8
Electrodynamics

The original approach to quantum electrodynamics was to take for granted Maxwell's classical theory of electromagnetism, and quantize it. It will probably not surprise the reader that this book will follow a different path. We shall first infer the need for a principle of gauge invariance from the peculiar difficulties that arise in formulating a quantum theory of massless particles with spin, and then deduce the main features of electrodynamics from the gauge invariance principle. After that we shall follow a more conventional modern approach, in which one takes gauge invariance as the starting point and uses it to deduce the existence of a vector potential describing massless particles of unit spin.

It is too soon to tell which of these two alternatives corresponds to the logical order of nature. Most theorists have tended to take gauge invariance as a starting point, but in modern string theories[1] the argument runs the other way; one first notices a state of mass zero and unit spin among the normal modes of a string, and then from that deduces the gauge invariance of the effective field theory that describes such particles. At any rate, as we shall see, using either approach one is led to the quantized version of Maxwell's theory, still the paradigmatic example of a successful quantum field theory.

8.1 Gauge Invariance

Let's start by recalling the problems encountered in constructing covariant free fields for a massless particle of helicity ± 1. We saw in Section 5.9 that there is no difficulty in constructing an antisymmetric tensor free field $f_{\mu\nu}(x)$ for such particles. This field can be expressed in terms of the four-potential $a_\mu(x)$, given by Eq. (5.9.23), through the familiar relation

$$f_{\mu\nu}(x) = \partial_\mu a_\nu(x) - \partial_\nu a_\mu(x) . \tag{8.1.1}$$

However, Eq. (5.9.23) shows that the $a_\mu(x)$ transforms as a four-vector only up to a gauge transformation

$$U_0(\Lambda)a_\mu(x)\,U_0^{-1}(\Lambda) = \Lambda_\mu{}^\nu a_\nu(\Lambda x) + \partial_\mu\Omega(x,\Lambda)\,. \tag{8.1.2}$$

There is, in fact, no way to construct a true four-vector as a linear combination of the creation and annihilation operators for helicity ± 1. This is one way of understanding the presence of singularities at $m = 0$ in the propagator of a massive vector field

$$\Delta_{\mu\nu}(x,y) = (2\pi)^{-4}\int d^4x\; e^{iq\cdot(x-y)}\,\frac{\eta_{\mu\nu} + q_\mu q_\nu/m^2}{q^2+m^2-i\epsilon}\,,$$

which prevent us from dealing with massless particles of helicity ± 1 by simply passing to the limit $m \to 0$ of the theory of a massive particle of spin one.

We could avoid these problems by demanding that all interactions involve only* $F_{\mu\nu}(x) \equiv \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)$ and its derivatives, not $A_\mu(x)$, but this is not the most general possibility, and not the one realized in nature. Instead of banishing $A_\mu(x)$ from the action, we shall require instead that the part of the action I_M for matter and its interaction with radiation be invariant under the general gauge transformation

$$A_\mu(x) \to A_\mu(x) + \partial_\mu\epsilon(x) \tag{8.1.3}$$

(at least when the matter fields satisfy the field equations) so that the extra term in Eq. (8.1.2) should have no effect. The change in the matter action under the transformations (8.1.3) may be written

$$\delta I_M = \int d^4x\,\frac{\delta I_M}{\delta A_\mu(x)}\partial_\mu\epsilon(x)\,. \tag{8.1.4}$$

Hence the Lorentz invariance of I_M requires that

$$\partial_\mu\frac{\delta I_M}{\delta A_\mu(x)} = 0\,. \tag{8.1.5}$$

This is trivially true if I_M involves only $F_{\mu\nu}(x)$ and its derivatives, along with matter fields. In this case

$$\frac{\delta I_M}{\delta A_\mu(x)} = 2\partial^\nu\,\frac{\delta I_M}{\delta F_{\mu\nu}(x)}\,.$$

But if I_M involves $A_\mu(x)$ itself then Eq. (8.1.5) is a non-trivial constraint on the theory.

Now, what sort of theory will provide conserved currents to which we can couple the field $A^\mu(x)$? We saw in Section 7.3 that infinitesimal internal

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* We now use A_μ and $F_{\mu\nu}$ for the electromagnetic potential vector and the field strength tensor because these are interacting fields.

symmetries of the action imply the existence of conserved currents. In particular, if the transformation**

$$\delta\Psi^\ell(x) = i\epsilon(x)\, q_\ell\, \Psi^\ell(x) \tag{8.1.6}$$

leaves the matter action invariant for a constant ϵ, then for general infinitesimal functions $\epsilon(x)$ the change in the matter action must take the form

$$\delta I_M = -\int d^4x\; J^\mu(x)\partial^\mu\epsilon(x)\,. \tag{8.1.7}$$

When the matter fields satisfy their field equations, the matter action is stationary with respect to *any* variation of the Ψ_ℓ, so in this case (8.1.7) must vanish, and hence

$$\partial_\mu J^\mu = 0\,. \tag{8.1.8}$$

In particular, we saw in Section 7.3 that if I_M is the integral of a function $\mathscr{L}_M$ of Ψ^ℓ and $\partial_\mu\Psi^\ell$, then the conserved current is given by†

$$J^\mu = -i\sum_\ell \frac{\partial\mathscr{L}_M}{\partial(\partial_\mu\Psi^\ell)}\, q_\ell\, \Psi^\ell\,,$$

and this generates the transformations (8.1.6) in the sense that

$$[Q\,,\, \Psi^\ell(x)] = -q_\ell\, \Psi^\ell(x)\,, \tag{8.1.9}$$

where Q is the time-independent charge operator

$$Q = \int d^3x\, J^0\,. \tag{8.1.10}$$

We can therefore construct a Lorentz-invariant theory by coupling the vector field A_μ to the conserved current J^μ, in the sense that $\delta I_M/\delta A_\mu(x)$ is taken to be proportional to $J^\mu(x)$. Any constant of proportionality may be absorbed into the definition of the overall scale of the charges q_ℓ, so we may simply set these quantities equal:

$$\frac{\delta I_M}{\delta A_\mu(x)} = J^\mu(x)\,. \tag{8.1.11}$$

The conservation of electric charge only allows us to fix the values of all charges in terms of the value of any one of them, conventionally taken to

** Because the field transformation matrix is taken now to be diagonal it is not convenient here to use the summation convention for sums over field indices, so there is no sum over ℓ in Eq. (8.1.6).

† Here Ψ^ℓ is understood to run over all independent fields other than A^μ. We use a capital psi to indicate that these are Heisenberg-picture fields, whose time-dependence includes the effects of interactions. Of course, this Ψ^ℓ is not to be confused with a state-vector or wave function.

be the electron charge, denoted $-e$. It is Eq. (8.1.11) that gives a definite meaning‡ to the value of e.

The requirement (8.1.11) may be restated as a principle of invariance:[1a] the matter action is invariant under the joint transformations

$$\delta\, A_\mu(x) = \partial_\mu \epsilon(x)\,, \tag{8.1.12}$$

$$\delta\, \Psi_\ell\,(x) = i\epsilon(x) q_\ell \Psi_\ell(x)\,. \tag{8.1.13}$$

A symmetry of this type with an arbitrary function $\epsilon(x)$ is called a *local symmetry*, or a gauge invariance of the second kind. A symmetry under a transformation with ϵ constant is called a *global* symmetry, or a gauge invariance of the first kind. Several exact local symmetries are now known, but the only purely global symmetries appear to be accidents enforced by other principles. (See Section 12.5.)

We have not yet said anything about the action for photons themselves. As a guess, we can take this to be the same as for massive vector fields, but with $m = 0$:

$$I_\gamma = -\tfrac{1}{4}\int d^4x\; F_{\mu\nu} F^{\mu\nu}\,. \tag{8.1.14}$$

This is the same as the action used in classical electrodynamics, but its real justification is that it is (up to a constant) the unique gauge-invariant functional that is quadratic in $F_{\mu\nu}$, without higher derivatives. Also, as we will see in the next section, it leads to a consistent quantum theory. If there are any terms in the action of with higher derivatives and/or of higher order in $F_{\mu\nu}$ they can be lumped into what we have called the matter action. Using Eqs. (8.1.11) and (8.1.14), the field equation for electromagnetism now reads

$$0 = \frac{\delta}{\delta A_\nu}\,[I_\gamma + I_M] = \partial_\mu F^{\mu\nu} + J^\nu\,. \tag{8.1.15}$$

We recognize these as the usual inhomogeneous Maxwell equations, with current J^ν. There are also other, homogeneous, Maxwell equations

$$0 = \partial_\mu F_{\nu\epsilon} + \partial_\epsilon F_{\mu\nu} + \partial_\nu F_{\epsilon\mu}\,, \tag{8.1.16}$$

which follow directly from the definition $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$.

In the above discussion, we have started with the existence of massless spin one particles, and have been led to infer the invariance of the matter action under a local gauge transformation (8.1.12), (8.1.13). As usually presented, the derivation runs in the opposite direction. That is, one starts

‡ Of course, Eq. (8.1.11) fixes the definition of e only after we have defined how we are normalizing $A_\mu(x)$. The question of electromagnetic field normalization is taken up in Section 10.4.

with a global internal symmetry

$$\delta\Psi^\ell(x) = i\epsilon\, q^\ell \Psi^\ell(x) \tag{8.1.17}$$

and asks what must be done to promote this to a local symmetry

$$\delta\Psi^\ell(x) = i\epsilon(x) q_\ell \Psi^\ell(x)\,. \tag{8.1.18}$$

If the Lagrange density $\mathscr{L}$ depended only on fields $\Psi^\ell(x)$ and not on their derivatives then it would make no difference whether ϵ is constant or not; invariance with ϵ constant would imply invariance with ϵ a function of spacetime position. But all realistic Lagrangians do involve field derivatives, and here we have the problem that derivatives of fields transform differently from fields themselves:

$$\delta\, \partial_\mu \Psi^\ell(x) = i\epsilon(x) q_\ell \partial_\mu \Psi^\ell(x) + i\, q_\ell \Psi^\ell(x) \partial_\mu \epsilon(x)\,. \tag{8.1.19}$$

In order to cancel the second term here, we 'invent' a vector field $A_\mu(x)$ with transformation rule

$$\delta A_\mu(x) = \partial_\mu \epsilon(x) \tag{8.1.20}$$

and require that the Lagrangian density depend on $\partial_\mu \Psi^\ell$ and A_μ only in the combination

$$D_\mu \Psi^\ell \equiv \partial_\mu \Psi^\ell - i q_\ell A_\mu \Psi^\ell\,, \tag{8.1.21}$$

which transforms just like Ψ^ℓ

$$\delta\, D_\mu \Psi^\ell(x) = i\epsilon(x) q_\ell D_\mu \Psi^\ell(x)\,. \tag{8.1.22}$$

A matter Lagrangian density $\mathscr{L}_M(\Psi, D\Psi)$ that is formed only out of Ψ^ℓ and $D_\mu \Psi^\ell$ will be invariant under the transformations (8.1.18), (8.1.20), with $\epsilon(x)$ an arbitrary function, if it is invariant with ϵ a constant. With the Lagrangian of this form, we have

$$\frac{\delta I_M}{\delta A_\mu} = \sum_\ell \frac{\partial \mathscr{L}_M}{\partial D_\mu \Psi^\ell}(-i q_\ell \Psi^\ell) = -i \sum_\ell \frac{\partial \mathscr{L}_M}{\partial\, \partial_\mu \Psi^\ell} q_\ell \Psi^\ell\,,$$

which is the same as Eq. (8.1.11). (We could also include $F_{\mu\nu}$ and its derivatives in $\mathscr{L}_M$.) From this point of view, the masslessness of the particles described by A_μ is a consequence of gauge invariance rather than an assumption: a term $-\frac{1}{2}m^2 A_\mu A^\mu$ in the Lagrangian density would violate gauge invariance.

8.2 Constraints and Gauge Conditions

There are aspects of electrodynamics that stand in the way of quantizing the theory as we did for various theories of massive particles in the previous chapter. As usual, we may define the canonical conjugates to the electromagnetic vector potential by

$$\Pi^\mu \equiv \frac{\partial \mathscr{L}}{\partial(\partial_0 A_\mu)} . \tag{8.2.1}$$

Quantization by the usual rules would give

$$[A_\mu(\mathbf{x},t), \Pi^\nu(\mathbf{y},t)] = i\delta_\mu^\nu \delta^3(\mathbf{x}-\mathbf{y}) .$$

But this is not possible here, because A_μ and Π^ν are subject to several constraints.

The first constraint arises from the fact that the Lagrangian density is independent* of the time-derivative of A_0, and therefore

$$\Pi^0(x) = 0 . \tag{8.2.2}$$

This is called a *primary constraint*, because it follows directly from the structure of the Lagrangian. There is also a *secondary constraint* here, which follows from the field equation for the quantity fixed by the primary constraint:**

$$\partial_i \Pi^i = -\partial_i \frac{\partial \mathscr{L}}{\partial F_{i0}} = -\frac{\partial \mathscr{L}}{\partial A_0} = -J^0 , \tag{8.2.3}$$

the time-derivative term dropping out because $F_{00} = 0$. Even though the matter Lagrangian may generally depend on A^0, the charge density depends only on the canonical matter fields† Q^n and their canonical conjugates P_n:

$$J^0 = -i\sum_\ell \frac{\partial \mathscr{L}}{\partial(\partial_0 \Psi^\ell)} q_\ell \Psi^\ell = -i\sum_n P_n q_n Q^n . \tag{8.2.4}$$

Hence Eq. (8.2.3) is a functional relation among canonical variables. Both Eq. (8.2.2) and Eq. (8.2.3) are inconsistent with the usual assumptions that $[A_\mu(\mathbf{x},t), \Pi^\nu(\mathbf{y},t)] = i\delta_\mu^\nu\delta^3(\mathbf{x}-\mathbf{y})$ and $[Q^n(\mathbf{x},t), \Pi^\nu(\mathbf{y},t)] = [P_n(\mathbf{x},t), \Pi^\nu(\mathbf{y},t)] = 0$.

We encountered a similar problem in the theory of the massive vector field. In that case we found two equivalent ways of dealing with it: either by the method of Dirac brackets or, more directly, by treating only

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* For $\mathscr{L}_\gamma = -F_{\mu\nu}F^{\mu\nu}/4$, we have $\partial\mathscr{L}_\gamma/\partial(\partial_0 A_\mu) = -F^{0\mu}$, which vanishes for $\mu = 0$ because $F^{\mu\nu}$ is antisymmetric. For matter Lagrangians $\mathscr{L}_M$ that involve only Ψ^ℓ and $D_\mu\Psi^\ell$, the prescription (8.1.21) tells us that $\mathscr{L}_M$ does not depend on any derivatives of any A^ν. Even if the matter Lagrangian depends also on $F_{\mu\nu}$, $\partial\mathscr{L}_M/\partial(\partial_\nu A_\mu)$ will be again antisymmetric in μ and ν, and therefore will vanish for $\mu = \nu = 0$.

** As usual, i, j, etc. run over the values 1, 2, 3.

† Due to exhaustion of alphabetic resources, I have had to adopt a notation here that is different from that of the previous chapter. The symbols Q^n and P_n are now reserved for the canonical matter fields and their canonical conjugates, respectively, while the canonical electromagnetic fields and canonical conjugates are A_i and Π_i.

A_i and Π^i as canonical variables, solving the analog of Eq. (8.2.3) to calculate A^0 in terms of these variables. It is clear that here we cannot use Dirac brackets; the constraint functions χ here are Π^0 and $\partial_i\Pi_i - J^0$ (as compared with $\partial_i\Pi_i - m^2A^0 - J^0$) and these obviously have vanishing Poisson brackets. In Dirac's terminology, the constraints (8.2.2) and (8.2.3) are *first class*. Nor can we eliminate A^0 as a dynamical variable by solving for it in terms of the other variables. Instead of giving A^0 for all time, Eq. (8.2.3) is a mere initial condition; if Eq. (8.2.3) is satisfied at one time, then it is satisfied for all times, because (using the field equations for the other fields A^i), we have

$$\partial_0\left[\partial_i\,\frac{\partial\mathscr{L}}{\partial F_{i0}} - J^0\right] = -\partial_i\partial_0\,\frac{\partial\mathscr{L}}{\partial F_{0i}} - \partial_0 J^0$$
$$= +\,\partial_i\partial_j\,\frac{\partial\mathscr{L}}{\partial F_{ji}} - \partial_i J_i - \partial_0 J^0$$

and the current conservation condition then gives

$$\partial_0\left[\partial_i\,\frac{\partial\mathscr{L}}{\partial F_{i0}} - J^0\right] = 0\,. \tag{8.2.5}$$

It should not be surprising that we still have four components of A^μ with only three field equations, because this theory has a local gauge symmetry that makes it, in principle, impossible to infer the values of the fields at arbitrary times from their values and rates of change at any one time. Given any solution $A_\mu(\mathbf{x},t)$ of the field equations, we can always find another solution $A_\mu(\mathbf{x},t) + \partial_\mu\epsilon(\mathbf{x},t)$ with the same value and time-derivative at $t=0$ (by choosing ϵ so that its first and second derivatives vanish there) but which differs from $A_\mu(\mathbf{x},t)$ at later times.

Because of this partial arbitrariness of $A_\mu(\mathbf{x},t)$, it is not possible to apply the canonical quantization procedure directly to A_μ (or, as for finite mass, to $\mathbf{A}$). Of the various approaches to this difficulty, two are particularly useful. One is the modern method of gauge-invariant quantization, to be discussed in Volume II. The other, which will be followed here, is to exploit the gauge invariance of the theory, to 'choose a gauge'. That is, we make a finite gauge transformation

$$A_\mu(x) \to A_\mu(x) + \partial_\mu\lambda(x)\,, \qquad \Psi_\ell(x) \to \exp\Big(iq_\ell\lambda(x)\Big)\Psi_\ell(x)$$

to impose a condition on $A_\mu(x)$ that will allow us to apply the methods of canonical quantization. There are various gauges that have been found useful in various applications:‡

‡ Here Φ is any complex scalar field with $q \neq 0$; this gauge condition is used when the gauge symmetry is spontaneously broken by a non-vanishing vacuum expectation value of Φ.

Lorentz (or Landau) gauge:	$\partial_\mu A^\mu = 0$
Coulomb gauge:	$\nabla \cdot \mathbf{A} = 0$
Temporal gauge:	$A^0 = 0$
Axial gauge:	$A^3 = 0$
Unitarity gauge:	Φ real

The canonical quantization procedure works most easily in the axial or Coulomb gauge, but of course Coulomb gauge keeps manifest rotation invariance in a way that axial gauge does not, so we will adopt Coulomb gauge here.[2]

To check that this is possible, note that if A^μ does not satisfy the Coulomb gauge condition, then the gauge-transformed field $A^\mu + \partial^\mu \lambda$ will, provided we choose λ so that $\nabla^2 \lambda = -\nabla \cdot \mathbf{A}$. From now on, we assume that this transformation has been made, so that

$$\nabla \cdot \mathbf{A} = 0\,. \tag{8.2.6}$$

It will be convenient henceforth to limit ourselves to theories in which the matter Lagrangian $\mathscr{L}_M$ may depend on matter fields and their time-derivatives and also on A^μ but not on derivatives of A^μ. (The standard theories of the electrodynamics of scalar and Dirac fields have Lagrangians of this type.) Then the only term in the Lagrangian that depends on $F_{\mu\nu}$ is the kinematic term $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$, and the constraint equation (8.2.3) reads

$$-\ \partial_i F^{i0} = J^0\,. \tag{8.2.7}$$

Together with the Coulomb gauge condition (8.2.6), this yields

$$-\nabla^2 A^0 = J^0\,, \tag{8.2.8}$$

which can be solved to give

$$A^0(\mathbf{x}, t) = \int d^3y\ \frac{J^0(\mathbf{y}, t)}{4\pi|\mathbf{x} - \mathbf{y}|}\,. \tag{8.2.9}$$

The remaining degrees of freedom are A^i, with $i = 1, 2, 3$, subject to the gauge condition $\nabla \cdot \mathbf{A} = 0$.

As mentioned earlier, the charge density depends only on the canonical matter fields Q^n and their canonical conjugates P_n, so Eq. (8.2.9) represents an explicit solution for the auxiliary field A^0.

8.3 Quantization in Coulomb Gauge

There is still an impediment to the canonical quantization of electrodynamics in the Coulomb gauge. Even after we use Eq. (8.2.9) to eliminate A^0 (and Π_0) from the list of canonical variables, we cannot apply the usual canonical commutation relations to A^i and Π_i, because there are two remaining constraints on these variables.* One of them is the Coulomb gauge condition

$$\chi_{1\mathbf{x}} \equiv \partial_i A^i(\mathbf{x}) = 0 \,. \tag{8.3.1}$$

The other is the secondary constraint Eq. (8.2.3), which requires that

$$\chi_{2\mathbf{x}} \equiv \partial_i \Pi^i(\mathbf{x}) + J^0(\mathbf{x}) = 0 \,. \tag{8.3.2}$$

Neither constraint is consistent with the usual commutation relations $[A_i(\mathbf{x}), \Pi_j(\mathbf{y})] = i\delta_{ij}\delta^3(\mathbf{x}-\mathbf{y})$, because operating on the right-hand side with either $\partial/\partial x^i$ or $\partial/\partial y^j$ does not give zero.

These constraints are of a type known as *second class*, for which there is a universal prescription for the commutation relations, discussed in Section 7.6. Note that the constraint functions have the Poisson brackets

$$\begin{aligned} C_{1\mathbf{x},2\mathbf{y}} &= -C_{2\mathbf{y},1\mathbf{x}} \equiv [\chi_{1\mathbf{x}}, \chi_{2\mathbf{y}}]_{\mathrm{P}} = -\nabla^2\delta^2(\mathbf{x}-\mathbf{y}) \,, \\ C_{1\mathbf{x},1\mathbf{y}} &\equiv [\chi_{1\mathbf{x}}, \chi_{1\mathbf{y}}]_{\mathrm{P}} = 0 \,, \\ C_{2\mathbf{x},2\mathbf{y}} &\equiv [\chi_{2\mathbf{x}}, \chi_{2\mathbf{y}}]_{\mathrm{P}} = 0 \,, \end{aligned} \tag{8.3.3}$$

where here, for any functionals U and V,

$$[U, V]_{\mathrm{P}} \equiv \int d^3x \left[\frac{\delta U}{\delta A^i(\mathbf{x})}\frac{\delta V}{\delta \Pi_i(\mathbf{x})} - \frac{\delta V}{\delta A^i(\mathbf{x})}\frac{\delta U}{\delta \Pi_i(\mathbf{x})}\right] .$$

The 'matrix' C_{NM} is non-singular, which identifies these as second class constraints. Also, the field variables A^i may be expressed in terms of independent canonical variables, which may, for instance, be taken as $Q_{1\mathbf{x}} = A^1(\mathbf{x})$, $Q_{2\mathbf{x}} = A^2(\mathbf{x})$, with A^3 given by the solution of Eq. (8.3.1):

$$A^3(\mathbf{x}) = -\int^{x^3} ds\, [\partial_1 A^1(x^1, x^2, s) + \partial_2 A^2(x^1, x^2, s)] \,.$$

Using Eq. (8.3.2), the canonical conjugates Π_i to A^i may likewise be expressed in terms of the canonical conjugates $P_{\mathbf{x}1}$ and $P_{2\mathbf{x}}$ to $Q_{1\mathbf{x}}$ and $Q_{2\mathbf{x}}$. In such cases, Part B of the Appendix to the previous chapter tells us that if the independent variables $Q_{1\mathbf{x}}$, $Q_{2\mathbf{x}}$, $P_{1\mathbf{x}}$, and $P_{2\mathbf{x}}$ satisfy the usual canonical commutation relations, then the commutators of the constrained variables and their canonical conjugates are given (aside from a factor i) by the corresponding Dirac brackets (7.6.20). This prescription has the great advantage that we do not have to do use explicit expressions for the dependent variables in terms of the independent ones.

* In this section i, j, etc. run over the values 1,2,3. We continue the practice of taking all operators at the same time, and omitting the time argument.

To calculate the Dirac brackets, we note that the matrix C has the inverse

$$(C^{-1})_{1\mathbf{x},2\mathbf{y}} = -(C^{-1})_{2\mathbf{y},1\mathbf{x}} = -\int \frac{d^3k}{(2\pi)^3}\frac{e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}}{\mathbf{k}^2} = -\frac{1}{4\pi|\mathbf{x}-\mathbf{y}|}\,,$$
$$(C^{-1})_{1\mathbf{x},1\mathbf{y}} = (C^{-1})_{2\mathbf{x},2\mathbf{y}} = 0\,. \tag{8.3.4}$$

Also, the non-vanishing Poisson brackets of the A^i and Π_i with the constraint functions are

$$[A^i(\mathbf{x}), \chi_{1\mathbf{y}}]_P = -\frac{\partial}{\partial x^i}\delta^3(\mathbf{x}-\mathbf{y})$$

and

$$[\Pi_i(\mathbf{x}), \chi_{2\mathbf{y}}]_P = +\frac{\partial}{\partial x^i}\delta^3(\mathbf{x}-\mathbf{y})\,.$$

Hence according to Eqs. (7.6.19) and (7.6.20), the equal-time commutators are

$$\Big[A^i(\mathbf{x}), \Pi_j(\mathbf{y})\Big] = i\delta^i_j\delta^3(\mathbf{x}-\mathbf{y}) + i\frac{\partial^2}{\partial x^j\partial x^i}\left(\frac{1}{4\pi|\mathbf{x}-\mathbf{y}|}\right)\,,$$
$$\Big[A^i(\mathbf{x})\,,\,A^j(\mathbf{y})\Big] = \big[\Pi_i(\mathbf{x})\,,\,\Pi_j(\mathbf{y})\big] = 0\,. \tag{8.3.5}$$

Note that these are consistent with the Coulomb gauge conditions (8.3.1) and (8.3.2), as is guaranteed by the general properties of the Dirac bracket.

Now, what is $\mathbf{\Pi}$ in electrodynamics? For the class of theories discussed in the previous section where only the kinematic term $-\frac{1}{4}\int d^3x F_{\mu\nu}F^{\mu\nu}$ in the Lagrangian depends on $\dot{\mathbf{A}}$, varying the Lagrangian with respect to $\dot{\mathbf{A}}$ without worrying about the constraint $\nabla\cdot\mathbf{A}=0$ gives

$$\Pi_i = \frac{\delta L}{\delta\dot{A}^j(\mathbf{x})} = \dot{A}^j(\mathbf{x}) + \frac{\partial}{\partial x^j}A^0(\mathbf{x})\,. \tag{8.3.6}$$

But with $\mathbf{A}$ constrained by the condition $\nabla\cdot\mathbf{A}=0$, variational derivatives with respect to $\dot{\mathbf{A}}$ are not really well defined. If the variation of L under a change $\delta\dot{\mathbf{A}}$ in $\dot{\mathbf{A}}$ is $\delta L = \int d^3x\,\mathscr{P}\cdot\delta\dot{\mathbf{A}}$, then since $\nabla\cdot\delta\dot{\mathbf{A}}=0$, we also have $\delta L = \int d^3x\,[\mathscr{P}+\nabla\mathscr{F}]\cdot\delta\dot{\mathbf{A}}$ for any scalar function $\mathscr{F}(\mathbf{x})$. Thus all we can conclude from inspection of the Lagrangian is that $\mathbf{\Pi}$ equals $\dot{\mathbf{A}}(\mathbf{x})+\nabla A^0(\mathbf{x})$ plus the gradient of some scalar. This ambiguity is removed by condition (8.3.2), which requires that $\nabla\cdot\mathbf{\Pi} = -J^0 = \nabla^2 A^0$. Because $\nabla\cdot\dot{\mathbf{A}}=0$, we conclude that Eq. (8.3.6) does indeed give the correct formula for Π^i.

Although the commutation relations (8.3.5) are reasonably simple, we must face the complication that $\mathbf{\Pi}$ does not commute with matter fields and their canonical conjugates. If F is any functional of these matter degrees of freedom, then its Dirac bracket with $\mathbf{A}$ vanishes, but its Dirac bracket with $\mathbf{\Pi}$ is

$$
\begin{aligned}
[F, \mathbf{\Pi}(\mathbf{z})]_{\rm D} &= -\int d^3x\, d^3y\; [F, \chi_{2\mathbf{x}}]_{\rm P} \frac{1}{4\pi|\mathbf{x}-\mathbf{y}|} [\chi_{1\mathbf{y}}, \mathbf{\Pi}(\mathbf{z})]_{\rm P} \\
&= -\int d^3x\, d^3y\; [F, J^0(\mathbf{x})]_{\rm P} \frac{1}{4\pi|\mathbf{x}-\mathbf{y}|} \nabla\delta^3(\mathbf{y}-\mathbf{z}) \\
&= -\int d^3y\; [F, A^0(\mathbf{y})]_{\rm P} \nabla\delta^3(\mathbf{y}-\mathbf{z}) \\
&= [F, \nabla A^0(\mathbf{z})]_{\rm P} = [F, \nabla A^0(\mathbf{z})]_{\rm D} \,.
\end{aligned}
$$

In order to facilitate the transition to the interaction picture, instead of expressing the Hamiltonian in terms of $\mathbf{A}$ and $\mathbf{\Pi}$, we shall write it in terms of $\mathbf{A}$ and $\mathbf{\Pi}_\perp$, where $\mathbf{\Pi}_\perp$ is the solenoidal part of $\mathbf{\Pi}$:

$$
\mathbf{\Pi}_\perp \equiv \mathbf{\Pi} - \nabla A^0 = \dot{\mathbf{A}} \,, \tag{8.3.7}
$$

for which $[F, \mathbf{\Pi}_\perp(\mathbf{z})]$ vanishes. By using the facts that $\mathbf{\Pi}_\perp(\mathbf{x})$ commutes with $\mathbf{\Pi}(\mathbf{y}) - \mathbf{\Pi}_\perp(\mathbf{y}) = \nabla A^0(\mathbf{y})$ and that $\partial_i A^0(\mathbf{x})$ commutes with $\partial_j A^0(\mathbf{y})$, it is easy to see that $\mathbf{\Pi}_\perp(\mathbf{x})$ satisfies the same commutation relations (8.3.5) as $\mathbf{\Pi}(\mathbf{x})$, and also the simple constraint

$$
\nabla \cdot \mathbf{\Pi}_\perp = 0 \,. \tag{8.3.8}
$$

Now we need to construct a Hamiltonian. According to the general results of the Appendix to Chapter 7, we can apply the usual relation between the Hamiltonian and Lagrangian using the constrained variables $\mathbf{A}$ and $\mathbf{\Pi}_\perp$, without first having explicitly to write the Hamiltonian in terms of the unconstrained Qs and Ps. In electrodynamics, this gives

$$
H = \int d^3x \left[\Pi_{\perp i} \dot{A}^i + P_n \dot{Q}^n - \mathscr{L} \right] \,, \tag{8.3.9}
$$

where, as mentioned earlier, Q^n and P_n are to be understood as the matter canonical fields and their canonical conjugates. (We can use $\mathbf{\Pi}_\perp$ in place of $\mathbf{\Pi}$ in Eq. (8.3.9) because $\nabla \cdot \mathbf{A} = 0$.)

To be specific, consider a theory with a Lagrangian density of the form

$$
\mathscr{L} = -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} + J_\mu A^\mu + \mathscr{L}_{\rm matter} \,, \tag{8.3.10}
$$

where J_μ is a current that does not involve A^μ, and $\mathscr{L}_{\rm matter}$ is the Lagrangian for whatever other fields do appear in J^μ, aside from their electromagnetic interactions, which are given explicitly by the term $J_\mu A^\mu$ in Eq. (8.3.10). (The electrodynamics of spin $\frac{1}{2}$ particles has a Lagrangian of this form, but the electrodynamics of spinless particles is more complicated.) Replacing $\dot{\mathbf{A}}$ everywhere with $\mathbf{\Pi}_\perp$, this gives a Hamiltonian (8.3.9) of the form

$$
H = \int d^3x \left[\mathbf{\Pi}_\perp^2 + \tfrac{1}{2}(\nabla \times \mathbf{A})^2 - \tfrac{1}{2}(\mathbf{\Pi}_\perp + \nabla A^0)^2 - \mathbf{J} \cdot \mathbf{A} + J^0 A^0 \right] + H_{\rm M} \,,
$$

where $H_{\rm M}$ is the Hamiltonian for matter fields, excluding their electromagnetic interactions

$$H_{\rm M} \equiv \int d^3x \, (P_n \dot{Q}^n - \mathscr{L}_{\rm matter}) \, .$$

Using the solution (8.2.9) for A^0, this is

$$H = \int d^3x \left[\tfrac{1}{2}\mathbf{\Pi}_\perp^2 + \tfrac{1}{2}(\nabla \times \mathbf{A})^2 - \mathbf{J} \cdot \mathbf{A} + \tfrac{1}{2}J^0 A^0 \right] + H_{\rm M} \, . \qquad (8.3.11)$$

The term $\frac{1}{2}J^0A^0$ may look peculiar, but this is nothing but the familiar Coulomb energy

$$V_{\rm Coul} = \tfrac{1}{2}\int d^3x \, J^0 \, A^0$$
$$= \tfrac{1}{2}\int d^3x \int d^3y \frac{J^0(\mathbf{x})J^0(\mathbf{y})}{4\pi|\mathbf{x}-\mathbf{y}|} \, . \qquad (8.3.12)$$

The reader can verify, using the commutation relations (8.3.5), that the rate of change of any operator function F of $\mathbf{A}$ and $\mathbf{\Pi}$ is given by $i\dot{F} = [F, H]$, as it should be.

8.4 Electrodynamics in the Interaction Picture

We now break up the Hamiltonian (8.3.11) into a free-particle term H_0 and an interaction V

$$H = H_0 + V \, , \qquad (8.4.1)$$

$$H_0 = \int d^3x \left[\frac{1}{2}\mathbf{\Pi}_\perp^2 + \frac{1}{2}(\nabla \times \mathbf{A})^2 \right] + H_{\rm matter,0} \, , \qquad (8.4.2)$$

$$V = -\int d^3x \, \mathbf{J} \cdot \mathbf{A} + V_{\rm Coul} + V_{\rm matter} \, , \qquad (8.4.3)$$

where $H_{\rm matter,0}$ and $V_{\rm matter}$ are the free-particle and interaction terms in $H_{\rm matter}$, and $V_{\rm Coul}$ is the Coulomb interaction (8.3.12). The total Hamiltonian (8.4.1) is time-independent, so Eqs. (8.4.2) and (8.4.3) can be evaluated at any time we like (as long as both are evaluated at the same time), in particular at $t = 0$. As in Chapter 7, the transition to the interaction picture is made by applying the similarity transformation

$$V(t) = \exp(iH_0 t) \, V[\mathbf{A}, \mathbf{\Pi}_\perp, Q, P]_{t=0} \exp(-iH_0 t)$$
$$= V\,[\mathbf{a}(t), \, \boldsymbol{\pi}(t), \, q(t), \, p(t)] \, , \qquad (8.4.4)$$

where P here denotes the canonical conjugates to the matter fields Q, and any operator $o(\mathbf{x}, t)$ in the interaction picture is related to its value $O(\mathbf{x}, 0)$

in the Heisenberg picture at $t = 0$ by

$$o(\mathbf{x}, t) = \exp(iH_0 t)\; O(\mathbf{x}, 0)\; \exp(-iH_0 t), \tag{8.4.5}$$

so that

$$i\,\dot{o}(\mathbf{x}, t) = [o(\mathbf{x}, t), H_0]\ . \tag{8.4.6}$$

(We are dropping the subscript $\perp$ on $\boldsymbol{\pi}(x)$.) Since Eq. (8.4.5) is a similarity transformation, the equal-time commutation relations are the same as is the Heisenberg picture:

$$\left[a^i(\mathbf{x}, t), \pi^i(\mathbf{y}, t)\right] = i\left[\delta_{ij}\delta^3(\mathbf{x}-\mathbf{y}) + \frac{\partial^2}{\partial x^i \partial x^j}\frac{1}{4\pi|\mathbf{x}-\mathbf{y}|}\right], \tag{8.4.7}$$

$$\left[a^i(\mathbf{x}, t), a^j(\mathbf{y}, t)\right] = 0, \tag{8.4.8}$$

$$\left[\pi^i(\mathbf{x}, t), \pi^j(\mathbf{y}, t)\right] = 0, \tag{8.4.9}$$

and likewise for the matter fields and their conjugates. For the same reason, the constraints (8.2.6) and (8.3.8) still apply

$$\nabla \cdot \mathbf{a} = 0\,, \tag{8.4.10}$$

$$\nabla \cdot \boldsymbol{\pi} = 0\,. \tag{8.4.11}$$

To establish the relation between $\boldsymbol{\pi}$ and $\dot{\mathbf{a}}$, we must use Eq. (8.4.6) to evaluate $\dot{\mathbf{a}}$:

$$\begin{aligned} i\dot{a}_i(\mathbf{x}, t) &= [a_i(\mathbf{x}, t), H_0] \\ &= i\int d^3y\ \left[\delta_{ij}\delta^3(\mathbf{x}-\mathbf{y}) + \frac{\partial^2}{\partial x^i \partial x^j}\frac{1}{4\pi|\mathbf{x}-\mathbf{y}|}\right]\ \pi_j(\mathbf{y}, t)\,. \end{aligned}$$

We can replace $\partial/\partial x^j$ in the second term with $-\partial/\partial y^j$, integrate by parts, and use Eq. (8.4.11), yielding

$$\dot{\mathbf{a}} = \boldsymbol{\pi} \tag{8.4.12}$$

just as in the Heisenberg picture. The field equation is likewise determined by

$$\begin{aligned} i\dot{\pi}_i(\mathbf{x}, t) &= [\pi_i(\mathbf{x}, t), H_0] \\ &= -i\int d^3y\ \left[\delta_{ij}\delta^3(\mathbf{x}-\mathbf{y}) + \frac{\partial^2}{\partial x^i \partial x^j}\frac{1}{4\pi|\mathbf{x}-\mathbf{y}|}\right] \\ &\quad \times (\nabla \times \nabla \times \mathbf{a}(\mathbf{y}, t))_j\ , \end{aligned}$$

which (using Eqs. (8.4.10) and (8.4.12)) just yields the usual wave equation

$$\Box\, \mathbf{a} = 0\,. \tag{8.4.13}$$

Since A^0 is not an independent Heisenberg-picture field variable, but rather a functional (8.2.9) of the matter fields and their canonical conjugates that vanishes in the limit of zero charges, we do not introduce any corresponding operator a^0 in the interaction picture, but rather take

$$a^0 = 0\,. \tag{8.4.14}$$

The most general real solution of Eqs. (8.4.10), (8.4.13), and (8.4.14) may be written

$$a^\mu(x) = (2\pi)^{-3/2}\int \frac{d^3p}{\sqrt{2p^0}}\sum_\sigma \Big[e^{ip\cdot x}e^\mu(\mathbf{p},\sigma)\;a(\mathbf{p},\sigma) + e^{-ip\cdot x}e^{\mu *}(\mathbf{p},\sigma)\;a^\dagger(\mathbf{p},\sigma)\Big], \tag{8.4.15}$$

where $p^0 \equiv |\mathbf{p}|$; $e^\mu(\mathbf{p},\sigma)$ are any two independent 'polarization vectors' satisfying

$$\mathbf{p}\cdot\;\mathbf{e}(\mathbf{p},\sigma) = 0\,, \tag{8.4.16}$$

$$e^0(\mathbf{p},\sigma) = 0\,, \tag{8.4.17}$$

and $a(\mathbf{p},\sigma)$ are a pair of operator coefficients, with σ a two-valued index. By adjusting the normalization of $a(\mathbf{p},\sigma)$, we can normalize the $e^\mu(\mathbf{p},\sigma)$ so that the completeness relation reads

$$\sum_\sigma e^i(\mathbf{p},\sigma)e^j(\mathbf{p},\sigma)^* = \delta_{ij} - p_i\;p_j/|\mathbf{p}|^2\,. \tag{8.4.18}$$

For instance, we could take the $\mathbf{e}(\mathbf{p},\sigma)$ to be the same polarization vectors that we encountered in Section 5.9:

$$e^\mu(\mathbf{p},\pm 1) = R(\hat{\mathbf{p}})\begin{bmatrix} 1/\sqrt{2} \\ \pm i/\sqrt{2} \\ 0 \\ 0 \end{bmatrix}\,, \tag{8.4.19}$$

where $R(\hat{\mathbf{p}})$ is a standard rotation that carries the three-axis into the direction of $\mathbf{p}$. Using Eqs. (8.4.18) and (8.4.12), we can easily see that the commutation relations (8.4.7)–(8.4.9) are satisfied if (and in fact only if) the operator coefficients in Eq. (8.4.15) satisfy

$$\Big[a(\mathbf{p},\sigma),\;a^\dagger(\mathbf{p}\,',\sigma')\Big] = \delta^3(\mathbf{p}-\mathbf{p}\,')\;\delta_{\sigma\sigma'}, \tag{8.4.20}$$

$$\big[a(\mathbf{p},\sigma),\;a(\mathbf{p}\,',\sigma')\big] = 0\,. \tag{8.4.21}$$

As remarked before for massive particles, this result should be regarded not so much as an alternative derivation of Eqs. (8.4.20) and (8.4.21), but rather as a verification that Eq. (8.4.2) gives the correct Hamiltonian for free massless particles of helicity ± 1. In the same spirit one can also use Eqs. (8.4.12) and (8.4.15) in Eq. (8.4.2) to calculate the free-photon

Hamiltonian

$$H_0 = \int d^3p \sum_\sigma \tfrac{1}{2}p^0 \left[a(\mathbf{p},\sigma)\,,\; a^\dagger(\mathbf{p},\sigma\right]_+$$
$$= \int d^3p \sum_\sigma p^0 \left(a^\dagger(\mathbf{p},\sigma)a(\mathbf{p},\sigma) + \tfrac{1}{2}\delta^3(\mathbf{p}-\mathbf{p})\right) \tag{8.4.22}$$

which (aside from an inconsequential infinite c-number term) is just what we should expect.

Finally, we record that the interaction (8.4.4) in the interaction picture is

$$V(t) = -\int d^3x\; j_\mu(\mathbf{x},t)\; a^\mu(\mathbf{x},t) + V_{\text{Coul}}(t) + V_{\text{matter}}(t)\,, \tag{8.4.23}$$

where in terms of the current J in the Heisenberg picture

$$j_\mu(\mathbf{x},t) \equiv \exp(iH_0t)\; J_\mu(\mathbf{x},0)\exp(-iH_0t)\,, \tag{8.4.24}$$

while $V_{\text{Coul}}(t)$ is the Coulomb term

$$V_{\text{Coul}}(t) = \exp(iH_0t)V_{\text{Coul}}\exp(-iH_0t)$$
$$= \int d^3x\; d^3y\; \frac{j^0(\mathbf{x},t)j^0(\mathbf{y},t)}{4\pi|\mathbf{x}-\mathbf{y}|} \tag{8.4.25}$$

and $V_{\text{matter}}(t)$ is the non-electromagnetic part of the matter field interaction in the interaction picture:

$$V_{\text{matter}}(t) = \exp(iH_0t)\; V_{\text{matter}}\exp(-iH_0t)\,. \tag{8.4.26}$$

We have written $j_\mu a^\mu$ instead of $\mathbf{j}\cdot\mathbf{a}$ in Eq. (8.4.23), but these are equal because a^μ has been defined to have $a^0 = 0$.

8.5 The Photon Propagator

The general Feynman rules described in Chapter 6 dictate that an internal photon line in a Feynman diagram contributes a factor to the corresponding term in the S-matrix, given by the propagator:

$$-\,i\Delta_{\mu\nu}(x-y) \equiv \left(\Phi_{\text{VAC}},\, T\left\{a_\mu(x)\,,\, a_\nu(y)\right\}\Phi_{\text{VAC}}\right)\,, \tag{8.5.1}$$

where T as usual denotes a time-ordered product. Inserting our formula (8.4.15) for the electromagnetic potential then yields

$$-\,i\Delta_{\mu\nu}(x-y) = \int \frac{d^3p}{(2\pi)^3 2|\mathbf{p}|} P_{\mu\nu}(\mathbf{p})\left[e^{ip\cdot(x-y)}\theta(x-y) + e^{ip\cdot(y-x)}\theta(y-x)\right]\,, \tag{8.5.2}$$

where

$$P_{\mu\nu}(\mathbf{p}) \equiv \sum_{\sigma=\pm 1} e_\mu(\mathbf{p},\sigma)\, e_\nu(\mathbf{p},\sigma)^* \tag{8.5.3}$$

and p^μ in the exponentials is taken with $p^0 = |\mathbf{p}|$. We recall from Eqs. (8.4.18) and (8.4.17) that

$$\begin{aligned} P_{ij}(\mathbf{p}) &= \delta_{ij} - \frac{p^i p^j}{|\mathbf{p}|^2}\,, \\ P_{0i}(\mathbf{p}) &= P_{i0}(\mathbf{p}) = P_{00}(\mathbf{p}) = 0\,. \end{aligned} \tag{8.5.4}$$

As we saw in Chapter 6, the theta functions in Eq. (8.5.2) may be expressed as integrals over an independent time-component q^0 of an off-shell four-momentum q^μ, so that Eq. (8.5.2) may be rewritten

$$\Delta_{\mu\nu}(x-y) = (2\pi)^{-4} \int d^4q\, \frac{P_{\mu\nu}(\mathbf{q})}{q^2 - i\epsilon}\, e^{iq\cdot(x-y)}\,. \tag{8.5.5}$$

Thus in using the Feynman rules in momentum space, the contribution of an internal photon line carrying four-momentum q that runs between vertices where the photon is created and destroyed by fields a^μ and a^ν is

$$\frac{-i}{(2\pi)^4} \frac{P_{\mu\nu}(\mathbf{q})}{q^2 - i\epsilon}\,. \tag{8.5.6}$$

It will be very useful (though apparently perverse) to rewrite Eq. (8.5.4) as

$$P_{\mu\nu}(\mathbf{q}) = \eta_{\mu\nu} + \frac{q^0 q_\mu n_\nu + q^0 q_\nu n_\mu - q_\mu q_\nu + q^2 n_\mu n_\nu}{|\mathbf{q}|^2}\,, \tag{8.5.7}$$

where $n^\mu \equiv (0,0,0,1)$ is a fixed time-like vector, q^2 as usual is $\mathbf{q}^2 - (q^0)^2$, but q^0 is here entirely arbitrary. We shall choose q^0 in Eq. (8.5.7) to be given by four-momentum conservation: it is the difference of the matter p^0s flowing in and out of the vertex where the photon line is created. The terms proportional to q_μ and/or q_ν then do not contribute to the S-matrix, because the factors q_μ or q_ν act like derivatives ∂_μ and ∂_ν, and the photon fields a_μ and a_ν are coupled to currents j^μ and j^ν that satisfy the conservation condition $\partial_\mu j^\mu = 0$.* The term proportional to $n_\mu n_\nu$ contains a factor q^2 that cancels the q^2 in the denominator of the propagator, yielding a term that is the same as would be produced by a term in the action:

$$-i\frac{1}{2} \int d^4x \int d^4y\, [-ij^0(x)][-ij^0(y)] \frac{-i}{(2\pi)^4} \int \frac{d^4q}{|\mathbf{q}|^2}\, e^{iq\cdot(x-y)}\,.$$

---

* This argument as given here is little better than hand-waving. The result has been justified by a detailed analysis of Feynman diagrams,[3] but the easiest way to treat this problem is by path-integral methods, as discussed in Section 9.6.

The integral over q^0 here yields a delta function in time, so this is equivalent to a correction to the interaction Hamiltonian $V(t)$, of the form

$$-\frac{1}{2}\int d^3x \int d^3y\, \frac{j^0(\mathbf{x},t)\, j^0(\mathbf{y},t)}{4\pi|\mathbf{x}-\mathbf{y}|}\ .$$

This is just right to cancel the Coulomb interaction (8.4.25). Our result is that the photon propagator can be taken effectively as the covariant quantity

$$\Delta^{\rm eff}_{\mu\nu}(x-y) = (2\pi)^{-4}\int d^4q\, \frac{\eta_{\mu\nu}}{q^2-i\epsilon} e^{iq\cdot(x-y)} \tag{8.5.8}$$

with the Coulomb interaction dropped from now on. We see that the apparent violation of Lorentz invariance in the instantaneous Coulomb interaction is cancelled by another apparent violation of Lorentz invariance, that as noted in Section 5.9 the fields $a^\mu(x)$ are not four-vectors, and therefore have a non-covariant propagator. From a practical point of view, the important point is that in the momentum space Feynman rules, the contribution of an internal photon line is simply given by

$$\frac{-i}{(2\pi)^4}\frac{\eta_{\mu\nu}}{q^2-i\epsilon} \tag{8.5.9}$$

and the Coulomb interaction is dropped.

8.6 Feynman Rules for Spinor Electrodynamics

We are now in a position to state the Feynman rules for calculating the S-matrix in quantum electrodynamics. For definiteness, we will consider the electrodynamics of a single species of spin $\frac{1}{2}$ particles of charge $q=-e$ and mass m. We will call these fermions electrons, but the same formalism applies to muons and other such particles. The simplest gauge- and Lorentz-invariant Lagrangian for this theory is*

$$\mathscr{L} = -\frac{1}{4}\, F_{\mu\nu}F^{\mu\nu} - \bar{\Psi}\left(\gamma^\mu[\partial_\mu + i\,e\,A_\mu] + m\right)\Psi\ . \tag{8.6.1}$$

The electric current four-vector is then simply

$$J^\mu = \frac{\partial\mathscr{L}}{\partial A_\mu} = -i\,e\,\bar{\Psi}\gamma^\mu\Psi\ . \tag{8.6.2}$$

* In Chapter 12 we will discuss reasons why more complicated terms are excluded from the Lagrangian density.