a nonzero temperature

$$T = \frac{\hbar \alpha}{2\pi c k_B}. \tag{9.120}$$

This result (Davies, 1975) is equivalent to the finding (Hawking, 1974) that a gravitational field of local acceleration g makes empty space radiate at a temperature

$$T = \frac{\hbar g}{2\pi c k_B}. \tag{9.121}$$

Black holes are not black.

9.6 Maximally symmetric spaces

The spheres S^2 and S^3 and the hyperboloids H^2 and H^3 are maximally symmetric spaces. A transformation $x \to x'$ is an **isometry** if $g'_{ik}(x') = g_{ik}(x')$ in which case the distances $g_{ik}(x) dx^i dx^k = g'_{ik}(x') dx'^i dx'^k = g_{ik}(x') dx'^i dx'^k$ are the same. To see what this symmetry condition means, we consider the infinitesimal transformation $x'^\ell = x^\ell + \epsilon y^\ell(x)$ under which to lowest order $g_{ik}(x') = g_{ik}(x) + g_{ik,\ell} \epsilon y^\ell$ and $dx'^i = dx^i + \epsilon y^i_{,j} dx^j$. The symmetry condition requires

$$g_{ik}(x) dx^i dx^k = (g_{ik}(x) + g_{ik,\ell} \epsilon y^\ell)(dx^i + \epsilon y^i_{,j} dx^j)(dx^k + \epsilon y^k_{,m} dx^m) \tag{9.122}$$

or

$$0 = g_{ik,\ell}\, y^\ell + g_{im}\, y^m_{,k} + g_{jk}\, y^j_{,i}. \tag{9.123}$$

The vector field $y^i(x)$ must satisfy this condition if $x'^i = x^i + \epsilon y^i(x)$ is to be a symmetry of the metric $g_{ik}(x)$. Since the covariant derivative of the metric tensor vanishes, $g_{ik;\ell} = 0$, we may write the condition on the symmetry vector $y^\ell(x)$ as

$$0 = y_{i;k} + y_{k;i}. \tag{9.124}$$

In more detail, we subtract $g_{ik;\ell}y^\ell$, which vanishes, from the condition (9.123) that y be a Killing vector:

$$\begin{aligned} 0 &= g_{ik,\ell}y^\ell + g_{im}y^m_{,k} + g_{jk}y^j_{,i} - g_{ik;\ell}y^\ell \\ &= g_{ik,\ell}y^\ell + g_{im}y^m_{,k} + g_{jk}y^j_{,i} - g_{ik,\ell}y^\ell + g_{im}\Gamma^m{}_{\ell k}y^\ell + g_{kj}\Gamma^j{}_{\ell i}y^\ell \\ &= g_{im}y^m_{,k} + g_{jk}y^j_{,i} + g_{im}\Gamma^m{}_{\ell k}y^\ell + g_{kj}\Gamma^j{}_{\ell i}y^\ell \\ &= g_{im}(y^m_{,k} + \Gamma^m{}_{\ell k}y^\ell) + g_{jk}(y^j_{,i} + \Gamma^j{}_{\ell i}y^\ell) \\ &= g_{im}y^m_{;k} + g_{jk}y^j_{;i} = y_{i;k} + y_{k;i}. \end{aligned} \tag{9.125}$$

Students leery of the last step may use the vanishing of the covariant derivative of the metric tensor and the derivation property

$$(AB)_{;k} = A_{;k}B + A\,B_{;k} \tag{9.126}$$

of covariant derivatives, to show that

$$0 = (g_{im}y^m)_{;k} + (g_{jk}y^j)_{;i} = y_{i;k} + y_{k;i}. \tag{9.127}$$

The symmetry vector y^ℓ is a **Killing** vector (Wilhelm Killing, 1847–1923). We may use symmetry condition (9.123) or (9.124) either to find the symmetries of a space with a known metric or to find a metric with given symmetries.

Example 9.1 (Killing vectors of the sphere S^2) The first Killing vector is $(y_1^\theta, y_1^\phi) = (0, 1)$. Since the components of y_1 are constants, the symmetry condition (9.123) says $g_{ik,\phi} = 0$ which tells us that the metric is independent of ϕ. The other two Killing vectors are $(y_2^\theta, y_2^\phi) = (\sin\phi, \cot\theta\cos\phi)$ and $(y_3^\theta, y_3^\phi) = (\cos\phi, -\cot\theta\sin\phi)$. The symmetry condition (9.123) for $i = k = \theta$ and Killing vectors y_2 and y_3 tell us that $g_{\theta\phi} = 0$ and that $g_{\theta\theta,\theta} = 0$. So $g_{\theta\theta}$ is a constant, which we set equal to unity. Finally, the symmetry condition (9.123) for $i = k = \phi$ and the Killing vectors y_2 and y_3 tell us that $g_{\phi\phi,\theta} = 2\cot\theta g_{\phi\phi}$ which we integrate to $g_{\phi\phi} = \sin^2\theta$. The 2-dimensional space with Killing vectors y_1, y_2, y_3 therefore has the metric

$$g_{ik} = \begin{pmatrix} g_{\theta\theta} & g_{\theta\phi} \\ g_{\phi\theta} & g_{\phi\phi} \end{pmatrix} = \begin{pmatrix} \boldsymbol{e}_\theta\cdot\boldsymbol{e}_\theta & \boldsymbol{e}_\theta\cdot\boldsymbol{e}_\phi \\ \boldsymbol{e}_\phi\cdot\boldsymbol{e}_\theta & \boldsymbol{e}_\phi\cdot\boldsymbol{e}_\phi \end{pmatrix} = \begin{pmatrix} R^2 & 0 \\ 0 & R^2\sin^2\theta \end{pmatrix} \tag{9.128}$$

of the sphere S^2. □

Example 9.2 (Killing vectors of the hyperboloid H^2) The metric

$$(g_{ij}) = R^2 \begin{pmatrix} 1 & 0 \\ 0 & \sinh^2\theta \end{pmatrix}. \tag{9.129}$$

of the hyperboloid H^2 is diagonal with $g_{\theta\theta} = R^2$ and $g_{\phi\phi} = R^2 \sinh^2\theta$. The Killing vector $(y_1^\theta, y_1^\phi) = (0, 1)$ satisfies the symmetry condition (9.123). Since $g_{\theta\theta}$ is independent of θ and ϕ, the $\theta\theta$ component of (9.123) implies that $y^\theta_{,\theta} = 0$. Since $g_{\phi\phi} = R^2 \sinh^2\theta$, the $\phi\phi$ component of (9.123) says that $y^\phi_{,\phi} = -\coth\theta \, y^\theta$. The $\theta\phi$ and $\phi\theta$ components of (9.123) give $y^\theta_{,\phi} = -\sinh^2\theta \, y^\phi_{,\theta}$. The vectors $y_2 = (y_2^\theta, y_2^\phi) = (\sin\phi, \coth\theta\cos\phi)$ and $y_3 = (y_3^\theta, y_3^\phi) = (\cos\phi, -\coth\theta\sin\phi)$ satisfy both of these equations. □

The **Lie derivative** $\mathcal{L}_y$ of a scalar field A is defined in terms of a vector field $y^\ell(x)$ as $\mathcal{L}_y A = y^\ell A_{,\ell}$. The Lie derivative $\mathcal{L}_y$ of a contravariant vector F^i is

$$\mathcal{L}_y F^i = y^\ell F^i_{,\ell} - F^\ell y^i_{,\ell} = y^\ell F^i_{;\ell} - F^\ell y^i_{;\ell} \tag{9.130}$$

in which the second equality follows from $y^\ell \Gamma^i_{\ell k} F^k = F^\ell \Gamma^i_{\ell k} y^k$. The Lie derivative $\mathcal{L}_y$ of a covariant vector V_i is

$$\mathcal{L}_y V_i = y^\ell V_{i,\ell} + V_\ell y^\ell_{,i} = y^\ell V_{i;\ell} + V_\ell y^\ell_{;i}. \tag{9.131}$$

Similarly, the Lie derivative $\mathcal{L}_y$ of a rank-2 covariant tensor T_{ik} is

$$\mathcal{L}_y T_{ik} = y^\ell T_{ik,\ell} + T_{\ell k} y^\ell_{,i} + T_{i\ell} y^\ell_{,k}. \tag{9.132}$$

We see now that the condition (9.123) that a vector field y^ℓ be a symmetry of a metric g_{jm} is that its Lie derivative

$$\mathcal{L}_y g_{ik} = g_{ik,\ell}\, y^\ell + g_{im}\, y^m_{,k} + g_{jk}\, y^j_{,i} = 0 \tag{9.133}$$

must vanish.

A maximally symmetric space (or spacetime) in d dimensions has d translation symmetries and $d(d-1)/2$ rotational symmetries which gives a total of $d(d+1)/2$ symmetries associated with $d(d+1)/2$ Killing vectors. Thus for $d = 2$, there is one rotation and two translations. For $d = 3$, there are three rotations and three translations. For $d = 4$, there are six rotations and four translations.

A maximally symmetric space has a curvature tensor that is simply related to its metric tensor

$$R_{ijk\ell} = c\,(g_{ik} g_{j\ell} - g_{i\ell} g_{jk}) \tag{9.134}$$

where c is a constant (Zee, 2013, IX.6). Since $g^{ki} g_{ik} = g^k_k = d$ is the number of dimensions of the space(time), the Ricci tensor and the curvature scalar of a maximally symmetric space are

$$R_{j\ell} = g^{ki} R_{ijk\ell} = c\,(d-1)\, g_{j\ell} \quad \text{and} \quad R = g^{\ell j} R_{j\ell} = c\, d(d-1). \tag{9.135}$$

9.7 Conformal algebra

An isometry requires that $g'_{ik}(x') = g_{ik}(x')$. The looser condition

$$g'_{ik}(x') = \Omega(x')\, g_{ik}(x') \quad \text{with} \quad \Omega(x') > 0 \tag{9.136}$$

says that two metrics are **conformally** related. An equivalent condition is

$$g'_{ik}(x') = x^{\ell}_{,i'} x^{m}_{,k'} g_{\ell m}(x) = \Omega(x') g_{ik}(x'). \tag{9.137}$$

This condition guarantees that angles do not change:

$$\frac{g'_{ik} u_i v_k}{\sqrt{g'_{ij} u_i u_j}\sqrt{g'_{ij} v_i v_j}} = \frac{g_{ik} u_i v_k}{\sqrt{g_{ij} u_i u_j}\sqrt{g_{ij} v_i v_j}}. \tag{9.138}$$

For the infinitesimal transformation

$$x'^{\ell} = x^{\ell} + \epsilon y^{\ell}, \tag{9.139}$$

we approximate the factor Ω as

$$\Omega^2(x') \approx 1 + \epsilon\omega(x'). \tag{9.140}$$

The conformal condition (9.137) then says that

$$g_{\ell m}(\delta^{\ell}_{i} - \epsilon y^{\ell}_{,i})(\delta^{m}_{k} - \epsilon y^{m}_{,k}) = (1 + \epsilon\omega)(g_{ik} + \epsilon y^{\ell} g_{ik,\ell}) \tag{9.141}$$

or

$$g_{im} y^{m}_{,k} + g_{\ell k} y^{\ell}_{,i} + y^{\ell} g_{ik,\ell} + \omega g_{ik} = 0. \tag{9.142}$$

This is Killing's conformal condition.

Multiplication by g^{ik} and summing over i and k gives

$$\delta^{k}_{m} y^{m}_{,k} + \delta^{i}_{\ell} y^{\ell}_{,i} + g^{ik} y^{\ell} g_{ik,\ell} + 4\omega = 0 \tag{9.143}$$

or

$$2 y^{k}_{,k} + y^{\ell} g^{ik} g_{ik,\ell} + 4\omega = 0. \tag{9.144}$$

The function $\omega(x)$ therefore

$$\omega = -\frac{1}{4}\left(2 y^{i}_{,i} + y^{\ell} g^{ik} g_{ik,\ell}\right). \tag{9.145}$$

Substituting this formula for ω into Killing's conformal condition (9.142) gives a condition on the metric g and on the vector y

$$g_{im} y^{m}_{,k} + g_{\ell k} y^{\ell}_{,i} + y^{\ell} g_{ik,\ell} - \left(\frac{1}{2} y^{j}_{,j} + \frac{1}{4} y^{\ell} g^{jn} g_{jn,\ell}\right) g_{ik} = 0. \tag{9.146}$$

When $\omega = 0$, Killing's conformal condition (9.142)

$$g_{im} y^m_{,k} + g_{\ell k} y^\ell_{,i} + y^\ell g_{ik,\ell} = 0 \tag{9.147}$$

is the same as his isometry condition (9.123) .

Using the vanishing of the covariant derivative of the metric tensor as in (9.125), we can write Killing's conformal condition (9.142) more succinctly as

$$y_{i;k} + y_{k;i} + \omega\, g_{ik} = 0 \tag{9.148}$$

or as

$$\mathcal{L}_y g_{ik} = -\omega\, g_{ik}. \tag{9.149}$$

9.8 Conformal algebra in flat space

To find Killing's conformal vectors y in flat space, we set $g_{ik} = \eta_{ik}$ and use the formula (9.145) for ω to find

$$\omega = -\frac{1}{2} y^i_{,i} \tag{9.150}$$

which in d dimensions of spacetime is $\omega = -2y^i_{,i}/d$. So in flat-space, Killing's conformal condition (9.142) is

$$y_{i,k} + y_{k,i} = \frac{1}{2}\eta_{ik}\, y^\ell_{,\ell} \quad \text{or} \quad y_{i,k} + y_{k,i} = \frac{2}{d}\eta_{ik}\, y^\ell_{,\ell}. \tag{9.151}$$

Infinitesimal transformations

$$x'^i = x^i + \epsilon\, y^i \tag{9.152}$$

that obey this rule generate the conformal algebra of Minkowski space.

We already know some of these vectors. The vector

$$y^\ell = a^\ell + b^\ell{}_k x^k \tag{9.153}$$

with a^i and $b^\ell{}_k$ constant and b antisymmetric, $b_{ik} = -\,b_{ki}$, has a vanishing divergence $y^\ell_{,\ell} = 0$ and satisfies Killing's conformal condition (9.151)

$$y_{i,k} + y_{k,i} = b_{ik} + b_{ki} = \frac{2}{d}\eta_{ik}\, y^\ell_{,\ell} = 0. \tag{9.154}$$

These y's generate the translations a^i and the Lorentz transformations b_{ik}.

A conformal transformation $x \to x'$ changes the metric by no more than an overall factor. That means that

$$\Omega(x')\, \eta_{ik} dx'^i dx'^k = \eta_{ik} dx^i dx^k. \tag{9.155}$$

So if space is just stretched by a constant factor σ, so that $x'^i = \sigma x^i$, then $dx'^i = \sigma dx^i$, and the stretch condition (9.155) is satisfied with $\Omega(x') = \sigma^{-2}$. This kind of conformal transformation is a **dilation**, which some call a dilatation. The Killing vector y is

$$y^i = \sigma x^i \tag{9.156}$$

with σ a constant. This vector obeys Killing's flat-space conformal condition (9.151) because $y^\ell_{,\ell} = c\,d$ in d dimensions, and so

$$y_{i,k} + y_{k,i} = \eta_{ij}\, c\, \delta^j_k + \eta_{kj}\, c\, \delta^j_i = 2\eta_{ik}\, c = \frac{2}{d}\, \eta_{ik}\, y^\ell_{,\ell}\,. \tag{9.157}$$

The **inversion**

$$x'^i = \frac{r^2\, x^i}{x^2} \tag{9.158}$$

where r is a length and $x^2 = x^k x_k$ also obeys Killing's conformal condition (9.151). The change in x'^i is

$$dx'^i = \frac{r^2}{x^2}\left(\delta^i_j - \frac{2x^i x_j}{x^2}\right) dx^j \tag{9.159}$$

so with $\Omega \equiv \Omega(x')$

$$\begin{aligned}
\Omega\eta_{ik} dx'^i dx'^k &= \Omega\eta_{ik}\left(\frac{r^2}{x^2}\right)^2 \left(\delta^i_j - \frac{2x^i x_j}{x^2}\right) dx^j \left(\delta^k_\ell - \frac{2x^k x_\ell}{x^2}\right) dx^\ell \\
&= \Omega\left(\frac{r^2}{x^2}\right)^2 \left[\eta_{j\ell} - \frac{2\eta_{jk}x^k x_\ell}{x^2} - \frac{2\eta_{i\ell}x^i x_j}{x^2} + \frac{4x^2 x_j x_\ell}{(x^2)^2}\right] dx^j dx^\ell \\
&= \Omega\left(\frac{r^2}{x^2}\right)^2 \eta_{j\ell}\, dx^j dx^\ell.
\end{aligned} \tag{9.160}$$

So the stretch condition (9.155) is satisfied with

$$\Omega(x') = \left(\frac{x^2}{r^2}\right)^2. \tag{9.161}$$

Since both inversions and translations obey Killing's conformal condition (9.151), we may combine them so as to shrink an inversion to its infinitesimal form. We invert, translate by a tiny vector a, and re-invert and so find as the legitimately infinitesimal form of an inversion the transformation

$$x^i \to \frac{x^i}{x^2} \to \frac{x^i}{x^2} + a^i \to \left(\frac{x^i}{x^2} + a^i\right) \Big/ \eta_{jk} \left(\frac{x^j}{x^2} + a^j\right)\left(\frac{x^k}{x^2} + a^k\right)$$

$$= \left(\frac{x^i}{x^2} + a^i\right) \Big/ \eta_{jk} \left(\frac{1}{x^2} + \frac{2a\cdot x}{x^2} + a^2\right) \tag{9.162}$$

$$= x^2 \left(\frac{x^i}{x^2} + a^i\right) / \left(1 + 2a\cdot x + a^2x^2\right) \simeq \left(x^i + a^i x^2\right)(1 - 2a\cdot x)$$

$$= x^i + a_k(\eta^{ik}x^2 - 2x^i x^k) \tag{9.163}$$

which is a **conformal transformation**.

If we use ∂^i to differentiate Killing's conformal condition (9.151), then we get

$$\partial^i y_{i,k} + \partial^i y_{k,i} = \frac{2}{d}\eta_{ik}\,\partial^i\, y^{\ell}_{,\ell} \tag{9.164}$$

or, with $\partial^2 = \partial^i\partial_i$ and $\partial\cdot y = \partial_i y^i$,

$$\left(1 - \frac{2}{d}\right)\partial_k\,\partial\cdot y = -\,\partial^2 y_k \tag{9.165}$$

which says that in two dimensions ($d = 2$) every Killing vector is a solution of Laplace's equation:

$$\partial^2 y_k = 0 \qquad \text{if} \quad d = 2. \tag{9.166}$$

There are infinitely many Killing vectors in two-dimensions, a fact exploited in string theory.

Differentiating the same equation (9.165) again, we get

$$(d-2)\,\partial_i\partial_k\,\partial\cdot y = -\,d\,\partial^2 y_{k,i} \tag{9.167}$$

which says that $\partial^2 y_{k,i} = \partial^2 y_{i,k}$. Applying this symmetry and ∂^2 to Killing's conformal condition (9.151), we get

$$\eta_{ik}\,\partial^2\partial\cdot y = d\,\partial^2 y_{k,i}. \tag{9.168}$$

Comparing the right-hand sides of the last two equations (9.167 & 9.168), we find

$$\left[(d-2)\,\partial_i\partial_k + \eta_{ik}\partial^2\right]\,\partial\cdot y = 0. \tag{9.169}$$

Differentiating the same equation (9.165) a third time, we have

$$\partial^k\,(d-2)\,\partial_k\,\partial\cdot y = -\,d\,\partial^2\partial^k y_k \tag{9.170}$$

or

$$(d-1)\,\partial^2\,\partial\cdot y = 0. \tag{9.171}$$

In the case of one dimension ($d = 1$), Killing's conformal condition (9.151) reduces to $2\partial_0 y_0 = 2\partial_0 y_0$ and so places no restriction on Killing's vector, as reflected equations (9.165 & 9.171). Every function $y^0(x^0) \equiv y(t)$ is a conformal Killing vector. The condition (9.155) for a finite one-dimensional conformal transformation is

$$\frac{dx'}{dx} = \Omega^{-1}(x'). \tag{9.172}$$

This condition constrains the transformation $x \to x' = x'(x)$ only by requiring that the derivative (9.172) be positive.

The vanishing (9.166) of the laplacian of the Killing vector in $d = 2$ dimensions means that every Killing vector is a solution of Laplace's equation:

$$\partial^2 y^k = 0 \qquad \text{if} \quad d = 2. \tag{9.173}$$

In two-dimensional euclidian space, we let the $d = 2$ space be the complex plane and let y^0 and y^1 be the real and imaginary parts of an analytic (or antianalytic) function:

$$z = x^0 + ix^1 \quad \text{and} \quad f(z) = y^0 + iy^1. \tag{9.174}$$

Then the Cauchy-Riemann conditions

$$y^0_{,0} = y^1_{,1} \quad \text{and} \quad y^0_{,1} = -\,y^1_{,0} \tag{9.175}$$

imply that the real and imaginary parts of every analytic function $f(z)$ are harmonic

$$y^0_{,00} + y^0_{,11} = 0 \quad \text{and} \quad y^1_{,00} + y^1_{,11} = 0 \tag{9.176}$$

because $y^0_{,00} = y^1_{10} = -y^0_{,11}$ and $y^1_{,00} = -y^0_{,10} = -y^1_{,11}$. The real and imaginary parts of every antianalytic function $f(z^*)$ also are harmonic, and so satisfy Killing's condition (9.173). There are infinitely many solutions for the Killing vectors of an infinitesimal conformal transformation in two-dimensional euclidian space.

What about finite conformal transformations in two-dimensional euclidian space? The definition (9.155) of a conformal transformation in flat space is

$$\eta'_{ik} = \Omega(x')\eta_{ik}. \tag{9.177}$$

But metrics transform like this:

$$\eta'_{ik} = \frac{\partial x^j}{\partial x'^i}\frac{\partial x^\ell}{\partial x'^k}\eta_{j\ell} \quad \text{and} \quad \eta'^{ik} = \frac{\partial x'^i}{\partial x^j}\frac{\partial x'^k}{\partial x^\ell}\eta^{j\ell}. \tag{9.178}$$

In two-dimensional euclidian space, η is the 2×2 matrix η is the 2×2 matrix

$$\eta = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{9.179}$$

so these equations say that

$$\begin{aligned} \Omega = \Omega\eta^{00} &= \frac{\partial x'^0}{\partial x^0}\frac{\partial x'^0}{\partial x^0}\eta^{00} + \frac{\partial x'^0}{\partial x^1}\frac{\partial x'^0}{\partial x^1}\eta^{11} = \left(\frac{\partial x'^0}{\partial x^0}\right)^2 + \left(\frac{\partial x'^0}{\partial x^1}\right)^2 \\ \Omega = \Omega\eta^{11} &= \frac{\partial x'^1}{\partial x^0}\frac{\partial x'^1}{\partial x^0}\eta^{00} + \frac{\partial x'^1}{\partial x^1}\frac{\partial x'^1}{\partial x^1}\eta^{11} = \left(\frac{\partial x'^1}{\partial x^0}\right)^2 + \left(\frac{\partial x'^1}{\partial x^1}\right)^2 \\ 0 = \Omega\eta^{01} &= \frac{\partial x'^0}{\partial x^0}\frac{\partial x'^1}{\partial x^0}\eta^{00} + \frac{\partial x'^0}{\partial x^1}\frac{\partial x'^1}{\partial x^1}\eta^{11} = \frac{\partial x'^0}{\partial x^0}\frac{\partial x'^1}{\partial x^0} + \frac{\partial x'^0}{\partial x^1}\frac{\partial x'^1}{\partial x^1}. \end{aligned} \tag{9.180}$$

More succinctly, these conditions are

$$\begin{aligned} \left(\frac{\partial x'^0}{\partial x^0}\right)^2 + \left(\frac{\partial x'^0}{\partial x^1}\right)^2 &= \left(\frac{\partial x'^1}{\partial x^0}\right)^2 + \left(\frac{\partial x'^1}{\partial x^1}\right)^2 \\ \frac{\partial x'^0}{\partial x^0}\frac{\partial x'^1}{\partial x^0} &= -\frac{\partial x'^0}{\partial x^1}\frac{\partial x'^1}{\partial x^1}. \end{aligned} \tag{9.181}$$

These conditions are equivalent either to

$$\frac{\partial x'^0}{\partial x^0} = \frac{\partial x'^1}{\partial x^1} \quad \text{and} \quad \frac{\partial x'^1}{\partial x^0} = -\frac{\partial x'^0}{\partial x^1} \tag{9.182}$$

which are the Cauchy-Riemann conditions $u_x = v_y$ and $v_x = -u_y$ for $f = u + iv$ to be an analytic function of $z = x + iy$ or to

$$\frac{\partial x'^0}{\partial x^0} = -\frac{\partial x'^1}{\partial x^1} \quad \text{and} \quad \frac{\partial x'^1}{\partial x^0} = \frac{\partial x'^0}{\partial x^1} \tag{9.183}$$

which are the Cauchy-Riemann conditions $u_x = -v_y$ and $v_x = u_y$ for $f(\bar{z}) = u + iv$ to be an analytic function of $\bar{z} = x - iy$.

9.8.1 Angles and analytic functions

An analytic function $f(z)$ maps curves in the z plane into curves in the $f(z)$ plane. In general, this mapping preserves angles. To see why, we consider the angle $d\theta$ between two tiny complex lines $dz = \epsilon\exp(i\theta)$ and $dz' = \epsilon\exp(i\theta')$ that radiate from the same point z. This angle $d\theta = \theta' - \theta$ is the phase of the ratio

$$\frac{dz'}{dz} = \frac{\epsilon e^{i\theta'}}{\epsilon e^{i\theta}} = e^{i(\theta'-\theta)}. \tag{9.184}$$

Let's use $w = \rho e^{i\phi}$ for $f(z)$. Then the analytic function $f(z)$ maps dz into

$$dw = f(z + dz) - f(z) \approx f'(z)\, dz \tag{9.185}$$

and dz' into

$$dw' = f(z + dz') - f(z) \approx f'(z)\, dz'. \tag{9.186}$$

The angle $d\phi = \phi' - \phi$ between dw and dw' is the phase of the ratio

$$\frac{dw'}{dw} = \frac{e^{i\phi'}}{e^{i\phi}} = \frac{f'(z)\, dz'}{f'(z)\, dz} = \frac{dz'}{dz} = \frac{e^{i\theta'}}{e^{i\theta}} = e^{i(\theta'-\theta)}. \tag{9.187}$$

So as long as the derivative $f'(z)$ does not vanish, the angle in the w-plane is the same as the angle in the z-plane

$$d\phi = d\theta. \tag{9.188}$$

Analytic functions preserve angles. They are **conformal** maps.

What if $f'(z) = 0$? In this case, $dw \approx f''(z)\, dz^2/2$ and $dw' \approx f''(z)\, dz'^2/2$, and so the angle $d\phi = d\phi' - d\phi$ between these two tiny complex lines is the phase of the ratio

$$\frac{dw'}{dw} = \frac{e^{i\phi'}}{e^{i\phi}} = \frac{f''(z)\, dz'^2}{f''(z)\, dz^2} = \frac{dz'^2}{dz^2} = e^{2i(\theta'-\theta)}. \tag{9.189}$$

So angles are doubled, $d\phi = 2d\theta$.

In general, if the first nonzero derivative is $f^{(n)}(z)$, then

$$\frac{dw'}{dw} = \frac{e^{i\phi'}}{e^{i\phi}} = \frac{f^{(n)}(z)\, dz'^n}{f^{(n)}(z)\, dz^n} = \frac{dz'^n}{dz^n} = e^{ni(\theta'-\theta)} \tag{9.190}$$

and so $d\phi = nd\theta$. The angles increase by a factor of n.

Example 9.3 (z^n) The function $f(z) = z^n$ has only one nonzero derivative $f^{(k)}(0) = n!\,\delta_{nk}$ at the origin $z = 0$. So at $z = 0$ the map $z \to z^n$ scales angles by n, $d\phi = n\, d\theta$, but at $z \neq 0$ the first derivative $f^{(1)}(z) = nz^{n-1}$ is not equal to zero. So z^n is conformal except at the origin. □

Example 9.4 (Möbius transformation) The function

$$f(z) = \frac{az + b}{cz + d} \tag{9.191}$$

maps (straight) lines into lines and circles and circles into circles and lines, unless $ad = bc$ in which case it is the constant b/d. □

But if we require that $f(z)$ be entire and invertible on the Riemann sphere (the complex plane with the "point" $z = \infty$ mapped onto the north pole) so that the transformations form the **special conformal group**, then the function $f(z)$ cannot vanish at more than one value of z or $f^{-1}(0)$ would not be unique. Similarly, $f(z)$ can't have more than one pole. So the only functions $f(z)$ that are entire and invertible on the Riemann sphere are those of the **projective** transformation

$$f(z) = \frac{az + b}{cz + d} \quad \text{with} \quad ad - bc = 1 \tag{9.192}$$

and a, b, c, and d complex, which is a special case of the fractional linear or Möbius transformation (9.193). These matrices form the group $SL(2, \mathbb{C})$ which is isomorphic to the Lorentz group $SO(3, 1)$. So the requirement that the finite transformations form a group has reduced the symmetry of the conformal group to that of the Lorentz group.

In two-dimensional Minkowski space, we must solve Killing's equation for the vector of an infinitesimal transformation

$$y^k_{,11} = y^k_{,00}. \tag{9.193}$$

So we use Dirac's light-cone coordinates. We set

$$x^{\pm} = t \pm x. \tag{9.194}$$

Then

$$\begin{aligned} ds^2 =& -dt^2 + dx^2 = -(dt + dx)(dt - dx) = \eta_{ik} dx^i dx^k \\ &= \eta_{--} dx^- dx^- + \eta_{++} dx^+ dx^+ + \eta_{-+} dx^+ dx^- + \eta_{+-} dx^- dx^+ \end{aligned} \tag{9.195}$$

with $\eta_{-+} = \eta_{+-} = -\frac{1}{2}$ and $\eta_{--} = \eta_{++} = 0$. So the inverse metric has $\eta^{-+} = \eta^{+-} = -2$. The light-cone derivatives are

$$\partial_{\pm} = \frac{1}{2} \left(\partial_t \pm \partial_x \right). \tag{9.196}$$

They act like this:

$$\begin{aligned} \partial_+ x^+ = 1 \quad &\text{and} \quad \partial_- x^- = 1 \\ \partial_+ x^- = 0 \quad &\text{and} \quad \partial_- x^+ = 0. \end{aligned} \tag{9.197}$$

In light-cone coordinates, the equation (9.195) for the $d = 2$ Killing vector is

$$\partial_- \partial_+ y^k = \partial_+ \partial_- y^k = 0. \tag{9.198}$$

Any vector $y^k(x^+)$ satisfies this equation because

$$\partial_- y^k(x^+) = y'^k \partial_- x^+ = 0. \tag{9.199}$$

Similarly, any function $y^k(x^-)$ obeys the same equation. So there are two sets of infinitely many solutions $y^k(x^+)$ and $y^k(x^-)$ of Killing's equation (9.200).

To understand finite conformal transformations in 2-d Minkowski space, we return to the definition (9.177)

$$\eta'_{ik} = \Omega(x')\eta_{ik}. \tag{9.200}$$

of a conformal transformation in flat space and recall how (9.203) metrics transform

$$\eta'_{ik} = \frac{\partial x^j}{\partial x'^i}\frac{\partial x^\ell}{\partial x'^k}\eta_{j\ell} \quad \text{and} \quad \eta'^{ik} = \frac{\partial x'^i}{\partial x^j}\frac{\partial x'^k}{\partial x^\ell}\eta^{j\ell}. \tag{9.201}$$

In two-dimensional Minkowski space, η is the 2×2 matrix

$$\eta = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{9.202}$$

so these equations say that

$$\begin{aligned} -\Omega = \Omega\eta^{00} &= \frac{\partial x'^0}{\partial x^0}\frac{\partial x'^0}{\partial x^0}\eta^{00} + \frac{\partial x'^0}{\partial x^1}\frac{\partial x'^0}{\partial x^1}\eta^{11} = -\left(\frac{\partial x'^0}{\partial x^0}\right)^2 + \left(\frac{\partial x'^0}{\partial x^1}\right)^2 \\ \Omega = \Omega\eta^{11} &= \frac{\partial x'^1}{\partial x^0}\frac{\partial x'^1}{\partial x^0}\eta^{00} + \frac{\partial x'^1}{\partial x^1}\frac{\partial x'^1}{\partial x^1}\eta^{11} = -\left(\frac{\partial x'^1}{\partial x^0}\right)^2 + \left(\frac{\partial x'^1}{\partial x^1}\right)^2 \\ 0 = \Omega\eta^{01} &= \frac{\partial x'^0}{\partial x^0}\frac{\partial x'^1}{\partial x^0}\eta^{00} + \frac{\partial x'^0}{\partial x^1}\frac{\partial x'^1}{\partial x^1}\eta^{11} = -\frac{\partial x'^0}{\partial x^0}\frac{\partial x'^1}{\partial x^0} + \frac{\partial x'^0}{\partial x^1}\frac{\partial x'^1}{\partial x^1}. \end{aligned} \tag{9.203}$$

More succinctly, these conditions are

$$\begin{aligned} \left(\frac{\partial x'^0}{\partial x^0}\right)^2 - \left(\frac{\partial x'^0}{\partial x^1}\right)^2 &= -\left(\frac{\partial x'^1}{\partial x^0}\right)^2 + \left(\frac{\partial x'^1}{\partial x^1}\right)^2 \\ \frac{\partial x'^0}{\partial x^0}\frac{\partial x'^1}{\partial x^0} &= \frac{\partial x'^0}{\partial x^1}\frac{\partial x'^1}{\partial x^1}. \end{aligned} \tag{9.204}$$

We can satisfy these equations by having x'^k be functions of either x^+ or x^-.

In more than two dimensions ($d \ge 3$), the vanishing (9.171) of the laplacian of the divergence $\partial \cdot y$ and the third-order equation (9.169) imply that the all the second derivatives of the divergence vanish

$$\partial_i\, \partial_k\, (\partial \cdot y) = 0. \tag{9.205}$$

So the divergence of Killing's vector must be at most linear in the coordinates

$$\partial \cdot y = a + b_i x^i \quad \text{for} \quad d \geq 3. \tag{9.206}$$

To see what this implies, we differentiate Killing's conformal condition (9.151)

$$y_{i,k\ell} + y_{k,i\ell} = \frac{2}{d}\eta_{ik}(\partial \cdot y)_{,\ell}, \tag{9.207}$$

add to it the same equation with k and ℓ interchanged

$$y_{i,\ell k} + y_{\ell,ik} = \frac{2}{d}\eta_{i\ell}(\partial \cdot y)_{,k}, \tag{9.208}$$

and then subtract the same equation with i, k, ℓ permuted

$$y_{k,\ell i} + y_{\ell,ki} = \frac{2}{d}\eta_{k\ell}(\partial \cdot y)_{,i} \tag{9.209}$$

so as to get

$$2y_{i,k\ell} = \frac{2}{d}\left[\eta_{ik}(\partial \cdot y)_{,\ell} + \eta_{i\ell}(\partial \cdot y)_{,k} - \eta_{k\ell}(\partial \cdot y)_{,i}\right]. \tag{9.210}$$

So if we substitute the at-most-linear condition (9.208) into this equation (9.212), then we find

$$2y_{i,k\ell} = \frac{2}{d}\left[\eta_{ik}b_\ell + \eta_{i\ell}b_k - \eta_{k\ell}b_i\right] \tag{9.211}$$

which says that y^i can be at most quadratic in the coordinates.

$$y_i = \alpha_i + \beta_{ij}x^j + \gamma_{ijk}x^j x^k \tag{9.212}$$

in which $\gamma_{ijk} = \gamma_{ikj}$.

Killing's conformal condition (9.151) must hold for all x's and is linear in the vector y, so we can impose it successively on a_i, b_{ij}, and c_{ijk} by differentiating with respect to the variables x^ℓ and setting them equal to zero. The vector α^i is an arbitrary translation. The condition on β_{ij} is

$$\beta_{ik} + \beta_{ki} = \frac{2}{d}\eta_{ik}\,\beta^\ell{}_\ell \tag{9.213}$$

which says that β_{ij} is the sum

$$\beta_{ij} = b_{ij} + c\,\eta_{ij} \tag{9.214}$$

of an antisymmetric part $b_{ij} = -b_{ji}$ that generates a Lorentz transformation and a term proportional to η_{ij} that generates a scale transformation. The conformal condition (9.151) makes the quadratic term satisfy the relations

$$\gamma_{ik\ell} + \gamma_{ki\ell} = 2\eta_{ik} d_\ell \tag{9.215}$$
$$\gamma_{i\ell k} + \gamma_{\ell ik} = 2\eta_{i\ell} d_k \tag{9.216}$$
$$\gamma_{k\ell i} + \gamma_{\ell ki} = 2\eta_{k\ell} d_i \tag{9.217}$$

in which

$$d_\ell = \frac{1}{d} \gamma^r{}_{r\ell}. \tag{9.218}$$

Subtracting the last equation (9.219) from the sum of the first two (9.217 & 9.218), we find

$$\gamma_{ik\ell} = \eta_{ik} d_\ell + \eta_{i\ell} d_k - \eta_{k\ell} d_i. \tag{9.219}$$

The corresponding infinitesimal transformation is

$$x'^i = x^i + 2(x \cdot d) x^i - d^i x^2 \tag{9.220}$$

which is called a **special conformal transformation** whose exponential or finite form is

$$x'^i = \frac{x^i - d^i x^2}{1 - 2d \cdot x + d^2 x^2}. \tag{9.221}$$

The general Killing vector and its infinitesimal transformation are

$$y^i = a^i + b^i{}_k x^k + c x^i + d_k (\eta^{ik} x^2 - 2x^i x^k) \tag{9.222}$$

and

$$x'^i = x^i + a^i + b^i{}_k x^k + c x^i + d_k (\eta^{ik} x^2 - 2x^i x^k) \tag{9.223}$$

in which we see a translation, a Lorentz transformation, a dilation, and a special conformal transformation.

The differential forms of the Poincaré transformations and the dilation D and the inversion k are

$$\begin{aligned} &P_i = \partial_i, \quad J_{ik} = (x_i \, \partial_k - x_k \, \partial_i), \\ &D = x^i \, \partial_i, \quad \text{and} \quad K^i = (\eta^{ik} \, x^j \, x_j - 2x^i \, x^k) \, \partial_k. \end{aligned} \tag{9.224}$$

The commutators of the conformal Lie algebra are

$$\begin{aligned}
&[P^i, P^j] = 0, \quad [K^i, K^j] = 0, \\
&[D, P^i] = -P^i, \quad [D, J_{ij}] = 0, \quad [D, K^i] = K^i, \\
&[J^{ij}, P^\ell] = -\eta^{i\ell} P^j + \eta^{j\ell} P^i, \quad [J^{ij}, K^\ell] = -\eta^{i\ell} K^j + \eta^{j\ell} K^i \\
&[J^{ij}, J^{\ell m}] = -\eta^{i\ell} J^{jm} - \eta^{jm} J^{i\ell} + \eta^{j\ell} J^{im} + \eta^{im} J^{j\ell}, \\
&[K^i, P^j] = 2J^{ij} + 2\eta^{ij} D.
\end{aligned} \tag{9.225}$$

These commutation relations say that D is a Lorentz scalar, and that K^i transforms as a vector.

There are $d\,P$'s, $d\,K$'s, $d(d-1)/2\,J$'s, and one D. That's $(d+2)(d+1)/2$ generators, which is the same number of generators as $SO(d+2)$. In fact, Minkowski space in d dimensions has $SO(d-1,1)$ as its Lorentz group and $SO(d,2)$ as its conformal algebra. The "Lorentz group" of a spacetime with d spatial dimensions and 2 time dimensions is also $SO(d,2)$.

9.9 Maxwell's action is conformally invariant for $d = 4$

Under a change of coordinates from x to x', a covariant vector field $A_i(x)$ changes this way

$$A'_i(x') = x^k_{,i'}\, A_k(x) = \frac{\partial x^k}{\partial x'^i}\, A_k(x). \tag{9.226}$$

So under the tiny change (with a convenient minus sign)

$$x' = x - y \quad \text{or} \quad x = x' + y \tag{9.227}$$

the change in A_i is

$$\begin{aligned}
\delta A_i(x) &\equiv A'_i(x) - A_i(x) \\
&= A'_i(x) - A'_i(x') + A'_i(x') - A_i(x) \\
&= y^j\, A'_{i,j}(x') + \left(\delta^j_i + y^j_{,i'}\right) A_j(x) - A_i(x) \\
&= y^j\, A'_{i,j}(x') + y^j_{,i'}\, A_j(x).
\end{aligned} \tag{9.228}$$

To lowest order in y, this is

$$\delta A_i(x) = y^j A_{i,j}(x) + y^j_{,i}\, A_j(x). \tag{9.229}$$

So the change in the Maxwell-Faraday tensor is

$$\begin{aligned}
\delta F_{ik} &= \delta\left(\partial_i A_k - \partial_k A_i\right) = \partial_i\left(\delta A_k\right) - \partial_k\left(\delta A_i\right) \\
&= \partial_i \left(y^j A_{k,j}(x) + y^j_{,k} A_j(x)\right) - \partial_k \left(y^j A_{i,j}(x) + y^j_{,i} A_j(x)\right) \\
&= y^j \partial_j F_{ik} + y^j_{,i} A_{k,j} - y^j_{,k} A_{i,j} + y^j_{,ki} A_j - y^j_{,ik} A_j + y^j_{,k} \partial_i A_j - y^j_{,i} \partial_k A_j \\
&= y^j \partial_j F_{ik} + y^j_{,i} F_{jk} + y^j_{,k} F_{ij}. \qquad (9.230)
\end{aligned}$$

Thus the change in Maxwell's action density is

$$\begin{aligned}
\delta\left(F^{ik} F_{ik}\right) &= 2\, F^{ik}\, \delta F_{ik} = 2\, F^{ik}\left(y^j \partial_j F_{ik} + y^j_{,i} F_{jk} + y^j_{,k} F_{ij}\right) \\
&= \partial_j\left(y^j F^{ik} F_{ik}\right) - (\partial\cdot y) F^{ik} F_{ik} + 2 y^j_{,i} F^{ik} F_{jk} + 2 y^j_{,k} F^{ik} F_{ij} \\
&= \partial_j\left(y^j F^{ik} F_{ik}\right) - (\partial\cdot y) F^{ik} F_{ik} + 2 y_{j,i} F^{ik} F^j{}_k + 2 y_{j,k} F^{ik} F_i{}^j \\
&= \partial_j\left(y^j F^{ik} F_{ik}\right) - (\partial\cdot y) F^{ik} F_{ik} + 4 y_{j,i} F^{ik} F^j{}_k \qquad (9.231) \\
&= \partial_j\left(y^j F^{ik} F_{ik}\right) - (\partial\cdot y) F^{ik} F_{ik} + 4 y_{k,i} F^{ij} F^k{}_j.
\end{aligned}$$

We can rewrite this as

$$\delta\left(F^{ik} F_{ik}\right) = \partial_j\left(y^j F^{ik} F_{ik}\right) + 2 F^{ij} F^k{}_j \left(y_{k,i} + y_{i,k} - \tfrac{1}{2}\eta_{ik} \partial\cdot y\right). \qquad (9.232)$$

But if y is a conformal Killing vector obeying (9.151), then this change in the Maxwell action density is

$$\delta\left(F^{ik} F_{ik}\right) = \partial_j\left(y^j F^{ik} F_{ik}\right) + 2 F^{ij} F^k{}_j \left(\frac{2}{d}\eta_{ik} y^\ell_{,\ell} - \frac{1}{2}\eta_{ik} \partial\cdot y\right). \qquad (9.233)$$

Iff $d = 4$, this change is a total derivative

$$\delta\left(F^{ik} F_{ik}\right) = \partial_j\left(y^j F^{ik} F_{ik}\right) \iff d = 4. \qquad (9.234)$$

The action density of pure (only gauge fields), classical Yang-Mills theory also is conformally invariant in four dimensions ($d = 4$). Quantum effects introduce a mass scale, however, and break conformal invariance.

9.10 Massless scalar field theory is conformally invariant if $d = 2$

Under a tiny change of coordinates $x = x' + y$, the change in a scalar field ϕ is

$$\delta\phi = y^j \phi_{,j} = y_j \phi^{,j}. \qquad (9.235)$$

So the change in the action density is

$$\delta\left(\frac{1}{2}\phi^{,i}\phi_{,i}\right) = \phi^{,i}\,\delta\phi_{,i} = \phi^{,i}\,\partial_i\delta\phi = \phi^{,i}\,\partial_i\left(y_j\phi^{,j}\right)$$
$$= \phi^{,i}\phi^{,j}y_{j,i} + \phi^{,i}y_j\phi^{,j}_{,i} \tag{9.236}$$
$$= \frac{1}{2}\phi^{,i}\phi^{,j}\left(y_{j,i}+y_{i,j}\right) + \phi^{,i}\phi^{,j}_{,i}\,y_j.$$

If y obeys the condition (9.151) for a conformal Killing vector, then the change (9.238) in the action density is a total divergence in two dimensions

$$\delta\left(\frac{1}{2}\phi^{,i}\phi_{,i}\right) = \frac{1}{2}\phi^{,i}\phi^{,j}\frac{2}{d}\eta_{ij}y^{\ell}_{,\ell} + \phi^{,i}\phi^{,j}_{,i}\,y_j$$
$$= \frac{1}{d}\phi^{,i}\phi_{,i}\,y^{\ell}_{,\ell} + \phi^{,i}\phi_{,i\ell}\,y^{\ell} = \frac{1}{d}\phi^{,i}\phi_{,i}\,y^{\ell}_{,\ell} + \frac{1}{2}\left(\phi^{,i}\phi_{,i}\right)_{,\ell}\,y^{\ell}$$
$$= \partial_\ell\left(\frac{1}{2}\phi^{,i}\phi_{,i}\,y^{\ell}\right) \iff d=2. \tag{9.237}$$

Thus the classical theory of a free massless scalar field is conformally invariant in two-dimensional spacetimes.

The rest of this chapter is at best a first draft, not ready for human consumption.

9.11 Christoffel symbols as nonabelian gauge fields

A contravariant vector V^i transforms like $dx'^i = x'^i_{,k}dx^k$ as

$$V'^i(x') = \frac{\partial x'^i}{\partial x^k}\,V^k(x) = x'^i_{,k}\,V^k(x) = E^i{}_k(x)\,V^k(x). \tag{9.238}$$

The 4×4 matrix $E^i{}_k(x) = x'^i_{,k}$ depends upon the spacetime point x and is a member of the huge noncompact group $GL(4,\mathbb{R})$.

The insight of Yang and Mills (section 5.1) lets us define a **covariant derivative** $D_\ell = \partial_\ell + A_\ell$ of a contravariant vector V^i

$$(D_\ell V)^k = (\partial_\ell\delta^k_j + A^k_{\ell j})V^j \tag{9.239}$$

that transforms as

$$\left[(D_\ell V)^k\right]' = \frac{\partial x^i}{\partial x'^\ell}\frac{\partial x'^k}{\partial x^m}(D_i V)^m. \tag{9.240}$$