n = x^k_{\ell'}x^m_{\ell}(V^m_{\ell i} + A^m_{\ell i}V^n)$$

(6.3)

or

$$V'^k_{\ell} + A^k_{\ell j}V'^j = x^k_{\ell'}x^m_{\ell}(V^m_{\ell i} + A^m_{\ell i}V^n).$$

(6.4)

Decoding the left-hand side, we get

$$\partial_\ell (x^m_{\ell'i}V^m) + A^k_{\ell j}V'^j = x^k_{\ell'}x^m_{\ell}(V^m_{\ell i} + A^m_{\ell i}V^n)$$

(6.5)

or

$$x^n_{\ell'i}\partial_\ell (x^m_{\ell'i}V^m) + A^k_{\ell j}V'^j = x^k_{\ell'}x^m_{\ell}(V^m_{\ell i} + A^m_{\ell i}V^n)$$

(6.6)

or

$$x^n_{\ell'i}x^m_{\ell'i}V^m + x^n_{\ell'i}x^m_{\ell'm}V^m + A^k_{\ell j}V'^j = x^k_{\ell'}x^m_{\ell}(V^m_{\ell i} + A^m_{\ell i}V^n).$$

(6.7)
After a cancelation, we get

\[ x^\ell_e x^j_m A^k_m V^n + A^k_j V^j = x^\ell_e x^j_m A^m_j V^n \]  \hspace{1cm} (6.8)

or

\[ A^k_j V^j = x^\ell_e x^j_m A^m_j V^n - x^\ell_e x^j_m V^n \]  \hspace{1cm} (6.9)

which is

\[ A^k_j x^j_m V^m = x^\ell_e x^j_m A^m_j V^n - x^\ell_e x^j_m V^n \]  \hspace{1cm} (6.10)

or

\[ A^k_j x^j_m V^m = x^\ell_e x^j_m A^m_j V^n - x^\ell_e x^j_m V^n \]  \hspace{1cm} (6.11)

or

\[ A^k_j x^j_m = x^\ell_e x^j_m A^m_j - x^\ell_e x^j_m \]  \hspace{1cm} (6.12)

or

\[ A^k_j x^j_m x^m_{j'} = x^\ell_e x^j_m A^m_j x^m_{j'} - x^\ell_e x^j_m x^m_{j'} \]  \hspace{1cm} (6.13)

or

\[ A^k_j = x^m_{j'} A^p_{nm} x^n_{j'} - x^p_{j'} x^m_{j'} \]  \hspace{1cm} (6.14)

The transformations are

\[ V^{\ell k} = x^k_j V^j \equiv E^k_j V^j \]  \hspace{1cm} and  \hspace{1cm} \[ V^j_\ell = x^j_\ell V_i \equiv E^{-1} i_j V_i = E^{-1} i_\ell V_i \]  \hspace{1cm} (6.15)

\[ A^k_j = x^m_{j'} A^p_{nm} x^n_{j'} - x^p_{j'} x^m_{j'} \]

\[ = E^{-1} \ell_p A^p_{nm} E^{-1} j_i - x^p_{j'} x^m_{j'} \]
\[ = E^{-1} \ell_p A^p_{nm} E^{-1} j_i - (\partial_p x^j_m) x^n_{j'} \]
\[ = E^{-1} \ell_p A^p_{nm} E^{-1} j_i - (\partial_p E^n_m) x^n_{j'} \]
\[ = E^{-1} \ell_p A^p_{nm} E^{-1} j_i - (\partial_p E^n_m) E^{-1} j_i \]
\[ = E^{-1} \ell_p A^p_{nm} E^{-1} j_i - E^{-1} j_i (\partial_e E^n_m) \]
\[ = E^{-1} \ell_p A^p_{nm} E^{-1} j_i + E^n_n (\partial_e E^{-1} j_i) \]  \hspace{1cm} (6.16)

So on a contravariant vector \( V' = E^{-1} V \), \( A' \) is

\[ A'_\ell = E^{-1} j_m E A_m E^{-1} + E \partial_e E^{-1} \]  \hspace{1cm} (6.17)

or

\[ A'^k_j = E^{-1} j_p A^p_{mn} E^{-1} j_n + E^n_n (\partial_e E^{-1} j_i) \]  \hspace{1cm} (6.18)
while on a covariant vector $V' = E^{-1}V$, $A'$ is
\[ A'_\ell = E^{-1m}_\ell E^{-1}A_mE - E^{-1}\partial_\ell E \]
(6.19)
or
\[ A'^{\ell k} = E^{-1m}_\ell E^{-1n}_k A^{p}_{nm}E^{k}_p - E^{-1n}_k(\partial_\ell E^{k}_n) \]
(6.20)

### 6.2 Spin connection

The Lorentz index $a$ on a tetrad $e^a_i$ is subject to local Lorentz transformations. A spin-one-half field $\psi^b_i$ also has an index $b$ that is subject to local Lorentz transformations. We can use the insight of Yang and Mills to introduce a gauge field, called the spin connection $\omega_{ab}^i$ and the generators $J_{ab}$ of the Lorentz group
\[ \omega^i = \omega_{ab}^i J_{ab}. \]
(6.21)
Since the generators $J_{ab}$ of the Lorentz group are antisymmetric, $J_{ab} = -J_{ba}$, so is the spin connection
\[ \omega_{ab}^i = -\omega_{ab}^i. \]
(6.22)

Under a general coordinate transformation and a local Lorentz transformation $L(x)$, we want
\[ [D_\ell(x)L\psi^i(x)]' = L(x)D_\ell(x)\psi^i(x) \]
(6.23)
where
\[ (D_\ell\psi)^{\alpha}_\beta(x) = [\partial_\ell \delta^{\alpha}_\beta + i\omega^{ab}_k(J_{ab})^{\alpha}_\beta]\psi^\beta(x) \]
(6.24)
so the spin connection should go as
\[ \omega^i_m(x') = x'^{mp}_i L_{wm}(x)L^{-1} + L\partial_p L^{-1}. \]
(6.25)
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