Why creation spinors are different from annihilation spinors

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Creation and annihilation operators transform like particles. Spinors—the coefficients of creation and annihilation operators in Fourier expansions of fields—make fields transform according to representations of the Poincaré group. This note explains why the Dirac spinors that are the coefficients of creation operators are so different from the Dirac spinors that are the coefficients of annihilation operators.

I. SPIN-ONE-HALF PARTICLES

A spin-one-half particle of kind n and 4-momentum $p = (\vec{p}, p^0)$ with $p^2 = -m^2$ and spin $s = \pm 1/2$ in the z direction is represented by a state $|\vec{p}, s, n\rangle$. For fixed kind and momentum, these states form a two-dimensional space spanned by $|\vec{p}, 1/2, n\rangle$ and $|\vec{p}, -1/2, n\rangle$.

If the momentum is in the z direction, then the state $|p\hat{z}, s, n\rangle$ is an eigenstate of the z component J_z of the angular momentum operator $\vec{J}$ with eigenvalue s

$$J_z|p\hat{z}, s, n\rangle = s|p\hat{z}, s, n\rangle. \tag{1}$$

So under an active right-handed rotation by angle θ about the z axis, the state $|p\hat{z}, s, n\rangle$ transforms as

$$e^{-i\theta J_z}|p\hat{z}, s, n\rangle = e^{-is\theta}|p\hat{z}, s, n\rangle. \tag{2}$$

The creation operator $a^\dagger(\vec{p}, s, n)$ adds such a particle to a state. It turns the vacuum state $|0\rangle$ into the state $|\vec{p}, s, n\rangle$

$$a^\dagger(\vec{p}, s, n)|0\rangle = |\vec{p}, s, n\rangle. \tag{3}$$

It turns a state of several particles $|\vec{p}_1, s_1, n_1, \ldots\rangle$ into a state

$$a^\dagger(\vec{p}, s, n)|\vec{p}_1, s_1, n_1; \ldots\rangle = |\vec{p}, s, n; \vec{p}_1, s_1, n_1; \ldots\rangle \tag{4}$$

that represents those particles plus a particle of 4-momentum $p = (\vec{p}, p^0)$, spin s in the z direction, and kind n.

The antiparticle of a particle of kind n is of kind n_c. For antiparticles, the particle equations (1–4) apply with $n \to n_c$:

$$J_z|p\hat{z}, s, n_c\rangle = s|p\hat{z}, s, n_c\rangle \tag{5}$$

$$e^{-i\theta J_z}|p\hat{z}, s, n_c\rangle = e^{-is\theta}|p\hat{z}, s, n_c\rangle \tag{6}$$

$$a^\dagger(\vec{p}, s, n_c)|0\rangle = |\vec{p}, s, n_c\rangle \tag{7}$$

$$a^\dagger(\vec{p}, s, n_c)|\vec{p}_1, s_1, n_1; \ldots\rangle = |\vec{p}, s, n_c; \vec{p}_1, s_1, n_1; \ldots\rangle. \tag{8}$$

Under a rotation by angle $\vec{\theta}$, the creation operator $a^\dagger(\vec{p}, s, n)$ transforms as

$$\begin{aligned} U(R_{\vec{\theta}})a^\dagger(\vec{p}, s, n)U^{-1}(R_{\vec{\theta}}) &= e^{-i\theta\cdot\vec{J}}a^\dagger(\vec{p}, s, n)e^{i\vec{\theta}\cdot\vec{J}} \\ &= D(R_{\vec{\theta}})_{s's}\; a^\dagger(R_{\vec{\theta}}\vec{p}, s', n) \end{aligned} \tag{9}$$

in which

$$D(R_{\vec{\theta}})_{s's} = \left[e^{-i\vec{\theta}\cdot\vec{\sigma}/2}\right]_{s's} = \delta_{s's}\,\cos(\theta/2) - i\vec{\theta}\cdot(\vec{\sigma})_{s's}\,\sin(\theta/2) \tag{10}$$

is the 2×2 representation of the rotation group, and the σ's are the Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \text{and} \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{11}$$

The annihilation operator goes as the adjoint of equation (9)

$$\begin{aligned} U(R_{\vec{\theta}})a(\vec{p}, s, n)U^{-1}(R_{\vec{\theta}}) &= e^{-i\theta\cdot\vec{J}}a(\vec{p}, s, n)e^{i\vec{\theta}\cdot\vec{J}} \\ &= D^*(R_{\vec{\theta}})_{s's}\; a(R_{\vec{\theta}}\vec{p}, s', n). \end{aligned} \tag{12}$$

For a rotation by angle θ about the z axis, the matrix $D(R_{\theta\hat{z}})$ is diagonal, and so the creation operator $a^\dagger(\vec{p}, s, n)$ goes as

$$\begin{aligned} U(R_{\theta\hat{z}})a^\dagger(\vec{p}, s, n)U^{-1}(R_{\theta\hat{z}}) &= e^{-i\theta J_z}a^\dagger(R_{\theta\hat{z}}\vec{p}, s, n)e^{i\theta J_z} \\ &= e^{-is\theta}a^\dagger(R_{\theta\hat{z}}\vec{p}, s, n) \end{aligned} \tag{13}$$

because $a^\dagger(R_{\theta\hat{z}}\vec{p}, s, n)$ adds s units of angular momentum to a state. The adjoint of this equation shows that under a rotation by angle θ about the z axis the annihilation operator $a(\vec{p}, s, n)$ transforms as

$$\begin{aligned} U(R_{\theta\hat{z}})a(\vec{p}, s, n)U^{-1}(R_{\theta\hat{z}}) &= e^{-i\theta J_z}a(R_{\theta\hat{z}}\vec{p}, s, n)e^{i\theta J_z} \\ &= e^{is\theta}a(R_{\theta\hat{z}}\vec{p}, s, n) \end{aligned} \tag{14}$$

because $a(R_{\theta\hat{z}}\vec{p}, s, n)$ subtracts s units of angular momentum from a state. Creation and annihilation operators transform differently under rotations. That's why the spinors u and v must be different.

II. THE SPINORS OF DIRAC FIELDS

Under a rotation $R_{\theta\hat{z}} = R(\theta\hat{z})$ about the $\hat{z}$ axis by angle θ, a Dirac field

$$\psi_a(x) = \sum_s \int \frac{d^3p}{(2\pi)^{3/2}}\; u_a(\vec{p}, s, n)\, e^{ip\cdot x}\, a(\vec{p}, s, n) + v_a(\vec{p}, s, n_c)\, e^{-ip\cdot x}\, a^\dagger(\vec{p}, s, n_c) \tag{15}$$

transforms as its creation and annihilation operators transform (13 and 14)

$$\begin{aligned} U(R_{\theta\hat{z}})\psi_a(x)U^{-1}(R_{\theta\hat{z}}) &= e^{-i\theta J_z}\psi_a(x)e^{i\theta J_z} \\ &= \sum_s \int \frac{d^3p}{(2\pi)^{3/2}}\Big[u_a(\vec{p}, s, n)e^{ip\cdot x}e^{-i\theta J_z}a(\vec{p}, s, n)e^{i\theta J_z} \\ &\qquad + v_a(\vec{p}, s, n_c)e^{-ip\cdot x}e^{-i\theta J_z}a^\dagger(\vec{p}, s, n_c)e^{i\theta J_z}\Big] \\ &= \sum_s \int \frac{d^3p}{(2\pi)^{3/2}}\Big[u_a(\vec{p}, s, n)e^{ip\cdot x}e^{is\theta}a(R_{\theta\hat{z}}\vec{p}, s, n) \\ &\qquad + v_a(\vec{p}, s, n_c)e^{-ip\cdot x}e^{-is\theta}a^\dagger(R_{\theta\hat{z}}\vec{p}, s, n_c)\Big]. \end{aligned} \tag{16}$$

Since $R\vec{p}\cdot R\vec{x} = \vec{p}\cdot\vec{x}$, and $d^3R\vec{p} = d^3\vec{p}$, this is

$$\begin{aligned} U(R_{\theta\hat{z}})\psi_a(x)U^{-1}(R_{\theta\hat{z}}) &= \sum_s \int \frac{d^3p}{(2\pi)^{3/2}}\Big[u_a(R_{\theta\hat{z}}^{-1}\vec{p}, s)e^{i\vec{p}\cdot R\vec{x}}e^{is\theta}a(\vec{p}, s) \\ &\qquad + v_a(R_{\theta\hat{z}}^{-1}\vec{p}, s)e^{-i\vec{p}\cdot R\vec{x}}e^{-is\theta}a_c^\dagger(\vec{p}, s)\Big]. \end{aligned} \tag{17}$$

Under the same rotation $R_{\theta\hat{z}}$ about the $\hat{z}$ axis by angle θ, a Dirac field transforms under the $(\frac{1}{2}, 0)\oplus(0, \frac{1}{2})$ representation D of the Lorentz group as

$$U(R_{\theta\hat{z}})\psi_a(x)U^{-1}(R_{\theta\hat{z}}) = e^{-i\theta J_z}\psi_a(x)e^{i\theta J_z} = D(R_{\theta\hat{z}}^{-1})_{ab}\psi_b(R_{\theta\hat{z}}x). \tag{18}$$

The two 2×2 representations $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ of the Lorentz group are the same for rotations, $D^{(1/2,0)}(R) = D^{(0,1/2)}(R)$. For rotations about the $\hat{z}$ axis, the matrix $D^{(1/2,0)\oplus(0,1/2)}(R_{\theta\hat{z}}^{-1})$ is diagonal

$$D(R_{\theta\hat{z}}^{-1}) = \begin{pmatrix} e^{i\theta\sigma_3/2} & 0 \\ 0 & e^{i\theta\sigma_3/2} \end{pmatrix} = \begin{pmatrix} e^{i\theta/2} & 0 & 0 & 0 \\ 0 & e^{-i\theta/2} & 0 & 0 \\ 0 & 0 & e^{i\theta/2} & 0 \\ 0 & 0 & 0 & e^{-i\theta/2} \end{pmatrix}. \tag{19}$$

Thus under a rotation $R_{\theta\hat{z}}$ by angle θ about the $\hat{z}$ axis, a Dirac field transforms as

$$U(R_{\theta\hat{z}})\psi_a(x^0,\vec{x})U^{-1}(R_{\theta\hat{z}}) = e^{-i\theta J_z}\psi_a(x^0,\vec{x})e^{i\theta J_z} = \begin{cases} e^{i\theta/2}\,\psi_a(x^0,R_{\theta\hat{z}}\vec{x}) & \text{if } a=1,3 \\ e^{-i\theta/2}\,\psi_a(x^0,R_{\theta\hat{z}}\vec{x}) & \text{if } a=2,4 \end{cases}. \tag{20}$$

Comparing this equation with (17), we see that for momentum in the $\hat{z}$ direction, the $u_a(p\hat{z},s)$ spinors satisfy

$$e^{is\theta}\,u_a(p\hat{z},s) = \begin{cases} e^{i\theta/2}\,u_a(p\hat{z},s) & \text{if } a=1,3 \\ e^{-i\theta/2}\,u_a(p\hat{z},s) & \text{if } a=2,4 \end{cases} \tag{21}$$

while the $v_a(p\hat{z},s)$ spinors satisfy

$$e^{-is\theta}\,v_a(p\hat{z},s) = \begin{cases} e^{i\theta/2}\,v_a(p\hat{z},s) & \text{if } a=1,3 \\ e^{-i\theta/2}\,v_a(p\hat{z},s) & \text{if } a=2,4 \end{cases}. \tag{22}$$

Thus for momentum in the $\hat{z}$ direction, the $s=\frac{1}{2}$ spinor $u_a(p\hat{z},\frac{1}{2})$ can have nonzero components only for $a=1$ and 3, while the $s=-\frac{1}{2}$ spinor $u_a(p\hat{z},-\frac{1}{2})$ can have nonzero components only for $a=2$ and 4

$$u(p\hat{z},\tfrac{1}{2}) = \begin{pmatrix} a \\ 0 \\ b \\ 0 \end{pmatrix} \quad \text{and} \quad u(p\hat{z},-\tfrac{1}{2}) = \begin{pmatrix} 0 \\ c \\ 0 \\ d \end{pmatrix}. \tag{23}$$

This is what one expects.

But we also see that the $s=\frac{1}{2}$ spinor $v_a(p\hat{z},\frac{1}{2})$ can have nonzero components only for $a=2$ and 4, while the $s=-\frac{1}{2}$ spinor $v_a(p\hat{z},-\frac{1}{2})$ can have nonzero components only for $a=1$ and 3.

$$v(p\hat{z},\tfrac{1}{2}) = \begin{pmatrix} 0 \\ e \\ 0 \\ f \end{pmatrix} \quad \text{and} \quad v(p\hat{z},-\tfrac{1}{2}) = \begin{pmatrix} g \\ 0 \\ h \\ 0 \end{pmatrix}. \tag{24}$$

These equations surprise some physicists. They are derived in a book by Steven Weinberg [1] and in various articles [2, 3], discussed by Srednicki [4], and stated correctly by Peskin and

Schroeder [5]. But they are glossed over or stated incorrectly in many textbooks on quantum field theory. In these books, it is assumed that spinors merely need to satisfy the Dirac equation $(\gamma^a \partial_a + m)\chi(x) = 0$, in which the γ matrices are

$$\gamma^0 = -i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \gamma^i = -i \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}. \tag{25}$$

The condition that $\chi(x)$ satisfy the Dirac equation is too weak because the combination $\chi(x) = (\gamma^a \partial_a - m)\, \psi\, e^{ip\cdot x}$ obeys it for *every* four-component spinor ψ as long as $p^2 = -m^2$ [6].

The zero-momentum Dirac equations $(-i\gamma^0 + 1)u(\vec{0}, s) = 0$ and $(i\gamma^0 + 1)v(\vec{0}, s) = 0$ require for $\vec{p} = \vec{0}$ that $b = a$, $d = c$, $f = -e$, and $h = -g$:

$$u(\vec{0}, \tfrac{1}{2}) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad v(\vec{0}, \tfrac{1}{2}) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}. \tag{26}$$

The $s = -\frac{1}{2}$ spinors are related to the $s = \frac{1}{2}$ spinors by a right-handed rotation by π about the $\hat{y}$ axis, $u(\vec{0}, -\frac{1}{2}) = D(R(\pi\hat{y}))u(\vec{0}, \frac{1}{2})$ and $v(\vec{0}, -\frac{1}{2}) = D(R(\pi\hat{y}))v(\vec{0}, \frac{1}{2})$,

$$\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \tag{27}$$

which gives us Weinberg's zero-momentum spinors [1]

$$\begin{aligned} u(\vec{0}, \tfrac{1}{2}) &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad u(\vec{0}, -\tfrac{1}{2}) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \\ v(\vec{0}, \tfrac{1}{2}) &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \quad \text{and} \quad v(\vec{0}, -\tfrac{1}{2}) = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}. \end{aligned} \tag{28}$$

These spinors (28) obey the Majorana conditions

$$u(\vec{0}, s) = \gamma^2 v^*(\vec{0}, s) \quad \text{and} \quad v(\vec{0}, s) = \gamma^2 u^*(\vec{0}, s). \tag{29}$$

The spinors for momentum $\vec{p}$ are [2]

$$u(\vec{p},s) = \frac{m - i\not{p}}{\sqrt{2p^0(p^0+m)}}\, u(\vec{0},s) \quad \text{and} \quad v(\vec{p},s) = \frac{m + i\not{p}}{\sqrt{2p^0(p^0+m)}}\, v(\vec{0},s) \tag{30}$$

or more explicitly

$$u(\vec{p},\tfrac{1}{2}) = \frac{1}{2\sqrt{p^0(p^0+m)}} \begin{pmatrix} m+p^0-p_3 \\ -p_1-ip_2 \\ m+p^0+p_3 \\ p_1+ip_2 \end{pmatrix}, \quad u(\vec{p},-\tfrac{1}{2}) = \frac{1}{2\sqrt{p^0(p^0+m)}} \begin{pmatrix} -p_1+ip_2 \\ m+p^0+p_3 \\ p_1-ip_2 \\ m+p^0-p_3 \end{pmatrix}$$

$$v(\vec{p},\tfrac{1}{2}) = \frac{1}{2\sqrt{p^0(p^0+m)}} \begin{pmatrix} -p_1+ip_2 \\ m+p^0+p_3 \\ -p_1+ip_2 \\ -m-p^0+p_3 \end{pmatrix}, \quad v(\vec{p},-\tfrac{1}{2}) = \frac{1}{2\sqrt{p^0(p^0+m)}} \begin{pmatrix} -m-p^0+p_3 \\ p_1+ip_2 \\ m+p^0+p_3 \\ p_1+ip_2 \end{pmatrix}. \tag{31}$$

They also obey the Majorana conditions

$$u(\vec{p},s) = \gamma^2 v^*(\vec{p},s) \quad \text{and} \quad v(\vec{p},s) = \gamma^2 u^*(\vec{p},s). \tag{32}$$

A field χ is a Majorana field if $n_c = n$ so that its creation operators are the adjoints of its annihilation operators

$$\psi_a(x) = \sum_s \int \frac{d^3p}{(2\pi)^{3/2}}\; u_a(\vec{p},s,n)\, e^{ip\cdot x}\, a(\vec{p},s,n) + v_a(\vec{p},s,n)\, e^{-ip\cdot x}\, a^\dagger(\vec{p},s,n). \tag{33}$$

The Majorana conditions (32) make a Majorana field obey the Majorana condition

$$\chi^*(x) = \gamma^2\, \chi(x). \tag{34}$$

III. CROSS-CHECKS

This final section is written for those who doubt the facts stated in the first two sections of this paper; others may want to skip it.

Comparing the two rules (17) and (20) that specify how a Dirac field transforms under a rotation $R_{\theta\hat{z}}$ about the z axis, we see that for momentum $\vec{p}$ the spinors obey the rules

$$e^{is\theta}\, u_a(R^{-1}_{\theta\hat{z}}\vec{p},s) = \begin{cases} e^{i\theta/2}\, u_a(\vec{p},s) & \text{if } a = 1,3 \\ e^{-i\theta/2}\, u_a(\vec{p},s) & \text{if } a = 2,4 \end{cases} \tag{35}$$

and

$$e^{-is\theta}\, v_a(R_{\theta\hat{z}}^{-1}\vec{p}, s) = \begin{cases} e^{i\theta/2}\, v_a(p\hat{z}, s) & \text{if } a = 1,3 \\ e^{-i\theta/2}\, v_a(p\hat{z}, s) & \text{if } a = 2,4 \end{cases} . \tag{36}$$

As a cross-check, we set $s = \frac{1}{2}$ and $a = 2$ in (35) and find

$$-p_1' - ip_2' = e^{-i\theta}(-p_1 - ip_2)$$

where the primes mean $\vec{p}' = R_{\theta\hat{z}}^{-1}\vec{p}$. That is,

$$p_1' = \cos\theta p_1 + \sin\theta p_2 \quad \text{and} \quad p_2' = \cos\theta p_2 - \sin\theta p_1$$

which is a left-handed rotation $R_{\theta\hat{z}}^{-1}$ about the z axis.

As a second cross-check, let's examine a state of one antiparticle at rest. The state $a^\dagger(0, s, n_c)|0\rangle$ is (summed over a and s')

$$\begin{aligned} \frac{1}{(2\pi)^{3/2}} \int d^3x\, \bar{v}_a(0, s)\psi_a(x)|0\rangle &= \bar{v}_a(0, s) \int \frac{d^3x d^3p}{(2\pi)^3}\, v(\vec{p}, s')\, e^{-ip\cdot x}\, a^\dagger(\vec{p}, s', n_c)|0\rangle \\ &= \bar{v}_a(0, s) \int d^3p\, \delta^3(\vec{p}) v(\vec{p}, s')\, a^\dagger(\vec{p}, s', n_c)|0\rangle \\ &= \bar{v}_a(0, s) v(0, s')\, a^\dagger(0, s', n_c)|0\rangle \\ &= \delta_{ss'}\, a^\dagger(0, s', n_c)|0\rangle = a^\dagger(0, s, n_c)|0\rangle. \end{aligned} \tag{37}$$

So

$$|0, s, n_c\rangle = \frac{1}{(2\pi)^{3/2}} \sum_a \int d^3x\, \bar{v}_a(0, s)\psi_a(x)|0\rangle. \tag{38}$$

To determine its spin, we act on it with the operator $e^{-i\theta J_3}$ that rotates states about the z axis by angle θ in a right-handed way

$$\begin{aligned} e^{-i\theta J_3}|0, s, n_c\rangle &= e^{-i\theta J_3} \frac{1}{(2\pi)^{3/2}} \sum_a \int d^3x\, \bar{v}_a(0, s)\psi_a(x)|0\rangle \\ &= \frac{1}{(2\pi)^{3/2}} \sum_a \int d^3x\, \bar{v}_a(0, s) e^{-i\theta J_3}\psi_a(x) e^{i\theta J_3} e^{-i\theta J_3}|0\rangle. \end{aligned} \tag{39}$$

Since the vacuum is invariant, this is

$$\begin{aligned} e^{-i\theta J_3}|0, s, n_c\rangle &= \frac{1}{(2\pi)^{3/2}} \sum_a \int d^3x\, \bar{v}_a(0, s) e^{-i\theta J_3}\psi_a(x) e^{i\theta J_3}|0\rangle \\ &= \frac{1}{(2\pi)^{3/2}} \sum_a \int d^3x\, \bar{v}_a(0, s) D(R_{\theta\hat{z}}^{-1})_{ab}\psi_b(Rx)|0\rangle. \end{aligned} \tag{40}$$

Because the jacobian of a rotation is unity, we have

$$e^{-i\theta J_3}|0, s, n_c\rangle = \frac{1}{(2\pi)^{3/2}} \sum_a \int d^3x \, \bar{v}_a(0, s) D(R^{-1}_{\theta\hat{z}})_{ab} \psi_b(x)|0\rangle \tag{41}$$

in which $D(R^{-1}_{\theta\hat{z}})_{ab}$ is given by (19). So the rotated state (41) is

$$e^{-i\theta J_3}|0, s, n_c\rangle = \frac{1}{(2\pi)^{3/2}} \sum_a \int d^3x \, \bar{v}_a(0, s) e^{i\sigma(a)\theta/2} \psi_a(x)|0\rangle \tag{42}$$

in which $\sigma(a) = 1$ for $a = 1$ and 3, and $\sigma(a) = -1$ for $a = 2$ and 4. The expressions (28) for the zero-momentum spinors show that for $s = 1/2$, slots $a = 2$ and 4 of the spinor $v(\vec{0}, s)$ are nonzero, while slots $a = 1$ and 3 are zero. Thus for $s = 1/2$

$$e^{-i\theta J_3}|0, 1/2, n_c\rangle = e^{-i\theta/2} \frac{1}{(2\pi)^{3/2}} \sum_a \int d^3x \, \bar{v}_a(0, s) \psi_a(x)|0\rangle \tag{43}$$

which means that the state has spin 1/2 in the z direction. Similarly, expressions (28) show that for $s = -1/2$, slots $a = 1$ and 3 of the spinor $v(\vec{0}, s)$ are nonzero, while slots $a = 2$ and 4 are zero. Thus for $s = -1/2$

$$e^{-i\theta J_3}|0, -1/2, n_c\rangle = e^{i\theta/2} \frac{1}{(2\pi)^{3/2}} \sum_a \int d^3x \, \bar{v}_a(0, s) \psi_a(x)|0\rangle \tag{44}$$

which means that the state has spin $-1/2$ in the z direction.

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[1] S. Weinberg, The quantum theory of fields (Cambridge University Press, 1995) pp. 219–224.

[2] P. Cahill and K. Cahill, Learning about spin-one-half fields, *Eur. J. Phys.* **27**, 29 (2006).

[3] H. K. Dreiner, H. E. Haber, and S. P. Martin, Two-component spinor techniques and Feynman rules for quantum field theory and supersymmetry, Phys. Rept. **494**, 1 (2010), arXiv:0812.1594 [hep-ph].

[4] M. Srednicki, Quantum field theory (Cambridge University Press, 2007) pp. 237–243.

[5] M. E. Peskin and D. V. Schroeder, An introduction to quantum field theory (Westview Press, Boulder (Colo.), 1995) pp. 803–804.

[6] K. Cahill, *Physical Mathematics* (Cambridge University Press, 2019) pp. 256–257, 2nd ed.