9.7 Conformal algebra

An isometry requires that \( g'_{ik}(x') = g_{ik}(x') \). The looser condition
\[
g'_{ik}(x') = \Omega(x') g_{ik}(x') \quad \text{with} \quad \Omega(x') > 0
\]
(9.136)
says that two metrics are **conformally** related. An equivalent condition is
\[
g'_{ik}(x') = x_{j'} x'_{i_k} g_{j m}(x) = \Omega(x') g_{ik}(x').
\]
(9.137)
This condition guarantees that angles do not change:
\[
g'_{ik} u_i v_k u_{i} v_{i} = g_{ik} u_i u_{i} v_{j} v_{j}.
\]
(9.138)
For the infinitesimal transformation
\[
x_{i}^\ell = x_{i}^\ell + \epsilon y_{i}^\ell,
\]
(9.139)
we approximate the factor \( \Omega \) as
\[
\Omega^2(x') \approx 1 + \epsilon \omega(x').
\]
(9.140)
The conformal condition (9.137) then says that
\[
g_{j m}(\delta_{i}^{\ell} - \epsilon y_{i}^{\ell})(\delta_{k}^{m} - \epsilon y_{k}^{m}) = (1 + \epsilon \omega)(g_{ik} + \epsilon y_{i}^{\ell} g_{ik,\ell})
\]
(9.141)
or
\[
g_{im} y_{k}^{m} + g_{ik} y_{i}^{\ell} + y_{i}^{\ell} g_{ik,\ell} + \omega g_{ik} = 0.
\]
(9.142)
This is Killing’s conformal condition.

Multiplication by \( g^{ik} \) and summing over \( i \) and \( k \) gives
\[
\delta_{m}^{k} y_{k}^{m} + \delta_{k}^{i} y_{i}^{\ell} + g^{ik} y_{i}^{\ell} g_{ik,\ell} + 4 \omega = 0
\]
(9.143)
or
\[
2 y_{i}^{k} + y^{\ell} g^{ik} g_{ik,\ell} + 4 \omega = 0.
\]
(9.144)
The function \( \omega(x) \) therefore
\[
\omega = - \frac{1}{4} \left( 2 y_{i}^{i} + y^{\ell} g^{ik} g_{ik,\ell} \right).
\]
(9.145)
Substituting this formula for \( \omega \) into Killing’s conformal condition (9.142) gives a condition on the metric \( g \) and on the vector \( y \)
\[
g_{im} y_{k}^{m} + g_{ik} y_{i}^{\ell} + y^{\ell} g_{ik,\ell} - \left( \frac{1}{2} y_{j}^{j} + \frac{1}{4} y^{\ell} g^{jn} g_{jn,\ell} \right) g_{ik} = 0.
\]
(9.146)
When \( \omega = 0 \), Killing’s conformal condition (9.142)

\[
g_{im}y_{ik}^m + g_{ik}y_{ij}^l + y_{ik,l} = 0 \quad (9.147)
\]

is the same as his isometry condition (9.123).

Using the vanishing of the covariant derivative of the metric tensor as in (9.125), we can write Killing’s conformal condition (9.142) more succinctly as

\[
y_{i;k} + y_{k;i} + \omega g_{ik} = 0 \quad (9.148)
\]
or as

\[
\mathcal{L}_y g_{ik} = -\omega g_{ik}. \quad (9.149)
\]

### 9.8 Conformal algebra in flat space

To find Killing’s conformal vectors \( y \) in flat space, we set \( g_{ik} = \eta_{ik} \) and use the formula (9.145) for \( \omega \) to find

\[
\omega = -\frac{1}{2} y_{i,i} \quad (9.150)
\]

which in \( d \) dimensions of spacetime is \( \omega = -2y_{i,i}/d \). So in flat-space, Killing’s conformal condition (9.142) is

\[
y_{i,k} + y_{k,i} = \frac{1}{2} \eta_{kk} y_{i,i} \quad \text{or} \quad y_{i,k} + y_{k,i} = \frac{2}{d} \eta_{kk} y_{i,i}. \quad (9.151)
\]

Infinitesimal transformations

\[
x'^i = x^i + \epsilon y^i \quad (9.152)
\]

that obey this rule generate the conformal algebra of Minkowski space.

We already know some of these vectors. The vector

\[
y^\ell = a^\ell + b^\ell_k x^k \quad (9.153)
\]

with \( a^i \) and \( b^\ell_k \) constant and \( b \) antisymmetric, \( b_{ik} = -b_{ki} \), has a vanishing divergence \( y^\ell_{,\ell} = 0 \) and satisfies Killing’s conformal condition (9.151)

\[
y_{i,k} + y_{k,i} = b_{ik} + b_{ki} = \frac{2}{d} \eta_{kk} y_{i,i} = 0. \quad (9.154)
\]

These \( y \)'s generate the translations \( a^i \) and the Lorentz transformations \( b_{ik} \).

A conformal transformation \( x \to x' \) changes the metric by no more than an overall factor. That means that

\[
\Omega(x') \eta_{ik} dx'^i dx'^k = \eta_{ik} dx^i dx^k. \quad (9.155)
\]
So if space is just stretched by a constant factor $\sigma$, so that $x'^i = \sigma x^i$, then $dx'^n = \sigma dx^i$, and the stretch condition (9.155) is satisfied with $\Omega(x') = \sigma^{-2}$.

This kind of conformal transformation is a dilation, which some call a dilatation. The Killing vector $y$ is

$$y^i = \sigma x^i$$

(9.156)

with $\sigma$ a constant. This vector obeys Killing’s flat-space conformal condition (9.151) because $y_{,\ell} = c d$ in $d$ dimensions, and so

$$y_{i,k} + y_{k,i} = \eta_{ij} c \delta^j_k + \eta_{kj} c \delta^j_i = 2\eta_{ik} c = 2\eta_{ik} y_{,\ell}/d.$$  

(9.157)

The inversion

$$x'^i = \frac{r^2 x^i}{x^2}$$

(9.158)

where $r$ is a length and $x^2 = x^k x_k$ also obeys Killing’s conformal condition (9.151). The change in $x'^i$ is

$$dx'^i = \frac{r^2}{x^2} \left( \delta^i_j - \frac{2x^i x_j}{x^2} \right) dx^j$$

(9.159)

so with $\Omega \equiv \Omega(x')$

$$\Omega \eta_{ik} dx'^i dx'^k = \Omega \eta_{ik} \left( \frac{r^2}{x^2} \right)^2 \left( \delta^i_j - \frac{2x^i x_j}{x^2} \right) dx^j \left( \delta^k_{\ell} - \frac{2x^k x_{\ell}}{x^2} \right) dx^\ell$$

$$= \Omega \left( \frac{r^2}{x^2} \right)^2 \left[ \eta_{j\ell} - \frac{2\eta_{ijk} x^k x_\ell}{x^2} - \frac{2\eta_{i\ell} x^i x_j}{x^2} + \frac{4x^2 x_j x_\ell}{(x^2)^2} \right] dx^j dx^\ell$$

$$= \Omega \left( \frac{r^2}{x^2} \right)^2 \eta_{j\ell} dx^j dx^\ell.$$  

(9.160)

So the stretch condition (9.155) is satisfied with

$$\Omega(x') = \left( \frac{x^2}{r^2} \right)^2.$$  

(9.161)

Since both inversions and translations obey Killing’s conformal condition (9.151), we may combine them so as to shrink an inversion to its infinitesimal form. We invert, translate by a tiny vector $a$, and re-invert and so find as
the legitimately infinitesimal form of an inversion the transformation

\[ x^i \to \frac{x^i}{x^2} \to \frac{x^i}{x^2} + a^i \to \left( \frac{x^i}{x^2} + a^i \right) \left/ \eta_{jk} \left( \frac{x^j}{x^2} + a^j \right) \right. \left( \frac{x^k}{x^2} + a^k \right) \]

\[ = \left( \frac{x^i}{x^2} + a^i \right) \left/ \eta_{jk} \left( \frac{1}{x^2} + \frac{2a \cdot x}{x^2} + a^2 \right) \right. \]

\[ = x^2 \left( \frac{x^i}{x^2} + a^i \right) \left/ (1 + 2a \cdot x + a^2 x^2) \right. \simeq (x^i + a^i x^2) \left( 1 - 2a \cdot x \right) \]

\[ = x^i + a_k (\eta_{ik} x^2 - 2x^i x^k) \]  

(9.163)

which is a conformal transformation.

If we use \( \partial^i \) to differentiate Killing’s conformal condition (9.151), then we get

\[ \partial^i y_{k,i} + \partial^j y_{k,i} = \frac{2}{d} \eta_{ik} \partial^j y_{\ell, \ell} \]  

(9.164)

or, with \( \partial^2 = \partial^i \partial_i \) and \( \partial \cdot y = \partial_i y^i \),

\[ \left( 1 - \frac{2}{d} \right) \partial_k \partial \cdot y = - \partial^2 y_k \]  

(9.165)

which says that in two dimensions \( (d = 2) \) every Killing vector is a solution of Laplace’s equation:

\[ \partial^2 y_k = 0 \quad \text{if} \quad d = 2. \]  

(9.166)

There are infinitely many Killing vectors in two-dimensions, a fact exploited in string theory.

Differentiating the same equation (9.165) again, we get

\[ (d - 2) \partial_i \partial_k \partial \cdot y = - d \partial^2 y_{k,i} \]  

(9.167)

which says that \( \partial^2 y_{k,i} = \partial^2 y_{i,k} \). Applying this symmetry and \( \partial^2 \) to Killing’s conformal condition (9.151), we get

\[ \eta_{ik} \partial^2 \partial \cdot y = d \partial^2 y_{k,i}. \]  

(9.168)

Comparing the right-hand sides of the last two equations (9.167 & 9.168), we find

\[ [(d - 2) \partial_i \partial_k + \eta_{ik} \partial^2] \partial \cdot y = 0. \]  

(9.169)

Differentiating the same equation (9.165) a third time, we have

\[ \partial^k (d - 2) \partial_k \partial \cdot y = - d \partial^2 \partial^k y_k \]  

(9.170)
or

\[(d - 1) \partial^2 \partial \cdot y = 0. \quad (9.171)\]

In the case of one dimension \(d = 1\), Killing’s conformal condition (9.151) reduces to \(2\partial_0 y_0 = 2\partial_0 y_0\) and so places no restriction on Killing’s vector, as reflected in the last equation (9.171). Every function \(y^0(x^0) = y(t)\) is a conformal Killing vector.

The vanishing (9.166) of the laplacian of the Killing vector in \(d = 2\) dimensions means that every Killing vector is a solution of Laplace’s equation:

\[\partial^2 y^k = 0 \quad \text{if} \quad d = 2. \quad (9.172)\]

In two-dimensional euclidian space, we let the \(d = 2\) space be the complex plane and to let \(y^0\) and \(y^1\) be the real and imaginary parts of an analytic (or antianalytic) function:

\[z = x^0 + ix^1 \quad \text{and} \quad f(z) = y^0 + iy^1. \quad (9.173)\]

Then the Cauchy-Riemann conditions

\[y^0_{,00} + y^0_{,11} = 0 \quad \text{and} \quad y^1_{,00} + y^1_{,11} = 0 \quad (9.174)\]

imply that the real and imaginary parts of every analytic function \(f(z)\) are harmonic

\[y^0_{,00} + y^0_{,11} = 0 \quad \text{and} \quad y^1_{,00} + y^1_{,11} = 0 \quad (9.175)\]

because \(y^0_{,00} = y^1_{,10} = -y^0_{,11}\) and \(y^1_{,00} = -y^0_{,10} = -y^1_{,11}\). The real and imaginary parts of every antianalytic function \(f(z^*)\) also are harmonic, and so satisfy Killing’s condition (9.172). There are infinitely many solutions.

But if we require that \(f(z)\) be entire and invertible, so that the transformations form the \textbf{special conformal group}, then the only solutions are the ratios

\[f(z) = \frac{az + b}{cz + d} \quad \text{with} \quad ad - bc = 1 \quad (9.176)\]

and \(a, b, c,\) and \(d\) complex.

In two-dimensional Minkowski space, we must solve

\[y^k_{,11} = y^k_{,00}. \quad (9.177)\]

So we use Dirac’s light-cone coordinates. We set

\[x^\pm = t \pm x. \quad (9.178)\]
Then
\[ ds^2 = -dt^2 + dx^2 = -(dt + dx)(dt - dx) = \eta_{ik} dx^i dx^k \]
\[ = \eta_{--} dx^- dx^- + \eta_{+-} dx^+ dx^- + \eta_{-+} dx^- dx^+ + \eta_{++} dx^+ dx^+ \]  
(9.179)

with \( \eta_{-+} = \eta_{+-} = -\frac{1}{2} \) and \( \eta_{--} = \eta_{++} = 0 \). So the inverse metric has \( \eta^{--} = \eta^{++} = -2 \). The light-cone derivatives are
\[ \partial_{\pm} = \frac{1}{2} (\partial_t \pm \partial_x) . \]  
(9.180)

They act like this:
\[ \partial_+ x^+ = 1 \quad \text{and} \quad \partial_- x^- = 1 \]
\[ \partial_+ x^- = 0 \quad \text{and} \quad \partial_- x^+ = 0 . \]  
(9.181)

In light-cone coordinates, the equation (9.177) for the \( d = 2 \) Killing vector is
\[ \partial_+ \partial_- y^k = 0 . \]  
(9.182)

Any vector \( y^k(x^+) \) satisfies this equation because
\[ \partial_- y^k(x^+) = y^k \partial_- x^+ = 0 . \]  
(9.183)

Similarly, any function \( y^k(x^-) \) obeys the same equation. So there are two sets of infinitely many solutions \( y^k(x^+) \) and \( y^k(x^-) \) of Killing’s equation (9.182).

In more than two dimensions \((d \geq 3)\), the vanishing (9.171) of the laplacian of the divergence \( \partial \cdot y \) together with the third-order equation (9.169) means that the all the second derivatives of the divergence vanish
\[ \partial_i \partial_k (\partial \cdot y) = 0 . \]  
(9.184)

So the divergence of Killing’s vector must be at most linear in the coordinates
\[ \partial \cdot y = a + b_i x^i \quad \text{for} \quad d \geq 3 . \]  
(9.185)

To see what this implies, we differentiate Killing’s conformal condition (9.151)
\[ y_{i,k\ell} + y_{k,i\ell} = \frac{2}{d} \eta_{ik} (\partial \cdot y)_{,\ell} , \]  
(9.186)

add to it the same equation with \( k \) and \( \ell \) interchanged
\[ y_{i,\ell k} + y_{\ell,i k} = \frac{2}{d} \eta_{i\ell} (\partial \cdot y)_{,k} , \]  
(9.187)

and then subtract the same equation with \( i, k, \ell \) permuted
\[ y_{k,\ell i} + y_{\ell,k i} = \frac{2}{d} \eta_{k\ell} (\partial \cdot y)_{,i} \]  
(9.188)
so as to get
\[
2y_{i,k\ell} = \frac{2}{d} \left[ \eta_{ik}(\partial \cdot y)_{,\ell} + \eta_{k\ell}(\partial \cdot y)_{,i} - \eta_{i\ell}(\partial \cdot y)_{,k} \right].
\] (9.189)

So if we substitute the at most linear condition (9.185) into this equation (9.189), then we find
\[
2y_{i,k\ell} = \frac{2}{d} \left[ \eta_{ik} b_{\ell} + \eta_{k\ell} b_i - \eta_{i\ell} b_k \right]
\] (9.190)

which says that \(y^i\) can be at most quadratic in the coordinates.
\[
y_i = \alpha_i + \beta_{ij} x^j + \gamma_{ijk} x^j x^k
\] (9.191)
in which \(\gamma_{ijk} = \gamma_{ikj}\).

Killing’s conformal condition (9.151) must hold for all \(x\)’s and is linear in the vector \(y\), so we can impose it successively on \(a_i, b_{ij}, \) and \(c_{ijk}\). The vector \(\alpha^i\) is an arbitrary translation. The condition on \(\beta_{ij}\) is
\[
\beta_{ik} + \beta_{ki} = \frac{2}{d} \eta_{ik} \beta_{\ell}\]
(9.192)

which says that \(\beta_{ij}\) is the sum
\[
\beta_{ij} = b_{ij} + c \eta_{ij}
\] (9.193)
of an antisymmetric part \(b_{ij} = -b_{ji}\) that generates a Lorentz transformation and a term proportional to \(\eta_{ij}\) that generates a scale transformation. The conformal condition (9.151) makes the quadratic term satisfy the relations
\[
\gamma_{ik\ell} + \gamma_{k\ell i} = 2\eta_{ik} d_{\ell}
\] (9.194)
\[
\gamma_{i\ell k} + \gamma_{\ell ik} = 2\eta_{i\ell} d_k
\] (9.195)
\[
\gamma_{k\ell i} + \gamma_{\ell ki} = 2\eta_{k\ell} d_i
\] (9.196)
in which
\[
d_{\ell} = \frac{1}{d} \gamma_{r\ell}.
\] (9.197)

Subtracting the last equation (9.196) from the sum of the first two (9.194 & gamma 2), we find
\[
\gamma_{ik\ell} = \eta_{ik} d_{\ell} + \eta_{i\ell} d_k - \eta_{k\ell} d_i.
\] (9.198)

The corresponding infinitesimal transformation is
\[
x'^i = x^i + 2(x \cdot d)x^i - d^i x^2
\] (9.199)
9.9 Maxwell’s action is conformally invariant for \( d = 4 \)

which is called a **special conformal transformation** whose exponential or finite form is

\[
x'^i = \frac{x^i - d^i x^2}{1 - 2d \cdot x + d^2 x^2}.
\] (9.200)

The general Killing vector and its infinitesimal transformation are

\[
y^i = a^i + b^j_k x^k + cx^i + d_k(\eta^{ik} x^2 - 2x^i x^k)
\] (9.201) and

\[
x'^i = x^i + a^i + b^j_k x^k + cx^i + d_k(\eta^{ik} x^2 - 2x^i x^k)
\] (9.202) in which we see a translation, a Lorentz transformation, a dilation, and a special conformal transformation.

The differential forms of the Poincaré transformations and the dilation \( D \) and the inversion \( k \) are

\[
P_i = \partial_i, \quad J_{ik} = (x_i \partial_k - x_k \partial_i),
\]

\[
D = x^i \partial_i, \quad \text{and} \quad K^i = (\eta^{ik} x^j x_j - 2x^i x^k) \partial_k.
\] (9.203)

The commutators of the conformal Lie algebra are

\[
[P^i, P^j] = 0, \quad [K^i, K^j] = 0,
\]

\[
[D, P^i] = - P^i, \quad [D, J_{ij}] = 0, \quad [D, K^i] = K^i,
\]

\[
[J^{ij}, P^i] = - \eta^{ij} P^j + \eta^{j\ell} P^i, \quad [J^{ij}, K^\ell] = -\eta^{\ell k} K^j + \eta^{i\ell} K^k,
\] (9.204)

\[
[J^{ij}, J^{j\ell}] = -\eta^{ij} J^{j\ell} - \eta^{ij} J^{j\ell} + \eta^{ij} J^{j\ell} + \eta^{ij} J^{j\ell},
\]

\[
[K^i, P^j] = 2J^{ij} + 2\eta^{ij} D.
\]

These commutation relations say that \( D \) is a Lorentz scalar, and that \( K^i \) transforms as a vector.

There are \( dP^i \)'s, \( dK^i \)'s, \( d(d-1)/2 \) \( J \)'s, and one \( D \). That’s \((d+2)(d+1)/2\) generators, which is the same number of generators as \( SO(d + 2) \). In fact, Minkowski space in \( d \) dimensions has \( SO(d - 1, 1) \) as its Lorentz group and \( SO(d, 2) \) as its conformal algebra. The “Lorentz group” of a spacetime with \( d \) spatial dimensions and 2 time dimensions is also \( SO(d, 2) \).

**9.9 Maxwell’s action is conformally invariant for \( d = 4 \)**

Under a change of coordinates from \( x \) to \( x' \), a covariant vector field \( A_i(x) \) changes this way

\[
A'_i(x') = x^k_{,i'} A_k(x) = \frac{\partial x^k}{\partial x'^i} A_k(x).
\] (9.205)
Thus the change in Maxwell’s action density is
\[ x' = x - y \quad \text{or} \quad x = x' + y \] (9.206)

the change in \( A_i \) is
\[
\delta A_i(x) = A'_i(x) - A_i(x) \\
= A'_i(x) - A'_i(x') + A'_i(x') - A_i(x) \\
= y^j A'_{i,j}(x') + \left( \delta^j_i + y^j_{i',i} \right) A_j(x) - A_i(x) \\
= y^j A'_{i,j}(x') + y^j_{i',i} A_j(x).
\] (9.207)

To lowest order in \( y \), this is
\[
\delta A_i(x) = y^j A_{i,j}(x) + y^j_{i,i} A_j(x). \quad (9.208)
\]

So the change in the Maxwell-Faraday tensor is
\[
\delta F_{ik} = \delta (\partial_i A_k - \partial_k A_i) = \partial_i (\delta A_k) - \partial_k (\delta A_i) \\
= \partial_i \left( y^j A_{k,j}(x) + y^j_{j,k} A_j(x) \right) - \partial_k \left( y^j A_{i,j}(x) + y^j_{i,i} A_j(x) \right) \\
= y^j \partial_j F_{ik} + y^j_{i,j} A_{k,j} - y^j_{j,k} A_i j + y^j_{j,i} A_{j,i} + y^j_{i,k} \partial_i A_j - y^j_{i,i} \partial_k A_j \\
= y^j \partial_j F_{ik} + y^j_{i,j} F_{jk} + y^j_{i,k} F_{ij}.
\] (9.209)

Thus the change in Maxwell’s action density is
\[
\delta \left( F^{ik} F_{ik} \right) = 2 F^{ik} \delta F_{ik} = 2 F^{ik} \left( y^j \partial_j F_{ik} + y^j_{i,j} F_{jk} + y^j_{i,k} F_{ij} \right) \\
= \partial_j \left( y^j F^{ik} F_{ik} \right) - (\partial \cdot y) F^{ik} F_{ik} + 2 y^j_{i} F^{ik} F_{jk} + 2 y^j_{k} F^{ik} F_{ij} \\
= \partial_j \left( y^j F^{ik} F_{ik} \right) - (\partial \cdot y) F^{ik} F_{ik} + 2 y^j_{i} F^{ik} F_{jk} + 2 y^j_{k} F^{ik} F_{ij} \\
= \partial_j \left( y^j F^{ik} F_{ik} \right) - (\partial \cdot y) F^{ik} F_{ik} + 4 y^j_{i} F^{ik} F_{jk} \quad (9.210)
\]

We can rewrite this as
\[
\delta \left( F^{ik} F_{ik} \right) = \partial_j \left( y^j F^{ik} F_{ik} \right) + 2 F^{ij} F_{jk} \left( y_{k,i} + y_{i,k} - \frac{1}{2} \eta_{ik} \partial \cdot y \right). \quad (9.211)
\]

But if \( y \) is a conformal Killing vector obeying (9.151), then this change in the Maxwell action density is
\[
\delta \left( F^{ik} F_{ik} \right) = \partial_j \left( y^j F^{ik} F_{ik} \right) + 2 F^{ij} F_{jk} \left( \frac{2}{3} \eta_{ik} y_{j,\ell} - \frac{1}{2} \eta_{ik} \partial \cdot y \right). \quad (9.212)
\]
9.10 Massless scalar field theory is conformally invariant if \( d = 2 \)

If \( d = 4 \), this change is a total derivative

\[
\delta \left( F^{ik} F_{ik} \right) = \partial_j \left( y^j F^{ik} F_{ik} \right) \iff d = 4. \tag{9.213}
\]

The action density of pure (only gauge fields), classical Yang-Mills theory also is conformally invariant in four dimensions (\( d = 4 \)). Quantum effects introduce a mass scale, however, and break conformal invariance.

9.10 Massless scalar field theory is conformally invariant if \( d = 2 \)

Under a tiny change of coordinates \( x = x' + y \), the change in a scalar field \( \phi \) is

\[
\delta \phi = y^j \phi_{,j} = y_j \phi^j. \tag{9.214}
\]

So the change in the action density is

\[
\delta \left( \frac{1}{2} \phi^{,i} \phi_{,i} \right) = \phi^{,i} \delta \phi_{,i} = \phi^{,i} \partial_i \delta \phi = \phi^{,i} \partial_i \left( y_j \phi^j \right) = \phi^{,i} \phi^j \delta y_{,j,i} + \phi^{,i} y_j \phi^{,j}_{,i} \tag{9.215}
\]

\[
= \frac{1}{2} \phi^{,i} \phi^j \left( y_{,j,i} + y_{,i,j} \right) + \phi^{,i} \phi^j y_j.
\]

If \( y \) obeys the condition (9.151) for a conformal Killing vector, then the change (9.215) in the action density is a total divergence in two dimensions

\[
\delta \left( \frac{1}{2} \phi^{,i} \phi_{,i} \right) = \frac{1}{2} \phi^{,i} \phi^j \partial_j \left( y_{,i} y_{,\ell} + \phi^{,i} \phi^j y_j \right) = \frac{1}{d} \phi^{,i} \phi_{,i} y_{,\ell}^\ell + \frac{1}{2} \left( \phi^{,i} \phi_{,i} \right)_{,\ell} y^\ell = \partial_\ell \left( \frac{1}{2} \phi^{,i} \phi_{,i} y^\ell \right) \iff d = 2. \tag{9.216}
\]

Thus the classical theory of a free massless scalar field is conformally invariant in two-dimensional spacetimes.

The rest of this chapter is at best a first draft, not ready for human consumption.