That is, we can always *define* the scale of the gauge fields (now dropping the tildes) so that g_m in Eq. (15.2.5) is unity:

$$g_{\alpha\beta} = \delta_{\alpha\beta} , \tag{15.2.9}$$

but then the transformation matrices t_α and the structure constants $C_{\alpha\beta\gamma}$ contain an unknown multiplicative factor g_m for each simple or $U(1)$ subalgebra. These factors are the *coupling constants* of the gauge theory. Alternatively, it is sometimes more convenient to adopt some fixed though arbitrary normalization for the t_α and structure constants within each simple or $U(1)$ subalgebra, in which case the coupling constants appear in the gauge-field Lagrangian (15.2.3) as the factors g_m^{-2} in Eq. (15.2.5).

15.3 Field Equations and Conservation Laws

Using Eq. (15.2.9) for the matrix $g_{\alpha\beta}$ in Eq. (15.2.3), the full Lagrangian density is

$$\mathscr{L} = -\tfrac{1}{4} F_{\alpha\mu\nu}F_\alpha{}^{\mu\nu} + \mathscr{L}_M(\psi, D_\mu\psi) , \tag{15.3.1}$$

where in the absence of gauge fields $\mathscr{L}_M(\psi, \partial_\mu\psi)$ would be the 'matter' Lagrangian density. We could, in principle, include a dependence of $\mathscr{L}_M$ on $F_{\alpha\mu\nu}$ as well as higher covariant derivatives $D_\nu D_\mu\psi$, $D_\lambda F_{\alpha\mu\nu}$, etc., but we exclude these non-renormalizable terms here for the same reason as in electrodynamics: as discussed in Section 12.3, such terms would be highly suppressed at ordinary energies by negative powers of some very large mass. For this reason the standard model of the weak, electromagnetic and strong interactions has a Lagrangian of the general form (15.3.1).

The equations of motion of the gauge field are here

$$\partial_\mu \frac{\partial\mathscr{L}}{\partial(\partial_\mu A_{\alpha\nu})} = -\partial_\mu F_\alpha{}^{\mu\nu} = \frac{\partial\mathscr{L}}{\partial A_{\alpha\nu}}$$
$$= -F_\gamma{}^{\nu\mu}C_{\gamma\alpha\beta}A_{\beta\mu} - i\,\frac{\partial\mathscr{L}_M}{\partial D_\nu\psi}\,t_\alpha\psi$$

and so

$$\partial_\mu F_\alpha{}^{\mu\nu} = -\mathscr{J}_\alpha{}^\nu , \tag{15.3.2}$$

where $\mathscr{J}_\alpha{}^\nu$ is the current:

$$\mathscr{J}_\alpha{}^\nu \equiv -F_\gamma{}^{\nu\mu}C_{\gamma\alpha\beta}A_{\beta\mu} - i\,\frac{\partial\mathscr{L}_M}{\partial D_\nu\psi}t_\alpha\psi . \tag{15.3.3}$$

The current $\mathscr{J}_\alpha{}^\nu$ is conserved in the ordinary sense

$$\partial_\nu\mathscr{J}_\alpha{}^\nu = 0 , \tag{15.3.4}$$

as can be seen either from the Euler–Lagrange equations for ψ and the invariance equivalent (15.2.2) or, more easily, directly from the field equations (15.3.2).

The derivatives in Eqs. (15.3.2) and (15.3.4) are ordinary derivatives, not the gauge-covariant derivatives D_ν, so the gauge invariance of these equations is somewhat obscure. It can be made manifest by rewriting Eq. (15.3.2) in terms of the gauge-covariant derivative of the field strength

$$\begin{aligned} D_\lambda F_\alpha{}^{\mu\nu} &\equiv \partial_\lambda F_\alpha{}^{\mu\nu} - i\,(t^A_\beta)_{\alpha\gamma} A_{\beta\lambda} F_\gamma{}^{\mu\nu} \\ &= \partial_\lambda F_\alpha{}^{\mu\nu} - C_{\alpha\gamma\beta} A_{\beta\lambda} F_\gamma{}^{\mu\nu}\,. \end{aligned} \tag{15.3.5}$$

Then Eq. (15.3.2) reads

$$D_\mu F_\alpha{}^{\mu\nu} = -J_\alpha{}^\nu\,, \tag{15.3.6}$$

where $J_\alpha{}^\nu$ is the current of the matter fields alone

$$J_\alpha{}^\nu \equiv -i \frac{\partial \mathscr{L}_M}{\partial D_\nu \psi}\, t_\alpha \psi\,. \tag{15.3.7}$$

This is gauge-covariant, if $\mathscr{L}_M$ is gauge-invariant. Also, by operating on Eq. (15.3.6) with D_ν, using the commutation relation

$$[D_\nu, D_\mu] F_\alpha^{\rho\sigma} = -i\,(t^A{}_\gamma)_{\alpha\beta} F_{\gamma\nu\mu} F_\beta{}^{\rho\sigma} = -C_{\gamma\alpha\beta} F_{\gamma\nu\mu} F_\beta{}^{\rho\sigma}\,,$$

we see that $J_\alpha{}^\nu$ satisfies a gauge-covariant conservation law

$$D_\nu J_\alpha{}^\nu = 0\,, \tag{15.3.8}$$

rather than the ordinary conservation law (15.3.4) obeyed by the full current $\mathscr{J}_\alpha{}^\nu$. Also, it is straightforward (using Eq. (15.1.5)) to derive the identities:

$$D_\mu\, F_{\alpha\nu\lambda} + D_\nu\, F_{\alpha\lambda\nu} + D_\lambda\, F_{\alpha\mu\nu} = 0\,, \tag{15.3.9}$$

which hold whether or not the gauge fields satisfy the field equations.

These results serve to underscore the profound analogy mentioned in Section 15.1 between non-Abelian gauge theories and general relativity. In general relativity there is a matter energy-momentum tensor $T^\nu{}_\mu$, analogous to J^μ, which satisfies a generally covariant conservation law $T^\nu{}_{\mu;\nu} = 0$, and stands on the right-hand side of the Einstein field equations in their generally covariant form, $R^\nu{}_\mu - \frac{1}{2}\delta^\nu{}_\mu R = -8\pi G T^\nu{}_\mu$. However, $T^\nu{}_\mu$ is not conserved in the ordinary sense: $\partial_\nu T^\nu{}_\mu$ does not vanish. On the other hand, moving the non-linear terms on the left-hand side of the Einstein equation to the right-hand side gives a field equation[8]

$$\left(R^\nu{}_\mu - \frac{1}{2}\,\delta^\nu{}_\mu R\right)_{\text{LINEAR}} = -8\pi G\, \tau^\nu{}_\mu\,,$$

where $\tau^\nu{}_\mu$ is the non-tensor

$$\tau^\nu{}_\mu \equiv T^\nu{}_\mu + \frac{1}{8\pi G}\left(R^\nu{}_\mu - \frac{1}{2}\,\delta^\nu{}_\mu R\right)_{\text{NONLINEAR}},$$

analogous to $\mathscr{J}_\alpha{}^\nu$. Like $\mathscr{J}_\alpha{}^\nu$, $\tau^\nu{}_\mu$ is conserved in the ordinary sense

$$\partial_\nu \tau^\nu{}_\mu = 0$$

and may be regarded as the current of energy and momentum:

$$P_\mu = \int \tau^0{}_\mu\, d^3x\,.$$

It contains a purely gravitational term, because gravitational fields carry energy and momentum; without this term, $\tau^\nu{}_\mu$ could not be conserved. Similarly, $\mathscr{J}^\nu_\alpha$ contains a gauge-field term (the first term on the right in Eq. (15.3.3)) because for non-Abelian groups (those with $C^\gamma_{\alpha\beta} \neq 0$) the gauge fields carry the quantum numbers with which they interact. Because $\mathscr{J}_\alpha{}^\nu$ is conserved in the ordinary sense, it can be regarded as the current of these quantum numbers, with the symmetry generators given by the time-independent quantities

$$T_\alpha = \int \mathscr{J}_\alpha{}^0 d^3x\,. \tag{15.3.10}$$

(Also, the homogeneous equations (15.3.9) involve covariant derivatives, just as do the Bianchi identities of general relativity.) In contrast, none of these complications arises in quantum electrodynamics, because photons do not carry the quantum number, electric charge, with which they interact.

15.4 Quantization

We now proceed to quantize the gauge theories described in the previous two sections. The Lagrangian density is taken in the form (15.3.1):

$$\mathscr{L} = -\tfrac{1}{4}\, F_{\alpha\mu\nu} F_\alpha{}^{\mu\nu} + \mathscr{L}_M(\psi, D_\mu\psi)\,, \tag{15.4.1}$$

with

$$F_{\alpha\mu\nu} \equiv \partial_\mu A_{\alpha\nu} - \partial_\nu A_{\alpha\mu} + C_{\alpha\beta\gamma} A_{\beta\mu} A_{\gamma\nu}\,,$$
$$D_{\mu\psi} \equiv \partial_\mu \psi - i t_\alpha A_{\alpha\mu} \psi\,.$$

We cannot immediately quantize this theory by setting commutators equal to i times the corresponding Poisson brackets. The problem is one of constraints. In the terminology of Dirac, described in Section 7.6, there is a primary constraint that

$$\Pi_{\alpha 0} \equiv \frac{\partial \mathscr{L}}{\partial(\partial_0 A^0_\alpha)} = 0 \tag{15.4.2}$$

and a secondary constraint provided by the field equation for A^0_α:

$$\begin{aligned} -\partial_\mu \frac{\partial \mathscr{L}}{\partial(\partial_\mu A_{\alpha 0})} + \frac{\partial \mathscr{L}}{\partial A_{\alpha 0}} &= \partial_\mu F_\alpha{}^{\mu 0} + F_\gamma{}^{\mu 0} C_{\gamma\alpha\beta} A_{\beta\mu} + J_\alpha{}^0 \\ &= \partial_k \Pi_\alpha{}^k + \Pi_\gamma{}^k C_{\gamma\alpha\beta} A_{\beta\, k} + J_\alpha{}^0 = 0\,, \end{aligned} \tag{15.4.3}$$

where $\Pi^k_\alpha \equiv \partial\mathscr{L}/\partial(\partial_0 A_{\alpha k}) = F^{k0}_\alpha$ is the 'momentum' conjugate to $A_{\alpha k}$, with k running over the values $1, 2, 3$. The Poisson brackets of $\Pi_{\alpha 0}$ and $\partial_k \Pi_\alpha{}^k + \Pi_\gamma{}^k C_{\gamma\alpha\beta} A_{\beta\, k} + J_\alpha{}^0$ vanish (because the latter quantity is independent of $A_\alpha{}^0$), so these are first class constraints, which cannot be dealt with by replacing Poisson brackets with Dirac brackets.

As in the case of electrodynamics, we deal with these constraints by choosing a gauge. The Coulomb gauge adopted for electrodynamics would lead to painful complications here,* so instead we will work in what is known as *axial gauge*, based on the condition

$$A_{\alpha 3} = 0\,. \tag{15.4.4}$$

The canonical variables of the gauge field are then $A_{\alpha i}$, with i now running over the values 1 and 2, together with their canonical conjugates

$$\Pi_{\alpha i} \equiv \frac{\partial \mathscr{L}}{\partial(\partial_0 A_{\alpha i})} = -F_\alpha{}^{0i} = \partial_0 A_{\alpha i} - \partial_i A_{\alpha 0} + C_{\alpha\beta\gamma} A_{\beta 0} A_{\gamma i}\,. \tag{15.4.5}$$

The field $A_{\alpha 0}$ is not an independent canonical variable, but rather is defined in terms of the other variables by the constraint (15.4.3). To see this, note that the 'electric' field strengths $F_\alpha{}^{\mu 0}$ are

$$F_\alpha{}^{i0} = \Pi_{\alpha i}\,, \quad F_\alpha{}^{30} = \partial_3 A_\alpha{}^0\,, \tag{15.4.6}$$

so the constraint (15.4.3) reads

$$-(\partial_3)^2 A^0_\alpha = \partial_i \Pi_{\alpha i} + \Pi_{\gamma i} C_{\gamma\alpha\beta} A_{\beta i} + J_\alpha{}^0\,, \tag{15.4.7}$$

which can easily be solved (with reasonable boundary conditions) to give $A_\alpha{}^0$ as a functional of $\Pi_{\gamma i}$, $A_{\beta i}$, and $J_\alpha{}^0$. (We are using a summation

* In addition to purely algebraic complications, Coulomb gauge (like many other gauges) has a problem known as the Gribov ambiguity:[9] even with the condition that $\mathbf{A}_\alpha$ vanishes at spatial infinity, for each solution of the Coulomb gauge condition $\nabla \cdot \mathbf{A}_\alpha = 0$ there are other solutions that differ by finite gauge transformations. The Gribov ambiguity will not bother us here, because we quantize in axial gauge where it is absent, and we shall use other gauges like Lorentz gauge only to generate a perturbation series.

convention, with indices i, j, etc. summed over the values 1 and 2.) It should be noted that the canonical conjugate to the matter field ψ_ℓ is

$$\pi_\ell = \frac{\partial \mathscr{L}}{\partial(\partial_0\psi_\ell)} = \frac{\partial \mathscr{L}_M}{\partial(D_0\psi_\ell)} , \tag{15.4.8}$$

so the time component of the matter current can be expressed in terms of the canonical variables of the matter fields *alone*

$$J_\alpha{}^0 = -i\,\frac{\partial \mathscr{L}_m}{\partial D_0\psi_\ell}\,(t_\alpha)_{\ell m}\psi_m = -i\,\pi_\ell(t_\alpha)_{\ell m}\psi_m\,. \tag{15.4.9}$$

Hence Eq. (15.4.7) defines $A_\alpha{}^0$ at a given time as a functional of the canonical variables $\Pi_{\gamma i}$, $A_{\beta i}$, π_ℓ, and ψ_m at the same time.

Now that we have identified the canonical variables in this gauge, we can proceed to the construction of a Hamiltonian. The Hamiltonian density is

$$\begin{aligned}\mathscr{H} &= \Pi_{\alpha i}\partial_0 A_{\alpha i} + \pi_\ell\partial_0\psi_\ell - \mathscr{L}\\ &= \Pi_{\alpha i}\left(F_{\alpha 0i} + \partial_i A_{\alpha 0} - C_{\alpha\beta\gamma}A_{\beta 0}A_{\gamma i}\right) + \pi_\ell\partial_0\psi_\ell\\ &\quad - \tfrac{1}{2}\,F_{\alpha 0i}F_{\alpha 0i} + \tfrac{1}{2}\,F_{\alpha ij}F_{\alpha ij} + \tfrac{1}{2}\,F_{\alpha i3}F_{\alpha i3}\\ &\quad - \tfrac{1}{2}\,F_{\alpha 03}F_{\alpha 03} - \mathscr{L}_M\,.\end{aligned} \tag{15.4.10}$$

Using Eqs. (15.4.4) and (15.4.6), this is

$$\begin{aligned}\mathscr{H} = \mathscr{H}_M &+ \Pi_{\alpha i}(\partial_i A_{\alpha 0} - C_{\alpha\beta\gamma}A_{\beta 0}A_{\gamma i}) + \tfrac{1}{2}\,\Pi_{\alpha i}\Pi_{\alpha i}\\ &+ \tfrac{1}{2}\,F_{\alpha ij}F_{\alpha ij} + \tfrac{1}{2}\,\partial_3 A_{\alpha i}\partial_3 A_{\alpha i} - \tfrac{1}{2}\partial_3 A_{\alpha 0}\partial_3 A_{\alpha 0}\,,\end{aligned} \tag{15.4.11}$$

where $\mathscr{H}_M$ is the matter Hamiltonian density:

$$\mathscr{H}_M \equiv \pi_\ell\,\partial_0\psi_\ell - \mathscr{L}_M\,. \tag{15.4.12}$$

Following the general rules derived in Section 9.2, we can now use this Hamiltonian density to calculate matrix elements as path integrals over $A_{\alpha i}$, $\Pi_{\alpha i}$, ψ_ℓ, and π_ℓ, with weighting factor $\exp(iI)$, where

$$I = \int d^4x\Big[\Pi_{\alpha i}\partial_0 A_{\alpha i} + \pi_\ell\partial_0\psi_\ell - \mathscr{H} + \epsilon \text{ terms}\Big]\,, \tag{15.4.13}$$

in which the 'ϵ terms' serve only to supply the correct imaginary infinitesimal terms in propagator denominators. (See Section 9.2.) We note that Eqs. (15.4.7) and (15.4.9) give A^0_α as a functional of the canonical variables, *linear* in $\Pi_{\alpha i}$ and π_ℓ. Inspection of Eq. (15.4.11) shows then (assuming $\mathscr{L}_M$ to be no more than quadratic in $D_\mu\psi$) that the integrand of the complete action (15.4.13) is no more than quadratic in $\Pi_{\alpha i}$ and π_α. We could therefore carry out the path integral over these canonical 'momenta' by the usual rules of Gaussian integration. The trouble with this procedure is that the coefficients of the terms in Eq. (15.4.13) of second order in $\Pi_{\alpha i}$ are functions of the $A_{\alpha i}$, so the Gaussian integral would yield an awkward field-dependent determinant factor. Also, the whole formalism at this point looks hopelessly non-Lorentz-invariant.

Instead of proceeding in this way, we will apply a trick like that used in the path integral formulation of electrodynamics in Section 9.6. Note that if for a moment we think of $A_{\alpha 0}$ as an independent variable, then the action (15.4.13) is evidently quadratic in $A_{\alpha 0}$, with the coefficient of the second-order term $A_{\alpha 0}(x)A_{\beta 0}(y)$ equal to the field-independent kernel $(\partial_3)^2\delta^4(x-y)$. As we saw in the appendix to Chapter 9, the integral of such a Gaussian over $A_{\alpha 0}(x)$ is, up to a constant factor, equal to the value of the integrand at the stationary 'point' of the argument of the exponential. But the variational derivative of the action here is

$$\frac{\delta I}{\delta A_{\alpha 0}} = -\frac{\partial \mathscr{H}}{\partial A_{\alpha 0}} = J_\alpha{}^0 + \partial_i \Pi_{\alpha i} + C_{\beta\alpha\gamma}\Pi_{\beta i}A_{\gamma i} - \partial_3^2 A_{\alpha 0}\ ,$$

so the stationary 'point' of the action is the solution of the constraint equation (15.4.7). Hence, instead of using for $A_{\alpha 0}$ the solution of Eq. (15.4.7), we can just as well treat it as an independent variable of integration.

With $A_{\alpha 0}$ now regarded as an independent variable, the Hamiltonian $\int d^3x \mathscr{H}$ is evidently quadratic in $\Pi_{\alpha i}$, with the coefficient of the second-order term $\Pi_{\alpha i}(x)\Pi_{\beta j}(y)$ given by the field-independent kernel $\frac{1}{2}\delta^4(x-y)\delta_{ij}$. Assuming that the same is true for the matter variable π_ℓ, we can evaluate path integrals over π_ℓ and $\Pi_{\alpha i}$ up to a constant factor by simply setting π_ℓ and $\Pi_{\alpha i}$ at the stationary 'points' of the action corresponding to Eq. (15.4.1):

$$0 = \frac{\delta I}{\delta \pi_\ell} = \partial_0 \psi_\ell - \frac{\partial \mathscr{H}_M}{\partial \pi_\ell}\ ,$$

$$0 = \frac{\delta I}{\delta \Pi_{\alpha i}} = \partial_0 A_{\alpha i} - \Pi_{\alpha i} - \partial_i A_{\alpha 0} + C_{\alpha\beta\gamma}A_{\beta 0}A_{\gamma i} = F_{\alpha 0 i} - \Pi_{\alpha i}\ .$$

Inserting these back into Eq. (15.4.13) gives

$$\begin{aligned} I = \int d^4x \Big[&\mathscr{L}_M + \tfrac{1}{2}F_{\alpha 0 i}F_{\alpha 0 i} \\ &- \tfrac{1}{2}F_{\alpha ij}F_{\alpha ij} - \tfrac{1}{2}\partial_3 A_{\alpha i}\partial_3 A_{\alpha i} + \tfrac{1}{2}(\partial_3 A_{\alpha 0})^2\Big] \\ = \int d^4x\, &\mathscr{L}\ , \end{aligned} \tag{15.4.14}$$

where $\mathscr{L}$ is the Lagrangian (15.3.1) with which we started! In other words, *we are to do path integrals over $\psi_\ell(x)$ and all four components of $A_{\alpha\mu}(x)$, with a manifestly covariant weighting factor* $\exp(iI)$ *given by Eqs. (15.4.14) and (15.3.1),* but with the axial-gauge condition enforced by inserting a factor

$$\prod_{x,\alpha} \delta\Big(A_{\alpha 3}(x)\Big) \,. \tag{15.4.15}$$

As long as $\mathscr{O}_A, \mathscr{O}_B \cdots$ are gauge-invariant, we have

$$\langle T\{\mathscr{O}_A \mathscr{O}_B \cdots\}\rangle_{\text{VACUUM}} \;\propto\; \int \left[\prod_{\ell,x} d\psi_\ell(x)\right] \left[\prod_{\alpha,\mu,x} dA_{\alpha\mu}(x)\right]$$
$$\times\; \mathscr{O}_A \mathscr{O}_B \cdots \; \exp\{iI + \epsilon \text{ terms}\} \prod_{x,\alpha} \delta\left(A_{\alpha 3}(x)\right) \,, \tag{15.4.16}$$

with Lorentz- and gauge-invariant action I given by Eq. (15.4.14).

* * *

For future reference, we note that the volume element $\prod_{\alpha,\mu,x} dA_{\alpha\mu}(x)$ for the integration over gauge fields in (15.4.16) is gauge-invariant, in the sense that

$$\prod_{\alpha,\mu,x} dA_{\Lambda\,\alpha\mu}(x) = \prod_{\alpha,\mu,x} dA_{\alpha\mu}(x) \,, \tag{15.4.17}$$

where $A_{\Lambda\,\alpha\mu}(x)$ is the result of acting on $A_{\alpha\mu}(x)$ with a gauge transformation having transformation parameters $\Lambda_\alpha(x)$. It will be enough to show that this is true for transformations near the identity, say with infinitesimal transformation parameters $\lambda_\alpha(x)$. In this case,

$$A^\mu_{\lambda\,\alpha} = A^\mu_\alpha + \partial^\mu \lambda_\alpha + C_{\alpha\beta\gamma}\, A^\mu_\beta \lambda_\gamma \,,$$

so the volume elements are related by

$$\prod_{\alpha,\mu,x} dA_{\lambda\,\alpha\mu}(x) = \text{Det}(\mathscr{N}) \prod_{\alpha,\mu,x} dA_{\alpha\mu}(x) \,,$$

where $\mathscr{N}$ is the 'matrix':

$$\mathscr{N}_{\alpha\mu x,\beta\nu y} = \frac{\delta A_{\lambda\,\alpha\mu}(x)}{\delta A_{\beta\nu}(y)} = \delta^4(x-y)\,\delta^\nu_\mu \left[\delta_{\alpha\beta} + C_{\alpha\beta\gamma}\lambda_\gamma(x)\right] .$$

The determinant of $\mathscr{N}$ is unity to first order in λ_γ because the trace $C_{\alpha\alpha\gamma}$ vanishes.

In this chapter we shall assume that the volume element $\prod_{n,x} d\psi_n(x)$ for the integration over matter fields is also gauge-invariant. There are important subtleties here, to which we shall return in Chapter 22, but as shown there this assumption turns out to be valid in our present non-Abelian gauge theories of strong and electroweak interactions.

15.5 The De Witt–Faddeev–Popov Method

Our formula (15.4.16) for the path integral was derived in a gauge that is convenient for canonical quantization, but the Feynman rules that would be derived from this formula would hide the underlying rotational and Lorentz invariance of the theory. In order to derive manifestly Lorentz-invariant Feynman rules, we need to change the gauge.

We first note that Eq. (15.4.16) is (up to an unimportant constant factor) a special case of a general class of functional integrals, of the form:

$$\mathscr{I} = \int \left[\prod_{n,x} d\phi_n(x)\right] \mathscr{G}[\phi]\, B\Big[f[\phi]\Big]\, \mathrm{Det}\,\mathscr{F}[\phi]\,, \qquad (15.5.1)$$

where $\phi_n(x)$ are a set of gauge and matter fields; $\prod_{n,x} d\phi_n(x)$ is a volume element; and $\mathscr{G}[\phi]$ is a functional of the $\phi_n(x)$, satisfying the gauge-invariance condition:

$$\mathscr{G}[\phi_\lambda] \prod_{n,x} d\phi_{\lambda n}(x) = \mathscr{G}[\phi] \prod_{n,x} d\phi_n(x)\,, \qquad (15.5.2)$$

where $\phi_{\lambda n}(x)$ is the result of operating on ϕ with a gauge transformation having parameters $\lambda_\alpha(x)$. (Usually when this is satisfied both the functional $\mathscr{G}$ and the volume element are separately invariant, but Eq. (15.5.2) is all we need here.) Also, $f_\alpha[\phi;x]$ is a non-gauge-invariant 'gauge-fixing functional' of these fields that also depends on x and α; $B[f]$ is some numerical functional defined for general functions $f_\alpha(x)$ of x and α; and $\mathscr{F}$ is the 'matrix':

$$\mathscr{F}_{\alpha x,\beta y}[\phi] \equiv \left.\frac{\delta f_\alpha[\phi_\lambda;x]}{\delta\lambda_\beta(y)}\right|_{\lambda=0}. \qquad (15.5.3)$$

(In accordance with our usual notation for functionals of functions or of functionals, $B\Big[f[\phi]\Big]$ is understood to depend on the values taken by $f_\alpha[\phi;x]$ for all values of the undisplayed variables α and x, with the displayed variable, the function $\phi_n(x)$, held fixed.) Eq. (15.5.1) does not represent the widest possible generalization of Eq. (15.4.16); we will see in Section 15.7 that there is a further generalization that is needed for some purposes. We start here with Eq. (15.5.1) because it will help to motivate the formalism of Section 15.7, and it is adequate for dealing with non-Abelian gauge theories in the most convenient gauges.

We now must check that the path integral (15.4.16) is in fact a special case of Eq. (15.5.1). In Eq. (15.4.16) the fields $\phi_n(x)$ consist of both $A_{\alpha\mu}(x)$ and matter fields $\psi_\ell(x)$, and

$$f_\alpha[A,\psi;x] \;=\; A_{\alpha 3}(x)\,, \qquad (15.5.4)$$

$$B[f] \;=\; \prod_{x,\alpha} \delta\Big(f_\alpha(x)\Big)\,, \qquad (15.5.5)$$

$$\mathscr{G}[A,\psi] = \exp\{iI + \epsilon \text{ terms }\}\, \mathscr{O}_A \mathscr{O}_B \cdots\,, \tag{15.5.6}$$

$$\prod_{n,x} d\phi_n(x) = \left[\prod_{\ell,x} d\psi_\ell(x)\right] \left[\prod_{\alpha,\mu,x} dA^\mu_\alpha(x)\right]\,. \tag{15.5.7}$$

(We are now dropping the distinction between upper and lower indices $\alpha, \beta, \cdots$.) Comparison of Eq. (15.4.16) with Eqs. (15.5.1)–(15.5.3) shows that these path integrals are indeed the same, aside from the factor $\text{Det}\,\mathscr{F}[\phi]$. For the particular gauge-fixing functional (15.5.4), this factor is field-independent: if $A^3_\alpha(x) = 0$, then the change in $A^3_\alpha(x)$ under a gauge transformation with parameters $\lambda_\alpha(x)$ is

$$A^3_{\lambda\alpha}(x) = \partial_3\, \lambda_\alpha(x) = \int d^4y\; \lambda_\alpha(y)\, \partial_3 \delta^4(x-y)\,,$$

so that here Eq. (15.5.3) is the field-independent 'matrix'

$$\mathscr{F}_{\alpha x, \beta y}[\phi] = \delta_{\alpha\beta}\, \partial_3 \delta^4(x-y)\,.$$

The determinant in Eq. (15.5.1) is therefore also field-independent in this gauge. As discussed in Chapter 9, field-independent factors in the functional integral affect only the vacuum-fluctuation part of expectation values and S-matrix elements, and so are irrelevant to the calculation of the connected parts of the S-matrix.

The point of recognizing the functional integral (15.4.16) for non-Abelian gauge theories as a special case of the general path integral (15.5.1) is that in this form we may freely change the gauge. Specifically, we have a theorem, that *the integral (15.5.1) is actually independent (within broad limits) of the gauge-fixing functional $f_\alpha[\phi;x]$, and depends on the choice of the functional $B[f]$ only through an irrelevant constant factor.*

Proof: Replace the integration variable ϕ everywhere in Eq. (15.5.1) with a new integration variable ϕ_Λ, with $\Lambda^\alpha(x)$ any arbitrary (but fixed) set of gauge transformation parameters:

$$\mathscr{I} = \int \left[\prod_{n,x} d\phi_{\Lambda n}(x)\right] \mathscr{G}[\phi_\Lambda] B\Big[f[\phi_\Lambda]\Big] \text{Det}\,\mathscr{F}[\phi_\Lambda]\,. \tag{15.5.8}$$

(This step is a mathematical triviality, like changing an integral $\int_{-\infty}^{\infty} f(x)dx$ to read $\int_{-\infty}^{\infty} f(y)dy$, and does not yet make use of our assumptions regarding gauge invariance.) Now use the assumed gauge invariance (15.5.2) of the measure $\Pi d\phi$ times the functional $\mathscr{G}[\phi]$ to rewrite this as

$$\mathscr{I} = \int \left[\prod_{n,x} d\phi_n(x)\right]\; \mathscr{G}[\phi] B\Big[f[\phi_\Lambda]\Big]\; \text{Det}\,\mathscr{F}[\phi_\Lambda]\,. \tag{15.5.9}$$

Since $\Lambda^\alpha(x)$ was arbitrary, the left-hand side here cannot depend on it. Integrating over $\Lambda^\alpha(x)$ with some suitable weight-functional $\rho[\Lambda]$ (to be chosen below) thus gives

$$\mathscr{I}\int\left[\prod_{\alpha,x}d\Lambda^\alpha(x)\right]\rho[\Lambda]=\int\left[\prod_{n,x}d\phi_n(x)\right]\mathscr{G}[\phi]C[\phi]\,, \qquad (15.5.10)$$

where

$$C[\phi]\equiv\int\left[\prod_{\alpha,x}d\Lambda^\alpha(x)\right]\rho[\Lambda]\,B\Big[f[\phi_\Lambda]\Big]\text{Det}\,\mathscr{F}[\phi_\Lambda]\,. \qquad (15.5.11)$$

Now, Eq. (15.5.3) gives

$$\mathscr{F}_{\alpha x,\beta y}[\phi_\Lambda]=\left.\frac{\delta f_\alpha[(\phi_\Lambda)_\lambda;x]}{\delta\lambda^\beta(y)}\right|_{\lambda=0}. \qquad (15.5.12)$$

We are assuming that these transformations form a group; that is, we may write the result of performing the gauge transformation with parameters $\Lambda^\alpha(x)$ followed by the gauge transformation with parameters $\lambda^\alpha(x)$ as the action of a single 'product' gauge transformation with parameters $\tilde{\Lambda}^\alpha(x;\Lambda,\lambda)$,

$$(\phi_\Lambda)_\lambda=\phi_{\tilde{\Lambda}(\Lambda,\lambda)}\,. \qquad (15.5.13)$$

Using the chain rule of partial (functional) differentiation, we have then

$$\mathscr{F}_{\alpha x,\beta y}[\phi_\Lambda]=\int\mathscr{J}_{\alpha x,\gamma z}[\phi,\Lambda]\mathscr{R}^{\gamma z}{}_{\beta y}[\Lambda]d^4z\,, \qquad (15.5.14)$$

where

$$\mathscr{J}_{\alpha x,\gamma z}[\phi,\Lambda]\equiv\left.\frac{\delta f_\alpha[\phi_{\tilde{\Lambda}};x]}{\delta\tilde{\Lambda}^\gamma(z)}\right|_{\tilde{\Lambda}=\Lambda}=\frac{\delta f_\alpha[\phi_\Lambda;x]}{\delta\Lambda^\gamma(z)} \qquad (15.5.15)$$

and

$$\mathscr{R}^{\gamma z}{}_{\beta y}[\Lambda]=\left.\frac{\delta\tilde{\Lambda}^\gamma(z;\Lambda,\lambda)}{\delta\lambda^\beta(y)}\right|_{\lambda=0}. \qquad (15.5.16)$$

It follows that

$$\text{Det}\,\mathscr{F}[\phi_\Lambda]=\text{Det}\,\mathscr{J}[\phi,\Lambda]\;\;\text{Det}\,\mathscr{R}[\Lambda]\,. \qquad (15.5.17)$$

We note that Det $\mathscr{J}[\phi,\Lambda]$ is nothing but the Jacobian of the transformation of integration variables from the $\Lambda^\alpha(x)$ to (for a fixed ϕ) the $f_\alpha[\phi_\Lambda;x]$. Hence, if we choose the weight-function $\rho(\Lambda)$ as

$$\rho(\Lambda)=1\Big/\text{Det}\,\mathscr{R}[\Lambda] \qquad (15.5.18)$$

then

$$C[\phi]=\int\left[\prod_{\alpha,x}d\Lambda^\alpha(x)\right]\;\text{Det}\,\mathscr{J}[\phi,\Lambda]\,B\Big[f[\phi_\Lambda]\Big]$$

$$=\int\left[\prod_{\alpha,x}df_\alpha(x)\right]\;B\,[f]\equiv C\,, \qquad (15.5.19)$$

which is clearly independent of ϕ. (Eq. (15.5.18) may be recognized by the reader as giving the invariant (Haar) measure on the space of group parameters.) We have then at last

$$\mathscr{I} = \frac{C \int \left[\prod_{n,x} d\phi_n(x) \right] \mathscr{G}[\phi]}{\int \left[\prod_{\alpha,x} d\Lambda^\alpha(x) \right] \rho[\Lambda]} \,. \tag{15.5.20}$$

This is clearly independent of our choice of $f_\alpha[\phi; x]$, which has been reduced to a mere variable of integration, and it depends on $B[f]$ only through the constant C, as was to be proved.

Before proceeding with the applications of this theorem, we should pause to note a tricky point in the derivation. The integrals in the numerator and denominator of Eq. (15.5.20) are both ill-defined for the same reason. Since $\mathscr{G}[\phi]$ is assumed to be gauge-invariant, its integral over ϕ cannot possibly converge; the integrand is constant along all 'orbits,' obtained by sending ϕ into ϕ_λ with all possible $\lambda^\alpha(x)$. Likewise, the integrand in the denominator is divergent, because $\rho(\Lambda)\Pi d\Lambda$ is nothing but the usual invariant volume element for integrating over the group, and this is also constant along 'orbits' $\Lambda \to \tilde{\Lambda}(\Lambda, \lambda)$. This divergence can be eliminated in both the numerator and denominator of Eq. (15.5.20) by formulating the theory on a finite spacetime lattice, in which case the volume of the gauge group is just the volume of the global Lie group itself times the number of lattice sites. Because the gauge-fixing factor $B[f]$ eliminates this divergence in the original definition (15.5.1) of the left-hand side of Eq. (15.5.20), we may presume that, as the number of lattice sites goes to infinity, it cancels between the numerator and denominator of the right-hand side of Eq. (15.5.20).

Now to the point. We have seen that the vacuum expectation value (15.4.16) in axial gauge is given by a functional integral of the general form (15.5.1). Armed with the above theorem, we conclude then that

$$\langle T\{\mathcal{O}_A \mathcal{O}_B \cdots\}\rangle_V \propto \int \left[\prod_{\ell,x} d\psi_\ell(x) \right] \left[\prod_{\alpha,\mu,x} dA^\mu{}_\alpha(x) \right]$$
$$\times\, \mathcal{O}_A \mathcal{O}_B \cdots \exp\{iI + \epsilon \text{ terms}\}\, B\Big[f[A,\psi]\Big] \,\text{Det}\, \mathscr{F}[A,\psi] \tag{15.5.21}$$

for (almost) any choice of $f_\alpha[A, \psi; x]$ and $B[f]$. We are now therefore free to use Eq. (15.5.21) to derive the Feynman rules in a more convenient gauge.

The path integrals that we understand how to calculate are of Gaussians times polynomials, so we will generally take

$$B[f] = \exp\left(-\frac{i}{2\xi} \int d^4x \, f_\alpha(x) f_\alpha(x) \right) \tag{15.5.22}$$

with arbitrary real parameter ξ. With this choice, the effect of the factor B in Eq. (15.5.21) is just to add a term to the effective Lagrangian

$$\mathscr{L}_{\text{EFF}} = \mathscr{L} - \frac{1}{2\xi} f_\alpha f_\alpha \,. \tag{15.5.23}$$

The simplest Lorentz-invariant choice of the gauge-fixing function f_α is the same as in electrodynamics:

$$f_\alpha = \partial_\mu A_\alpha^\mu \,. \tag{15.5.24}$$

The bare gauge-field propagator can then be calculated just as in quantum electrodynamics. The free-vector-boson part of the effective action can be written

$$\begin{aligned} I_{0A} = -\int d^4x \Big[&\frac{1}{4}(\partial_\mu A_{\alpha\nu} - \partial_\nu A_{\alpha\mu})(\partial^\mu A_\alpha{}^\nu - \partial^\nu A_\alpha{}^\mu) \\ &+ \frac{1}{2\xi}(\partial_\mu A_\alpha{}^\mu)(\partial_\nu A_\alpha{}^\nu) \;+\; \epsilon \text{ terms} \Big] \\ = -\frac{1}{2}\int & d^4x \; \mathscr{D}_{\alpha\mu x,\beta\nu y} \, A_\alpha{}^\mu(x) A_\beta{}^\nu(y) \,, \end{aligned}$$

where

$$\begin{aligned} \mathscr{D}_{\alpha\mu x,\beta\nu y} &= \eta_{\mu\nu} \, \frac{\partial^2}{\partial x^\lambda \partial y_\lambda} \, \delta^4(x-y) \\ &\quad - \left(1 - \frac{1}{\xi}\right) \frac{\partial^2}{\partial x^\mu \partial y^\nu} \, \delta^4(x-y) \;+\; \epsilon \text{ terms} \\ &= (2\pi)^{-4} \int d^4p \left[\eta_{\mu\nu}(p^2 - i\epsilon) - \left(1 - \frac{1}{\xi}\right) p_\mu p_\nu \right] e^{ip\cdot(x-y)} \,. \end{aligned}$$

Taking the reciprocal of the matrix in square brackets, we find the propagator:

$$\begin{aligned} \Delta_{\alpha\mu,\beta\nu}(x,y) &= (\mathscr{D}^{-1})_{\alpha\mu x,\beta\nu y} \\ &= (2\pi)^{-4} \int d^4p \left[\eta_{\mu\nu} + (\xi - 1)\frac{p_\mu p_\nu}{p^2} \right] \frac{e^{ip\cdot(x-y)}}{p^2 - i\epsilon} \,. \end{aligned} \tag{15.5.25}$$

This is a generalization of both Landau and Feynman gauges, which are recovered by taking $\xi = 0$ and $\xi = 1$, respectively. For $\xi \to 0$, the functional (15.5.22) oscillates very rapidly except near $f_\alpha = 0$, so this functional acts like a delta-function imposing the Landau gauge condition $\partial_\mu A^\mu = 0$, leading naturally to a propagator satisfying the corresponding condition $\partial^\mu \Delta_{\alpha\mu,\beta\nu} = 0$. For non-zero values of ξ the functional $B[f]$ does not pick out gauge fields satisfying any specific gauge condition on the field $A_{\alpha\mu}$, but it is common to refer to the propagator (15.5.25) as being in a 'generalized Feynman gauge' or 'generalized ξ-gauge'. It is often a good strategy to calculate physical amplitudes with ξ left arbitrary, and then at the end of the calculation check that the results are ξ-independent.

With one qualification, the Feynman rules are now obvious: the contributions of vertices are to be read off from the interaction terms in the original Lagrangian $\mathscr{L}$, with gauge-field propagators given by Eq. (15.5.25), and matter-field propagators calculated as before. To be specific, the trilinear interaction term in $\mathscr{L}$

$$-\tfrac{1}{2}\, C_{\alpha\beta\gamma}(\partial_\mu A_{\alpha\nu} - \partial_\nu A_{\alpha\mu})A_\beta{}^\mu A_\gamma{}^\nu$$

corresponds to a vertex to which are attached three vector boson lines. If these lines carry (incoming) momenta p, q, k and Lorentz and gauge-field indices $\mu\alpha, \nu\beta, \rho\gamma$, then according to the momentum-space Feynman rules, the contribution of such a vertex to the integrand is

$$i(2\pi)^4\delta^4(p+q+k)\,[-i\, C_{\alpha\beta\gamma}]\left[p_\nu\eta_{\mu\lambda} - p_\lambda\eta_{\mu\nu} + q_\lambda\eta_{\nu\mu} - q_\mu\eta_{\nu\lambda} + k_\mu\eta_{\lambda\nu} - k_\nu\eta_{\lambda\mu}\right]. \tag{15.5.26}$$

Also, the A^4 interaction term in $\mathscr{L}$,

$$-\tfrac{1}{4}\, C_{\epsilon\alpha\beta}C_{\epsilon\gamma\delta}\, A_{\alpha\mu}A_{\beta\nu}A_\gamma{}^\mu A_\delta{}^\nu\,,$$

corresponds to a vertex to which are attached four vector boson lines. If these lines carry (incoming) momenta p, q, k, ℓ, and Lorentz and gauge indices $\mu\alpha$, $\nu\beta$, $\rho\gamma$, and $\sigma\delta$, then the contribution of such a vertex to the integrand is

$$i(2\pi)^4\delta^4(p+q+k+\ell)\;\times\;\Big[-C_{\epsilon\alpha\beta}\,C_{\epsilon\gamma\delta}(\eta_{\mu\rho}\eta_{\nu\sigma} - \eta_{\mu\sigma}\eta_{\nu\rho})$$
$$-C_{\epsilon\alpha\gamma}\,C_{\epsilon\delta\beta}(\eta_{\mu\sigma}\eta_{\rho\nu} - \eta_{\mu\nu}\eta_{\sigma\rho}) - C_{\epsilon\alpha\delta}\,C_{\epsilon\beta\gamma}(\eta_{\mu\nu}\eta_{\rho\sigma} - \eta_{\mu\rho}\eta_{\sigma\nu})\Big]\,. \tag{15.5.27}$$

(Recall that the structure constants $C_{\alpha\beta\gamma}$ contain coupling constant factors, so the factors (15.5.26) and (15.5.27) are respectively of first and second order in coupling constants.)

The one complication in the Feynman rules with which we have not yet dealt is the presence in Eq. (15.5.21) of the factor Det $\mathscr{F}$, which for general gauges is *not* a constant. We now turn to a consideration of this factor.

15.6 Ghosts

We now consider the effect of the factor Det $\mathscr{F}$ in Eq. (15.5.22) on the Feynman rules for a non-Abelian gauge theory. In order to be able to treat this effect as a modification of the Feynman rules, recall that as shown in

Section 9.5, the determinant of any matrix $\mathscr{F}_{\alpha x,\beta y}$ may be expressed as a path integral

$$\text{Det}\,\mathscr{F} \;\propto\; \int \left[\prod_{\alpha,x} d\omega^*_\alpha(x)\right]\left[\prod_{\alpha,x} d\omega_\alpha(x)\right]\,\exp(iI_{GH})\,, \qquad (15.6.1)$$

where

$$I_{GH} \equiv \int d^4x\,d^4y\;\omega^*_\alpha(x)\,\omega_\beta(y)\,\mathscr{F}_{\alpha x,\beta y}\,. \qquad (15.6.2)$$

Here ω^*_α and ω_α are a set of independent anticommuting classical variables, and the constant of proportionality is field-independent. (We have to choose the ω_α and ω^*_α field variables to be fermionic in order to reproduce the factor Det $\mathscr{F}$; had we chosen these field variables to be bosonic, the path integral (15.6.1) would have been proportional to (Det $\mathscr{F})^{-1}$.) The fields ω^*_α and ω_α are not necessarily related by complex conjugation; indeed, in Section 15.7 we shall see that for some purposes we need to assume that ω^*_α and ω_α are independent *real* variables. The whole effect of the factor Det $\mathscr{F}$ is the same as that of including $I_{GH}(\omega,\omega^*)$ in the full effective action, and integrating over 'fields' ω and ω^*. That is, for arbitrary gauge-fixing functionals $f_\alpha(x)$,

$$\langle T\{\mathscr{O}_A\cdots\}\rangle_V \;\propto\; \int \left[\prod_{n,x} d\psi_n(x)\right]\left[\prod_{\alpha,\mu,x} dA_{\alpha\mu}(x)\right]$$
$$\times\left[\prod_{\alpha,x} d\omega_\alpha(x)\,d\omega^*_\alpha(x)\right]\;\exp\Big(iI_{\text{MOD}}[\psi,A,\omega,\omega^*]\Big)\;\mathscr{O}_A\cdots\,, \qquad (15.6.3)$$

where I_{MOD} is a modified action

$$I_{\text{MOD}} = \int d^4x\,\left[\mathscr{L} - \frac{1}{2\xi}\,f_\alpha f_\alpha\right] + I_{GH}\,. \qquad (15.6.4)$$

The fields ω_α and ω^*_α are Lorentz scalars (at least in covariant gauges) but satisfy Fermi statistics. The connection between spin and statistics is not really violated here, because there are no particles described by these fields that can appear in initial or final states. For that reason, ω_α and ω^*_α are called the fields of 'ghost' and 'antighost' particles. Inspection of Eq. (15.6.2) shows that the action respects the conservation of a quantity known as 'ghost number,' equal to $+1$ for ω_α, -1 for ω^*_α, and zero for all other fields.

The Feynman rules for the ghosts are simplest in the case in which the 'matrix' $\mathscr{F}$ may be expressed as

$$\mathscr{F} = \mathscr{F}_0 + \mathscr{F}_1\,, \qquad (15.6.5)$$

where $\mathscr{F}_0$ is field-independent and of zeroth order in coupling constants, while $\mathscr{F}_1$ is field-dependent and proportional to one or more coupling constant factors. In this case, the ghost propagator is just

$$\Delta_{\alpha\beta}(x,y) = (\mathscr{F}_0^{-1})_{\alpha x,\beta y} \tag{15.6.6}$$

and the ghost vertices are to be read off from the interaction term

$$I'_{GH} = \int d^4x\, d^4y\; \omega^*_\alpha(x)\, \omega_\beta(y) (\mathscr{F}_1)_{\alpha x,\beta y}\ . \tag{15.6.7}$$

For instance, in the generalized ξ-gauge discussed in the previous section, we have

$$f_\alpha = \partial_\mu A^\mu_\alpha \tag{15.6.8}$$

and for infinitesimal gauge parameters λ_α, Eq. (15.1.9) gives:

$$A^\mu_{\alpha\lambda} = A^\mu_\alpha + \partial^\mu \lambda_\alpha + C_{\alpha\gamma\beta}\lambda_\beta\, A^\mu_\gamma$$

so that

$$\begin{aligned}\mathscr{F}_{\alpha x,\beta y} &= \left.\frac{\delta \partial_\mu A^\mu_{\alpha\lambda}(x)}{\delta\lambda_\beta(y)}\right|_{\lambda=0} \\ &= \Box\, \delta^4(x-y) + C_{\alpha\gamma\beta}\frac{\partial}{\partial x^\mu}\left[A^\mu_\gamma(x)\, \delta^4(x-y)\right]\ . \end{aligned} \tag{15.6.9}$$

This is of the form (15.6.5), with

$$(\mathscr{F}_0)_{\alpha x,\beta y} = \Box \delta^4(x-y)\, \delta_{\alpha\beta}\ , \tag{15.6.10}$$

$$(\mathscr{F}_1)_{\alpha x,\beta y} = -C_{\alpha\beta\gamma}\frac{\partial}{\partial x^\mu}\left[A^\mu_\gamma(x)\delta^4(x-y)\right]\ . \tag{15.6.11}$$

From Eqs. (15.6.6) and (15.6.10), we see that the ghost propagator is

$$\Delta_{\alpha\beta}(x,y) = \delta_{\alpha\beta}(2\pi)^{-4}\int d^4p\; (p^2 - i\epsilon)^{-1}\, e^{ip\cdot(x-y)}\ , \tag{15.6.12}$$

so in this gauge the ghosts behave like spinless fermions of zero mass, transforming according to the adjoint representation of the gauge group. Using Eqs. (15.6.7) and (15.6.11) and integrating by parts, we find that the ghost interaction term in the action is now

$$I'_{\rm GH} = \int d^4x\; C_{\alpha\beta\gamma}\,\frac{\partial\omega^*_\alpha}{\partial x^\mu}\, A^\mu_\gamma\, \omega_\beta\ . \tag{15.6.13}$$

This interaction corresponds to vertices to which are attached one outgoing ghost line, one incoming ghost line, and one vector boson line. If these lines carry (incoming) momenta p, q, k respectively and gauge group indices α, β, γ respectively, and the gauge field carries a vector index μ, then the contribution of such a vertex to the integrand is given by the momentum-space Feynman rules as

$$i(2\pi)^4\delta^4(p+q+k)\ \times\ ip_\mu C_{\alpha\beta\gamma}\ . \tag{15.6.14}$$

The ghosts propagate around loops, with single vector boson lines attached at each vertex along the loops, and with an extra minus sign supplied for each loop as is usual for fermionic field variables.

The extra minus sign for ghost loops suggests that each ghost field ω_α together with the associated antighost field ω^*_α represents something like a negative degree of freedom. These negative degrees of freedom are necessary because in using covariant gauge field propagators we are really over-counting; the physical degrees of freedom are the components of $A^\mu_\alpha(x)$, *less* the parameters $\Lambda_\alpha(x)$ needed to describe a gauge transformation.

In summary, the modified action (15.6.4) may be written in generalized ξ-gauge as

$$I_{\rm MOD} = \int d^4x \, \mathscr{L}_{\rm MOD} \qquad (15.6.15)$$

with a modified Lagrangian density:

$$\mathscr{L}_{\rm MOD} = \mathscr{L}_{\rm M} - \frac{1}{4} F^{\mu\nu}_\alpha F_{\alpha\mu\nu} - \frac{1}{2\xi} (\partial_\mu A^\mu_\alpha)(\partial_\nu A^\nu_\alpha)$$
$$- \partial_\mu \omega^*_\alpha \, \partial^\mu \omega_\alpha + C_{\alpha\beta\gamma} (\partial_\mu \omega^*_\alpha) A^\mu_\gamma \, \omega_\beta \, . \qquad (15.6.16)$$

It is important that this Lagrangian is *renormalizable* (if the matter Lagrangian $\mathscr{L}_{\rm M}$ is), in the elementary sense that its terms involve products of fields and their derivatives of total dimensionality (in powers of mass) four or less. (The kinematic term $-\partial_\mu \omega^*_\alpha \, \partial^\mu \omega_\alpha$ in Eq. (15.6.16) fixes the dimensionality of the fields ω and ω^* to be mass to the power unity, just like ordinary scalar and gauge fields.) However, there is more to renormalizability than power counting; it is necessary also that there be a counterterm to absorb every divergence. In the next section we shall consider a remarkable symmetry that will be used in Section 17.2 to show that non-Abelian gauge theories are indeed renormalizable in this sense, and that can even take the place of the Faddeev–Popov–De Witt approach that we have been following.

15.7 BRST Symmetry

Although the Faddeev–Popov–De Witt method described in the previous two sections makes the Lorentz invariance of the theory manifest, it still rests on a choice of gauge, and hence naturally it hides the underlying gauge invariance of the theory. This is a serious problem in trying to prove the renormalizability of the theory — gauge invariance restricts the form of the terms in the Lagrangian that are available as counterterms to absorb ultraviolet divergences, but once we choose a gauge, how do we know that gauge invariance still restricts the ways that the infinities can appear?

Remarkably, however, even after we choose a gauge, the path integral still does have a symmetry related to gauge invariance. This symmetry was discovered by Becchi, Rouet, and Stora,[10] (and independently by Tyutin,[11]) in 1975, several years after the work of Faddeev and Popov and De Witt, and is known in honor of its discoverers as BRST symmetry. This symmetry will be presented more-or-less as it was originally discovered, as a by-product of the method of Faddeev, Popov, and De Witt, but as we shall see it can also be regarded as a replacement for the Faddeev–Popov–De Witt approach.

We have seen in Eqs. (15.6.3) and (15.6.4) that the Feynman rules for a non-Abelian gauge theory may be obtained from a path integral over matter, gauge, and ghost fields, with a modified action, which we may write

$$I_{\rm MOD} = I_{\rm EFF} + I_{GH} = \int d^4x\, \mathscr{L}_{\rm MOD}\,, \tag{15.7.1}$$

$$\mathscr{L}_{\rm MOD} \equiv \mathscr{L} - \frac{1}{2\xi} f_\alpha f_\alpha + \omega^*_\alpha \Delta_\alpha\,, \tag{15.7.2}$$

where we have now introduced the quantity

$$\Delta_\alpha(x) \equiv \int d^4y\, \mathscr{F}_{\alpha x,\beta y}[A,\psi]\, \omega_\beta(y)\,. \tag{15.7.3}$$

This is for the choice

$$B[f] \propto \exp\left(-\frac{i}{2\xi}\int d^4x\, f_\alpha f_\alpha\right) \tag{15.7.4}$$

of the gauge-fixing functional in Eq. (15.5.21). For our present purposes, it will be helpful to rewrite $B[f]$ as a Fourier integral:

$$B[f] = \int \left[\prod_{\alpha,x} dh_\alpha(x)\right] \exp\left[\frac{i\xi}{2}\int h_\alpha h_\alpha\right] \exp\left[i\int d^4x f_\alpha h_\alpha\right]\,. \tag{15.7.5}$$

We must now do our path integrals over the field h_α (often known as a 'Nakanishi–Lautrup' field[11a]) as well as over matter, gauge, ghost and antighost fields, with a new modified action

$$I_{\rm NEW} = \int d^4x \left(\mathscr{L} + \omega^*_\alpha \Delta_\alpha + h_\alpha f_\alpha + \tfrac{1}{2}\xi h_\alpha h_\alpha\right)\,. \tag{15.7.6}$$

This modified action is not gauge-invariant — indeed, it had better not be, if we are to be able to use it in path integrals. However, it *is* invariant under a 'BRST' symmetry transformation, parameterized by an infinitesimal constant θ that *anticommutes* with ω_α, ω^*_α, and all fermionic matter fields. For a given θ, the BRST transformation is

$$\delta_\theta \psi = it_\alpha \theta \omega_\alpha \psi \,, \tag{15.7.7}$$

$$\delta_\theta A_{\alpha\mu} = \theta D_\mu \omega_\alpha = \theta[\partial_\mu \omega_\alpha + C_{\alpha\beta\gamma} A_{\beta\mu} \omega_\gamma] \,, \tag{15.7.8}$$

$$\delta_\theta \omega^*_\alpha = -\theta h_\alpha \,, \tag{15.7.9}$$

$$\delta_\theta \omega_\alpha = -\tfrac{1}{2}\theta\, C_{\alpha\beta\gamma} \omega_\beta \omega_\gamma \,, \tag{15.7.10}$$

$$\delta_\theta h_\alpha = 0 \,. \tag{15.7.11}$$

(Recall that in fermionic path integrals, there is no connection between ω_α and ω^*_α, so that Eq. (15.7.9) does not need to be the adjoint of Eq. (15.7.10).) Because h_α is BRST-invariant, we could if we like replace the Gaussian factor $\exp(\frac{1}{2}i\xi \int h_\alpha h_\alpha)$ in Eq. (15.7.5) with an arbitrary smooth functional of h_α, yielding an arbitrary functional $B[f]$, without affecting the BRST invariance of the action. However, for the purposes of diagrammatic calculation and renormalization it will help to leave $B[f]$ as a Gaussian.

In checking the invariance of the action (15.7.1), it will be very useful first to note that the transformation (15.7.7)–(15.7.11) is *nilpotent*; that is, if F is any functional of $\psi, A, \omega, \omega^*$, and h, and we define sF by

$$\delta_\theta F \equiv \theta sF \tag{15.7.12}$$

then*

$$\delta_\theta(sF) = 0 \tag{15.7.13}$$

or equivalently

$$s(sF) = 0 \,. \tag{15.7.14}$$

It is straightforward to verify this nilpotence when δ_θ acts on a single field. First, acting on a matter field,

$$\begin{aligned} \delta_\theta s\psi = it_\alpha\, \delta_\theta(\omega_\alpha \psi) &= -\tfrac{1}{2}i\, C_{\alpha\beta\gamma} t_\alpha \theta \omega_\beta \omega_\gamma \psi - t_\alpha t_\beta \omega_\alpha \theta \omega_\beta \psi \\ &= -\tfrac{1}{2}i\, C_{\alpha\beta\gamma} t_\alpha \theta \omega_\beta \omega_\gamma \psi + t_\alpha t_\beta \theta \omega_\alpha \omega_\beta \psi \,. \end{aligned}$$

The product $\omega_\alpha \omega_\beta$ in the second term on the right is antisymmetric in α and β, so we can replace $t_\alpha t_\beta$ in this term with $\frac{1}{2}[t_\alpha, t_\beta]$, and this term thus cancels the first term:

$$ss\psi = 0 \,. \tag{15.7.15}$$

* In the original work on BRST symmetry the functional $B[f]$ was left in the form (15.7.4), so that h_α was replaced in Eq. (15.7.9) with $-f_\alpha/\xi$, and the BRST transformation was only nilpotent when acting on functions of ω_α and the gauge and matter fields, but not of ω^*_α.

Next, acting on a gauge field, we have

$$\begin{aligned}
\delta_\theta s A_{\alpha\mu} &= \delta_\theta D_\mu \omega_\alpha \\
&= \partial_\mu \delta_\theta \omega_\alpha + C_{\alpha\beta\gamma}\delta_\theta A_{\beta\mu}\omega_\gamma + C_{\alpha\beta\gamma}A_{\beta\mu}\delta_\theta\omega_\gamma \\
&= \theta\Big(-\tfrac{1}{2}C_{\alpha\beta\gamma}\partial_\mu(\omega_\beta\omega_\gamma) + C_{\alpha\beta\gamma}(\partial_\mu\omega_\beta)\omega_\gamma \\
&\quad + C_{\alpha\beta\gamma}C_{\beta\delta\epsilon}A_{\delta\mu}\omega_\epsilon\omega_\gamma - \tfrac{1}{2}C_{\alpha\beta\gamma}C_{\gamma\delta\epsilon}A_{\beta\mu}\omega_\delta\omega_\epsilon\Big) \\
&= \theta\Big(\tfrac{1}{2}C_{\alpha\beta\gamma}(\partial_\mu\omega_\beta)\omega_\gamma + \tfrac{1}{2}C_{\alpha\beta\gamma}(\partial_\mu\omega_\gamma)\omega_\beta \\
&\quad - C_{\alpha\beta\gamma}C_{\gamma\delta\epsilon}A_{\delta\mu}\omega_\epsilon\omega_\beta - \tfrac{1}{2}C_{\alpha\beta\gamma}C_{\gamma\delta\epsilon}A_{\beta\mu}\omega_\delta\omega_\epsilon\Big)\,.
\end{aligned}$$

The first two terms of the final expression cancel because $C_{\alpha\beta\gamma}$ is antisymmetric in β and γ, and the third and fourth terms cancel because of the Jacobi identity (15.1.5), so

$$ssA_{\alpha\mu} = 0\,. \tag{15.7.16}$$

Eqs. (15.7.9) and (15.7.11) show immediately that

$$ss\omega^*_\alpha = 0 \tag{15.7.17}$$

and

$$ssh_\alpha = 0\,. \tag{15.7.18}$$

Finally,

$$\begin{aligned}
\delta_\theta s\omega_\alpha &= -\tfrac{1}{2}C_{\alpha\beta\gamma}\delta_\theta(\omega_\beta\omega_\gamma) \\
&= \tfrac{1}{4}\theta\left(C_{\alpha\beta\gamma}C_{\beta\delta\epsilon}\,\omega_\delta\omega_\epsilon\omega_\gamma + C_{\alpha\beta\gamma}C_{\gamma\delta\epsilon}\,\omega_\beta\omega_\delta\omega_\epsilon\right) \\
&= \tfrac{1}{4}\theta\, C_{\alpha\beta\gamma}C_{\gamma\delta\epsilon}\Big[-\omega_\delta\omega_\epsilon\omega_\beta + \omega_\beta\omega_\delta\omega_\epsilon\Big]\,.
\end{aligned}$$

But ω_β commutes with $\omega_\delta\omega_\epsilon$, so this too vanishes

$$ss\omega_\alpha = 0\,. \tag{15.7.19}$$

Now consider a product of two fields ϕ_1 and ϕ_2, either or both of which may be ψ, A, ω, ω^*, or h, not necessarily at the same point in spacetime. Then

$$\delta_\theta(\phi_1\phi_2) = \theta(s\phi_1)\phi_2 + \phi_1\theta(s\phi_2) = \theta\Big[(s\phi_1)\phi_2 \pm \phi_1 s\phi_2\Big]\,,$$

where the sign $\pm$ is plus if ϕ_1 is bosonic, minus if ϕ_1 is fermionic. That is,

$$s(\phi_1\phi_2) = (s\phi_1)\phi_2 \pm \phi_1 s\phi_2\,.$$

Since as we have seen $\delta_\theta(s\phi_1) = \delta_\theta(s\phi_2) = 0$, the effect of a BRST transformation on $s(\phi_1\phi_2)$ is

$$\delta_\theta s(\phi_1\phi_2) = (s\phi_1)\theta(s\phi_2) \pm \theta(s\phi_1)(s\phi_2)\,.$$

But $s\phi$ always has statistics opposite to ϕ, so moving θ to the left in the first term on the right-hand side introduces a sign factor $\mp$:

$$\delta_\theta s(\phi_1\phi_2) = \theta\Big[\mp (s\phi_1)(s\phi_2) \pm (s\phi_1)(s\phi_2)\Big] = 0\,.$$

Continuing in this way, we see that BRST transformations are nilpotent acting on any product of fields at arbitrary spacetime points:

$$\delta_\theta\, s(\phi_1\phi_2\phi_3\cdots) = 0\,.$$

Any functional $F[\phi]$ can be written as a sum of multiple integrals of such products with c-number coefficients, so likewise

$$\delta_\theta\, sF[\phi] = \theta ssF[\phi] = 0\,. \tag{15.7.20}$$

This completes the proof of the nilpotency of the BRST transformation.

Now let us return to the verification of the BRST invariance of the action (15.7.6). First note that for any functional of matter and gauge fields alone, the BRST transformation is just a gauge transformation with infinitesimal gauge parameter

$$\lambda_\alpha(x) = \theta\omega_\alpha(x)\,. \tag{15.7.21}$$

Therefore the first term in Eq. (15.7.6) is automatically BRST-invariant:

$$\delta_\theta \int d^4x\, \mathscr{L} = 0\,. \tag{15.7.22}$$

To calculate the effect of a BRST transformation on the rest of the action (15.7.6), note that its effect on the gauge-fixing function is just the gauge transformation (15.7.21), so

$$\begin{aligned}\delta_\theta f_\alpha[x;A,\psi] &= \int \frac{\delta f_\alpha[x;A_\lambda,\psi_\lambda]}{\delta\lambda^\beta(y)}\bigg|_{\lambda=0} \theta\,\omega_\beta(y)\,d^4y \\ &= \theta\int \mathscr{F}_{\alpha x,\beta y}[A,\psi]\,\omega_\beta(y)\,d^4y\end{aligned}$$

or in terms of the quantity (15.7.3)

$$\delta_\theta f_\alpha[x;A,\psi] = \theta\Delta_\alpha(x;A,\psi,\omega)\,. \tag{15.7.23}$$

(Note that $\mathscr{F}$ is a bosonic quantity, so there is no sign change in moving θ to the left here.) Also recall that $\delta_\theta\omega_\alpha = -\theta h_\alpha$ and $\delta_\theta h_\alpha = 0$. Therefore the terms in the integrand of the 'new' action (15.7.6) other than $\mathscr{L}$ may be written

$$\omega^*_\alpha\Delta_\alpha + h_\alpha f_\alpha + \tfrac{1}{2}\xi h_\alpha h_\alpha = s\Big(\omega^*_\alpha f_\alpha + \tfrac{1}{2}\xi\omega^*_\alpha h_\alpha\Big) \tag{15.7.24}$$

or in other words

$$I_{\rm NEW} = \int d^4x\, \mathscr{L} + s\Psi\,, \tag{15.7.25}$$

where

$$\Psi \equiv \int d^4x \left(\omega^*_\alpha f_\alpha + \tfrac{1}{2}\xi\omega^*_\alpha h_\alpha\right) . \tag{15.7.26}$$

The nilpotence of the BRST transformation tells us immediately that the term $s\Psi$ as well as $\int d^4x\, \mathscr{L}$ is BRST-invariant.

In a sense the converse of this result also applies: we shall see in Section 17.2 that a renormalizable Lagrangian that obeys BRST invariance and the other symmetries of the Lagrangian (15.7.25) must take the form of Eq. (15.7.25), aside from changes in the values of various constant coefficients. But this is not enough to establish the renormalizability of these theories. BRST symmetry transformations act non-linearly on the fields, and in this case there is no simple connection between the symmetries of the Lagrangian and the symmetries of matrix elements and Greens functions. Using the external field methods developed in the next chapter, it will be shown in Section 17.2 that the ultraviolet divergent terms in Feynman amplitudes (though not the finite parts) do obey a sort of renormalized BRST invariance, which allows the proof of renormalizability to be completed.

Eq. (15.7.25) shows that the physical content of any gauge theory is contained in the *kernel* of the BRST operator (that is, in a general BRST-invariant term $\int d^4x\, \mathscr{L} + s\Psi$), modulo terms in the *image* of the BRST transformation (that is, terms of the form $s\Psi$). The kernel modulo the image of any nilpotent transformation is said to form the *cohomology* of the transformation. There is another sense in which the physical content of a gauge theory may be identified with the cohomology of the BRST operator.[12] It is a fundamental physical requirement that matrix elements between physical states should be independent of our choice of the gauge-fixing function f_α, or in other words, of the functional Ψ in Eq. (15.7.25). The change in any matrix element $\langle\alpha|\beta\rangle$ due to a change $\tilde{\delta}\Psi$ in Ψ is

$$\tilde{\delta}\langle\alpha|\beta\rangle = i\langle\alpha|\tilde{\delta}I_{\rm NEW}|\beta\rangle = i\langle\alpha|s\tilde{\delta}\Psi|\beta\rangle . \tag{15.7.27}$$

(We use a tilde here to distinguish this arbitrary change in the gauge-fixing function from a BRST transformation or a gauge transformation.) We can introduce a fermionic BRST 'charge' Q, defined so that for any field operator Φ,

$$\delta_\theta\Phi = i[\theta\, Q, \Phi] = i\theta\, [Q, \Phi]_\mp ,$$

or in other words,

$$[Q, \Phi]_\mp = is\Phi , \tag{15.7.28}$$

the sign being $-$ or $+$ according as Φ is bosonic or fermionic. The nilpotence of the BRST transformation then gives

$$0 = -ss\Phi = [Q, [Q, \Phi]_\mp]_\pm = [Q^2, \Phi]_- .$$

For this to be satisfied for all operators Φ, it is necessary for Q^2 either to vanish or be proportional to the unit operator. But Q^2 cannot be proportional to the unit operator, since it has a non-vanishing ghost quantum number**, so it must vanish:

$$Q^2 = 0 . \tag{15.7.29}$$

From Eqs. (15.7.27) and (15.7.28), we have

$$\tilde{\delta}\langle\alpha|\beta\rangle = \langle\alpha|[Q, \tilde{\delta}\Psi]|\beta\rangle . \tag{15.7.30}$$

In order for this to vanish for all changes $\tilde{\delta}\Psi$ in Ψ, it is necessary that

$$\langle\alpha|Q = Q|\beta\rangle = 0 . \tag{15.7.31}$$

Thus physical states are in the kernel of the nilpotent operator Q. Two physical states that differ only by a state vector in the image of Q, that is, of form $Q|\cdots\rangle$, evidently have the same matrix element with all other physical states, and are therefore physically equivalent. Hence *independent* physical states correspond to states in the kernel of Q, modulo the image of Q — that is, they correspond to the cohomology of Q.

To see how this works in practice, let us consider the simple example of pure electrodynamics.† Taking the gauge-fixing function as $f = \partial_\mu A^\mu$ and integrating over the auxiliary field h, the BRST transformation (15.7.8)–(15.7.10) is here

$$s A_\mu = \partial_\mu \omega , \qquad s\,\omega^* = \partial_\mu A^\mu/\xi , \qquad s\,\omega = 0 . \tag{15.7.32}$$

We expand the fields in normal modes††

$$\begin{aligned} A^\mu(x) &= (2\pi)^{-3/2} \int \frac{d^3p}{\sqrt{2p^0}} \left[a^\mu(\mathbf{p})\, e^{ip\cdot x} + a^{\mu *}(\mathbf{p})\, e^{-ip\cdot x}\right] , \\ \omega(x) &= (2\pi)^{-3/2} \int \frac{d^3p}{\sqrt{2p^0}} \left[c(\mathbf{p})\, e^{ip\cdot x} + c^*(\mathbf{p})\, e^{-ip\cdot x}\right] , \\ \omega^*(x) &= (2\pi)^{-3/2} \int \frac{d^3p}{\sqrt{2p^0}} \left[b(\mathbf{p})\, e^{ip\cdot x} + b^*(\mathbf{p})\, e^{-ip\cdot x}\right] . \end{aligned} \tag{15.7.33}$$

** Recall that the ghost quantum number is defined as +1 for ω_α, −1 for ω^*_α, and 0 for all gauge and matter fields.

† Eqs. (15.6.11) and (15.6.7) show that because the structure constants vanish in electrodynamics, the ghosts here are not coupled to other fields. Nevertheless, electrodynamics provides a good example of the use of BRST symmetry in identifying physical states. Indeed, in analyzing the physicality conditions on 'in' and 'out' states we ignore interactions, so for this purpose a non-Abelian gauge theory is treated like several copies of quantum electrodynamics.

†† Just as $\omega^*(x)$ is not to be thought of as the Hermitian adjoint of $\omega(x)$, b^* and c^* are not the adjoints of c and b. But since $A^\mu(x)$ is Hermitian, $\omega(x)$ is Hermitian if Q is.

Matching coefficients of $e^{\pm ip\cdot x}$ on both sides of Eq. (15.7.28) yields

$$\begin{aligned} &[Q, a^\mu(\mathbf{p})]_- = -p^\mu c(\mathbf{p})\,, \qquad [Q, a^{\mu *}(\mathbf{p})]_- = p^\mu c^*(\mathbf{p})\,, \\ &[Q, b(\mathbf{p})]_+ = p^\mu a_\mu(\mathbf{p})/\xi\,, \qquad [Q, b^*(\mathbf{p})]_+ = p^\mu a^*_\mu(\mathbf{p})/\xi\,, \\ &[Q, c(\mathbf{p})]_+ = [Q, c^*(\mathbf{p})]_+ = 0\,. \end{aligned} \tag{15.7.34}$$

Consider any state $|\psi\rangle$ satisfying the physicality condition (15.7.31):

$$Q|\psi\rangle = 0\,. \tag{15.7.35}$$

The states $|e,\psi\rangle = e_\mu a^*(\mathbf{p})|\psi\rangle$ with one additional photon then satisfy the physicality condition $Q|e,\psi\rangle = 0$ if $e_\mu p^\mu = 0$. Also, the state $|\psi\rangle' \equiv b^*(\mathbf{p})|\psi\rangle$ satisfies

$$Q|\psi\rangle' = p^\mu a^*_\mu(\mathbf{p})|\psi\rangle/\xi\,, \tag{15.7.36}$$

so $|e+\alpha p,\psi\rangle = |e,\psi\rangle + \xi\alpha Q|\psi\rangle'$, and is therefore physically equivalent to $|e,\psi\rangle$. From this we conclude that e^μ is physically equivalent to $e^\mu + \alpha p^\mu$, which is the usual 'gauge-invariance' condition on photon polarization vectors. On the other hand,

$$Qb^*(\mathbf{p})|\psi\rangle = p^\mu a^*(\mathbf{p})|\psi\rangle \neq 0\,,$$

so $b^*|\psi\rangle$ does not satisfy the physicality condition (15.7.31) Also, for any e_μ with $e\cdot p \neq 0$,

$$c^*(\mathbf{p})|\psi\rangle = Qe_\mu a^{*\mu}(\mathbf{p})|\psi\rangle/e\cdot p$$

so $c^*|\psi\rangle$ is BRST-exact, and hence equivalent to zero. *Thus the physical Hilbert space is free of ghosts and antighosts.*

To maintain Lorentz invariance, we must interpret all four components of $a^\mu(\mathbf{p})$ as annihilation operators, in the sense that

$$0 = a_\mu(\mathbf{p})|0\rangle\,. \tag{15.7.37}$$

where $|0\rangle$ is the BRST-invariant vacuum state. But the canonical commutation relations derived from the BRST-invariant action (say, with $\xi = 1$) give

$$[a_\mu(\mathbf{p}), a^*_\nu(\mathbf{p}')]_- = \eta_{\mu\nu}\delta^3(\mathbf{p}-\mathbf{p}')\,, \tag{15.7.38}$$

corresponding to the propagator in Feynman gauge. This violates the usual positivity rules of quantum mechanics, because Eqs. (15.7.37) and (15.7.38) yield[13]

$$\langle 0|a_0(\mathbf{p})\,a^*_0(\mathbf{p}')|0\rangle = -\langle 0|0\rangle\,. \tag{15.7.39}$$

Nevertheless we can rest assured that all amplitudes among physical states satisfy the usual positivity conditions, because these states satisfy Eq. (15.7.31), and for such states the transition amplitudes are the same as they would be in a more physical gauge like Coulomb or axial gauge, where there is no problem of positivity or unitarity.

The Faddeev–Popov–De Witt formalism described so far necessarily yields an action that is bilinear in the ghost fields ω^*_α and ω_α. This is adequate for renormalizable Yang–Mills theories with the gauge-fixing function $f_\alpha = \partial_\mu A^\mu_\alpha$, but not in more general cases. For instance, as we shall see in Section 17.2, in other gauges renormalizable Yang–Mills theories need $\omega^*\omega^*\omega\omega$ terms in the action Lagrangian density to serve as counterterms for the ultraviolet divergences in loop graphs with four external ghost lines.

Fortunately the Faddeev–Popov–De Witt formalism represents only one way of generating a class of equivalent Lagrangians that yield the same unitary S-matrix. The BRST formalism provides a more general approach, that dispenses altogether with the Faddeev–Popov–De Witt formalism. In this approach, one takes the action to be the most general local functional of matter, gauge, ω^A, ω^{*A} and h^A fields with ghost number zero that is invariant under the BRST transformation (15.7.7)–(15.7.11) and under any other global symmetries of the theory. (For renormalizable theories one would also limit the Lagrangian density to operators of dimensionality four or less, but this restriction plays no role in the following discussion.) In the next section we shall prove, in a context more general than Yang–Mills theories, that the most general action of this sort is the sum of a functional of the matter and gauge fields (collectively called ϕ) alone, plus a term given by the action of the BRST operator s on an arbitrary functional Ψ of ghost number -1:

$$I_{\rm NEW}[\phi, \omega, \omega^*, h] = I_0[\phi] + s\,\Psi[\phi, \omega, \omega^*, h]\,, \tag{15.7.40}$$

as for instance in the Faddeev–Popov–De Witt action (15.7.25), but with $s\Psi$ now not necessarily bilinear in ghost and antighost fields.

By the same argument as before, the S-matrix elements for states that are annihilated by the BRST generator Q are independent of the choice of Ψ in Eq. (15.7.40), so if there is any choice of Ψ for which the ghosts decouple, then the ghosts decouple in general. In Yang–Mills theories, such a Ψ is provided by quantization of the theory in axial gauge, so in such theories ghosts decouple for arbitrary choices of the functional $\Psi[\phi, \omega, \omega^*, h]$, not just those choices like (15.7.25) that are generated by the Faddeev–Popov De Witt formalism.

We can go further, and free ourselves of all dependence on canonical quantization in Lorentz-non-invariant gauges like axial gauge. Again, take the action to be the most general functional of gauge, matter, ω^A, ω^{*A} and h^A fields with ghost number zero, that is invariant under the BRST transformation (15.7.7)–(15.7.11) and under any other global symmetries of the theory, including Lorentz invariance. From the BRST invariance of the action we can infer the existence of a conserved nilpotent BRST generator Q. With the ghost and antighost fields treated as Hermitian, Q is also Hermitian. The space of physical states is defined as above as consisting of states annihilated by Q, with two states treated as equivalent if their difference is Q acting on another state. It has been shown that for Yang–Mills theories this space is free of ghosts and antighosts and has a positive-definite norm, and that the S-matrix in this space is unitary.[13a]

This procedure is known as *BRST quantization.* It has been extended to theories with other local symmetries, such as general relativity and string theories. Unfortunately, it seems so far to be necessary to give separate proofs in each case that the BRST-cohomology is ghost-free and that the S-matrix acting in this space is unitary. The key point in these proofs is that, for each negative-norm degree of freedom, such as the time components of the gauge fields in Yang–Mills theories, there is one local symmetry that allows this degree of freedom to be transformed away.

* * *

Although we shall not use it here, there is a beautiful geometric interpretation[14] of the ghosts and the BRST symmetry that should be mentioned. The gauge fields A^μ_α may be written as one-forms $A_\alpha \equiv A_{\alpha\mu} dx^\mu$, where dx^μ are a set of anticommuting c-numbers. (See Section 5.8.) This can be combined with the ghost to compose a one-form $\mathscr{A}_\alpha \equiv A_\alpha + \omega_\alpha$ in an extended space. Also, the ordinary exterior derivative $d \equiv dx^\mu\, \partial/\partial x^\mu$ may be combined with the BRST operator s to form an exterior derivative $\mathscr{D} \equiv d + s$ in this space, which is nilpotent because $s^2 = d^2 = sd + ds = 0$.

The next chapter will introduce external field methods, which will be used along with the BRST symmetry in Chapter 17 to complete the proof of the renormalizability of non-Abelian gauge theories.

15.8 Generalizations of BRST Symmetry*

The BRST symmetry described in the previous section has a useful generalization to the quantization of a wide class of theories, including general relativity and string theories. In all these cases, we deal with an action $I[\phi]$ and measure $[d\phi] \equiv \prod_r d\phi^r$ that are invariant under a set of infinitesimal transformations

$$\phi^r \to \phi^r + \epsilon^A \delta_A \phi^r \,. \tag{15.8.1}$$

* This section lies somewhat out of the book's main line of development, and may be omitted in a first reading.