factor, which cancels. But if before integrating over all gauge transformations, we shift Λ so that $\triangle\Lambda$ changes to $\triangle\Lambda - \dot{A}^0$, then the exponential factor is $\exp[-i\frac{1}{2}\alpha\int(\dot{A}^0 - \triangle\Lambda)^2\, d^4x]$. Now when we integrate over $\Lambda(x)$, the delta function $\delta(\boldsymbol{\nabla}\cdot\boldsymbol{A} + \triangle\Lambda)$ replaces $\triangle\Lambda$ by $-\boldsymbol{\nabla}\cdot\boldsymbol{A}$ in the inserted exponential, converting it to $\exp[-i\frac{1}{2}\alpha\int(\dot{A}^0 + \boldsymbol{\nabla}\cdot\boldsymbol{A})^2\, d^4x]$. This term changes the gauge-invariant action (20.194) to the gauge-fixed action

$$S_\alpha = \int -\frac{1}{4}F_{ab}\,F^{ab} - \frac{\alpha}{2}(\partial_b A^b)^2 + A^b j_b + \mathcal{L}_m\; d^4x. \tag{20.199}$$

This Lorentz-invariant, gauge-fixed action is much easier to use than the Coulomb-gauge action (20.188) with the Coulomb potential (20.186). We can use it to compute scattering amplitudes perturbatively. The mean value of a time-ordered product of operators in the ground state $|0\rangle$ of the free theory is

$$\langle 0|\mathcal{T}\left[\mathcal{O}_1\ldots\mathcal{O}_n\right]|0\rangle = \frac{\displaystyle\int \mathcal{O}_1\ldots\mathcal{O}_n\, e^{iS_\alpha}\, DA\, D\psi}{\displaystyle\int e^{iS_\alpha}\, DA\, D\psi}. \tag{20.200}$$

By following steps analogous to those that led to (20.177), one may show (exercise 20.30) that in Feynman's gauge, $\alpha = 1$, the photon propagator is

$$\langle 0|\mathcal{T}\left[A_\mu(x)A_\nu(y)\right]|0\rangle = -i\triangle_{\mu\nu}(x-y) = -i\int \frac{\eta_{\mu\nu}}{q^2 - i\epsilon}\; e^{iq\cdot(x-y)}\,\frac{d^4q}{(2\pi)^4}. \tag{20.201}$$

20.12 Fermionic path integrals

In our brief introduction (1.11–1.12) and (1.44–1.46), to Grassmann variables, we learned that because $\theta^2 = 0$ the most general function $f(\theta)$ of a single Grassmann variable θ is $f(\theta) = a + b\,\theta$. So a complete integral table consists of the integral of this linear function

$$\int f(\theta)\, d\theta = \int a + b\,\theta\, d\theta = a\int d\theta + b\int \theta\, d\theta. \tag{20.202}$$

This equation has two unknowns, the integral $\int d\theta$ of unity and the integral $\int \theta\, d\theta$ of θ. We choose them so that the integral of $f(\theta + \zeta)$

$$\int f(\theta + \zeta)\, d\theta = \int a + b\,(\theta + \zeta)\, d\theta = (a + b\,\zeta)\int d\theta + b\int \theta\, d\theta \tag{20.203}$$

is the same as the integral (20.202) of $f(\theta)$. Thus the integral $\int d\theta$ of unity must vanish, while the integral $\int \theta \, d\theta$ of θ can be any constant, which we choose to be unity. Our complete table of integrals is then

$$\int d\theta = 0 \quad \text{and} \quad \int \theta \, d\theta = 1. \tag{20.204}$$

The anticommutation relations for a fermionic degree of freedom ψ are

$$\{\psi, \psi^\dagger\} \equiv \psi \, \psi^\dagger + \psi^\dagger \psi = 1 \quad \text{and} \quad \{\psi, \psi\} = \{\psi^\dagger, \psi^\dagger\} = 0. \tag{20.205}$$

Because ψ has $\psi^\dagger$, it is conventional to introduce a variable $\theta^* = \theta^\dagger$ that anti-commutes with itself and with θ

$$\{\theta^*, \theta^*\} = \{\theta^*, \theta\} = \{\theta, \theta\} = 0. \tag{20.206}$$

The logic that led to (20.204) now gives

$$\int d\theta^* = 0 \quad \text{and} \quad \int \theta^* \, d\theta^* = 1. \tag{20.207}$$

We define the reference state $|0\rangle$ as $|0\rangle \equiv \psi|s\rangle$ for a state $|s\rangle$ that is not annihilated by ψ. Since $\psi^2 = 0$, the operator ψ annihilates the state $|0\rangle$

$$\psi|0\rangle = \psi^2|s\rangle = 0. \tag{20.208}$$

The effect of the operator ψ on the state

$$|\theta\rangle = \exp\left(\psi^\dagger\theta - \tfrac{1}{2}\theta^*\theta\right)|0\rangle = \left(1 + \psi^\dagger\theta - \tfrac{1}{2}\theta^*\theta\right)|0\rangle \tag{20.209}$$

is

$$\psi|\theta\rangle = \psi(1 + \psi^\dagger\theta - \tfrac{1}{2}\theta^*\theta)|0\rangle = \psi\psi^\dagger\theta|0\rangle = (1 - \psi^\dagger\psi)\theta|0\rangle = \theta|0\rangle \tag{20.210}$$

while that of θ on $|\theta\rangle$ is

$$\theta|\theta\rangle = \theta(1 + \psi^\dagger\theta - \tfrac{1}{2}\theta^*\theta)|0\rangle = \theta|0\rangle. \tag{20.211}$$

The state $|\theta\rangle$ therefore is an eigenstate of ψ with eigenvalue θ

$$\psi|\theta\rangle = \theta|\theta\rangle. \tag{20.212}$$

The bra corresponding to the ket $|\zeta\rangle$ is

$$\langle\zeta| = \langle 0|\left(1 + \zeta^*\psi - \frac{1}{2}\zeta^*\zeta\right) \tag{20.213}$$

and the inner product $\langle\zeta|\theta\rangle$ is (exercise 20.31)

$$\begin{aligned}\langle\zeta|\theta\rangle &= \langle 0|\left(1+\zeta^*\psi-\frac{1}{2}\zeta^*\zeta\right)\left(1+\psi^\dagger\theta-\frac{1}{2}\theta^*\theta\right)|0\rangle \\ &= \langle 0|1+\zeta^*\psi\psi^\dagger\theta-\frac{1}{2}\zeta^*\zeta-\frac{1}{2}\theta^*\theta+\frac{1}{4}\zeta^*\zeta\theta^*\theta|0\rangle \\ &= \langle 0|1+\zeta^*\theta-\frac{1}{2}\zeta^*\zeta-\frac{1}{2}\theta^*\theta+\frac{1}{4}\zeta^*\zeta\theta^*\theta|0\rangle \\ &= \exp\left[\zeta^*\theta-\frac{1}{2}\left(\zeta^*\zeta+\theta^*\theta\right)\right]. \end{aligned} \tag{20.214}$$

Example 20.9 (A gaussian integral) For any number c, we can compute the integral of $\exp(c\,\theta^*\theta)$ by expanding the exponential

$$\int e^{c\,\theta^*\theta}\,d\theta^*d\theta = \int (1+c\,\theta^*\theta)\,d\theta^*d\theta = \int (1-c\,\theta\,\theta^*)\,d\theta^*d\theta = -c. \tag{20.215}$$

□

The identity operator for the space of states

$$c|0\rangle + d|1\rangle \equiv c|0\rangle + d\psi^\dagger|0\rangle \tag{20.216}$$

is (exercise 20.32) the integral

$$I = \int |\theta\rangle\langle\theta|\,d\theta^*d\theta = |0\rangle\langle 0| + |1\rangle\langle 1| \tag{20.217}$$

in which the differentials anti-commute with each other and with other fermionic variables: $\{d\theta, d\theta^*\} = 0$, $\{d\theta, \theta\} = 0$, $\{d\theta, \psi\} = 0$, and so forth.

The case of several Grassmann variables $\theta_1, \theta_2, \ldots, \theta_n$ and several Fermi operators $\psi_1, \psi_2, \ldots, \psi_n$ is similar. The θ_k anticommute among themselves and with the Fermi operators

$$\{\theta_i, \theta_j\} = \{\theta_i, \theta_j^*\} = \{\theta_i^*, \theta_j^*\} = 0 \quad \text{and} \quad \{\theta_i, \psi_k\} = \{\theta_i^*, \psi_k\} = 0 \tag{20.218}$$

while the ψ_k satisfy

$$\{\psi_k, \psi_\ell^\dagger\} = \delta_{k\ell} \quad \text{and} \quad \{\psi_k, \psi_l\} = \{\psi_k^\dagger, \psi_\ell^\dagger\} = 0. \tag{20.219}$$

The reference state $|0\rangle$ is

$$|0\rangle = \left(\prod_{k=1}^{n}\psi_k\right)|s\rangle \tag{20.220}$$

in which $|s\rangle$ is any state not annihilated by any ψ_k (so the resulting $|0\rangle$ isn't zero). The direct-product state

$$|\theta\rangle \equiv \exp\left(\sum_{k=1}^{n} \psi_k^\dagger \theta_k - \frac{1}{2}\theta_k^* \theta_k\right)|0\rangle = \left[\prod_{k=1}^{n}\left(1 + \psi_k^\dagger \theta_k - \frac{1}{2}\theta_k^* \theta_k\right)\right]|0\rangle \tag{20.221}$$

is (exercise 20.33) a simultaneous eigenstate $\psi_k|\theta\rangle = \theta_k|\theta\rangle$ of each ψ_k. It follows that

$$\psi_\ell \psi_k|\theta\rangle = \psi_\ell \theta_k|\theta\rangle = -\theta_k \psi_\ell|\theta\rangle = -\theta_k \theta_\ell|\theta\rangle = \theta_\ell \theta_k|\theta\rangle \tag{20.222}$$

and so too $\psi_k \psi_\ell|\theta\rangle = \theta_k \theta_\ell|\theta\rangle$. Since the ψ's anticommute, their eigenvalues must also

$$\theta_\ell \theta_k|\theta\rangle = \psi_\ell \psi_k|\theta\rangle = -\psi_k \psi_\ell|\theta\rangle = -\theta_k \theta_\ell|\theta\rangle. \tag{20.223}$$

The inner product $\langle\zeta|\theta\rangle$ is

$$\begin{aligned}\langle\zeta|\theta\rangle &= \langle 0|\left[\prod_{k=1}^{n}(1 + \zeta_k^* \psi_k - \frac{1}{2}\zeta_k^* \zeta_k)\right]\left[\prod_{\ell=1}^{n}(1 + \psi_\ell^\dagger \theta_\ell - \frac{1}{2}\theta_\ell^* \theta_\ell)\right]|0\rangle \\ &= \exp\left[\sum_{k=1}^{n} \zeta_k^* \theta_k - \frac{1}{2}\left(\zeta_k^* \zeta_k + \theta_k^* \theta_k\right)\right] = e^{\zeta^\dagger \theta - (\zeta^\dagger \zeta + \theta^\dagger \theta)/2}. \end{aligned} \tag{20.224}$$

The identity operator is

$$I = \int |\theta\rangle\langle\theta| \prod_{k=1}^{n} d\theta_k^* d\theta_k. \tag{20.225}$$

Example 20.10 (Gaussian Grassmann integral) For any 2×2 matrix A, we may compute the gaussian integral

$$g(A) = \int e^{-\theta^\dagger A \theta}\, d\theta_1^* d\theta_1 d\theta_2^* d\theta_2 \tag{20.226}$$

by expanding the exponential. The only terms that survive are the ones that have exactly one of each of the four variables θ_1, θ_2, θ_1^*, and θ_2^*. Thus the integral is the determinant of the matrix A

$$\begin{aligned} g(A) &= \int \frac{1}{2}\left(\theta_k^* A_{k\ell} \theta_\ell\right)^2\, d\theta_1^* d\theta_1 d\theta_2^* d\theta_2 \\ &= \int \left(\theta_1^* A_{11} \theta_1\, \theta_2^* A_{22} \theta_2 + \theta_1^* A_{12} \theta_2\, \theta_2^* A_{21} \theta_1\right)\, d\theta_1^* d\theta_1 d\theta_2^* d\theta_2 \\ &= A_{11} A_{22} - A_{12} A_{21} = \det A. \end{aligned} \tag{20.227}$$

The natural generalization to n dimensions is

$$\int e^{-\theta^\dagger A \theta} \prod_{k=1}^{n} d\theta_k^* d\theta_k = \det A \tag{20.228}$$

and is true for any $n \times n$ matrix A. If A is invertible, then the invariance of Grassmann integrals under translations implies that

$$\begin{aligned}
\int e^{-\theta^\dagger A\theta + \theta^\dagger \zeta + \zeta^\dagger \theta} \prod_{k=1}^{n} d\theta_k^* d\theta_k &= \int e^{-\theta^\dagger A(\theta + A^{-1}\zeta) + \theta^\dagger \zeta + \zeta^\dagger(\theta + A^{-1}\zeta)} \prod_{k=1}^{n} d\theta_k^* d\theta_k \\
&= \int e^{-\theta^\dagger A\theta + \zeta^\dagger \theta + \zeta^\dagger A^{-1}\zeta} \prod_{k=1}^{n} d\theta_k^* d\theta_k \\
&= \int e^{-(\theta^\dagger + \zeta^\dagger A^{-1})A\theta + \zeta^\dagger \theta + \zeta^\dagger A^{-1}\zeta} \prod_{k=1}^{n} d\theta_k^* d\theta_k \\
&= \int e^{-\theta^\dagger A\theta + \zeta^\dagger A^{-1}\zeta} \prod_{k=1}^{n} d\theta_k^* d\theta_k \\
&= \det A \; e^{\zeta^\dagger A^{-1}\zeta}.
\end{aligned} \tag{20.229}$$

The values of θ and $\theta^\dagger$ that make the argument $-\theta^\dagger A\theta + \theta^\dagger\zeta + \zeta^\dagger\theta$ of the exponential stationary are $\overline{\theta} = A^{-1}\zeta$ and $\overline{\theta^\dagger} = \zeta^\dagger A^{-1}$. So a gaussian Grassmann integral is equal to its exponential evaluated at its stationary point, apart from a prefactor involving the determinant $\det A$. Exercises (20.2 & 20.4) are about the bosonic versions (20.3 & 20.4) of this result. □

One may further extend these definitions to a Grassmann field $\chi_m(x)$ and an associated Dirac field $\psi_m(x)$. The $\chi_m(x)$'s anticommute among themselves and with all fermionic variables at all points of spacetime

$$\{\chi_m(x), \chi_n(x')\} = \{\chi_m^*(x), \chi_n(x')\} = \{\chi_m^*(x), \chi_n^*(x')\} = 0 \tag{20.230}$$

and the Dirac field $\psi_m(x)$ obeys the equal-time anticommutation relations

$$\begin{aligned}
\{\psi_m(\boldsymbol{x}, t), \psi_n^\dagger(\boldsymbol{x'}, t)\} &= \delta_{mn}\, \delta(\boldsymbol{x} - \boldsymbol{x'}) \quad (n, m = 1, \ldots, 4) \\
\{\psi_m(\boldsymbol{x}, t), \psi_n(\boldsymbol{x'}, t)\} &= \{\psi_m^\dagger(\boldsymbol{x}, t), \psi_n^\dagger(\boldsymbol{x'}, t)\} = 0.
\end{aligned} \tag{20.231}$$

As in (20.220), we use eigenstates of the field ψ at $t = 0$. If $|0\rangle$ is defined in terms of a state $|s\rangle$ that is not annihilated by any $\psi_m(\boldsymbol{x}, 0)$ as

$$|0\rangle = \left[\prod_{m,\boldsymbol{x}} \psi_m(\boldsymbol{x}, 0)\right] |s\rangle \tag{20.232}$$

then (exercise 20.34) the state

$$|\chi\rangle = \exp\left(\int \sum_m \psi^\dagger_m(\boldsymbol{x},0)\,\chi_m(\boldsymbol{x}) - \frac{1}{2}\chi^*_m(\boldsymbol{x})\chi_m(\boldsymbol{x})\,d^3x\right)|0\rangle$$

$$= \exp\left(\int \psi^\dagger\chi - \tfrac{1}{2}\chi^\dagger\chi\,d^3x\right)|0\rangle \tag{20.233}$$

is an eigenstate of the operator $\psi_m(\boldsymbol{x},0)$ with eigenvalue $\chi_m(\boldsymbol{x})$

$$\psi_m(\boldsymbol{x},0)|\chi\rangle = \chi_m(\boldsymbol{x})|\chi\rangle. \tag{20.234}$$

The inner product of two such states is (exercise 20.35)

$$\langle\chi'|\chi\rangle = \exp\left[\int \chi'^\dagger\chi - \frac{1}{2}\chi'^\dagger\chi' - \frac{1}{2}\chi^\dagger\chi\;d^3x\right]. \tag{20.235}$$

The identity operator is the integral

$$I = \int |\chi\rangle\langle\chi|\,D\chi^* D\chi \tag{20.236}$$

in which

$$D\chi^* D\chi \equiv \prod_{m,\boldsymbol{x}} d\chi^*_m(\boldsymbol{x})d\chi_m(\boldsymbol{x}). \tag{20.237}$$

The hamiltonian for a free Dirac field ψ of mass m is the spatial integral

$$H_0 = \int \overline{\psi}\,(\boldsymbol{\gamma}\cdot\boldsymbol{\nabla} + m)\,\psi\;d^3x \tag{20.238}$$

in which $\overline{\psi} \equiv i\psi^\dagger\gamma^0$ and the gamma matrices (11.327) satisfy

$$\{\gamma^a,\gamma^b\} = 2\,\eta^{ab} \tag{20.239}$$

where η is the 4×4 diagonal matrix with entries $(-1,1,1,1)$. Since $\psi|\chi\rangle = \chi|\chi\rangle$ and $\langle\chi'|\psi^\dagger = \langle\chi'|\chi'^\dagger$, the quantity $\langle\chi'|\exp(-i\epsilon H_0)|\chi\rangle$ is by (20.235)

$$\langle\chi'|e^{-i\epsilon H_0}|\chi\rangle = \langle\chi'|\chi\rangle\exp\left[-i\epsilon\int\overline{\chi}'\,(\boldsymbol{\gamma}\cdot\boldsymbol{\nabla}+m)\,\chi\;d^3x\right] \tag{20.240}$$

$$= \exp\left[\int\tfrac{1}{2}(\chi'^\dagger - \chi^\dagger)\chi - \tfrac{1}{2}\chi'^\dagger(\chi'-\chi) - i\epsilon\overline{\chi}'(\boldsymbol{\gamma}\cdot\boldsymbol{\nabla}+m)\chi d^3x\right]$$

$$= \exp\left\{\epsilon\int\left[\tfrac{1}{2}\dot{\chi}^\dagger\chi - \tfrac{1}{2}\chi'^\dagger\dot{\chi} - i\overline{\chi}'\,(\boldsymbol{\gamma}\cdot\boldsymbol{\nabla}+m)\,\chi\right]\;d^3x\right\}$$

in which $\chi'^\dagger - \chi^\dagger = \epsilon\dot{\chi}^\dagger$ and $\chi' - \chi = \epsilon\dot{\chi}$. Everything within the square brackets is multiplied by ϵ, so we may replace $\chi'^\dagger$ by $\chi^\dagger$ and $\overline{\chi}'$ by $\overline{\chi}$ so as to write to first order in ϵ

$$\langle\chi'|e^{-i\epsilon H_0}|\chi\rangle = \exp\left[\epsilon\int \tfrac{1}{2}\dot{\chi}^\dagger\chi - \tfrac{1}{2}\chi^\dagger\dot{\chi} - i\overline{\chi}\,(\boldsymbol{\gamma}\cdot\boldsymbol{\nabla} + m)\,\chi\; d^3x\right] \tag{20.241}$$

in which the dependence upon χ' is through the time derivatives.

Putting together $n = 2t/\epsilon$ such matrix elements, integrating over all intermediate-state dyadics $|\chi\rangle\langle\chi|$, and using our formula (20.236), we find

$$\langle\chi_t|e^{-2itH_0}|\chi_{-t}\rangle = \int\exp\left[\int \tfrac{1}{2}\dot{\chi}^\dagger\chi - \tfrac{1}{2}\chi^\dagger\dot{\chi} - i\overline{\chi}\,(\boldsymbol{\gamma}\cdot\boldsymbol{\nabla} + m)\,\chi d^4x\right] D\chi^* D\chi. \tag{20.242}$$

Integrating $\dot{\chi}^\dagger\chi$ by parts and dropping the surface term, we get

$$\langle\chi_t|e^{-2itH_0}|\chi_{-t}\rangle = \int\exp\left[\int - \chi^\dagger\dot{\chi} - i\overline{\chi}\,(\boldsymbol{\gamma}\cdot\boldsymbol{\nabla} + m)\,\chi\, d^4x\right] D\chi^* D\chi. \tag{20.243}$$

Since $-\chi^\dagger\dot{\chi} = -i\overline{\chi}\gamma^0\dot{\chi}$, the argument of the exponential is

$$i\int - \overline{\chi}\gamma^0\dot{\chi} - \overline{\chi}\,(\boldsymbol{\gamma}\cdot\boldsymbol{\nabla} + m)\,\chi\; d^4x = i\int - \overline{\chi}\,(\gamma^\mu\partial_\mu + m)\,\chi\; d^4x. \tag{20.244}$$

We then have

$$\langle\chi_t|e^{-2itH_0}|\chi_{-t}\rangle = \int\exp\left(i\int \mathcal{L}_0(\chi)\, d^4x\right) D\chi^* D\chi \tag{20.245}$$

in which $\mathcal{L}_0(\chi) = -\overline{\chi}\,(\gamma^\mu\partial_\mu + m)\,\chi$ is the action density (11.329) for a free Dirac field. Thus the amplitude is a path integral with phases given by the classical action $S_0[\chi]$

$$\langle\chi_t|e^{-2itH_0}|\chi_{-t}\rangle = \int e^{i\int\mathcal{L}_0(\chi)\, d^4x} D\chi^* D\chi = \int e^{iS_0[\chi]} D\chi^* D\chi \tag{20.246}$$

and the integral is over all fields that go from $\chi(\boldsymbol{x}, -t) = \chi_{-t}(\boldsymbol{x})$ to $\chi(\boldsymbol{x}, t) = \chi_t(\boldsymbol{x})$. Any normalization factor will cancel in ratios of such integrals.

Since Fermi fields anticommute, their time-ordered product has an extra minus sign

$$\mathcal{T}\left[\overline{\psi}(x_1)\psi(x_2)\right] = \theta(x_1^0 - x_2^0)\,\overline{\psi}(x_1)\,\psi(x_2) - \theta(x_2^0 - x_1^0)\,\psi(x_2)\,\overline{\psi}(x_1). \tag{20.247}$$

The logic behind our formulas (20.140) and (20.158) for the time-ordered product of bosonic fields now leads to an expression for the time-ordered product of $2n$ Dirac fields (with $D\chi''$ and $D\chi'$ and so forth suppressed)

$$\langle 0|\mathcal{T}\left[\overline{\psi}(x_1)\dots\psi(x_{2n})\right]|0\rangle = \frac{\displaystyle\int \langle 0|\chi''\rangle\, \overline{\chi}(x_1)\dots\chi(x_{2n})\, e^{iS_0[\chi]}\langle\chi'|0\rangle\, D\chi^* D\chi}{\displaystyle\int \langle 0|\chi''\rangle\, e^{iS_0[\chi]}\langle\chi'|0\rangle\, D\chi^* D\chi}. \tag{20.248}$$

As in (20.169), the effect of the inner products $\langle 0|\chi''\rangle$ and $\langle\chi'|0\rangle$ is to insert ϵ-terms which modify the Dirac propagators

$$\langle 0|\mathcal{T}\left[\overline{\psi}(x_1)\dots\psi(x_{2n})\right]|0\rangle = \frac{\displaystyle\int \overline{\chi}(x_1)\dots\chi(x_{2n})\, e^{iS_0[\chi,\epsilon]}\, D\chi^* D\chi}{\displaystyle\int e^{iS_0[\chi,\epsilon]}\, D\chi^* D\chi}. \tag{20.249}$$

Imitating (20.170), we introduce a Grassmann external current $\zeta(x)$ and define a fermionic analog of $Z_0[j]$

$$Z_0[\zeta] \equiv \langle 0|\,\mathcal{T}\left[e^{\int \overline{\zeta}\psi+\overline{\psi}\zeta\, d^4x}\right]|0\rangle = \frac{\displaystyle\int e^{\int\overline{\zeta}\chi+\overline{\chi}\zeta\, d^4x} e^{iS_0[\chi,\epsilon]} D\chi^* D\chi}{\displaystyle\int e^{iS_0[\chi,\epsilon]} D\chi^* D\chi}. \tag{20.250}$$

Example 20.11 (Feynman's fermion propagator) Since

$$\begin{aligned} i\left(\gamma^\mu\partial_\mu + m\right)\Delta(x-y) &\equiv i\left(\gamma^\mu\partial_\mu + m\right)\int \frac{d^4p}{(2\pi)^4} e^{ip(x-y)}\, \frac{-i\left(-i\gamma^\nu p_\nu + m\right)}{p^2+m^2-i\epsilon} \\ &= \int \frac{d^4p}{(2\pi)^4} e^{ip(x-y)}\left(i\gamma^\mu p_\mu + m\right)\frac{\left(-i\gamma^\nu p_\nu + m\right)}{p^2+m^2-i\epsilon} \\ &= \int \frac{d^4p}{(2\pi)^4} e^{ip(x-y)}\, \frac{p^2+m^2}{p^2+m^2-i\epsilon} = \delta^4(x-y), \end{aligned} \tag{20.251}$$

the function $\Delta(x-y)$ is the inverse of the differential operator $i(\gamma^\mu\partial_\mu + m)$. Thus the Grassmann identity (20.229) implies that $Z_0[\zeta]$ is

$$\begin{aligned} \langle 0|\,\mathcal{T}\left[e^{\int \overline{\zeta}\psi+\overline{\psi}\zeta\, d^4x}\right]|0\rangle &= \frac{\displaystyle\int e^{\int [\overline{\zeta}\chi+\overline{\chi}\zeta - i\,\overline{\chi}(\gamma^\mu\partial_\mu+m)\chi] d^4x} D\chi^* D\chi}{\displaystyle\int e^{iS_0[\chi,\epsilon]} D\chi^* D\chi} \\ &= \exp\left[\int \overline{\zeta}(x)\Delta(x-y)\zeta(y)\, d^4x d^4y\right]. \end{aligned} \tag{20.252}$$

Differentiating we get

$$\langle 0|\mathcal{T}\left[\psi(x)\overline{\psi}(y)\right]|0\rangle = \Delta(x-y) = -i\int \frac{d^4p}{(2\pi)^4} e^{ip(x-y)} \frac{-i\gamma^\nu p_\nu + m}{p^2+m^2-i\epsilon}. \quad (20.253)$$

□

20.13 Application to nonabelian gauge theories

The action of a generic nonabelian gauge theory is

$$S = \int -\tfrac{1}{4} F_{a\mu\nu} F_a^{\mu\nu} - \overline{\psi}\left(\gamma^\mu D_\mu + m\right)\psi \; d^4x \quad (20.254)$$

in which the Maxwell field is

$$F_{a\mu\nu} \equiv \partial_\mu A_{a\nu} - \partial_\nu A_{a\mu} + g\, f_{abc}\, A_{b\mu}\, A_{c\nu} \quad (20.255)$$

and the covariant derivative is

$$D_\mu \psi \equiv \partial_\mu \psi - ig\, t_a\, A_{a\mu}\, \psi. \quad (20.256)$$

Here g is a coupling constant, f_{abc} is a structure constant (11.68), and t_a is a generator (11.57) of the Lie algebra (section 11.16) of the gauge group.

One may show (Weinberg, 1996, pp. 14–18) that the analog of equation (20.195) for quantum electrodynamics is

$$\langle \Omega|\mathcal{T}\left[\mathcal{O}_1 \dots \mathcal{O}_n\right]|\Omega\rangle = \frac{\displaystyle\int \mathcal{O}_1 \dots \mathcal{O}_n\, e^{iS}\, \delta[A_{a3}]\, DA\, D\psi}{\displaystyle\int e^{iS}\, \delta[A_{a3}]\, DA\, D\psi} \quad (20.257)$$

in which the functional delta function

$$\delta[A_{a3}] \equiv \prod_{x,b} \delta(A_{a3}(x)) \quad (20.258)$$

enforces the axial-gauge condition, and $D\psi$ stands for $D\psi^* D\psi$.

Initially, physicists had trouble computing nonabelian amplitudes beyond the lowest order of perturbation theory. Then DeWitt showed how to compute to second order (DeWitt, 1967), and Faddeev and Popov, using path integrals, showed how to compute to all orders (Faddeev and Popov, 1967).