10

Group theory

10.1 What is a group?

A group G is a set of elements f, g, h, ... and an operation called multiplication such that for all elements f, g, and h in the group G:

1. The product fg is in the group G (**closure**);
2. $f(gh) = (fg)h$ (**associativity**);
3. there is an **identity** element e in the group G such that $ge = eg = g$;
4. every g in G has an **inverse** g^{-1} in G such that $gg^{-1} = g^{-1}g = e$.

Physical transformations naturally form groups. The elements of a group might be all physical transformations on a given set of objects that leave invariant a chosen property of the set of objects. For instance, the objects might be the points (x, y) in a plane. The chosen property could be their distances $\sqrt{x^2 + y^2}$ from the origin. The physical transformations that leave unchanged these distances are the rotations about the origin

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \tag{10.1}$$

These rotations form the special orthogonal group in 2 dimensions, $SO(2)$.

More generally, suppose the transformations $T, T', T'', \ldots$ change a set of objects in ways that leave invariant a chosen property property of the objects. Suppose the product $T'\,T$ of the transformations T and T' represents the action of T followed by the action of T' on the objects. Since both T and T' leave the chosen property unchanged, so will their product $T'\,T$. Thus the closure condition is satisfied. The triple products $T''\,(T'\,T)$ and $(T''\,T')\,T$ both represent the action of T followed by the action of T' followed by action of T''. Thus the action of $T''\,(T'\,T)$ is the same as the action of

$(T'' T') T$, and so the transformations are associative. The identity element e is the null transformation, the one that does nothing. The inverse T^{-1} is the transformation that reverses the action of T. Thus physical transformations that leave a chosen property unchanged naturally form a group.

Example 10.1 (Permutations) A **permutation** of an ordered set of n objects changes the order but leaves the set unchanged. □

Example 10.2 (Groups of coordinate transformations) The set of all transformations that leave invariant the distance from the origin of every point in n-dimensional space is the group $O(n)$ of **rotations** and **reflections**. The rotations in n-space form the special orthogonal group $SO(n)$.

Linear transformations $\boldsymbol{x}' = \boldsymbol{x} + \boldsymbol{a}$, for different n-dimensional vectors $\boldsymbol{a}$, leave invariant the spatial difference $\boldsymbol{x} - \boldsymbol{y}$ between every pair of points $\boldsymbol{x}$ and $\boldsymbol{y}$ in n-dimensional space. They form the group of **translations**. Here, group multiplication is vector addition.

The set of all linear transformations that leave invariant the square of the Minkowski distance $x_1^2 + x_2^2 + x_3^2 - x_0^2$ between any 4-vector x and the origin is the **Lorentz group** (Hermann Minkowski 1864–1909, Hendrik Lorentz 1853–1928).

The set of all linear transformations that leave invariant the square of the Minkowski distance $(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2 - (x^0 - y^0)^2$ between any two 4-vectors x and y is the **Poincaré group**, which includes Lorentz transformations and translations (Henri Poincaré 1854–1912). □

In the group of translations, the order of multiplication (which is vector addition) does not matter. A group whose elements all **commute**

$$[g, h] \equiv g\,h - h\,g = 0 \tag{10.2}$$

is said to be **abelian**. Except for the group of translations and the group $SO(2)$, the order of the physical transformations in these examples does matter: the transformation $T' T$ is not in general the same as $T T'$. Such groups are **nonabelian** (Niels Abel 1802–1829).

Matrices naturally form groups with group multiplication defined as matrix multiplication. Since matrix multiplication is associative, any set of $n \times n$ non-singular matrices D that includes the inverse D^{-1} of every matrix in the set as well as the identity matrix I automatically satisfies three of the four properties that characterize a group, leaving only the closure property uncertain. Such a set $\{D\}$ of matrices will form a group as long as the prod-

uct of any two matrices is in the set. As with physical transformations, one way to ensure closure is to have every matrix leave something unchanged.

Example 10.3 (Orthogonal groups) The group of all real matrices that leave unchanged the squared distance $x_1^2+\cdots+x_n^2$ of a point $x = (x_1, \ldots, x_n)$ from the origin is the group $O(n)$ of all $n \times n$ orthogonal (1.37) matrices (exercises 10.1 & 10.2). The group $O(2n)$ leaves unchanged the anticommutation relations (section 10.18) of the real and imaginary parts of n complex fermionic operators $\psi_1, \ldots, \psi_n$. The $n \times n$ orthogonal matrices of unit determinant form the special orthogonal group $SO(n)$. The group $SO(3)$ describes rotations in 3-space.

□

Example 10.4 (Unitary groups) The set of all $n \times n$ complex matrices that leave invariant the quadratic form $z_1^* z_1 + z_2^* z_2 + \cdots + z_n^* z_n$ forms the unitary group $U(n)$ of all $n \times n$ unitary (1.36) matrices (exercises 10.3 & 10.4). Those of unit determinant form the special unitary group $SU(n)$ (exercise 10.5).

Like $SO(3)$, the group $SU(2)$ represents rotations. The group $SU(3)$ is the symmetry group of the strong interactions, quantum chromodynamics. Physicists have used the groups $SU(5)$ and $SO(10)$ to unify the electroweak and strong interactions; whether nature also does so is unclear. □

Example 10.5 (Symplectic groups) The set of all $2n \times 2n$ real matrices R that leave invariant the commutation relations $[q_i, p_k] = i\hbar\delta_{ik}$, $[q_i, q_k] = 0$, and $[p_i, p_k] = 0$ of quantum mechanics is the symplectic group $Sp(2n, \mathbb{R})$. □

The number of elements in a group is the **order** of the group. A **finite group** is a group with a finite number of elements, or equivalently a group of finite order.

Example 10.6 (Z_2 and Z_n) The **parity** group whose elements are 1 and -1 under ordinary multiplication is the finite group $\boldsymbol{Z_2}$. It is abelian and of order 2. The group Z_n for any positive integer n is made of the phases $\exp(i2k\pi/n)$ for $k = 1, 2 \ldots n$. It is abelian and of order n. □

A group whose elements $g = g(\{\alpha\})$ depend continuously upon a set of parameters α_k is a **continuous group** or a **Lie group**. Continuous groups are of infinite order.

A group G of matrices D is **compact** if the (squared) norm as given by the trace

$$\mathrm{Tr}\left(D^\dagger D\right) \leq M \tag{10.3}$$

is bounded for all the $D \in G$.

Example 10.7 ($SO(n)$, $O(n)$, $SU(n)$, and $U(n)$) The groups $SO(n)$, $O(n)$, $SU(n)$, and $U(n)$ are continuous Lie groups of infinite order. Since for any matrix D in one of these groups

$$\mathrm{Tr}\left(D^\dagger D\right) = \mathrm{Tr}\, I = n \leq M \tag{10.4}$$

these groups also are compact. □

Example 10.8 (Noncompact groups) The set of all real $n \times n$ matrices forms the general linear group $GL(n, \mathbb{R})$; those of unit determinant form the special linear group $SL(n, \mathbb{R})$. The corresponding groups of matrices with complex entries are $GL(n, \mathbb{C})$ and $SL(n, \mathbb{C})$. These four groups and the symplectic groups $Sp(2n, \mathbb{R})$ and $Sp(2n, \mathbb{C})$ have matrix elements that are unbounded; they are noncompact. They are continuous Lie groups of infinite order like the orthogonal and unitary groups. The group $SL(2, \mathbb{C})$ represents Lorentz transformations. □

Incidentally, a **semigroup** is a set of elements $G = \{f, g, h, \dots\}$ and an operation called multiplication that is closed, $fg \in G$, and associative, $f(gh) = (fg)h$, but that may lack an identity element and inverses.

10.2 Representations of Groups

If one can associate with every element g of a group G a square matrix $D(g)$ and have matrix multiplication imitate group multiplication

$$D(f)\, D(g) = D(fg) \tag{10.5}$$

for all elements f and g of the group G, then the set of matrices $D(g)$ is said to form a **representation** of the group G. If the matrices of the representation are $n \times n$, then n is the **dimension** of the representation. The dimension of a representation also is the dimension of the vector space on which the matrices act. If the matrices $D(g)$ are unitary $D^\dagger(g) = D^{-1}(g)$, then they form a **unitary representation** of the group.

Example 10.9 (Representations of the groups $SU(2)$ and $SO(3)$) The defining representations of $SU(2)$ and $SO(3)$ are the 2×2 complex matrix

$$D(\boldsymbol{\theta}) = \begin{pmatrix} \cos\frac{1}{2}\theta + i\hat{\theta}_3 \sin\frac{1}{2}\theta & i(\hat{\theta}_1 - i\hat{\theta}_2) \sin\frac{1}{2}\theta \\ i(\hat{\theta}_1 + i\hat{\theta}_2) \sin\frac{1}{2}\theta & \cos\frac{1}{2}\theta - i\hat{\theta}_3 \sin\frac{\theta}{2} \end{pmatrix} \tag{10.6}$$

and the 3×3 real matrix

$$D(\boldsymbol{\theta}) = \begin{pmatrix} c + \hat{\theta}_1^2(1-c) & \hat{\theta}_1\hat{\theta}_2(1-c) - \hat{\theta}_3 s & \hat{\theta}_1\hat{\theta}_3(1-c) + \hat{\theta}_2 s \\ \hat{\theta}_2\hat{\theta}_1(1-c) + \hat{\theta}_3 s & c + \hat{\theta}_2^2(1-c) & \hat{\theta}_2\hat{\theta}_3(1-c) - \hat{\theta}_1 s \\ \hat{\theta}_3\hat{\theta}_1(1-c) - \hat{\theta}_2 s & \hat{\theta}_3\hat{\theta}_2(1-c) + \hat{\theta}_1 s & c + \hat{\theta}_3^2(1-c) \end{pmatrix} \tag{10.7}$$

in which $\theta = |\boldsymbol{\theta}|$, $\hat{\theta}_i = \theta_i/\theta$, $c = \cos\theta$, and $s = \sin\theta$. □

Compact groups possess finite-dimensional unitary representations; noncompact groups do not. A group of bounded (10.3) matrices is compact. An abstract group of elements $g(\{\alpha\})$ is compact if its space of parameters $\{\alpha\}$ is **closed and bounded**. (A set is **closed** if the limit of every convergent sequence of its points lies in the set. A set is **open** if each of its elements lies in a neighborhood that lies in the set. For example, the interval $[a, b] \equiv \{x|a \le x \le b\}$ is closed, while $(a, b) \equiv \{x|a < x < b\}$ is open.) The group of rotations is compact, but the group of translations and the Lorentz group are noncompact.

Every $n \times n$ matrix S that is nonsingular ($\det S \neq 0$) maps any $n \times n$ representation $D(g)$ of a group G into an **equivalent representation** $D'(g)$ through the **similarity transformation**

$$D'(g) = S^{-1} D(g) S \tag{10.8}$$

which preserves the law of multiplication

$$\begin{aligned} D'(f)\, D'(g) &= S^{-1} D(f) S\; S^{-1} D(g) S \\ &= S^{-1} D(f)\, D(g) S = S^{-1} D(fg) S = D'(fg). \end{aligned} \tag{10.9}$$

A **proper subspace** W of a vector space V is a subspace of lower (but not zero) dimension. A proper subspace W is **invariant** under the action of a representation $D(g)$ if $D(g)$ maps every vector $v \in W$ to a vector $D(g)\, v = v' \in W$. A representation that has a **proper invariant subspace** is **reducible**. A representation that is not reducible is **irreducible**.

A representation $D(g)$ is **completely reducible** if it is equivalent to a representation whose matrices are in **block-diagonal form**

$$S^{-1} D(g)\, S = \begin{pmatrix} D_1(g) & 0 & \dots \\ 0 & D_2(g) & \dots \\ \vdots & \vdots & \vdots \end{pmatrix} \tag{10.10}$$

in which each representation $D_i(g)$ irreducible. A representation in block-

diagonal form is said to be a **direct sum** of its irreducible representations

$$S^{-1} D\, S = D_1 \oplus D_2 \oplus \dots . \tag{10.11}$$

10.3 Representations Acting in Hilbert Space

A symmetry transformation g is a map (1.187) of states $\psi \to \psi'$ that preserves probabilities

$$|\langle \phi' | \psi' \rangle|^2 = |\langle \phi | \psi \rangle|^2. \tag{10.12}$$

The action of a group G of symmetry transformations g on the Hilbert space of a quantum theory can be represented either by operators $U(g)$ that are linear and unitary (the usual case) or by ones $K(g)$ that are antilinear (1.185) and antiunitary (1.186), as in the case of time reversal. Wigner proved this theorem in the 1930's, and Weinberg improved it in his 1995 classic (Weinberg, 1995, p. 51) (Eugene Wigner, 1902–1995; Steven Weinberg, 1933–).

Two operators F_1 and F_2 that commute $F_1\, F_2 = F_2\, F_1$ are **compatible** (1.374). A set of compatible operators F_1, F_2, ... is **complete** if to every set of eigenvalues there belongs only a single eigenvector (section 1.31).

Example 10.10 (Rotation Operators) Suppose that the hamiltonian H, the square of the angular momentum $\boldsymbol{J}^2$, and its z-component J_z form a complete set of compatible observables, so that the identity operator can be expressed as a sum over the eigenvectors of these operators

$$I = \sum_{E,j,m} |E, j, m\rangle\langle E, j, m|. \tag{10.13}$$

Then the matrix element of a unitary operator $U(g)$ between two states $|\psi\rangle$ and $|\phi\rangle$ is

$$\langle \phi | U(g) | \psi \rangle = \langle \phi | \sum_{E',j',m'} |E', j', m'\rangle\langle E', j', m'|\, U(g) \sum_{E,j,m} |E, j, m\rangle\langle E, j, m | \psi \rangle. \tag{10.14}$$

Let H and $\boldsymbol{J}^2$ be invariant under the action of $U(g)$ so that $U^\dagger(g) H U(g) = H$ and $U^\dagger(g) \boldsymbol{J}^2 U(g) = \boldsymbol{J}^2$. Then $HU(g) = U(g)H$ and $\boldsymbol{J}^2 U(g) = U(g)\boldsymbol{J}^2$, and so if $H|E, j, m\rangle = E|E, j, m\rangle$ and $\boldsymbol{J}^2|E, j, m\rangle = j(j+1)|E, j, m\rangle$, we have

$$\begin{aligned} HU(g)|E, j, m\rangle &= U(g)H|E, j, m\rangle = EU(g)|E, j, m\rangle \\ \boldsymbol{J}^2 U(g)|E, j, m\rangle &= U(g)\boldsymbol{J}^2|E, j, m\rangle = j(j+1)U(g)|E, j, m\rangle. \end{aligned} \tag{10.15}$$

Thus $U(g)$ cannot change E or j, and so

$$\langle E', j', m'|U(g)|E, j, m\rangle = \delta_{E'E}\delta_{j'j}\langle j, m'|U(g)|j, m\rangle = \delta_{E'E}\delta_{j'j}D^{(j)}_{m'm}(g). \tag{10.16}$$

The matrix element (10.14) is a single sum over E and j in which the irreducible representations $D^{(j)}_{m'm}(g)$ of the rotation group $SU(2)$ appear

$$\langle\phi|U(g)|\psi\rangle = \sum_{E,j,m',m} \langle\phi|E, j, m'\rangle D^{(j)}_{m'm}(g)\langle E, j, m|\psi\rangle. \tag{10.17}$$

This is how the block-diagonal form (10.10) usually appears in calculations. The matrices $D^{(j)}_{m'm}(g)$ inherit the unitarity of the operator $U(g)$. □

10.4 Subgroups

If all the elements of a group S also are elements of a group G, then S is a **subgroup** of G. Every group G has two **trivial subgroups**—the identity element e and the whole group G itself. Many groups have more interesting subgroups. For example, the rotations about a fixed axis is an abelian subgroup of the group of all rotations in 3-dimensional space.

A subgroup $S \subset G$ is an **invariant** subgroup if every element s of the subgroup S is left inside the subgroup under the **action** of every element g of the whole group G, that is, if

$$g^{-1}s\,g = s' \in S \quad \text{for all} \quad g \in G. \tag{10.18}$$

This condition often is written as $g^{-1}Sg = S$ for all $g \in G$ or as

$$S\,g = g\,S \quad \text{for all } g \in G. \tag{10.19}$$

Invariant subgroups also are called **normal subgroups**.

A set $C \subset G$ is called a **conjugacy class** if it's invariant under the action of the whole group G, that is, if $Cg = g\,C$ or

$$g^{-1}C\,g = C \quad \text{for all } g \in G. \tag{10.20}$$

A subgroup that is the union of a set of conjugacy classes is invariant.

The **center** C of a group G is the set of all elements $c \in G$ that **commute** with every element g of the group, that is, their **commutators**

$$[c, g] \equiv cg - gc = 0 \tag{10.21}$$

vanish for all $g \in G$.

Example 10.11 (Centers Are Abelian Subgroups) Does the center C always form an abelian subgroup of its group G? The product c_1c_2 of any two elements c_1 and c_2 of the center commutes with every element g of G since $c_1c_2g = c_1gc_2 = gc_1c_2$. So the center is closed under multiplication. The identity element e commutes with every $g \in G$, so $e \in C$. If $c' \in C$, then $c'g = gc'$ for all $g \in G$, and so multiplication of this equation from the left and the right by c'^{-1} gives $gc'^{-1} = c'^{-1}g$, which shows that $c'^{-1} \in C$. The subgroup C is abelian because each of its elements commutes with all the elements of G including those of C itself. □

So the center of any group always is one of its abelian invariant subgroups. The center may be trivial, however, consisting either of the identity or of the whole group. But a group with a nontrivial center can not be simple or semisimple (section 10.27).

10.5 Cosets

If H is a subgroup of a group G, then for every element $g \in G$ the set of elements $Hg \equiv \{hg|h \in H\}$ is a **right coset of the subgroup $H \subset G$**. (Here $\subset$ means *is a subset of* or equivalently *is contained in*.)

If H is a subgroup of a group G, then for every element $g \in G$ the set of elements gH is a **left coset of the subgroup $H \subset G$**.

The number of elements in a coset is the same as the number of elements of H, which is the order of H.

An element g of a group G is in one and only one right coset (and in one and only one left coset) of the subgroup $H \subset G$. For suppose instead that g were in two right cosets $g \in Hg_1$ and $g \in Hg_2$, so that $g = h_1g_1 = h_2g_2$ for suitable $h_1, h_2 \in H$ and $g_1, g_2 \in G$. Then since H is a (sub)group, we have $g_2 = h_2^{-1}h_1g_1 = h_3g_1$, which says that $g_2 \in Hg_1$. But this means that every element $hg_2 \in Hg_2$ is of the form $hg_2 = hh_3g_1 = h_4g_1 \in Hg_1$. So every element $hg_2 \in Hg_2$ is in Hg_1: the two right cosets are identical, $Hg_1 = Hg_2$.

The right (or left) cosets are the points of the **quotient coset space** G/H.

If H is an invariant subgroup of G, then by definition (10.19) $Hg = gH$ for all $g \in G$, and so the left cosets are the same sets as the right cosets. In this case, the coset space G/H is itself a group with multiplication defined

by

$$\begin{aligned}(Hg_1)\,(Hg_2) &= \{h_i g_1 h_j g_2 | h_i, h_j \in H\} \\ &= \left\{h_i g_1 h_j g_1^{-1} g_1 g_2 | h_i, h_j \in H\right\} \\ &= \{h_i h_k g_1 g_2 | h_i, h_k \in H\} \\ &= \{h_\ell g_1 g_2 | h_\ell \in H\} = H g_1 g_2 \end{aligned} \tag{10.22}$$

which is the multiplication rule of the group G. This group G/H is called the **quotient** or **factor** group of $\boldsymbol{G}$ by $\boldsymbol{H}$

10.6 Morphisms

An **isomorphism** is a one-to-one map between groups that respects their multiplication laws. For example, a similarity transformation (10.8) relates two equivalent representations

$$D'(g) = S^{-1} D(g) S \tag{10.23}$$

and is an isomorphism (exercise 10.8). An **automorphism** is an isomorphism between a group and itself. The map $g_i \to g\, g_i\, g^{-1}$ is one to one because $g\, g_1\, g^{-1} = g\, g_2\, g^{-1}$ implies that $g\, g_1 = g\, g_2$, and so that $g_1 = g_2$. This map also preserves the law of multiplication since $g\, g_1\, g^{-1}\, g\, g_2\, g^{-1} = g\, g_1\, g_2\, g^{-1}$. So the map

$$G \to g G g^{-1} \tag{10.24}$$

is an automorphism. It is called an **inner automorphism** because g is an element of G. An automorphism not of this form (10.24) is an **outer automorphism**.

10.7 Schur's Lemma

Part 1: If D_1 and D_2 are inequivalent, irreducible representations of a group G, and if $D_1(g)A = AD_2(g)$ for some matrix A and for all $g \in G$, then the matrix A must vanish, $A = 0$.

Proof: First suppose that A annihilates some vector $|x\rangle$, that is, $A|x\rangle = 0$. Let P be the projection operator into the subspace that A annihilates, which is of at least one dimension. This subspace, incidentally, is called the **null space** $\mathcal{N}(A)$ or the **kernel** of the matrix A. The representation D_2 must leave this null space $\mathcal{N}(A)$ invariant since

$$AD_2(g)P = D_1(g)AP = 0. \tag{10.25}$$

If $\mathcal{N}(A)$ were a proper subspace, then it would be a proper invariant subspace of the representation D_2, and so D_2 would be reducible, which is contrary to our assumption that D_1 and D_2 are irreducible. So the null space $\mathcal{N}(A)$ must be the whole space upon which A acts, that is, $A = 0$.

A similar argument shows that if $\langle y|A = 0$ for some bra $\langle y|$, then $A = 0$.

So either A is zero or it annihilates no ket and no bra. In the latter case, A must be square and invertible, which would imply that $D_2(g) = A^{-1}D_1(g)A$, that is, that D_1 and D_2 are equivalent representations, which is contrary to our assumption that they are inequivalent. The only way out is that A vanishes.

Part 2: If for a finite-dimensional, irreducible representation $D(g)$ of a group G, we have $D(g)A = AD(g)$ for some matrix A and for all $g \in G$, then $A = cI$. That is, any matrix that commutes with every element of a finite-dimensional, irreducible representation must be a multiple of the identity matrix.

Proof: Every square matrix A has at least one eigenvector $|x\rangle$ and eigenvalue c so that $A|x\rangle = c|x\rangle$ because its characteristic equation $\det(A-cI) = 0$ always has at least one root by the fundamental theorem of algebra (5.79). So the null space $\mathcal{N}(A - cI)$ has dimension greater than zero. The assumption $D(g)A = AD(g)$ for all $g \in G$ implies that $D(g)(A - cI) = (A - cI)D(g)$ for all $g \in G$. Let P be the projection operator onto the null space $\mathcal{N}(A - cI)$. Then we have $(A - cI)D(g)P = D(g)(A - cI)P = 0$ for all $g \in G$ which implies that $D(g)P$ maps vectors into the null space $\mathcal{N}(A - cI)$. This null space therefore is a subspace that is invariant under $D(g)$, which means that D is reducible unless the null space $\mathcal{N}(A - cI)$ is the whole space. Since by assumption D is irreducible, it follows that $\mathcal{N}(A - cI)$ is the whole space, that is, that $A = cI$. (Issai Schur, 1875–1941)

Example 10.12 (Schur, Wigner, and Eckart) Suppose an arbitrary observable O is invariant under the action of the rotation group $SU(2)$ represented by unitary operators $U(g)$ for $g \in SU(2)$

$$U^\dagger(g)\, O\, U(g) = O \quad \text{or} \quad [O, U(g)] = 0. \tag{10.26}$$

These unitary rotation operators commute with the square $\boldsymbol{J}^2$ of the angular momentum $[\boldsymbol{J}^2, U] = 0$. Suppose that they also leave the hamiltonian H unchanged $[H, U] = 0$. Then as shown in example 10.10, the state $U|E, j, m\rangle$

is a sum of states all with the same values of j and E. It follows that

$$\begin{aligned}\sum_{m'}&\langle E,j,m|O|E',j',m'\rangle\langle E',j',m'|U(g)|E',j',m''\rangle\\ &=\sum_{m'}\langle E,j,m|U(g)|E,j,m'\rangle\langle E,j,m'|O|E',j',m''\rangle\end{aligned}\tag{10.27}$$

or in the notation of (10.16)

$$\sum_{m'}\langle E,j,m|O|E',j',m'\rangle D^{(j')}(g)_{m'm''}=\sum_{m'}D^{(j)}(g)_{mm'}\langle E,j,m'|O|E',j',m''\rangle.\tag{10.28}$$

Now Part 1 of Schur's lemma tells us that the matrix $\langle E,j,m|O|E',j',m'\rangle$ must vanish unless the representations are equivalent, which is to say unless $j=j'$. So we have

$$\sum_{m'}\langle E,j,m|O|E',j,m'\rangle D^{(j)}(g)_{m'm''}=\sum_{m'}D^{(j)}(g)_{mm'}\langle E,j,m'|O|E',j,m''\rangle.\tag{10.29}$$

Now Part 2 of Schur's lemma tells us that the matrix $\langle E,j,m|O|E',j,m'\rangle$ must be a multiple of the identity. Thus the symmetry of O under rotations simplifies the matrix element to

$$\langle E,j,m|O|E',j',m'\rangle=\delta_{jj'}\delta_{mm'}O_j(E,E').\tag{10.30}$$

This result is a special case of the **Wigner-Eckart theorem** (Eugene Wigner 1902–1995, Carl Eckart 1902–1973). □

10.8 Characters

Suppose the $n\times n$ matrices $D_{ij}(g)$ form a representation of a group $G\ni g$. The **character** $\chi_D(g)$ of the matrix $D(g)$ is the trace

$$\chi_D(g)=\mathrm{Tr}D(g)=\sum_{i=1}^{n}D_{ii}(g).\tag{10.31}$$

Traces are cyclic, that is, $\mathrm{Tr}ABC=\mathrm{Tr}BCA=\mathrm{Tr}CAB$. So if two representations D and D' are equivalent, so that $D'(g)=S^{-1}D(g)S$, then they have the same characters because

$$\chi_{D'}(g)=\mathrm{Tr}D'(g)=\mathrm{Tr}\left(S^{-1}D(g)S\right)=\mathrm{Tr}\left(D(g)SS^{-1}\right)=\mathrm{Tr}D(g)=\chi_D(g).\tag{10.32}$$

If two group elements g_1 and g_2 are in the same conjugacy class, that is,

if $g_2 = g g_1 g^{-1}$ for all $g \in G$, then they have the same character in a given representation $D(g)$ because

$$\begin{aligned}\chi_D(g_2) &= \operatorname{Tr} D(g_2) = \operatorname{Tr} D(g g_1 g^{-1}) = \operatorname{Tr}\big(D(g)D(g_1)D(g^{-1})\big)\\ &= \operatorname{Tr}\big(D(g_1)D^{-1}(g)D(g)\big) = \operatorname{Tr} D(g_1) = \chi_D(g_1).\end{aligned} \tag{10.33}$$

10.9 Direct products

Suppose $D^{(a)}(g)$ is a k-dimensional representation of a group G, and $D^{(b)}(g)$ is an n-dimensional representation of the same group. Then their product

$$D^{(a,b)}_{im,j\ell}(g) = D^{(a)}_{ij}(g)\, D^{(b)}_{m\ell}(g) \tag{10.34}$$

is a (kn)-dimensional **direct-product** representation of the group G. Direct products are also called **tensor products**. They occur in quantum systems that have two or more parts, each described by a different space of vectors.

Suppose the vectors $|i\rangle$ for $i = 1 \ldots k$ are the basis vectors of the k-dimensional space V_k on which the representation $D^{(a)}(g)$ acts, and that the vectors $|m\rangle$ for $m = 1 \ldots n$ are the basis vectors of the n-dimensional space V_n on which $D^{(b)}(g)$ acts. The kn vectors $|i, m\rangle$ are basis vectors for the kn-dimensional tensor-product space V_{kn}. The matrices $D^{(a,b)}(g)$ defined as

$$\langle i, m|D(g)^{(a,b)}|j, \ell\rangle = \langle i|D^{(a)}(g)|j\rangle\langle m|D^{(b)}(g)|\ell\rangle \tag{10.35}$$

act in this kn-dimensional space V_{kn} and form a representation of the group G; this direct-product representation usually is reducible. Many tricks help one to decompose reducible tensor-product representations into direct sums of irreducible representations (Georgi, 1999; Zee, 2016).

Example 10.13 (Adding Angular Momenta) The addition of angular momenta illustrates both the direct product and its reduction to a direct sum of irreducible representations. Let $D^{(j_1)}(g)$ and $D^{(j_2)}(g)$ respectively be the $(2j_1 + 1) \times (2j_1 + 1)$ and the $(2j_2 + 1) \times (2j_2 + 1)$ representations of the rotation group $SU(2)$. The direct-product representation $D^{(j_1,j_2)}$

$$\langle m_1', m_2'|D^{(j_1,j_2)}|m_1, m_2\rangle = \langle m_1'|D^{(j_1)}(g)|m_1\rangle\langle m_2'|D^{(j_2)}(g)|m_2\rangle \tag{10.36}$$

is reducible into a direct sum of all the irreducible representations of $SU(2)$ from $D^{(j_1+j_2)}(g)$ down to $D^{(|j_1-j_2|)}(g)$ in integer steps:

$$D^{(j_1,j_2)} = D^{(j_1+j_2)} \oplus D^{(j_1+j_2-1)} \oplus \cdots \oplus D^{(|j_1-j_2|+1)} \oplus D^{(|j_1-j_2|)} \tag{10.37}$$

each irreducible representation occurring once in the direct sum. □

Example 10.14 (Adding Two Spins) When one adds $j_1 = 1/2$ to $j_2 = 1/2$, one finds that the tensor-product matrix $D^{(1/2,1/2)}$ is equivalent to the direct sum $D^{(1)} \oplus D^{(0)}$

$$D^{(1/2,1/2)}(\boldsymbol{\theta}) = S^{-1} \begin{pmatrix} D^{(1)}(\boldsymbol{\theta}) & 0 \\ 0 & D^{(0)}(\boldsymbol{\theta}) \end{pmatrix} S \tag{10.38}$$

where the matrices S, $D^{(1)}$, and $D^{(0)}$ are 4×4, 3×3, and 1×1. □

10.10 Finite Groups

A **finite group** is one that has a finite number of elements. The number of elements in a group is the **order** of the group.

Example 10.15 (Z_2) The group $\boldsymbol{Z_2}$ consists of two elements e and p with multiplication rules

$$ee = e, \quad ep = pe = p, \quad \text{and} \quad pp = e. \tag{10.39}$$

Clearly, Z_2 is abelian, and its order is 2. The identification $e \to 1$ and $p \to -1$ gives a 1-dimensional representation of the group Z_2 in terms of 1×1 matrices, which are just numbers. □

It is tedious to write the multiplication rules as individual equations. Normally people compress them into a multiplication table like this:

$\times$	e	p
e	e	p
p	p	e

(10.40)

A simple generalization of Z_2 is the group $\boldsymbol{Z_n}$ whose elements may be represented as $\exp(i2\pi m/n)$ for $m = 1, \ldots, n$. This group is also abelian, and its order is n.

Example 10.16 (Z_3) The multiplication table for $\boldsymbol{Z_3}$ is

$\times$	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

(10.41)

which says that $a^2 = b$, $b^2 = a$, and $ab = ba = e$. □

10.11 The Regular Representation

For any finite group G we can associate an orthonormal vector $|g_i\rangle$ with each element g_i of the group. So $\langle g_i|g_j\rangle = \delta_{ij}$. These orthonormal vectors $|g_i\rangle$ form a basis for a vector space whose dimension is the order of the group. The matrix $D(g_k)$ of the regular representation of G is defined to map any vector $|g_i\rangle$ into the vector $|g_k g_i\rangle$ associated with the product $g_k g_i$

$$D(g_k)|g_i\rangle = |g_k g_i\rangle. \tag{10.42}$$

Since group multiplication is associative, we have

$$D(g_j)D(g_k)|g_i\rangle = D(g_j)|g_k g_i\rangle = |g_j(g_k g_i)\rangle = |(g_j g_k)g_i)\rangle = D(g_j g_k)|g_i\rangle. \tag{10.43}$$

Because the vector $|g_i\rangle$ was an arbitrary basis vector, it follows that

$$D(g_j)D(g_k) = D(g_j g_k) \tag{10.44}$$

which means that the matrices $D(g)$ satisfy the criterion (10.5) for their being a representation of the group G. The matrix $D(g)$ has entries

$$[D(g)]_{ij} = \langle g_i|D(g)|g_j\rangle. \tag{10.45}$$

The sum of dyadics $|g_\ell\rangle\langle g_\ell|$ over all the elements g_ℓ of a finite group G is the unit matrix

$$\sum_{g_\ell \in G} |g_\ell\rangle\langle g_\ell| = I_n \tag{10.46}$$

in which n is the order of G, that is, the number of elements in G. So by taking the m, n matrix element of the multiplication law (10.44), we find

$$\begin{aligned}[D(g_j g_k)]_{m,n} &= \langle g_m|D(g_j g_k)|g_n\rangle = \langle g_m|D(g_j)D(g_k)|g_n\rangle \\ &= \sum_{g_\ell \in G} \langle g_m|D(g_j)|g_\ell\rangle\langle g_\ell|D(g_k)|g_n\rangle = \sum_{g_\ell \in G} [D(g_j)]_{m,\ell}[D(g_k)]_{\ell,n}.\end{aligned} \tag{10.47}$$

Example 10.17 (Z_3's Regular Representation) The regular representation of Z_3 is

$$D(e) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad D(a) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad D(b) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \tag{10.48}$$

so $D(a)^2 = D(b)$, $D(b)^2 = D(a)$, and $D(a)D(b) = D(b)D(a) = D(e)$. □

10.12 Properties of Finite Groups

In his book (Georgi, 1999, ch. 1), Georgi proves the following theorems:

1. Every representation of a finite group is equivalent to a unitary representation.
2. Every representation of a finite group is completely reducible.
3. The irreducible representations of a finite abelian group are one dimensional.
4. If $D^{(a)}(g)$ and $D^{(b)}(g)$ are two unitary irreducible representations of dimensions n_a and n_b of a group G of N elements $g_1, \ldots, g_N$, then the functions

$$\sqrt{\frac{n_a}{N}}\, D^{(a)}_{jk}(g) \tag{10.49}$$

are orthonormal and complete in the sense that

$$\frac{n_a}{N} \sum_{j=1}^{N} D^{(a)*}_{ik}(g_j) D^{(b)}_{\ell m}(g_j) = \delta_{ab}\delta_{i\ell}\delta_{km}. \tag{10.50}$$

5. The order N of a finite group is the sum of the squares of the dimensions of its inequivalent irreducible representations

$$N = \sum_a n_a^2. \tag{10.51}$$

Example 10.18 (Z_N) The abelian cyclic group Z_N with elements

$$g_j = e^{2\pi i j/N} \tag{10.52}$$

has N one-dimensional irreducible representations

$$D^{(a)}(g_j) = e^{2\pi i a j/N} \tag{10.53}$$

for $a = 1, 2, \ldots, N$. Their orthonormality relation (10.50) is the Fourier formula

$$\frac{1}{N} \sum_{j=1}^{N} e^{-2\pi i a j/N} e^{2\pi i b j/N} = \delta_{ab}. \tag{10.54}$$

The n_a are all unity, there are N of them, and the sum of the n_a^2 is N as required by the sum rule (10.51). □

10.13 Permutations

The permutation group on n objects is called S_n. Permutations are made of **cycles** that change the order of some of the n objects. For instance, the permutation $(1\,2) = (2\,1)$ is a 2-cycle that means $x_1 \to x_2 \to x_1$; the unitary operator $U((1\,2))$ that represents it interchanges states like this:

$$U((1\,2))|+,-\rangle = U((1\,2))|+,1\rangle\,|-,2\rangle = |-,1\rangle, |+,2\rangle = |-,+\rangle. \tag{10.55}$$

The 2-cycle $(3\,4)$ means $x_3 \to x_4 \to x_3$, it changes (a,b,c,d) into (a,b,d,c). The 3-cycle $(1\,2\,3) = (2\,3\,1) = (3\,1\,2)$ means $x_1 \to x_2 \to x_3 \to x_1$, it changes (a,b,c,d) into (b,c,a,d). The 4-cycle $(1\,3\,2\,4)$ means $x_1 \to x_3 \to x_2 \to x_4 \to x_1$ and changes (a,b,c,d) into (c,d,b,a). The 1-cycle (2) means $x_2 \to x_2$ and leaves everything unchanged.

The identity element of S_n is the product of 1-cycles $e = (1)(2)\ldots(n)$. The inverse of the cycle $(1\,3\,2\,4)$ must invert $x_1 \to x_3 \to x_2 \to x_4 \to x_1$, so it must be $(1\,4\,2\,3)$ which means $x_1 \to x_4 \to x_2 \to x_3 \to x_1$ so that it changes (c,d,b,a) back into (a,b,c,d). Every element of S_n has each integer from 1 to n in one and only one cycle. So an arbitrary element of S_n with ℓ_k k-cycles must satisfy

$$\sum_{k=1}^{n} k\,\ell_k = n. \tag{10.56}$$

10.14 Compact and Noncompact Lie Groups

Imagine rotating an object repeatedly. Notice that the biggest rotation is by an angle of $\pm\pi$ about some axis. The possible angles form a circle; the space of parameters is a circle. The parameter space of a compact group is compact—closed and bounded. The rotations form a **compact group**.

Now consider the translations. Imagine moving a pebble to the Sun, then moving it to the next-nearest star, then moving it to the nearest galaxy. If space is flat, then there is no limit to how far one can move a pebble. The parameter space of a noncompact group is not compact. The translations form a **noncompact group**.

We'll see that compact Lie groups possess unitary representations, with $n \times n$ unitary matrices $D(\alpha)$, while noncompact ones don't. Here α stands for the parameters $\alpha_1, \ldots, \alpha_n$ that label the elements of the group, three for the rotation group. The α's usually are real, but can be complex.

10.15 Generators

To study continuous groups, we will use calculus and algebra, and we will focus on the simplest part of the group—the elements $g(\alpha)$ for $\alpha \approx 0$ which are near the identity $e = g(0)$ for which all $\alpha_a = 0$. Each element $g(\alpha)$ of the group is represented by a matrix $D(\alpha) \equiv D(g(\alpha))$ in the D representation of the group and by another matrix $D'(\alpha) \equiv D'(g(\alpha))$ in any other D' representation of the group. Every representation respects the multiplication law of the group. So if $g(\beta)g(\alpha) = g(\gamma)$, then the matrices of the D representation must satisfy $D(\beta)D(\alpha) = D(\gamma)$, and those of any other representation D' must satisfy $D'(\beta)D'(\alpha) = D'(\gamma)$.

A **generator** t_a of a representation D is the partial derivative of the matrix $D(\alpha)$ with respect to the component α_a of α evaluated at $\alpha = 0$

$$t_a = -i \left.\frac{\partial D(\alpha)}{\partial \alpha_a}\right|_{\alpha=0}. \tag{10.57}$$

When all the parameters α_a are infinitesimal, $|\alpha_a| \ll 1$, the matrix $D(\alpha)$ is very close to the identity matrix I

$$D(\alpha) \simeq I + i \sum_a \alpha_a \, t_a. \tag{10.58}$$

Replacing α by α/n, we get a relation that becomes exact as $n \to \infty$

$$D\left(\frac{\alpha}{n}\right) = I + i \sum_a \frac{\alpha_a}{n} \, t_a. \tag{10.59}$$

The nth power of this equation is the matrix $D(\alpha)$ that represents the group element $g(\alpha)$ in the **exponential parametrization**

$$D(\alpha) = D\left(\frac{\alpha}{n}\right)^n = \lim_{n\to\infty} \left(I + i \sum_a \frac{\alpha_a}{n} \, t_a\right)^n = \exp\left(i \sum_a \alpha_a \, t_a\right). \tag{10.60}$$

The i's appear in these equations so that when the generators t_a are hermitian matrices, $(t_a)^\dagger = t_a$, and the α's are real, the matrices $D(\alpha)$ are unitary

$$D^{-1}(\alpha) = \exp\left(-i \sum_a \alpha_a \, t_a\right) = D^\dagger(\alpha) = \exp\left(-i \sum_a \alpha_a \, t_a\right). \tag{10.61}$$

Compact groups have finite-dimensional, unitary representations with hermitian generators.

10.16 Lie algebra

If t_a and t_b are any two generators of a representation D, then the matrices

$$D(\alpha) = e^{i\epsilon t_a} \quad \text{and} \quad D(\beta) = e^{i\epsilon t_b} \tag{10.62}$$

represent the group elements $g(\alpha)$ and $g(\beta)$ with infinitesimal exponential parameters $\alpha_i = \epsilon\,\delta_{ia}$ and $\beta_i = \epsilon\,\delta_{ib}$. The inverses of these group elements $g^{-1}(\alpha) = g(-\alpha)$ and $g^{-1}(\beta) = g(-\beta)$ are represented by the matrices $D(-\alpha) = e^{-i\epsilon t_a}$ and $D(-\beta) = e^{-i\epsilon t_b}$. The multiplication law of the group determines the parameters $\gamma(\alpha,\beta)$ of the product

$$g(\beta)\,g(\alpha)\,g(-\beta)\,g(-\alpha) = g(\gamma(\alpha,\beta)). \tag{10.63}$$

The matrices of any two representations D with generators t_a and D' with generators t'_a obey the same multiplication law

$$\begin{aligned} D(\beta)\,D(\alpha)\,D(-\beta)\,D(-\alpha) &= D(\gamma(\alpha,\beta)) \\ D'(\beta)\,D'(\alpha)\,D'(-\beta)\,D'(-\alpha) &= D'(\gamma(\alpha,\beta)) \end{aligned} \tag{10.64}$$

with the same infinitesimal exponential parameters α, β, and $\gamma(\alpha,\beta)$. To order ϵ^2, the product of the four D's is

$$\begin{aligned} e^{i\epsilon t_b}\,e^{i\epsilon t_a}\,e^{-i\epsilon t_b}\,e^{-i\epsilon t_a} &\approx (1 + i\epsilon\,t_b - \frac{\epsilon^2}{2}\,t_b^2)(1 + i\epsilon\,t_a - \frac{\epsilon^2}{2}\,t_a^2) \\ &\quad \times (1 - i\epsilon\,t_b - \frac{\epsilon^2}{2}\,t_b^2)(1 - i\epsilon\,t_a - \frac{\epsilon^2}{2}\,t_a^2) \\ &\approx 1 + \epsilon^2(t_a\,t_b - t_b\,t_a) = 1 + \epsilon^2[t_a, t_b]. \end{aligned} \tag{10.65}$$

The other representation gives the same result but with primes

$$e^{i\epsilon t'_b}\,e^{i\epsilon t'_a}\,e^{-i\epsilon t'_b}\,e^{-i\epsilon t'_a} \approx 1 + \epsilon^2[t'_a, t'_b]. \tag{10.66}$$

The products (10.65 & 10.66) represent the same group element $g(\gamma(\alpha,\beta))$, so they have the same infinitesimal parameters $\gamma(\alpha,\beta)$ and therefore are the same linear combinations of their respective generators t_c and t'_c

$$\begin{aligned} D(\gamma(\alpha,\beta)) &\approx 1 + \epsilon^2[t_a, t_b] = 1 + i\epsilon^2 \sum_{c=1}^{n} f_{ab}^c\,t_c \\ D'(\gamma(\alpha,\beta)) &\approx 1 + \epsilon^2[t'_a, t'_b] = 1 + i\epsilon^2 \sum_{c=1}^{n} f_{ab}^c\,t'_c. \end{aligned} \tag{10.67}$$

which in turn imply the **Lie algebra** formulas

$$[t_a, t_b] = \sum_{c=1}^{n} f_{ab}^c\,t_c \quad \text{and} \quad [t'_a, t'_b] = \sum_{c=1}^{n} f_{ab}^c\,t'_c. \tag{10.68}$$

The commutator of any two generators is a linear combination of the generators. The coefficients $f^c{}_{ab}$ are the structure constants of the group. They are the same for all representations of the group.

Unless the parameters α_a are redundant, the generators are linearly independent. They span a vector space, and any linear combination may be called a generator. By using the Gram-Schmidt procedure (section 1.10), we may make the generators t_a orthogonal with respect to the inner product (1.91)

$$(t_a, t_b) = \mathrm{Tr}\left(t_a^\dagger t_b\right) = k\,\delta_{ab} \tag{10.69}$$

in which k is a nonnegative normalization constant that depends upon the representation. We can't normalize the generators, making k unity, because the structure constants f^c_{ab} are the same in all representations.

In what follows, I will often omit the summation symbol $\sum$ when an index is repeated. In this notation, the structure-constant formulas (10.68) are

$$[t_a, t_b] = f^c_{ab}\, t_c \quad \text{and} \quad [t'_a, t'_b] = f^c_{ab}\, t'_c. \tag{10.70}$$

This summation convention avoids unnecessary summation symbols.

By multiplying both sides of the first of the two Lie algebra formulas (10.68) by $t_d^\dagger$ and using the orthogonality (10.69) of the generators, we find

$$\mathrm{Tr}\left([t_a, t_b]\, t_d^\dagger\right) = i\, f^c{}_{ab}\, \mathrm{Tr}\left(t_c\, t_d^\dagger\right) = i\, f^c{}_{ab}\, k\, \delta_{cd} = ik\, f^d{}_{ab} \tag{10.71}$$

which implies that the structure constant $f^c{}_{ab}$ is the trace

$$f^c{}_{ab} = -\frac{i}{k}\, \mathrm{Tr}\left([t_a, t_b]\, t_c^\dagger\right). \tag{10.72}$$

Because of the antisymmetry of the commutator $[t_a, t_b]$, structure constants are **antisymmetric in their lower indices**

$$f^c{}_{ab} = -f^c{}_{ba}. \tag{10.73}$$

From any $n \times n$ matrix A, one may make a hermitian matrix $A + A^\dagger$ and an antihermitian one $A - A^\dagger$. Thus, one may separate the n_G generators into a set that are hermitian $t_a^{(h)}$ and a set that are antihermitian $t_a^{(ah)}$. The exponential of any imaginary linear combination of $n \times n$ hermitian generators $D(\alpha) = \exp\left(i\alpha_a\, t_a^{(h)}\right)$ is an $n \times n$ unitary matrix since

$$D^\dagger(\alpha) = \exp\left(-i\alpha_a\, t_a^{\dagger(h)}\right) = \exp\left(-i\alpha_a\, t_a^{(h)}\right) = D^{-1}(\alpha). \tag{10.74}$$

A group with only hermitian generators is **compact** and has finite-dimensional unitary representations.

On the other hand, the exponential of any imaginary linear combination of antihermitian generators $D(\alpha) = \exp\left(i\alpha_a\, t_a^{(ah)}\right)$ is a real exponential of their hermitian counterparts $i\, t_a^{(ah)}$ whose squared norm

$$\|D(\alpha)\|^2 = \mathrm{Tr}\left[D(\alpha)^\dagger D(\alpha)\right] = \mathrm{Tr}\left[\exp\left(2\alpha_a\, it_a^{(ah)}\right)\right] \tag{10.75}$$

grows exponentially and without limit as the parameters $\alpha_a \to \pm\infty$. A group with some antihermitian generators is **noncompact** and does not have finite-dimensional unitary representations. (The unitary representations of the translations and of the Lorentz and Poincaré groups are infinite dimensional.)

Compact Lie groups have hermitian generators, and so the structure-constant formula (10.72) reduces in this case to

$$f^c{}_{ab} = (-i/k)\mathrm{Tr}\left([t_a, t_b]\, t_c^\dagger\right) = (-i/k)\mathrm{Tr}\left([t_a, t_b]\, t_c\right). \tag{10.76}$$

Now, since the trace is cyclic, we have

$$\begin{aligned} f^b{}_{ac} &= (-i/k)\mathrm{Tr}\left([t_a, t_c]\, t_b\right) = (-i/k)\mathrm{Tr}\left(t_a t_c t_b - t_c t_a t_b\right) \\ &= (-i/k)\mathrm{Tr}\left(t_b t_a t_c - t_a t_b t_c\right) \\ &= (-i/k)\mathrm{Tr}\left([t_b, t_a]\, t_c\right) = f^c{}_{ba} = -f^c{}_{ab}. \end{aligned} \tag{10.77}$$

Interchanging a and b, we get

$$f^a{}_{bc} = f^c{}_{ab} = -f^c{}_{ba}. \tag{10.78}$$

Finally, interchanging b and c in (10.77) gives

$$f^c{}_{ab} = f^b{}_{ca} = -f^b{}_{ac}. \tag{10.79}$$

Combining (10.77, 10.78, & 10.79), we see that **the structure constants of a compact Lie group are totally antisymmetric**

$$f^b{}_{ac} = -f^b{}_{ca} = f^c{}_{ba} = -f^c{}_{ab} = -f^a{}_{bc} = f^a{}_{cb}. \tag{10.80}$$

Because of this antisymmetry, it is usual to lower the upper index

$$f^c{}_{ab} = f_{cab} = f_{abc} \tag{10.81}$$

and write the antisymmetry of the structure constants of compact Lie groups as

$$f_{acb} = -f_{cab} = f_{bac} = -f_{abc} = -f_{bca} = f_{cba}. \tag{10.82}$$

For compact Lie groups, the generators are hermitian, and so the **structure constants f_{abc} are real**, as we may see by taking the complex conjugate of the formula (10.76) for f_{abc}

$$f^*_{abc} = (i/k)\mathrm{Tr}\,(t_c\,[t_b, t_a]) = (-i/k)\mathrm{Tr}\,([t_a, t_b]\,t_c) = f_{abc}. \tag{10.83}$$

It follows from (10.68 & 10.81–10.83) that **the commutator of any two generators of a Lie group is a linear combination**

$$[t_a, t_b] = i\, f^c_{ab}\, t_c \tag{10.84}$$

of its generators t_c, and that the structure constants $f_{abc} \equiv f^c_{ab}$ are real and totally antisymmetric if the group is compact.

Example 10.19 (Gauge Transformation) The action density of a Yang-Mills theory is unchanged when a spacetime dependent unitary matrix $U(x)$ changes a vector $\psi(x)$ of matter fields to $\psi'(x) = U(x)\psi(x)$. Terms like $\psi^\dagger\psi$ are invariant because $\psi^\dagger(x)U^\dagger(x)U(x)\psi(x) = \psi^\dagger(x)\psi(x)$, but how can kinetic terms like $\partial_i\psi^\dagger\,\partial^i\psi$ be made invariant? Yang and Mills introduced matrices A_i of gauge fields, replaced ordinary derivatives ∂_i by **covariant derivatives** $D_i \equiv \partial_i + A_i$, and required that $D'_i\psi' = UD_i\psi$ or that

$$\left(\partial_i + A'_i\right) U\psi = \left(\partial_i U + U\partial_i + A'_i U\right)\psi = U\left(\partial_i + A_i\right)\psi. \tag{10.85}$$

Their nonabelian gauge transformation is

$$A'_i(x) = U(x)A_i(x)U^\dagger(x) - \left(\partial_i U(x)\right)U^\dagger(x). \tag{10.86}$$

One can write the unitary matrix as $U(x) = \exp(-ig\,\theta_a(x)\,t_a)$ in which g is a coupling constant, the functions $\theta_a(x)$ parametrize the gauge transformation, and the generators t_a belong to the representation that acts on the vector $\psi(x)$ of matter fields. □

10.17 The rotation group

The rotations and reflections in three-dimensional space form a compact group $O(3)$ whose elements R are 3×3 real matrices that leave invariant the dot product of any two three vectors

$$(R\boldsymbol{x}) \cdot (R\boldsymbol{y}) = \boldsymbol{x}^\mathsf{T} R^\mathsf{T} R\,\boldsymbol{y} = \boldsymbol{x}^\mathsf{T} I\boldsymbol{y} = \boldsymbol{x} \cdot \boldsymbol{y}. \tag{10.87}$$

These matrices therefore are orthogonal (1.181)

$$R^{\mathsf{T}} R = I. \tag{10.88}$$

Taking the determinant of both sides and using the transpose (1.209) and product (1.222) rules, we have

$$(\det R)^2 = 1 \tag{10.89}$$

whence $\det R = \pm 1$. The group $O(3)$ contains reflections as well as rotations and is disjoint. The subgroup with $\det R = 1$ is the group $SO(3)$. An $SO(3)$ element near the identity $R = I + \omega$ must satisfy

$$(I + \omega)^{\mathsf{T}} (I + \omega) = I. \tag{10.90}$$

Neglecting the tiny quadratic term, we find that the infinitesimal matrix ω is antisymmetric

$$\omega^{\mathsf{T}} = -\omega. \tag{10.91}$$

One complete set of real 3×3 antisymmetric matrices is

$$\omega_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \omega_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \omega_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{10.92}$$

which we may write as

$$[\omega_b]_{ac} = \epsilon_{abc} \tag{10.93}$$

in which ϵ_{abc} is the **Levi-Civita symbol** which is totally antisymmetric with $\epsilon_{123} = 1$ (Tullio Levi-Civita 1873–1941). The ω_b are antihermitian, but we make them hermitian by multiplying by i

$$t_b = i\,\omega_b \quad \text{so that} \quad [t_b]_{ac} = i\epsilon_{abc} \tag{10.94}$$

and $R = I - i\theta_b\, t_b$.

The three hermitian generators t_a satisfy (exercise 10.15) the commutation relations

$$[t_a, t_b] = i\, f_{abc}\, t_c \tag{10.95}$$

in which the structure constants are given by the Levi-Civita symbol ϵ_{abc}

$$f_{abc} = \epsilon_{abc} \tag{10.96}$$

so that

$$[t_a, t_b] = i\, \epsilon_{abc}\, t_c. \tag{10.97}$$

They are the generators of the **defining representation** of $SO(3)$ (and also of the **adjoint representation** of $SU(2)$ (section 10.24)).

Physicists usually scale the generators by $\hbar$ and define the angular-momentum generator L_a as

$$L_a = \hbar\, t_a \tag{10.98}$$

so that the eigenvalues of the angular-momentum operators are the physical values of the angular momenta. With $\hbar$, the commutation relations are

$$[L_a, L_b] = i\,\hbar\,\epsilon_{abc}\,L_c. \tag{10.99}$$

The matrix that represents a right-handed rotation (of an object) by an angle $\theta = |\boldsymbol{\theta}|$ about an axis $\boldsymbol{\theta}$ is

$$D(\boldsymbol{\theta}) = e^{-i\boldsymbol{\theta}\cdot\boldsymbol{t}} = e^{-i\boldsymbol{\theta}\cdot\boldsymbol{L}/\hbar}. \tag{10.100}$$

By using the fact (1.290) that a matrix obeys its characteristic equation, one may show (exercise 10.17) that the 3×3 matrix $D(\boldsymbol{\theta})$ that represents a right-handed rotation of θ radians about the axis $\boldsymbol{\theta}$ is the matrix (10.7) whose i, jth entry is

$$D_{ij}(\boldsymbol{\theta}) = \cos\theta\,\delta_{ij} - \sin\theta\,\epsilon_{ijk}\,\theta_k/\theta + (1-\cos\theta)\,\theta_i\theta_j/\theta^2 \tag{10.101}$$

in which a sum over $k = 1, 2, 3$ is understood.

A set of generators J_a equivalent to the antisymmetric ω's (10.92) but with J_3 diagonal is

$$J_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix},\quad J_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}0&-i&0\\i&0&-i\\0&i&0\end{pmatrix},\quad J_3 = \begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}. \tag{10.102}$$

Example 10.20 (Demonstration of commutation relations) Take a big sphere with a distinguished point and orient the sphere so that the point lies in the y-direction from the center of the sphere. Now rotate the sphere by a small angle, say 15 degrees or $\epsilon = \pi/12$, right-handedly about the x-axis, then right-handedly about the y-axis by the same angle, then left-handedly about the x-axis and then left-handedly about the y-axis. Using the approximations (10.65 & 10.67) for the product of these four rotation matrices and the definitions (10.98) of the generators and of their structure constants (10.99), we have $\hbar t_a = L_1 = L_x$, $\hbar t_b = L_2 = L_y$, $\hbar f_{abc} t_c = \epsilon_{12c} L_c = L_3 = L_z$, and

$$e^{i\epsilon L_y/\hbar}\,e^{i\epsilon L_x/\hbar}\,e^{-i\epsilon L_y/\hbar}\,e^{-i\epsilon L_x/\hbar} \approx 1 + \frac{\epsilon^2}{\hbar^2}[L_x, L_y] = 1 + i\frac{\epsilon^2}{\hbar}L_z \approx e^{i\epsilon^2 L_z/\hbar} \tag{10.103}$$

which is a left-handed rotation about the (vertical) z-axis. The magnitude of that rotation should be about $\epsilon^2 = (\pi/12)^2 \approx 0.069$ or about 3.9 degrees. Photographs of an actual demonstration are displayed in Fig. 10.1.

The demonstrated equation (10.103) shows (exercise 10.16) that the generators L_x and L_y satisfy the commutation relation

$$[L_x, L_y] = i\hbar L_z \tag{10.104}$$

of the rotation group. □

10.18 Rotations and reflections in $2n$ dimensions

The orthogonal group $O(2n)$ of rotations and reflections in $2n$ dimensions is the group of all real $2n \times 2n$ matrices O whose transposes O^T are their inverses

$$O^\mathsf{T} O = O\, O^\mathsf{T} = I \tag{10.105}$$

in which I is the $2n \times 2n$ identity matrix. These orthogonal matrices leave unchanged the distances from the origin of points in $2n$ dimensions. Those with unit determinant, $\det O = 1$, constitute the subgroup $SO(2n)$ of rotations in $2n$ dimensions.

A symmetric sum $\{A, B\} = AB + BA$ is called an **anticommutator**. Complex fermionic variables ψ_i obey the anticommutation relations

$$\{\psi_i, \psi_k^\dagger\} = \hbar\, \delta_{ik}, \quad \{\psi_i, \psi_k\} = 0, \quad \text{and} \quad \{\psi_i^\dagger, \psi_k^\dagger\} = 0. \tag{10.106}$$

Their real x_i and imaginary y_i parts

$$x_i = \frac{1}{\sqrt{2}}(\psi_i + \psi_i^\dagger) \quad \text{and} \quad y_i = \frac{1}{i\sqrt{2}}(\psi_i - \psi_i^\dagger) \tag{10.107}$$

obey the anticommutation relations

$$\{x_i, x_k\} = \hbar\, \delta_{ik}, \quad \{y_i, y_k\} = \hbar\, \delta_{ik}, \quad \text{and} \quad \{x_i, y_k\} = 0. \tag{10.108}$$

More simply, the anticommutation relations of these $2n$ hermitian variables $v = (x_1, \ldots, x_n, y_1, \ldots, y_n)$ are

$$\{v_i, v_k\} = \hbar\, \delta_{ik}. \tag{10.109}$$

If the real linear transformation $v_i' = L_{i1}\, v_1 + L_{i2}\, v_2 + \cdots + L_{i2n}\, v_{2n}$ preserves these anticommutation relations, then the matrix L must satisfy

$$\hbar\, \delta_{ik} = \{v_i', v_k'\} = L_{ij}\, L_{k\ell} \{v_j, v_\ell\} = L_{ij}\, L_{k\ell}\, \hbar\, \delta_{j\ell} = \hbar L_{ij}\, L_{kj} \tag{10.110}$$

Physical Demonstration of the Commutation Relations

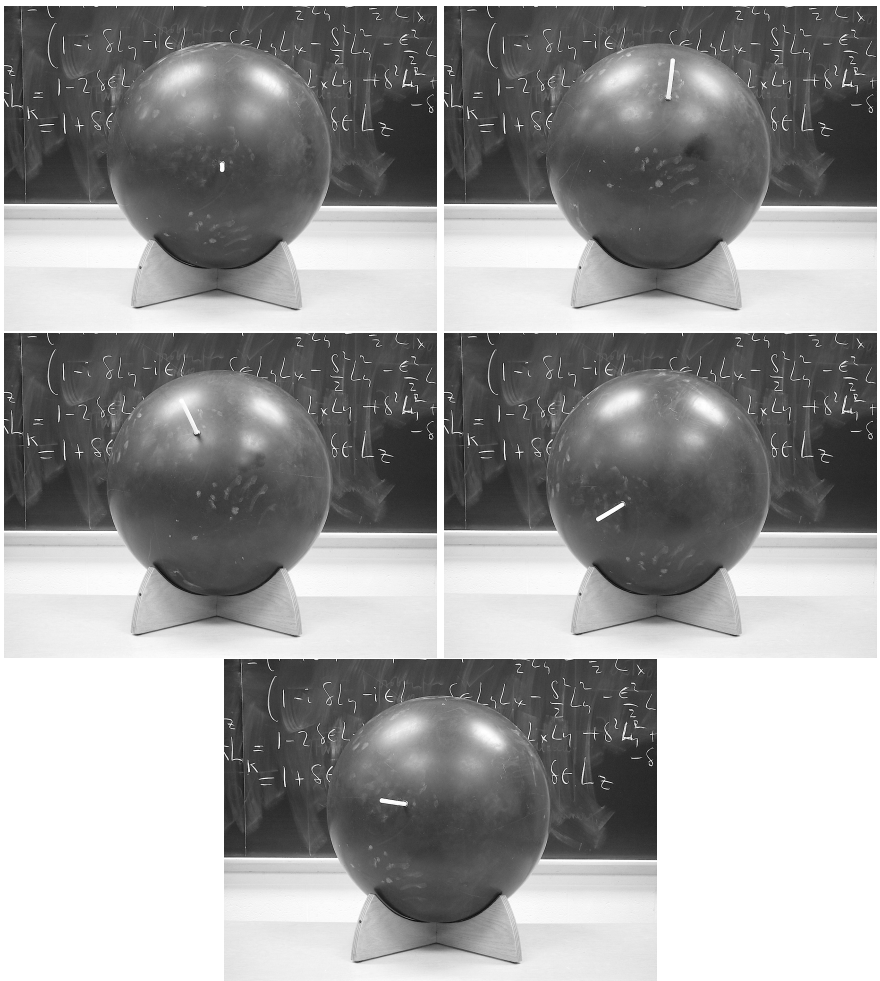

Figure 10.1 Demonstration of equation (10.103) and the commutation relation (10.104). Upper left: black ball with a white stick pointing in the y-direction; the x-axis is to the reader's left, the z-axis is vertical. Upper right: ball after a small right-handed rotation about the x-axis. Center left: ball after that rotation is followed by a small right-handed rotation about the y-axis. Center right: ball after these rotations are followed by a small left-handed rotation about the x-axis. Bottom: ball after these rotations are followed by a small left-handed rotation about the y-axis. The net effect is approximately a small left-handed rotation about the z-axis.

which is the statement that it is orthogonal, $LL^{\mathsf{T}} = I$. Thus the group $O(2n)$ is the largest group of linear transformations that preserve the anticommu-

tation relations of the $2n$ hermitian real and imaginary parts v_i of n complex fermionic variables ψ_i.

10.19 The defining representation of $SU(2)$

The smallest positive value of angular momentum is $\hbar/2$. The spin-one-half angular momentum operators are represented by three 2×2 matrices

$$S_a = \frac{\hbar}{2}\,\sigma_a \tag{10.111}$$

in which the σ_a are the **Pauli matrices**

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \text{and} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{10.112}$$

which obey the multiplication law

$$\sigma_i\,\sigma_j = \delta_{ij} + i\epsilon_{ijk}\,\sigma_k \tag{10.113}$$

summed over k from 1 to 3. Since the symbol ϵ_{ijk} is totally antisymmetric in i, j, and k, the Pauli matrices obey the commutation and anticommutation relations

$$\begin{aligned} [\sigma_i, \sigma_j] &\equiv \sigma_i\sigma_j - \sigma_j\sigma_i = 2i\epsilon_{ijk}\,\sigma_k \\ \{\sigma_i, \sigma_j\} &\equiv \sigma_i\,\sigma_j + \sigma_j\,\sigma_i = 2\delta_{ij}. \end{aligned} \tag{10.114}$$

The Pauli matrices divided by 2 satisfy the commutation relations (10.97) of the rotation group

$$\left[\tfrac{1}{2}\sigma_a, \tfrac{1}{2}\sigma_b\right] = i\,\epsilon_{abc}\,\tfrac{1}{2}\sigma_c \tag{10.115}$$

and generate the elements of the group $SU(2)$

$$\exp\left(i\;\boldsymbol{\theta}\cdot\frac{\boldsymbol{\sigma}}{2}\right) = I\,\cos\frac{\theta}{2} + i\,\hat{\boldsymbol{\theta}}\cdot\boldsymbol{\sigma}\,\sin\frac{\theta}{2} \tag{10.116}$$

in which I is the 2×2 identity matrix, $\theta = \sqrt{\boldsymbol{\theta}^2}$ and $\hat{\boldsymbol{\theta}} = \boldsymbol{\theta}/\theta$.

It follows from (10.115) that the spin operators (10.111) satisfy

$$[S_a, S_b] = i\,\hbar\,\epsilon_{abc}\,S_c. \tag{10.117}$$

10.20 The Lie Algebra and Representations of *SU(2)*

The three generators of $SU(2)$ in its 2×2 defining representation are the Pauli matrices divided by 2, $t_a = \sigma_a/2$. The structure constants of $SU(2)$

are $f_{abc} = \epsilon_{abc}$ which is totally antisymmetric with $\epsilon_{123} = 1$

$$[t_a, t_b] = i f_{abc} t_c = \left[\tfrac{1}{2}\sigma_a, \tfrac{1}{2}\sigma_b\right] = i\epsilon_{abc}\,\tfrac{1}{2}\sigma_c. \tag{10.118}$$

For every half-integer

$$j = \frac{n}{2} \quad \text{for} \quad n = 0, 1, 2, 3, \ldots \tag{10.119}$$

there is an irreducible representation of $SU(2)$

$$D^{(j)}(\boldsymbol{\theta}) = e^{-i\boldsymbol{\theta}\cdot\boldsymbol{J}^{(j)}} \tag{10.120}$$

in which the three generators $t_a^{(j)} \equiv J_a^{(j)}$ are $(2j+1) \times (2j+1)$ square hermitian matrices. In a basis in which $J_3^{(j)}$ is diagonal, the matrix elements of the complex linear combinations $J_\pm^{(j)} \equiv J_1^{(j)} \pm i J_2^{(j)}$ are

$$\left[J_1^{(j)} \pm i J_2^{(j)}\right]_{s',s} = \delta_{s',s\pm1}\sqrt{(j \mp s)(j \pm s + 1)} \tag{10.121}$$

where s and s' run from $-j$ to j in integer steps and those of $J_3^{(j)}$ are

$$\left[J_3^{(j)}\right]_{s',s} = s\,\delta_{s',s}. \tag{10.122}$$

Borrowing a trick from section 10.25, one may show that the commutator of the square $\boldsymbol{J}^{(j)} \cdot \boldsymbol{J}^{(j)}$ of the angular momentum matrix commutes with every generator $J_a^{(j)}$. Thus $\boldsymbol{J}^{(j)2}$ commutes with $D^{(j)}(\boldsymbol{\theta})$ for every element of the group. Part 2 of Schur's lemma (section 10.7) then implies that $\boldsymbol{J}^{(j)2}$ must be a multiple of the $(2j+1) \times (2j+1)$ identity matrix. The coefficient turns out to be $j(j+1)$

$$\boldsymbol{J}^{(j)} \cdot \boldsymbol{J}^{(j)} = j(j+1)\,I. \tag{10.123}$$

Combinations of generators that are a multiple of the identity are called **Casimir operators**.

Example 10.21 (Spin 2) For $j = 2$, the spin-two matrices $J_+^{(2)}$ and $J_3^{(2)}$ are

$$J_+^{(2)} = \begin{pmatrix} 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{6} & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad J_3^{(2)} = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{pmatrix} \tag{10.124}$$

and $J_- = \left(J_+^{(2)}\right)^\dagger$. □

The tensor product of any two irreducible representations $D^{(j)}$ and $D^{(k)}$ of $SU(2)$ is equivalent to the direct sum of all the irreducible representations $D^{(\ell)}$ for $|j-k| \le \ell \le j+k$

$$D^{(j)} \otimes D^{(k)} = \bigoplus_{\ell=|j-k|}^{j+k} D^{(\ell)} \tag{10.125}$$

each $D^{(\ell)}$ occurring once.

Example 10.22 (Addition theorem) The spherical harmonics $Y_{\ell m}(\theta,\phi) = \langle\theta,\phi|\ell,m\rangle$ of section 8.13 transform under the $(2\ell+1)$-dimensional representation D^ℓ of the rotation group. If a rotation R takes θ,ϕ into the vector θ',ϕ', so that $|\theta',\phi'\rangle = U(R)|\theta,\phi\rangle$, then summing over m' from $-\ell$ to ℓ, we get

$$\begin{aligned} Y^*_{\ell,m}(\theta',\phi') &= \langle\ell,m|\theta',\phi'\rangle = \langle\ell,m|U(R)|\theta,\phi\rangle \\ &= \langle\ell,m|U(R)|\ell,m'\rangle\langle\ell,m'|\theta,\phi\rangle = D^\ell(R)_{m,m'}Y^*_{\ell,m'}(\theta,\phi). \end{aligned}$$

Suppose now that a rotation R maps $|\theta_1,\phi_1\rangle$ and $|\theta_2,\phi_2\rangle$ into $|\theta_1',\phi_1'\rangle = U(R)\,|\theta_1,\phi_1\rangle$ and $|\theta_2',\phi_2'\rangle = U(R)\,|\theta_2,\phi_2\rangle$. Then summing over the repeated indices m, m', and m'' from $-\ell$ to ℓ, we find

$$Y_{\ell,m}(\theta_1',\phi_1')Y^*_{\ell,m}(\theta_2',\phi_2') = D^\ell(R)^*_{m,m'}Y_{\ell,m'}(\theta_1,\phi_1)D^\ell(R)_{m,m''}Y^*_{\ell,m''}(\theta_2,\phi_2).$$

In this equation, the matrix element $D^\ell(R)^*_{m,m'}$ is

$$D^\ell(R)^*_{m,m'} = \langle\ell,m|U(R)|\ell,m'\rangle^* = \langle\ell,m'|U^\dagger(R)|\ell,m\rangle = D^\ell(R^{-1})_{m',m}.$$

Thus since D^ℓ is a representation of the rotation group, the product of the two D^ℓ's in (10.22) is

$$\begin{aligned} D^\ell(R)^*_{m,m'}D^\ell(R)_{m,m''} &= D^\ell(R^{-1})_{m',m}D^\ell(R)_{m,m''} \\ &= D^\ell(R^{-1}R)_{m',m''} = D^\ell(I)_{m',m''} = \delta_{m',m''}. \end{aligned}$$

So as long as the same rotation R maps θ_1,ϕ_1 into θ_1',ϕ_1' and θ_2,ϕ_2 into θ_2',ϕ_2', then we have

$$\sum_{m=-\ell}^{\ell} Y_{\ell,m}(\theta_1',\phi_1')Y^*_{\ell,m}(\theta_2',\phi_2') = \sum_{m=-\ell}^{\ell} Y_{\ell,m}(\theta_1,\phi_1)Y^*_{\ell,m}(\theta_2,\phi_2).$$

We choose the rotation R as the product of a rotation that maps the unit vector $\hat{n}(\theta_2,\phi_2)$ into $\hat{n}(\theta_2',\phi_2') = \hat{\boldsymbol{z}} = (0,0,1)$ and a rotation about the z axis that maps $\hat{n}(\theta_1,\phi_1)$ into $\hat{n}(\theta_1',\phi_1') = (\sin\theta,0,\cos\theta)$ in the x-z plane where it makes an angle θ with $\hat{n}(\theta_2',\phi_2') = \hat{\boldsymbol{z}}$. We then have $Y^*_{\ell,m}(\theta_2',\phi_2') =$

$Y^*_{\ell,m}(0,0)$ and $Y_{\ell,m}(\theta'_1,\phi'_1) = Y_{\ell,m}(\theta,0)$ in which θ is the angle between the unit vectors $\hat{n}(\theta'_1,\phi'_1)$ and $\hat{n}(\theta'_2,\phi'_2)$, which is the same as the angle between the unit vectors $\hat{n}(\theta_1,\phi_1)$ and $\hat{n}(\theta_2,\phi_2)$. The vanishing (8.108) at $\theta = 0$ of the associated Legendre functions $P_{\ell,m}$ for $m \neq 0$ and the definitions (8.4, 8.101, & 8.112–8.114) say that $Y^*_{\ell,m}(0,0) = \sqrt{(2\ell+1)/4\pi}\,\delta_{m,0}$, and that $Y_{\ell,0}(\theta,0) = \sqrt{(2\ell+1)/4\pi}\,P_\ell(\cos\theta)$. Thus, our identity (10.22) gives us the for the spherical harmonics **addition theorem** (8.123)

$$P_\ell(\cos\theta) = \frac{2\ell+1}{4\pi}\sum_{m=-\ell}^{\ell} Y_{\ell,m}(\theta_1,\phi_1)Y^*_{\ell,m}(\theta_2,\phi_2).$$

□

10.21 How a field transforms under a rotation

Under a rotation R, a field $\psi_s(x)$ that transforms under the $D^{(j)}$ representation of $SU(2)$ responds as

$$U(R)\,\psi_s(x)\,U^{-1}(R) = D^{(j)}_{s,s'}(R^{-1})\,\psi_{s'}(Rx). \tag{10.126}$$

Example 10.23 (Spin and Statistics) Suppose $|a,m\rangle$ and $|b,m\rangle$ are any eigenstates of the rotation operator J_3 with eigenvalue m (in units with $\hbar = c = 1$). If u and v are any two space-like points, then in some Lorentz frame they have spacetime coordinates $u = (t,x,0,0)$ and $v = (t,-x,0,0)$. Let U be the unitary operator that represents a right-handed rotation by π about the 3-axis or z-axis of this Lorentz frame. Then

$$U|a,m\rangle = e^{-im\pi}|a,m\rangle \quad \text{and} \quad \langle b,m|U^{-1} = \langle b,m|e^{im\pi}. \tag{10.127}$$

And by (10.126), U transforms a field ψ of spin j with $\boldsymbol{x} \equiv (x,0,0)$ to

$$U(R)\,\psi_s(t,\boldsymbol{x})\,U^{-1}(R) = D^{(j)}_{ss'}(R^{-1})\,\psi_{s'}(t,-\boldsymbol{x}) = e^{i\pi s}\psi_s(t,-\boldsymbol{x}). \tag{10.128}$$

Thus by inserting the identity operator in the form $I = U^{-1}U$ and using both (10.127) and (10.128), we find, since the phase factors $\exp(-im\pi)$ and $\exp(im\pi)$ cancel,

$$\begin{aligned}\langle b,m|\psi_s(t,\boldsymbol{x})\,\psi_s(t,-\boldsymbol{x})|a,m\rangle &= \langle b,m|U\psi_s(t,\boldsymbol{x})U^{-1}U\psi_s(t,-\boldsymbol{x})U^{-1}|a,m\rangle \\ &= e^{2i\pi s}\,\langle b,m|\psi_s(t,-\boldsymbol{x})\psi_s(t,\boldsymbol{x})|a,m\rangle. \end{aligned}\tag{10.129}$$

Now if j is an integer, then so is s, and the phase factor $\exp(2i\pi s) = 1$ is

unity. In this case, we find that the mean value of the equal-time commutator vanishes

$$\langle b, m|[\psi_s(t, \boldsymbol{x}), \psi_s(t, -\boldsymbol{x})]|a, m\rangle = 0 \tag{10.130}$$

which suggests that fields of integral spin commute at space-like separations. They represent bosons. On the other hand, if j is half an odd integer, that is, $j = (2n+1)/2$, where n is an integer, then the phase factor $\exp(2i\pi s) = -1$ is minus one. In this case, the mean value of the equal-time anticommutator vanishes

$$\langle b, m|\{\psi_s(t, \boldsymbol{x}), \psi_s(t, -\boldsymbol{x})\}|a, m\rangle = 0 \tag{10.131}$$

which suggests that fields of half-odd-integral spin anticommute at space-like separations. They represent fermions. This argument shows that the behavior of fields under rotations is related to their equal-time commutation or anticommutation relations

$$\psi_s(t, \boldsymbol{x})\psi_{s'}(t, \boldsymbol{x'}) + (-1)^{2j}\psi_{s'}(t, \boldsymbol{x'})\psi_s(t, \boldsymbol{x}) = 0 \tag{10.132}$$

and their statistics. □

10.22 The addition of two spin-one-half systems

The spin operators (10.111)

$$S_a = \frac{\hbar}{2}\sigma_a \tag{10.133}$$

obey the commutation relation (10.117)

$$[S_a, S_b] = i\,\hbar\,\epsilon_{abc}\,S_c. \tag{10.134}$$

The raising and lowering operators

$$S_\pm = S_1 \pm iS_2 \tag{10.135}$$

have simple commutators with S_3

$$[S_3, S_\pm] = \pm\,\hbar\,S_\pm. \tag{10.136}$$

This relation implies that if the state $|\frac{1}{2}, m\rangle$ is an eigenstate of S_3 with eigenvalue $\hbar m$, then the states $S_\pm|\frac{1}{2}, m\rangle$ either vanish or are eigenstates of S_3 with eigenvalues $\hbar(m \pm 1)$

$$S_3S_\pm|\tfrac{1}{2}, m\rangle = S_\pm S_3|\tfrac{1}{2}, m\rangle \pm \hbar S_\pm|\tfrac{1}{2}, m\rangle = \hbar(m \pm 1)S_\pm|\tfrac{1}{2}, m\rangle. \tag{10.137}$$

Thus the raising and lowering operators raise and lower the eigenvalues of S_3. The eigenvalues of $S_3 = \hbar\sigma_3/2$ are $\pm\hbar/2$. So with the usual sign and normalization conventions

$$S_+|-\rangle = \hbar|+\rangle \quad \text{and} \quad S_-|+\rangle = \hbar|-\rangle \tag{10.138}$$

while

$$S_+|+\rangle = 0 \quad \text{and} \quad S_-|-\rangle = 0. \tag{10.139}$$

The square of the total spin operator is simply related to the raising and lowering operators and to S_3

$$\boldsymbol{S}^2 = S_1^2 + S_2^2 + S_3^2 = \frac{1}{2}S_+S_- + \frac{1}{2}S_-S_+ + S_3^2. \tag{10.140}$$

But the squares of the Pauli matrices are unity, and so $S_a^2 = (\hbar/2)^2$ for all three values of a. Thus

$$\boldsymbol{S}^2 = \frac{3}{4}\,\hbar^2 \tag{10.141}$$

is a Casimir operator (10.123) for a spin one-half system.

Consider two spin operators $\boldsymbol{S}^{(1)}$ and $\boldsymbol{S}^{(2)}$ as defined by (10.111) acting on two spin-one-half systems. Let the tensor-product states

$$|\pm,\pm\rangle = |\pm\rangle_1|\pm\rangle_2 = |\pm\rangle_1 \otimes |\pm\rangle_2 \tag{10.142}$$

be eigenstates of $S_3^{(1)}$ and $S_3^{(2)}$ so that

$$\begin{aligned} S_3^{(1)}|+,\pm\rangle &= \frac{\hbar}{2}\,|+,\pm\rangle \quad \text{and} \quad S_3^{(2)}|\pm,+\rangle = \frac{\hbar}{2}\,|\pm,+\rangle \\ S_3^{(1)}|-,\pm\rangle &= -\frac{\hbar}{2}\,|-,\pm\rangle \quad \text{and} \quad S_3^{(2)}|\pm,-\rangle = -\frac{\hbar}{2}\,|\pm,-\rangle \end{aligned} \tag{10.143}$$

The total spin of the system is the sum of the two spins $\boldsymbol{S} = \boldsymbol{S}^{(1)} + \boldsymbol{S}^{(2)}$, so

$$\boldsymbol{S}^2 = \left(\boldsymbol{S}^{(1)} + \boldsymbol{S}^{(2)}\right)^2 \quad \text{and} \quad S_3 = S_3^{(1)} + S_3^{(2)}. \tag{10.144}$$

The state $|+,+\rangle$ is an eigenstate of S_3 with eigenvalue $\hbar$

$$S_3|+,+\rangle = S_3^{(1)}|+,+\rangle + S_3^{(2)}|+,+\rangle = \frac{\hbar}{2}|+,+\rangle + \frac{\hbar}{2}|+,+\rangle = \hbar|+,+\rangle. \tag{10.145}$$

So the state of angular momentum $\hbar$ in the 3-direction is $|1,1\rangle = |+,+\rangle$. Similarly, the state $|-,-\rangle$ is an eigenstate of S_3 with eigenvalue $-\hbar$

$$S_3|-,-\rangle = S_3^{(1)}|-,-\rangle + S_3^{(2)}|-,-\rangle = -\frac{\hbar}{2}|-,-\rangle - \frac{\hbar}{2}|-,-\rangle = -\hbar|-,-\rangle \tag{10.146}$$

and so the state of angular momentum $\hbar$ in the negative 3-direction is $|1,-1\rangle = |-,-\rangle$. The states $|+,-\rangle$ and $|-,+\rangle$ are eigenstates of S_3 with eigenvalue 0

$$\begin{aligned} S_3|+,-\rangle &= S_3^{(1)}|+,-\rangle + S_3^{(2)}|+,-\rangle = \frac{\hbar}{2}|+,-\rangle - \frac{\hbar}{2}|+,-\rangle = 0 \\ S_3|-,+\rangle &= S_3^{(1)}|-,+\rangle + S_3^{(2)}|-,+\rangle = -\frac{\hbar}{2}|-,+\rangle + \frac{\hbar}{2}|-,+\rangle = 0. \end{aligned} \tag{10.147}$$

To see which states are eigenstates of $\boldsymbol{S}^2$, we use the lowering operator for the combined system $S_- = S_-^{(1)} + S_-^{(2)}$ and the rules (10.121, 10.138, & 10.139) to lower the state $|1,1\rangle$

$$S_-|+,+\rangle = \left(S_-^{(1)} + S_-^{(2)}\right)|+,+\rangle = \hbar\left(|-,+\rangle + |+,-\rangle\right) = \hbar\sqrt{2}\,|1,0\rangle.$$

Thus the state $|1,0\rangle$ is

$$|1,0\rangle = \frac{1}{\sqrt{2}}\left(|+,-\rangle + |-,+\rangle\right). \tag{10.148}$$

The orthogonal and normalized combination of $|+,-\rangle$ and $|-,+\rangle$ must be the state of spin zero

$$|0,0\rangle = \frac{1}{\sqrt{2}}\left(|+,-\rangle - |-,+\rangle\right) \tag{10.149}$$

with the usual sign convention.

To check that the states $|1,0\rangle$ and $|0,0\rangle$ really are eigenstates of $\boldsymbol{S}^2$, we use (10.140 & 10.141) to write $\boldsymbol{S}^2$ as

$$\begin{aligned} \boldsymbol{S}^2 &= \left(\boldsymbol{S}^{(1)} + \boldsymbol{S}^{(2)}\right)^2 = \frac{3}{2}\hbar^2 + 2\boldsymbol{S}^{(1)}\cdot\boldsymbol{S}^{(2)} \\ &= \frac{3}{2}\hbar^2 + S_+^{(1)}S_-^{(2)} + S_-^{(1)}S_+^{(2)} + 2S_3^{(1)}S_3^{(2)}. \end{aligned} \tag{10.150}$$

Now the sum $S_+^{(1)}S_-^{(2)} + S_-^{(1)}S_+^{(2)}$ merely interchanges the states $|+,-\rangle$ and $|-,+\rangle$ and multiplies them by $\hbar^2$, so

$$\begin{aligned} \boldsymbol{S}^2|1,0\rangle &= \frac{3}{2}\hbar^2|1,0\rangle + \hbar^2|1,0\rangle - \frac{2}{4}\hbar^2|1,0\rangle \\ &= 2\hbar^2|1,0\rangle = s(s+1)\hbar^2|1,0\rangle \end{aligned} \tag{10.151}$$

which confirms that $s = 1$. Because of the relative minus sign in formula (10.149) for the state $|0,0\rangle$, we have

$$\begin{aligned} \boldsymbol{S}^2|0,0\rangle &= \frac{3}{2}\hbar^2|0,0\rangle - \hbar^2|1,0\rangle - \frac{1}{2}\hbar^2|1,0\rangle \\ &= 0\hbar^2|1,0\rangle = s(s+1)\hbar^2|1,0\rangle \end{aligned} \tag{10.152}$$

which confirms that $s = 0$.

Example 10.24 (Two equivalent representations of $SU(2)$) The identity

$$\left[\exp\left(i\boldsymbol{\theta}\cdot\frac{\boldsymbol{\sigma}}{2}\right)\right]^* = \sigma_2 \exp\left(i\boldsymbol{\theta}\cdot\frac{\boldsymbol{\sigma}}{2}\right)\sigma_2 \tag{10.153}$$

shows that the defining representation of $SU(2)$ (section 10.19) and its complex conjugate are equivalent (10.8) representations. To prove this identity, we expand the exponential on the right-hand side in powers of its argument

$$\sigma_2 \exp\left(i\boldsymbol{\theta}\cdot\frac{\boldsymbol{\sigma}}{2}\right)\sigma_2 = \sigma_2\left[\sum_{n=0}^{\infty}\frac{1}{n!}\left(i\boldsymbol{\theta}\cdot\frac{\boldsymbol{\sigma}}{2}\right)^n\right]\sigma_2 \tag{10.154}$$

and use the fact that σ_2 is its own inverse to get

$$\sigma_2 \exp\left(i\boldsymbol{\theta}\cdot\frac{\boldsymbol{\sigma}}{2}\right)\sigma_2 = \sum_{n=0}^{\infty}\frac{1}{n!}\left[\sigma_2\left(i\boldsymbol{\theta}\cdot\frac{\boldsymbol{\sigma}}{2}\right)\sigma_2\right]^n. \tag{10.155}$$

Since the Pauli matrices obey the anticommutation relation (10.114), and since both σ_1 and σ_3 are real, while σ_2 is imaginary, we can write the 2×2 matrix within the square brackets as

$$\sigma_2\left(i\boldsymbol{\theta}\cdot\frac{\boldsymbol{\sigma}}{2}\right)\sigma_2 = -i\theta_1\frac{\sigma_1}{2} + i\theta_2\frac{\sigma_2}{2} - i\theta_3\frac{\sigma_3}{2} = \left(i\boldsymbol{\theta}\cdot\frac{\boldsymbol{\sigma}}{2}\right)^* \tag{10.156}$$

which implies the identity (10.153)

$$\sigma_2 \exp\left(i\boldsymbol{\theta}\cdot\frac{\boldsymbol{\sigma}}{2}\right)\sigma_2 = \sum_{n=0}^{\infty}\frac{1}{n!}\left[\left(i\boldsymbol{\theta}\cdot\frac{\boldsymbol{\sigma}}{2}\right)^*\right]^n = \left[\exp\left(i\boldsymbol{\theta}\cdot\frac{\boldsymbol{\sigma}}{2}\right)\right]^*. \tag{10.157}$$

□

10.23 The Jacobi Identity

Any three square matrices A, B, and C satisfy the commutator-product rule

$$\begin{aligned}[A, BC] &= ABC - BCA = ABC - BAC + BAC - BCA \\ &= [A, B]C + B[A, C].\end{aligned} \tag{10.158}$$

Interchanging B and C gives

$$[A, CB] = [A, C]B + C[A, B]. \tag{10.159}$$

Subtracting the second equation from the first, we get the Jacobi identity

$$[A, [B, C]] = [[A, B], C] + [B, [A, C]] \tag{10.160}$$

and its equivalent cyclic form

$$[A,[B,C]]+[B,[C,A]]+[C,[A,B]]=0. \tag{10.161}$$

Another Jacobi identity uses the anticommutator $\{A,B\}\equiv AB+BA$

$$\{[A,B],C\}+\{[A,C],B\}+[\{B,C\},A]=0. \tag{10.162}$$

10.24 The Adjoint Representation

Any three generators t_a, t_b, and t_c satisfy the Jacobi identity (10.161)

$$[t_a,[t_b,t_c]]+[t_b,[t_c,t_a]]+[t_c,[t_a,t_b]]=0. \tag{10.163}$$

By using the structure-constant formula (10.84), we may express each of these double commutators as a linear combination of the generators

$$\begin{aligned}
[t_a,[t_b,t_c]]=[t_a,if^d_{bc}t_d]&=-f^d_{bc}f^e_{ad}t_e\\
[t_b,[t_c,t_a]]]=[t_b,if^d_{ca}t_d]&=-f^d_{ca}f^e_{bd}t_e\\
[t_c,[t_a,t_b]]=[t_c,if^d_{ab}t_d]&=-f^d_{ab}f^e_{cd}t_e.
\end{aligned} \tag{10.164}$$

So the Jacobi identity (10.163) implies that

$$\left(f^d_{bc}f^e_{ad}+f^d_{ca}f^e_{bd}+f^d_{ab}f^e_{cd}\right)t_e=0 \tag{10.165}$$

or since the generators are linearly independent

$$f^d_{bc}f^e_{ad}+f^d_{ca}f^e_{bd}+f^d_{ab}f^e_{cd}=0. \tag{10.166}$$

If we define a set of matrices T_a by

$$(T_b)_{ac}=if^c_{ab} \tag{10.167}$$

then, since the structure constants are antisymmetric in their lower indices, we may write the three terms in the preceding equation (10.166) as

$$f^d_{bc}f^e_{ad}=f^d_{cb}f^e_{da}=(-T_bT_a)_{ce} \tag{10.168}$$

$$f^d_{ca}f^e_{bd}=-f^d_{ca}f^e_{db}=(T_aT_b)_{ce} \tag{10.169}$$

and

$$f^d_{ab}f^e_{cd}=-if^d_{ab}(T_d)_{ce} \tag{10.170}$$

or in matrix notation

$$[T_a,T_b]=if^c_{ab}T_c. \tag{10.171}$$

So the matrices T_a, which we made out of the structure constants by the rule $(T_b)_{ac} = i f^c_{ab}$ (10.167), obey the same algebra (10.68) as do the generators t_a. They are the **generators in the adjoint representation** of the Lie algebra. If the Lie algebra has N generators t_a, then the N generators T_a in the adjoint representation are $N \times N$ matrices.

10.25 Casimir operators

For any compact Lie algebra, the sum of the squares of all the generators

$$C = \sum_{a=1}^{N} t_a t_a \equiv t_a t_a \tag{10.172}$$

commutes with every generator t_b

$$\begin{aligned}[C, t_b] &= [t_a t_a, t_b] = [t_a, t_b] t_a + t_a [t_a, t_b] \\ &= i f_{abc} t_c t_a + t_a i f_{abc} t_c = i \left(f_{abc} + f_{cba} \right) t_c t_a = 0 \end{aligned} \tag{10.173}$$

because of the total antisymmetry (10.82) of the structure constants. This sum, called a **Casimir operator**, commutes with every matrix

$$[C, D(\alpha)] = [C, \exp(i\alpha_a t_a)] = 0 \tag{10.174}$$

of the representation generated by the t_a's. Thus by part 2 of Schur's lemma (section 10.7), it must be a multiple of the identity matrix

$$C = t_a t_a = cI. \tag{10.175}$$

The constant c depends upon the representation $D(\alpha)$ and is called the **quadratic Casimir**

$$C_2(D) = \mathrm{Tr}\left(t_a^2\right) / \mathrm{Tr} I \,. \tag{10.176}$$

Example 10.25 (Quadratic Casimirs of $SU(2)$) The quadratic Casimir $C(\mathbf{2})$ of the defining representation of $SU(2)$ with generators $t_a = \sigma_a/2$ (10.112) is

$$C_2(\mathbf{2}) = \mathrm{Tr}\left(\sum_{a=1}^{3} \left(\tfrac{1}{2}\sigma_a\right)^2 \right) / \mathrm{Tr}\,(I) = \frac{3 \cdot 2 \cdot \left(\frac{1}{2}\right)^2}{2} = \frac{3}{4}. \tag{10.177}$$

That of the adjoint representation (10.94) is

$$C_2(\mathbf{3}) = \mathrm{Tr}\left(\sum_{b=1}^{3} t_b^2 \right) / \mathrm{Tr}\,(I) = \sum_{a,b,c=1}^{3} \frac{i\epsilon_{abc} i\epsilon_{cba}}{3} = 2. \tag{10.178}$$

□

The generators of some noncompact groups come in pairs t_a and it_a, and so the sum of the squares of these generators vanishes, $C = t_a t_a - t_a t_a = 0$.

10.26 Tensor operators for the rotation group

Suppose $A_m^{(j)}$ is a set of $2j+1$ operators whose commutation relations with the generators J_i of rotations are

$$[J_i, A_m^{(j)}] = A_s^{(j)} (J_i^{(j)})_{sm} \tag{10.179}$$

in which the sum over s runs from $-j$ to j. Then $A^{(j)}$ is said to be a **spin-j tensor operator** for the group $SU(2)$.

Example 10.26 (A Spin-One Tensor Operator) For instance, if $j = 1$, then $(J_i^{(1)})_{sm} = i\hbar\epsilon_{sim}$, and so a spin-1 tensor operator of $SU(2)$ is a vector $A_m^{(1)}$ that transforms as

$$[J_i, A_m^{(1)}] = A_s^{(1)}\, i\,\hbar\,\epsilon_{sim} = i\,\hbar\,\epsilon_{ims}\, A_s^{(1)} \tag{10.180}$$

under rotations. □

Let's rewrite the definition (10.179) as

$$J_i A_m^{(j)} = A_s^{(j)} (J_i^{(j)})_{sm} + A_m^{(j)} J_i \tag{10.181}$$

and specialize to the case $i = 3$ so that $(J_3^{(j)})_{sm}$ is diagonal, $(J_3^{(j)})_{sm} = \hbar m \delta_{sm}$

$$J_3 A_m^{(j)} = A_s^{(j)} (J_3^{(j)})_{sm} + A_m^{(j)} J_3 = A_s^{(j)} \hbar m \delta_{sm} + A_m^{(j)} J_3 = A_m^{(j)} (\hbar m + J_3) \tag{10.182}$$

Thus if the state $|j, s, E\rangle$ is an eigenstate of J_3 with eigenvalue $\hbar s$, then the state $A_m^{(j)}|j, s, E\rangle$ is an eigenstate of J_3 with eigenvalue $\hbar(m+s)$

$$J_3 A_m^{(j)} |j, s, E\rangle = A_m^{(j)} (\hbar m + J_3)\, |j, s, E\rangle = \hbar\,(m+s)\, A_m^{(j)} |j, s, E\rangle. \tag{10.183}$$

The J_3 eigenvalues of the tensor operator $A_m^{(j)}$ and the state $|j, s, E\rangle$ add.

10.27 Simple and semisimple Lie algebras

An **invariant subalgebra** is a set of generators $t_a^{(i)}$ whose commutator with every generator t_b of the group is a linear combination of the generators $t_c^{(i)}$ of the invariant subalgebra

$$[t_a^{(i)}, t_b] = i f_{abc} t_c^{(i)}. \tag{10.184}$$

The whole algebra and the null algebra are trivial invariant subalgebras.

An algebra with no nontrivial invariant subalgebras is a **simple** algebra. A simple algebra generates a **simple group**. An algebra that has no nontrivial abelian invariant subalgebras is a **semisimple** algebra. A semisimple algebra generates a **semisimple group**.

Example 10.27 (Some Simple Lie Groups) The groups of unitary matrices of unit determinant $SU(2)$, $SU(3)$, ... are simple. So are the groups of orthogonal matrices of unit determinant $SO(n)$ (except $SO(4)$, which is semisimple) and the groups of symplectic matrices $Sp(2n)$ (section 10.31). □

Example 10.28 (Unification and Grand Unification) The symmetry group of the **standard model of particle physics** is a **direct product** of an $SU(3)$ group that acts on colored fields, an $SU(2)$ group that acts on **left-handed** quark and lepton fields, and a $U(1)$ group that acts on fields that carry hypercharge. Each of these three groups, is an invariant subgroup of the full symmetry group $SU(3)_c \otimes SU(2)_\ell \otimes U(1)_Y$, and the last one is abelian. Thus the symmetry group of the standard model is neither simple nor semisimple. In theories of **grand unification**, the strong and electroweak interactions unify at very high energies and are described by a simple group which makes all its charges simple multiples of each other. Georgi and Glashow suggested the group $SU(5)$ in 1976 (Howard Georgi, 1947– ; Sheldon Glashow, 1932–). Others have proposed $SO(10)$ and even bigger groups. □

10.28 $SU(3)$

The Gell-Mann matrices are

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \text{and} \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \tag{10.185}$$

The generators t_a of the 3×3 defining representation of $SU(3)$ are these Gell-Mann matrices divided by 2

$$t_a = \lambda_a/2 \tag{10.186}$$

(Murray Gell-Mann, 1929–).

The eight generators t_a are orthogonal with $k = 1/2$

$$\mathrm{Tr}\,(t_a t_b) = \frac{1}{2}\delta_{ab} \tag{10.187}$$

and satisfy the commutation relation

$$[t_a, t_b] = i f_{abc}\, t_c. \tag{10.188}$$

The trace formula (10.72) gives us the ***SU*(3) structure constants** as

$$f_{abc} = -2i\,\mathrm{Tr}\,([t_a, t_b]t_c)\,. \tag{10.189}$$

They are real and totally antisymmetric with $f_{123} = 1$, $f_{458} = f_{678} = \sqrt{3}/2$, and $f_{147} = -f_{156} = f_{246} = f_{257} = f_{345} = -f_{367} = 1/2$.

While no two generators of $SU(2)$ commute, two generators of $SU(3)$ do. In the representation (10.185,10.186), t_3 and t_8 are diagonal and so commute

$$[t_3, t_8] = 0. \tag{10.190}$$

They generate the **Cartan subalgebra** (section 10.30) of $SU(3)$.

10.29 $SU(3)$ and quarks

The generators defined by Eqs.(10.186 & 10.185) give us the 3×3 representation

$$D(\alpha) = \exp\,(i\alpha_a t_a) \tag{10.191}$$

in which the sum $a = 1, 2, \ldots 8$ is over the eight generators t_a. This representation acts on complex 3-vectors and is called the **3**.

Note that if

$$D(\alpha_1)D(\alpha_2) = D(\alpha_3) \tag{10.192}$$

then the complex conjugates of these matrices obey the same multiplication rule

$$D^*(\alpha_1)D^*(\alpha_2) = D^*(\alpha_3) \tag{10.193}$$

and so form another representation of $SU(3)$. It turns out that (unlike in $SU(2)$) this representation is inequivalent to the **3**; it is the $\overline{\mathbf{3}}$.

There are three quarks with masses less than about 100 MeV/c^2—the

u, d, and s quarks. The other three quarks c, b, and t are more massive; $m_c = 1.28$ GeV, $m_b = 4.18$ GeV, and $m_t = 173.1$ GeV. Nobody knows why. Gell-Mann and Zweig suggested that the low-energy strong interactions were approximately invariant under unitary transformations of the three light quarks, which they represented by a $\mathbf{3}$, and of the three light antiquarks, which they represented by a $\overline{\mathbf{3}}$. They imagined that the eight light pseudoscalar mesons, that is, the three pions π^-, π^0, π^+, the neutral η, and the four kaons K^0, K^+, K^- $\overline{K}^0$, were composed of a quark and an antiquark. So they should transform as the tensor product

$$\mathbf{3} \otimes \overline{\mathbf{3}} = \mathbf{8} \oplus \mathbf{1}. \tag{10.194}$$

They put the eight pseudoscalar mesons into an $\mathbf{8}$.

They imagined that the eight light baryons — the two nucleons N and P, the three sigmas Σ^-, Σ^0, Σ^+, the neutral lambda Λ, and the two cascades Ξ^- and Ξ^0 were each made of three quarks. They should transform as the tensor product

$$\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = \mathbf{10} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{1}. \tag{10.195}$$

They put the eight light baryons into one of these $\mathbf{8}$'s. When they were writing these papers, there were nine spin-3/2 resonances with masses somewhat heavier than 1200 MeV/c^2 — four Δ's, three Σ^*'s, and two Ξ^*'s. They put these into the $\mathbf{10}$ and predicted the tenth and its mass. In 1964, a tenth spin-3/2 resonance, the Ω^-, was found with a mass close to their prediction of 1680 MeV/c^2, and by 1973 an MIT-SLAC team had discovered quarks inside protons and neutrons. (George Zweig, 1937–)

10.30 Cartan Subalgebra

In any Lie group, the maximum set of mutually commuting generators H_a generate the **Cartan subalgebra**

$$[H_a, H_b] = 0 \tag{10.196}$$

which is an abelian subalgebra. The number of generators in the Cartan subalgebra is the **rank** of the Lie algebra. The Cartan generators H_a can be simultaneously diagonalized, and their eigenvalues or diagonal elements are the **weights**

$$H_a|\mu, x, D\rangle = \mu_a|\mu, x, D\rangle \tag{10.197}$$

in which D labels the representation and x whatever other variables are needed to specify the state. The vector μ is the **weight vector**. The **roots** are the weights of the adjoint representation.

10.31 The Symplectic Group *Sp(2n)*

The real symplectic group $Sp(2n, \mathbb{R})$ is the group of linear transformations that preserve the canonical commutation relations of quantum mechanics

$$[q_i, p_k] = i\hbar\delta_{ik}, \quad [q_i, q_k] = 0, \quad \text{and} \quad [p_i, p_k] = 0 \tag{10.198}$$

for $i, k = 1, \ldots, n$. In terms of the $2n$ vector $v = (q_1, \ldots, q_n, p_1, \ldots, p_n)$ of quantum variables, these commutation relations are $[v_i, v_k] = i\hbar J_{ik}$ where J is the $2n \times 2n$ real matrix

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \tag{10.199}$$

in which I is the $n \times n$ identity matrix. The real linear transformation

$$v'_i = \sum_{\ell=1}^{2n} R_{i\ell}\, v_\ell \tag{10.200}$$

will preserve the quantum-mechanical commutation relations (10.198) if

$$[v'_i, v'_k] = \left[\sum_{\ell=1}^{2n} R_{i\ell}\, v_\ell, \sum_{m=1}^{2n} R_{km}\, v_m\right] = i\hbar \sum_{\ell,k=1}^{2n} R_{i\ell}\, J_{\ell m}\, R_{km} = i\hbar J_{ik} \tag{10.201}$$

which in matrix notation is just the condition

$$R\, J\, R^{\mathsf{T}} = J \tag{10.202}$$

that the matrix R be in the real symplectic group $Sp(2n, \mathbb{R})$. The transpose and product rules (1.209 & 1.219) for determinants imply that $\det(R) = \pm 1$, but the condition (10.202) itself implies that $\det(R) = 1$ (Zee, 2016, p. 281).

In terms of the matrix J and the hamiltonian $H(v) = H(q, p)$, Hamilton's equations have the symplectic form

$$\dot{q}_i = \frac{\partial H(q,p)}{\partial p_i} \quad \text{and} \quad \dot{p}_i = -\frac{\partial H(q,p)}{\partial q_i} \quad \text{are} \quad \frac{dv_i}{dt} = \sum_{\ell=1}^{2n} J_{i\ell}\, \frac{\partial H(v)}{\partial v_\ell}. \tag{10.203}$$

A matrix $R = e^t$ obeys the defining condition (10.202) if $t\, J = -J\, t^{\mathsf{T}}$ (exercise 10.22) or equivalently if $JtJ = t^{\mathsf{T}}$. It follows (exercise 10.23) that the generator t must be

$$t = \begin{pmatrix} b & s_1 \\ s_2 & -b^{\mathsf{T}} \end{pmatrix} \tag{10.204}$$

in which the matrices b, s_1, s_2 are real, and both s_1 and s_2 are symmetric.

The group $Sp(2n, \mathbb{R})$ is noncompact.

Example 10.29 (Squeezed states) A coherent state $|\alpha\rangle$ is an eigenstate of the annihilation operator $a = (\lambda q + ip/\lambda)/\sqrt{2\hbar}$ with a complex eigenvalue α (2.146), $a|\alpha\rangle = \alpha|\alpha\rangle$. In a coherent state with $\lambda = \sqrt{m\omega}$, the variances are $(\Delta q)^2 = \langle\alpha|(q - \bar{q})^2|\alpha\rangle = \hbar/(2m\omega)$ and $(\Delta p)^2 = \langle\alpha|(p - \bar{p})^2|\alpha\rangle = \hbar m\omega/2$. Thus coherent states have minimum uncertainty, $\Delta q\, \Delta p = \hbar/2$.

A squeezed state $|\alpha\rangle'$ is an eigenstate of $a' = (\lambda q' + ip'/\lambda)/\sqrt{2\hbar}$ in which q' and p' are related to q and p by an $Sp(2)$ transformation

$$\begin{pmatrix} q' \\ p' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix} \text{ with inverse } \begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} q' \\ p' \end{pmatrix} \tag{10.205}$$

in which $ad - bc = 1$. The standard deviations of the variables q and p in the squeezed state $|\alpha'\rangle$ are

$$\Delta q = \sqrt{\frac{\hbar}{2}} \sqrt{\frac{d^2}{m\omega} + m\omega\, b^2} \quad \text{and} \quad \Delta p = \sqrt{\frac{\hbar}{2}} \sqrt{\frac{c^2}{m\omega} + m\omega\, a^2}. \tag{10.206}$$

Thus by making b and d tiny, one can reduce the uncertainty Δq by any factor, but then Δp will increase by the same factor since the determinant of the $Sp(2)$ transformation must remain equal to unity, $ad - bc = 1$. □

Example 10.30 ($Sp(2, \mathbb{R})$) The matrices (exercise 10.27)

$$T = \pm \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix} \tag{10.207}$$

are elements of the noncompact symplectic group $Sp(2, \mathbb{R})$. □

A dynamical map $\mathcal{M}$ that takes a $2n$ vector $v = (q_1, \ldots, q_n, p_1, \ldots, p_n)$ from $v(t_1)$ to $v(t_2)$ has a jacobian (section 1.21)

$$M_{ab} = \frac{\partial z_a(t_2)}{\partial z_b(t_1)} \tag{10.208}$$

in $Sp(2n, \mathbb{R})$ if and only if its dynamics are hamiltonian (10.203, section 17.1) (Carl Jacobi 1804–1851, William Hamilton 1805–1865).

The complex symplectic group $Sp(2n, \mathbb{C})$ consists of all $2n \times 2n$ complex matrices C that satisfy the condition

$$C\, J\, C^{\mathsf{T}} = J. \tag{10.209}$$

The group $Sp(2n, \mathbb{C})$ also is noncompact.

The unitary symplectic group $USp(2n)$ consists of all $2n \times 2n$ complex unitary matrices U that satisfy the condition

$$U\,J\,U^{\mathsf{T}} = J. \tag{10.210}$$

It is compact.

10.32 Quaternions

If z and w are any two complex numbers, then the 2×2 matrix

$$q = \begin{pmatrix} z & w \\ -w^* & z^* \end{pmatrix} \tag{10.211}$$

is a quaternion. The quaternions are closed under addition and multiplication and under multiplication by a real number (exercise 10.21), but not under multiplication by an arbitrary complex number. The squared norm of q is its determinant

$$\|q\|^2 = |z|^2 + |w|^2 = \det q. \tag{10.212}$$

The matrix products $q^\dagger q$ and $q\,q^\dagger$ are the squared norm $\|q\|^2$ multiplied by the 2×2 identity matrix

$$q^\dagger q = q\,q^\dagger = \|q\|^2\,I \tag{10.213}$$

The 2×2 matrix

$$i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{10.214}$$

provides another expression for $\|q\|^2$ in terms of q and its transpose q^{T}

$$q^{\mathsf{T}} i\sigma_2\,q = \|q\|^2\,i\sigma_2. \tag{10.215}$$

Clearly $\|q\| = 0$ implies $q = 0$. The norm of a product of quaternions is the product of their norms

$$\|q_1 q_2\| = \sqrt{\det(q_1 q_2)} = \sqrt{\det q_1 \det q_2} = \|q_1\|\|q_2\|. \tag{10.216}$$

The quaternions therefore form an **associative division algebra** (over the real numbers); the only others are the real numbers and the complex numbers; the **octonions** are a nonassociative division algebra.

One may use the Pauli matrices to define for any real 4-vector x a quaternion $q(x)$ as

$$\begin{aligned} q(x) &= x_0 + i\,\sigma_k\,x_k = x_0 + i\,\boldsymbol{\sigma}\cdot\boldsymbol{x} \\ &= \begin{pmatrix} x_0 + i\,x_3 & x_2 + i\,x_1 \\ -x_2 + i\,x_1 & x_0 - i\,x_3 \end{pmatrix} \end{aligned} \tag{10.217}$$

with squared norm

$$\|q(x)\|^2 = x_0^2 + x_1^2 + x_2^2 + x_3^2. \tag{10.218}$$

The product rule (10.113) for the Pauli matrices tells us that the product of two quaternions is

$$\begin{aligned} q(x)\, q(y) = &\ (x_0 + i\,\boldsymbol{\sigma}\cdot\boldsymbol{x})(y_0 + i\,\boldsymbol{\sigma}\cdot\boldsymbol{y}) \\ = &\ x_0 y_0 + i\,\boldsymbol{\sigma}\cdot(y_0\,\boldsymbol{x} + x_0\,\boldsymbol{y}) - i\,(\boldsymbol{x}\times\boldsymbol{y})\cdot\boldsymbol{\sigma} - \boldsymbol{x}\cdot\boldsymbol{y} \end{aligned} \tag{10.219}$$

so their commutator is

$$[q(x), q(y)] = -2\,i\,(\boldsymbol{x}\times\boldsymbol{y})\cdot\boldsymbol{\sigma}. \tag{10.220}$$

Example 10.31 (Lack of Analyticity) One may define a function $f(q)$ of a quaternionic variable and then ask what functions are analytic in the sense that the (one-sided) derivative

$$f'(q) = \lim_{q'\to 0} \big[f(q+q') - f(q)\big]\, q'^{-1} \tag{10.221}$$

exists and is independent of the direction through which $q' \to 0$. This space of functions is extremely limited and does not even include the function $f(q) = q^2$ (exercise 10.24). □

10.33 Quaternions and symplectic groups

This section is optional on a first reading.

One may regard the unitary symplectic group $USp(2n)$ as made of $2n\times 2n$ matrices W that map n-tuples q of quaternions into n-tuples $q' = Wq$ of quaternions with the same value of the quadratic quaternionic form

$$\|q'\|^2 = \|q_1'\|^2 + \|q_2'\|^2 + \ldots + \|q_n'\|^2 = \|q_1\|^2 + \|q_2\|^2 + \ldots + \|q_n\|^2 = \|q\|^2. \tag{10.222}$$

By (10.213), the quadratic form $\|q'\|^2$ times the 2×2 identity matrix I is equal to the hermitian form $q'^\dagger q'$

$$\|q'\|^2\, I = q'^\dagger q' = q_1'^\dagger q_1' + \ldots + q_n'^\dagger q_n' = q^\dagger W^\dagger W q \tag{10.223}$$

and so any matrix W that is both a $2n\times 2n$ unitary matrix and an $n\times n$ matrix of quaternions keeps $\|q'\|^2 = \|q\|^2$

$$\|q'\|^2\, I = q^\dagger W^\dagger W q = q^\dagger q = \|q\|^2\, I. \tag{10.224}$$

The group $USp(2n)$ thus consists of all $2n\times 2n$ unitary matrices that also

are $n \times n$ matrices of quaternions. (This last requirement is needed so that $q' = Wq$ is an n-tuple of quaternions.)

The generators t_a of the symplectic group $USp(2n)$ are $2n \times 2n$ direct-product matrices of the form

$$I \otimes A, \quad \sigma_1 \otimes S_1, \quad \sigma_2 \otimes S_2, \quad \text{and} \quad \sigma_3 \otimes S_3 \tag{10.225}$$

in which I is the 2×2 identity matrix, the three σ_i's are the Pauli matrices, A is an imaginary $n \times n$ anti-symmetric matrix, and the S_i are $n \times n$ real symmetric matrices. These generators t_a close under commutation

$$[t_a, t_b] = i f_{abc} t_c. \tag{10.226}$$

Any imaginary linear combination $i\alpha_a t_a$ of these generators is not only a $2n \times 2n$ antihermitian matrix but also an $n \times n$ matrix of quaternions. Thus the matrices

$$D(\alpha) = e^{i\alpha_a t_a} \tag{10.227}$$

are both unitary $2n \times 2n$ matrices and $n \times n$ quaternionic matrices and so are elements of the group $Sp(2n)$.

Example 10.32 ($USp(2) \cong SU(2)$) There is no 1×1 anti-symmetric matrix, and there is only one 1×1 symmetric matrix. So the generators t_a of the group $Sp(2)$ are the Pauli matrices $t_a = \sigma_a$, and $Sp(2) = SU(2)$. The elements $g(\alpha)$ of $SU(2)$ are quaternions of unit norm (exercise 10.20), and so the product $g(\alpha)q$ is a quaternion

$$\|g(\alpha)q\|^2 = \det(g(\alpha)q) = \det(g(\alpha)) \det q = \det q = \|q\|^2 \tag{10.228}$$

with the same squared norm. □

Example 10.33 ($SO(4) \cong SU(2) \otimes SU(2)$) If g and h are any two elements of the group $SU(2)$, then the squared norm (10.218) of the quaternion $q(x) = x_0 + i\boldsymbol{\sigma} \cdot \boldsymbol{x}$ is invariant under the transformation $q(x') = g\, q(x)\, h^{-1}$, that is, $x_0'^2 + x_1'^2 + x_2'^2 + x_3'^2 = x_0^2 + x_1^2 + x_2^2 + x_3^2$. So $x \to x'$ is an $SO(4)$ rotation of the 4-vector x. The Lie algebra of $SO(4)$ thus contains two commuting invariant $SU(2)$ subalgebras and so is semisimple. □

Example 10.34 ($USp(4) \cong SO(5)$) Apart from scale factors, there are three real symmetric 2×2 matrices $S_1 = \sigma_1$, $S_2 = I$, and $S_3 = \sigma_3$ and one imaginary anti-symmetric 2×2 matrix $A = \sigma_2$. So there are 10 generators

of $USp(4) = SO(5)$

$$\begin{aligned} t_1 &= I \otimes \sigma_2 = \begin{pmatrix} 0 & -iI \\ iI & 0 \end{pmatrix}, \quad t_{k1} = \sigma_k \otimes \sigma_1 = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix} \\ t_{k2} &= \sigma_k \otimes I = \begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix}, \quad t_{k3} = \sigma_k \otimes \sigma_3 = \begin{pmatrix} \sigma_k & 0 \\ 0 & -\sigma_k \end{pmatrix} \end{aligned} \tag{10.229}$$

where k runs from 1 to 3. □

Another way of looking at $USp(2n)$ is to use (10.215) to write the quadratic form $\|q\|^2$ as

$$\|q\|^2 \mathcal{J} = q^\mathsf{T} \mathcal{J} q \tag{10.230}$$

in which the $2n \times 2n$ matrix $\mathcal{J}$ has n copies of $i\sigma_2$ on its 2×2 diagonal

$$\mathcal{J} = \begin{pmatrix} i\sigma_2 & 0 & 0 & 0 & \dots & 0 \\ 0 & i\sigma_2 & 0 & 0 & \dots & 0 \\ 0 & 0 & i\sigma_2 & 0 & \dots & 0 \\ 0 & 0 & 0 & i\sigma_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & i\sigma_2 \end{pmatrix} \tag{10.231}$$

and is the matrix J (10.199) in a different basis. Thus any $n \times n$ matrix of quaternions W that satisfies

$$W^\mathsf{T} \mathcal{J} W = \mathcal{J} \tag{10.232}$$

also satisfies

$$\|Wq\|^2 \mathcal{J} = q^\mathsf{T} W^\mathsf{T} \mathcal{J} W q = q^\mathsf{T} \mathcal{J} q = \|q\|^2 \mathcal{J} \tag{10.233}$$

and so leaves invariant the quadratic form (10.222). The group $USp(2n)$ therefore consists of all $2n \times 2n$ matrices W that satisfy (10.232) and that also are $n \times n$ matrices of quaternions.

10.34 Compact Simple Lie Groups

Élie Cartan (1869–1951) showed that all compact, simple Lie groups fall into four infinite classes and five discrete cases. For $n = 1, 2, \dots$, his four classes are

- $A_n = SU(n+1)$ which are $(n+1) \times (n+1)$ unitary matrices with unit determinant,

- $B_n = SO(2n+1)$ which are $(2n+1) \times (2n+1)$ orthogonal matrices with unit determinant,
- $C_n = Sp(2n)$ which are $2n \times 2n$ symplectic matrices, and
- $D_n = SO(2n)$ which are $2n \times 2n$ orthogonal matrices with unit determinant.

The five discrete cases are the **exceptional groups** G_2, F_4, E_6, E_7, and E_8.

The exceptional groups are associated with the **octonians**

$$a + b_\alpha i_\alpha \tag{10.234}$$

where the α-sum runs from 1 to 7; the eight numbers a and b_α are real; and the seven i_α's obey the multiplication law

$$i_\alpha \, i_\beta = -\delta_{\alpha\beta} + g_{\alpha\beta\gamma} \, i_\gamma \tag{10.235}$$

in which $g_{\alpha\beta\gamma}$ is totally antisymmetric with

$$g_{123} = g_{247} = g_{451} = g_{562} = g_{634} = g_{375} = g_{716} = 1. \tag{10.236}$$

Like the quaternions and the complex numbers, the octonians form a **division algebra** with an absolute value

$$|a + b_\alpha i_\alpha| = \left(a^2 + b_\alpha^2\right)^{1/2} \tag{10.237}$$

that satisfies

$$|AB| = |A||B| \tag{10.238}$$

but they lack associativity.

The group G_2 is the subgroup of $SO(7)$ that leaves the $g_{\alpha\beta\gamma}$'s of (10.235) invariant.

10.35 Group Integration

Suppose we need to integrate some function $f(g)$ over a group. Naturally, we want to do so in a way that gives equal weight to every element of the group. In particular, if g' is any group element, we want the integral of the shifted function $f(g'g)$ to be the same as the integral of $f(g)$

$$\int f(g) \, dg = \int f(g'g) \, dg. \tag{10.239}$$

Such a measure dg is said to be **left invariant** (Creutz, 1983, chap. 8).

Let's use the letters $a = a_1, \ldots, a_n$, $b = b_1, \ldots, b_n$, and so forth to label the elements $g(a)$, $g(b)$, so that an integral over the group is

$$\int f(g)\, dg = \int f(g(a))\, m(a)\, d^n a \tag{10.240}$$

in which $m(a)$ is the left-invariant measure and the integration is over the n-space of a's that label all the elements of the group.

To find the left-invariant measure $m(a)$, we use the multiplication law of the group

$$g(a(c,b)) \equiv g(c)\, g(b) \tag{10.241}$$

and impose the requirement (10.239) of left invariance with $g' \equiv g(c)$

$$\int f(g(b))\, m(b)\, d^n b = \int f(g(c)g(b))\, m(b)\, d^n b = \int f(g(a(c,b)))\, m(b)\, d^n b. \tag{10.242}$$

We change variables from b to $a = a(c,b)$ by using the jacobian $\det(\partial b/\partial a)$ which gives us $d^n b = \det(\partial b/\partial a)\, d^n a$

$$\int f(g(b))\, m(b)\, d^n b = \int f(g(a))\, \det(\partial b/\partial a)\, m(b)\, d^n a. \tag{10.243}$$

Replacing b by $a = a(c,b)$ on the left-hand side of this equation, we find

$$m(a) = \det(\partial b/\partial a)\, m(b) \tag{10.244}$$

or since $\det(\partial b/\partial a) = 1/\det(\partial a(c,b)/\partial b)$

$$m(a(c,b)) = m(b)/\det(\partial a(c,b)/\partial b). \tag{10.245}$$

So if we let $g(b) \to g(0) = e$, the identity element of the group, and set $m(e) = 1$, then we find for the measure

$$m(a) = m(c) = m(a(c,b))|_{b=0} = 1/\det(\partial a(c,b)/\partial b)|_{b=0}\,. \tag{10.246}$$

Example 10.35 (The Invariant Measure for SU(2)) A general element of the group $SU(2)$ is given by (10.116) as

$$\exp\left(i\, \boldsymbol{\theta} \cdot \frac{\boldsymbol{\sigma}}{2}\right) = I\, \cos\frac{\theta}{2} + i\, \hat{\boldsymbol{\theta}} \cdot \boldsymbol{\sigma}\, \sin\frac{\theta}{2}. \tag{10.247}$$

Setting $a_0 = \cos(\theta/2)$ and $\boldsymbol{a} = \hat{\boldsymbol{\theta}}\, \sin(\theta/2)$, we have

$$g(a) = a_0 + i\, \boldsymbol{a} \cdot \boldsymbol{\sigma} \tag{10.248}$$

in which $a^2 \equiv a_0^2 + \boldsymbol{a} \cdot \boldsymbol{a} = 1$. Thus, the parameter space for $SU(2)$ is the

unit sphere S_3 in four dimensions. Its invariant measure is

$$\int \delta(1-a^2)\,d^4a = \int \delta(1-a_0^2-\boldsymbol{a}^2)\,d^4a = \int (1-\boldsymbol{a}^2)^{-1/2}\,d^3a \tag{10.249}$$

or

$$m(\boldsymbol{a}) = (1-\boldsymbol{a}^2)^{-1/2} = \frac{1}{|\cos(\theta/2)|}. \tag{10.250}$$

We also can write the arbitrary element (10.248) of $SU(2)$ as

$$g(\boldsymbol{a}) = \pm\sqrt{1-\boldsymbol{a}^2} + i\,\boldsymbol{a}\cdot\boldsymbol{\sigma} \tag{10.251}$$

and the group-multiplication law (10.241) as

$$\sqrt{1-\boldsymbol{a}^2} + i\,\boldsymbol{a}\cdot\boldsymbol{\sigma} = \left(\sqrt{1-\boldsymbol{c}^2} + i\,\boldsymbol{c}\cdot\boldsymbol{\sigma}\right)\left(\sqrt{1-\boldsymbol{b}^2} + i\,\boldsymbol{b}\cdot\boldsymbol{\sigma}\right). \tag{10.252}$$

Thus, by multiplying both sides of this equation by σ_i and taking the trace, we find (exercise 10.28) that the parameters $\boldsymbol{a}(\boldsymbol{c},\boldsymbol{b})$ that describe the product $g(\boldsymbol{c})\,g(\boldsymbol{b})$ are

$$\boldsymbol{a}(\boldsymbol{c},\boldsymbol{b}) = \sqrt{1-\boldsymbol{c}^2}\,\boldsymbol{b} + \sqrt{1-\boldsymbol{b}^2}\,\boldsymbol{c} - \boldsymbol{c}\times\boldsymbol{b}. \tag{10.253}$$

To compute the jacobian of our formula (10.246) for the invariant measure, we differentiate this expression (10.253) at $\boldsymbol{b}=\boldsymbol{0}$ and so find (exercise 10.29)

$$m(\boldsymbol{a}) = 1/\det(\partial a(c,b)/\partial b)|_{b=0} = (1-\boldsymbol{a}^2)^{-1/2} \tag{10.254}$$

as the left-invariant measure in agreement with (10.250). □

10.36 The Lorentz group

The Lorentz group $O(3,1)$ is the set of all linear transformations L that leave invariant the Minkowski inner product

$$xy \equiv \boldsymbol{x}\cdot\boldsymbol{y} - x^0y^0 = x^\mathsf{T}\eta y \tag{10.255}$$

in which η is the diagonal matrix

$$\eta = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{10.256}$$

So L is in $O(3,1)$ if for all 4-vectors x and y

$$(Lx)^\mathsf{T}\,\eta L\,y = x^\mathsf{T}L^\mathsf{T}\,\eta\,Ly = x^\mathsf{T}\eta\,y. \tag{10.257}$$

Since x and y are arbitrary, this condition amounts to

$$L^{\mathsf{T}} \eta\, L = \eta. \tag{10.258}$$

Taking the determinant of both sides and using the transpose (1.209) and product (1.222) rules, we have

$$(\det L)^2 = 1. \tag{10.259}$$

So $\det L = \pm 1$, and every Lorentz transformation L has an inverse. Multiplying (10.258) by η, we get

$$\eta L^{\mathsf{T}} \eta L = \eta^2 = I \tag{10.260}$$

which identifies L^{-1} as

$$L^{-1} = \eta L^{\mathsf{T}} \eta. \tag{10.261}$$

The subgroup of $O(3,1)$ with $\det L = 1$ is the proper Lorentz group $SO(3,1)$. The subgroup of $SO(3,1)$ that leaves invariant the sign of the time component of timelike vectors is the **proper orthochronous Lorentz group** $SO^+(3,1)$.

To find the Lie algebra of $SO^+(3,1)$, we take a Lorentz matrix $L = I + \omega$ that differs from the identity matrix I by a tiny matrix ω and require L to obey the condition (10.258) for membership in the Lorentz group

$$\left(I + \omega^{\mathsf{T}}\right) \eta\, (I + \omega) = \eta + \omega^{\mathsf{T}} \eta + \eta\, \omega + \omega^{\mathsf{T}} \omega = \eta. \tag{10.262}$$

Neglecting $\omega^{\mathsf{T}} \omega$, we have $\omega^{\mathsf{T}} \eta = -\eta\, \omega$ or since $\eta^2 = I$

$$\omega^{\mathsf{T}} = -\, \eta\, \omega\, \eta. \tag{10.263}$$

This equation says (exercise 10.31) that under transposition the time-time and space-space elements of ω change sign, while the time-space and space-time elements do not. That is, the tiny matrix ω is for infinitesimal $\boldsymbol{\theta}$ and $\boldsymbol{\lambda}$ a linear combination

$$\omega = \boldsymbol{\theta} \cdot \boldsymbol{R} + \boldsymbol{\lambda} \cdot \boldsymbol{B} \tag{10.264}$$

of three antisymmetric space-space matrices

$$R_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad R_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad R_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{10.265}$$

and of three symmetric time-space matrices

$$B_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad B_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad B_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{10.266}$$

all of which satisfy condition (10.263). The three R_ℓ are 4×4 versions of the rotation generators (10.92); the three B_ℓ generate Lorentz boosts.

If we write $L = I + \omega$ as

$$L = I - i\theta_\ell \, iR_\ell - i\lambda_\ell \, iB_\ell \equiv I - i\theta_\ell J_\ell - i\lambda_\ell K_\ell \tag{10.267}$$

then the three matrices $J_\ell = iR_\ell$ are imaginary and antisymmetric, and therefore hermitian. But the three matrices $K_\ell = iB_\ell$ are imaginary and symmetric, and so are antihermitian. The 4×4 matrix $L = \exp(i\theta_\ell J_\ell - i\lambda_\ell K_\ell)$ is **not unitary** because the Lorentz group is **not compact**.

One may verify (exercise 10.32) that the six generators J_ℓ and K_ℓ satisfy three sets of commutation relations:

$$[J_i, J_j] = i\epsilon_{ijk} J_k \tag{10.268}$$

$$[J_i, K_j] = i\epsilon_{ijk} K_k \tag{10.269}$$

$$[K_i, K_j] = -i\epsilon_{ijk} J_k. \tag{10.270}$$

The first (10.268) says that the three J_ℓ generate the rotation group $SO(3)$; the second (10.269) says that the three boost generators transform as a 3-vector under $SO(3)$; and the third (10.270) implies that four canceling infinitesimal boosts can amount to a rotation. These three sets of commutation relations form the Lie algebra of the Lorentz group $SO(3,1)$. Incidentally, one may show (exercise 10.33) that if $\boldsymbol{J}$ and $\boldsymbol{K}$ satisfy these commutation relations (10.268–10.270), then so do

$$\boldsymbol{J} \quad \text{and} \quad -\boldsymbol{K}. \tag{10.271}$$

The infinitesimal Lorentz transformation (10.267) is the 4×4 matrix

$$L = I + \omega = I + \theta_\ell R_\ell + \lambda_\ell B_\ell = \begin{pmatrix} 1 & \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1 & 1 & -\theta_3 & \theta_2 \\ \lambda_2 & \theta_3 & 1 & -\theta_1 \\ \lambda_3 & -\theta_2 & \theta_1 & 1 \end{pmatrix}. \tag{10.272}$$

It moves any 4-vector x to $x' = L\,x$ or in components $x'^a = L^a{}_b\,x^b$

$$\begin{aligned} x'^0 &= x^0 + \lambda_1 x^1 + \lambda_2 x^2 + \lambda_3 x^3 \\ x'^1 &= \lambda_1 x^0 + x^1 - \theta_3 x^2 + \theta_2 x^3 \\ x'^2 &= \lambda_2 x^0 + \theta_3 x^1 + x^2 - \theta_1 x^3 \\ x'^3 &= \lambda_3 x^0 - \theta_2 x^1 + \theta_1 x^2 + x^3. \end{aligned} \tag{10.273}$$

More succinctly with $t = x^0$, this is

$$\begin{aligned} t' &= t + \boldsymbol{\lambda}\cdot\boldsymbol{x} \\ \boldsymbol{x}' &= \boldsymbol{x} + t\boldsymbol{\lambda} + \boldsymbol{\theta}\wedge\boldsymbol{x} \end{aligned} \tag{10.274}$$

in which $\wedge \equiv \times$ means cross-product.

For arbitrary real $\boldsymbol{\theta}$ and $\boldsymbol{\lambda}$, the matrices

$$L = e^{-i\boldsymbol{\theta}\cdot\boldsymbol{J} - i\boldsymbol{\lambda}\cdot\boldsymbol{K}} \tag{10.275}$$

form the subgroup of $O(3,1)$ that is connected to the identity matrix I. The matrices of this subgroup have unit determinant and preserve the sign of the time of time-like vectors, that is, if $x^2 < 0$, and $y = Lx$, then $y^0 x^0 > 0$. This is the proper orthochronous Lorentz group $SO^+(3,1)$. The rest of the (homogeneous) Lorentz group can be obtained from it by space $\mathcal{P}$, time $\mathcal{T}$, and spacetime $\mathcal{PT}$ reflections.

The task of finding all the finite-dimensional irreducible representations of the proper orthochronous homogeneous Lorentz group becomes vastly simpler when we write the commutation relations (10.268–10.270) in terms of the hermitian matrices

$$J_\ell^\pm = \frac{1}{2}\left(J_\ell \pm iK_\ell\right) \tag{10.276}$$

which generate two independent rotation groups

$$\begin{aligned} [J_i^+, J_j^+] &= i\epsilon_{ijk} J_k^+ \\ [J_i^-, J_j^-] &= i\epsilon_{ijk} J_k^- \\ [J_i^+, J_j^-] &= 0. \end{aligned} \tag{10.277}$$

Thus the Lie algebra of the Lorentz group is equivalent to two copies of the Lie algebra (10.118) of $SU(2)$.

The hermitian generators of the rotation subgroup $SU(2)$ are by (10.276)

$$\boldsymbol{J} = \boldsymbol{J}^+ + \boldsymbol{J}^-. \tag{10.278}$$

The antihermitian generators of the boosts are (also by 10.276)

$$\boldsymbol{K} = -i\left(\boldsymbol{J}^+ - \boldsymbol{J}^-\right). \tag{10.279}$$

Since $\boldsymbol{J}^+$ and $\boldsymbol{J}^-$ commute, the finite-dimensional irreducible representations of the Lorentz group are the direct products

$$\begin{aligned} D(j,j')(\boldsymbol{\theta},\boldsymbol{\lambda}) &= e^{-i\boldsymbol{\theta}\cdot\boldsymbol{J}-i\boldsymbol{\lambda}\cdot\boldsymbol{K}} = e^{(-i\boldsymbol{\theta}-\boldsymbol{\lambda})\cdot\boldsymbol{J}^+ + (-i\boldsymbol{\theta}+\boldsymbol{\lambda})\cdot\boldsymbol{J}^-} \\ &= e^{(-i\boldsymbol{\theta}-\boldsymbol{\lambda})\cdot\boldsymbol{J}^+}\, e^{(-i\boldsymbol{\theta}+\boldsymbol{\lambda})\cdot\boldsymbol{J}^-} \end{aligned} \tag{10.280}$$

of the nonunitary representations

$$D^{(j,0)}(\boldsymbol{\theta},\boldsymbol{\lambda}) = e^{(-i\boldsymbol{\theta}-\boldsymbol{\lambda})\cdot\boldsymbol{J}^+} \quad \text{and} \quad D^{(0,j')}(\boldsymbol{\theta},\boldsymbol{\lambda}) = e^{(-i\boldsymbol{\theta}+\boldsymbol{\lambda})\cdot\boldsymbol{J}^-} \tag{10.281}$$

generated by the three $(2j+1)\times(2j+1)$ matrices J_ℓ^+ and by the three $(2j'+1)\times(2j'+1)$ matrices J_ℓ^-.

Under a Lorentz transformation L, a field $\psi^{(j,j')}_{m,m'}(x)$ that transforms under the $D^{(j,j')}$ representation of the Lorentz group responds as

$$U(L)\,\psi^{(j,j')}_{m,m'}(x)\,U^{-1}(L) = D^{(j,0)}_{mm''}(L^{-1})\,D^{(0,j')}_{m'm'''}(L^{-1})\,\psi^{(j,j')}_{m'',m'''}(Lx). \tag{10.282}$$

The representation $D^{(j,j')}$ describes objects of the spins s that can arise from the direct product of spin-j with spin-j' (Weinberg, 1995, p. 231)

$$s = j+j',\, j+j'-1,\ldots,\, |j-j'|. \tag{10.283}$$

For instance, $D^{(0,0)}$ describes a spinless field or particle, while $D^{(1/2,0)}$ and $D^{(0,1/2)}$ respectively describe left-handed and right-handed spin-1/2 fields or particles. The representation $D^{(1/2,1/2)}$ describes objects of spin 1 and spin 0—the spatial and time components of a 4-vector. The interchange of $\boldsymbol{J}^+$ and $\boldsymbol{J}^-$ replaces the generators $\boldsymbol{J}$ and $\boldsymbol{K}$ with $\boldsymbol{J}$ and $-\boldsymbol{K}$, a substitution that we know (10.271) is legitimate.

10.37 Two-dimensional left-handed representation of the Lorentz group

The generators of the representation $D^{(1/2,0)}$ with $j=1/2$ and $j'=0$ are given by (10.278 & 10.279) with $\boldsymbol{J}^+ = \boldsymbol{\sigma}/2$ and $\boldsymbol{J}^- = 0$. They are

$$\boldsymbol{J} = \frac{1}{2}\boldsymbol{\sigma} \quad \text{and} \quad \boldsymbol{K} = -i\frac{1}{2}\boldsymbol{\sigma}. \tag{10.284}$$

The 2×2 matrix $D^{(1/2,0)}$ that represents the Lorentz transformation (10.275)

$$L = e^{-i\boldsymbol{\theta}\cdot\boldsymbol{J}-i\boldsymbol{\lambda}\cdot\boldsymbol{K}} \tag{10.285}$$

is

$$D^{(1/2,0)}(\boldsymbol{\theta},\boldsymbol{\lambda}) = \exp\left(-i\boldsymbol{\theta}\cdot\boldsymbol{\sigma}/2 - \boldsymbol{\lambda}\cdot\boldsymbol{\sigma}/2\right). \tag{10.286}$$

And so the generic $D^{(1/2,0)}$ matrix is

$$D^{(1/2,0)}(\boldsymbol{\theta},\boldsymbol{\lambda}) = e^{-\boldsymbol{z}\cdot\boldsymbol{\sigma}/2} \tag{10.287}$$

with $\boldsymbol{\lambda} = \mathrm{Re}\boldsymbol{z}$ and $\boldsymbol{\theta} = \mathrm{Im}\boldsymbol{z}$. It is nonunitary and of unit determinant; it is a member of the group $SL(2,C)$ of complex unimodular 2×2 matrices. The (covering) group $SL(2,C)$ relates to the Lorentz group $SO(3,1)$ as $SU(2)$ relates to the rotation group $SO(3)$.

Example 10.36 (The standard left-handed boost) For a particle of mass $m > 0$, the standard boost that takes the 4-vector $k = (m, \mathbf{0})$ to $p = (p^0, \boldsymbol{p})$, where $p^0 = \sqrt{m^2 + \boldsymbol{p}^2}$ is a boost in the $\hat{\boldsymbol{p}}$ direction. It is the 4×4 matrix

$$B(p) = R(\hat{\boldsymbol{p}})\, B_3(p^0)\, R^{-1}(\hat{\boldsymbol{p}}) = \exp\left(\alpha\, \hat{\boldsymbol{p}}\cdot\boldsymbol{B}\right) \tag{10.288}$$

in which $\cosh\alpha = p^0/m$ and $\sinh\alpha = |\boldsymbol{p}|/m$, as one may show by expanding the exponential (exercise 10.35). This standard boost is represented by $D^{(1/2,0)}(\mathbf{0},\boldsymbol{\lambda})$, the 2×2 matrix (10.285), with $\boldsymbol{\lambda} = \alpha\,\hat{\boldsymbol{p}}$. The power-series expansion of this matrix is (exercise 10.36)

$$\begin{aligned} D^{(1/2,0)}(\mathbf{0},\alpha\,\hat{\boldsymbol{p}}) &= e^{-\alpha\hat{\boldsymbol{p}}\cdot\boldsymbol{\sigma}/2} = I\,\cosh(\alpha/2) - \hat{\boldsymbol{p}}\cdot\boldsymbol{\sigma}\,\sinh(\alpha/2) \\ &= I\,\sqrt{(p^0+m)/(2m)} - \hat{\boldsymbol{p}}\cdot\boldsymbol{\sigma}\,\sqrt{(p^0-m)/(2m)} \\ &= \frac{(p^0+m)I - \boldsymbol{p}\cdot\boldsymbol{\sigma}}{\sqrt{2m(p^0+m)}} \end{aligned} \tag{10.289}$$

in which I is the 2×2 identity matrix. □

Under $D^{(1/2,0)}$, the vector $(-I,\boldsymbol{\sigma})$ transforms like a 4-vector. For tiny $\boldsymbol{\theta}$ and $\boldsymbol{\lambda}$, one may show (exercise 10.38) that the vector $(-I,\boldsymbol{\sigma})$ transforms as

$$\begin{aligned} D^{\dagger(1/2,0)}(\boldsymbol{\theta},\boldsymbol{\lambda})(-I)D^{(1/2,0)}(\boldsymbol{\theta},\boldsymbol{\lambda}) &= -I + \boldsymbol{\lambda}\cdot\boldsymbol{\sigma} \\ D^{\dagger(1/2,0)}(\boldsymbol{\theta},\boldsymbol{\lambda})\,\boldsymbol{\sigma}\,D^{(1/2,0)}(\boldsymbol{\theta},\boldsymbol{\lambda}) &= \boldsymbol{\sigma} + (-I)\boldsymbol{\lambda} + \boldsymbol{\theta}\wedge\boldsymbol{\sigma} \end{aligned} \tag{10.290}$$

which is how the 4-vector $(t,\boldsymbol{x})$ transforms (10.274). Under a finite Lorentz transformation L, the 4-vector $S^a \equiv (-I,\boldsymbol{\sigma})$ goes to

$$D^{\dagger(1/2,0)}(L)\,S^a\,D^{(1/2,0)}(L) = L^a{}_b S^b. \tag{10.291}$$

A massless field $u(x)$ that responds to a unitary Lorentz transformation $U(L)$ like

$$U(L)\,u(x)\,U^{-1}(L) = D^{(1/2,0)}(L^{-1})\,u(Lx) \tag{10.292}$$

is called a **left-handed Weyl spinor**. Its action density

$$\mathcal{L}_\ell(x) = i\,u^\dagger(x)\,(\partial_0 I - \boldsymbol{\nabla}\cdot\boldsymbol{\sigma})\,u(x) \tag{10.293}$$

is Lorentz covariant, that is

$$U(L)\,\mathcal{L}_\ell(x)\,U^{-1}(L) = \mathcal{L}_\ell(Lx). \tag{10.294}$$

Example 10.37 (Why $\mathcal{L}_\ell$ is Lorentz covariant) We first note that the derivatives ∂'_b in $\mathcal{L}_\ell(Lx)$ are with respect to $x' = Lx$. Since the inverse matrix L^{-1} takes x' back to $x = L^{-1}x'$ or in tensor notation $x^a = L^{-1a}{}_b\, x'^b$, the derivative ∂'_b is

$$\partial'_b = \frac{\partial}{\partial x'^b} = \frac{\partial x^a}{\partial x'^b}\frac{\partial}{\partial x^a} = L^{-1a}{}_b\,\frac{\partial}{\partial x^a} = \partial_a\, L^{-1a}{}_b. \tag{10.295}$$

Now using the abbreviation $\partial_0 I - \boldsymbol{\nabla}\cdot\boldsymbol{\sigma} \equiv -\partial_a S^a$ and the transformation laws (10.291 & 10.292), we have

$$\begin{aligned} U(L)\,\mathcal{L}_\ell(x)\,U^{-1}(L) &= i\,u^\dagger(Lx)D^{(1/2,0)\dagger}(L^{-1})(-\partial_a S^a)D^{(1/2,0)}(L^{-1})\,u(Lx) \\ &= i\,u^\dagger(Lx)(-\partial_a L^{-1a}{}_b S^b)\,u(Lx) \\ &= i\,u^\dagger(Lx)(-\partial'_b S^b)\,u(Lx) = \mathcal{L}_\ell(Lx) \end{aligned} \tag{10.296}$$

which shows that $\mathcal{L}_\ell$ is Lorentz covariant. □

Incidentally, the rule (10.295) ensures, among other things, that the divergence $\partial_a V^a$ is invariant

$$(\partial_a V^a)' = \partial'_a V'^a = \partial_b\, L^{-1b}{}_a\, L^a{}_c V^c = \partial_b\,\delta^b{}_c\, V^c = \partial_b\, V^b. \tag{10.297}$$

Example 10.38 (Why u is left handed) The spacetime integral S of the action density $\mathcal{L}_\ell$ is stationary when $u(x)$ satisfies the wave equation

$$(\partial_0 I - \boldsymbol{\nabla}\cdot\boldsymbol{\sigma})\,u(x) = 0 \tag{10.298}$$

or in momentum space

$$(E + \boldsymbol{p}\cdot\boldsymbol{\sigma})\,u(p) = 0. \tag{10.299}$$

Multiplying from the left by $(E - \boldsymbol{p}\cdot\boldsymbol{\sigma})$, we see that the energy of a particle created or annihilated by the field u is the same as its momentum, $E = |\boldsymbol{p}|$. The particles of the field u are massless because the action density $\mathcal{L}_\ell$ has no mass term. The spin of the particle is represented by the matrix $\boldsymbol{J} = \boldsymbol{\sigma}/2$, so the momentum-space relation (10.299) says that $u(p)$ is an eigenvector of $\hat{\boldsymbol{p}}\cdot\boldsymbol{J}$ with eigenvalue $-1/2$

$$\hat{\boldsymbol{p}}\cdot\boldsymbol{J}\,u(p) = -\frac{1}{2}\,u(p). \tag{10.300}$$

A particle whose spin is opposite to its momentum is said to have **negative helicity** or to be **left handed**. Nearly massless neutrinos are nearly left handed. □

One may add to this action density the **Majorana mass term**

$$\mathcal{L}_M(x) = -\tfrac{1}{2} m\, u^\dagger(x)\, \sigma_2\, u^*(x) - \tfrac{1}{2} m^*\, u^\mathsf{T}(x)\, \sigma_2\, u(x) \tag{10.301}$$

which is Lorentz covariant because the matrices σ_1 and σ_3 anti-commute with σ_2 which is antisymmetric (exercise 10.41). This term would vanish if $u_1 u_2$ were equal to $u_2 u_1$. Since charge is conserved, only neutral fields like neutrinos can have Majorana mass terms. The action density of a left-handed field of mass m is the sum $\mathcal{L} = \mathcal{L}_\ell + \mathcal{L}_M$ of the kinetic one (10.293) and the Majorana mass term (10.301). The resulting equations of motion

$$\begin{aligned} 0 &= i\,(\partial_0 - \nabla \cdot \boldsymbol{\sigma})\, u - m\sigma_2 u^* \\ 0 &= \left(\partial_0^2 - \nabla^2 + |m|^2\right) u \end{aligned} \tag{10.302}$$

show that the field u represents particles of mass $|m|$.

10.38 Two-dimensional right-handed representation of the Lorentz group

The generators of the representation $D^{(0,1/2)}$ with $j = 0$ and $j' = 1/2$ are given by (10.278 & 10.279) with $\boldsymbol{J}^+ = 0$ and $\boldsymbol{J}^- = \boldsymbol{\sigma}/2$; they are

$$\boldsymbol{J} = \frac{1}{2}\boldsymbol{\sigma} \quad \text{and} \quad \boldsymbol{K} = i\frac{1}{2}\boldsymbol{\sigma}. \tag{10.303}$$

Thus 2×2 matrix $D^{(0,1/2)}(\boldsymbol{\theta}, \boldsymbol{\lambda})$ that represents the Lorentz transformation (10.275)

$$L = e^{-i\boldsymbol{\theta}\cdot\boldsymbol{J} - i\boldsymbol{\lambda}\cdot\boldsymbol{K}} \tag{10.304}$$

is

$$D^{(0,1/2)}(\boldsymbol{\theta}, \boldsymbol{\lambda}) = \exp\left(-i\boldsymbol{\theta}\cdot\boldsymbol{\sigma}/2 + \boldsymbol{\lambda}\cdot\boldsymbol{\sigma}/2\right) = D^{(1/2,0)}(\boldsymbol{\theta}, -\boldsymbol{\lambda}) \tag{10.305}$$

which differs from $D^{(1/2,0)}(\boldsymbol{\theta}, \boldsymbol{\lambda})$ only by the sign of $\boldsymbol{\lambda}$. The generic $D^{(0,1/2)}$ matrix is the complex unimodular 2×2 matrix

$$D^{(0,1/2)}(\boldsymbol{\theta}, \boldsymbol{\lambda}) = e^{\boldsymbol{z}^*\cdot\boldsymbol{\sigma}/2} \tag{10.306}$$

with $\boldsymbol{\lambda} = \mathrm{Re}\boldsymbol{z}$ and $\boldsymbol{\theta} = \mathrm{Im}\boldsymbol{z}$.

Example 10.39 (The standard right-handed boost) For a particle of mass $m > 0$, the "standard" boost (10.288) that transforms $k = (m, \mathbf{0})$ to $p = (p^0, \boldsymbol{p})$ is the 4×4 matrix $B(p) = \exp\left(\alpha\, \hat{\boldsymbol{p}} \cdot \boldsymbol{B}\right)$ in which $\cosh\alpha = p^0/m$

and $\sinh\alpha = |\boldsymbol{p}|/m$. This Lorentz transformation with $\boldsymbol{\theta} = \mathbf{0}$ and $\boldsymbol{\lambda} = \alpha\,\hat{\boldsymbol{p}}$ is represented by the matrix (exercise 10.37)

$$\begin{aligned} D^{(0,1/2)}(\mathbf{0}, \alpha\,\hat{\boldsymbol{p}}) &= e^{\alpha\hat{\boldsymbol{p}}\cdot\boldsymbol{\sigma}/2} = I\,\cosh(\alpha/2) + \hat{\boldsymbol{p}}\cdot\boldsymbol{\sigma}\,\sinh(\alpha/2) \\ &= I\,\sqrt{(p^0+m)/(2m)} + \hat{\boldsymbol{p}}\cdot\boldsymbol{\sigma}\,\sqrt{(p^0-m)/(2m)} \\ &= \frac{p^0 + m + \boldsymbol{p}\cdot\boldsymbol{\sigma}}{\sqrt{2m(p^0+m)}} \end{aligned} \tag{10.307}$$

in the third line of which the 2×2 identity matrix I is suppressed. □

Under $D^{(0,1/2)}$, the vector $(I, \boldsymbol{\sigma})$ transforms as a 4-vector; for tiny $\boldsymbol{z}$

$$\begin{aligned} D^{\dagger(0,1/2)}(\boldsymbol{\theta},\boldsymbol{\lambda})\,I\,D^{(0,1/2)}(\boldsymbol{\theta},\boldsymbol{\lambda}) &= I + \boldsymbol{\lambda}\cdot\boldsymbol{\sigma} \\ D^{\dagger(0,1/2)}(\boldsymbol{\theta},\boldsymbol{\lambda})\,\boldsymbol{\sigma}\,D^{(0,1/2)}(\boldsymbol{\theta},\boldsymbol{\lambda}) &= \boldsymbol{\sigma} + I\boldsymbol{\lambda} + \boldsymbol{\theta}\wedge\boldsymbol{\sigma} \end{aligned} \tag{10.308}$$

as in (10.274).

A massless field $v(x)$ that responds to a unitary Lorentz transformation $U(L)$ as

$$U(L)\,v(x)\,U^{-1}(L) = D^{(0,1/2)}(L^{-1})\,v(Lx) \tag{10.309}$$

is called a **right-handed Weyl spinor**. One may show (exercise 10.40) that the action density

$$\mathcal{L}_r(x) = i\,v^\dagger(x)\,(\partial_0 I + \boldsymbol{\nabla}\cdot\boldsymbol{\sigma})\,v(x) \tag{10.310}$$

is Lorentz covariant

$$U(L)\,\mathcal{L}_r(x)\,U^{-1}(L) = \mathcal{L}_r(Lx). \tag{10.311}$$

Example 10.40 (Why v is right handed) An argument like that of example (10.38) shows that the field $v(x)$ satisfies the wave equation

$$(\partial_0 I + \boldsymbol{\nabla}\cdot\boldsymbol{\sigma})\,v(x) = 0 \tag{10.312}$$

or in momentum space

$$(E - \boldsymbol{p}\cdot\boldsymbol{\sigma})\,v(p) = 0. \tag{10.313}$$

Thus, $E = |\boldsymbol{p}|$, and $v(p)$ is an eigenvector of $\hat{\boldsymbol{p}}\cdot\boldsymbol{J}$

$$\hat{\boldsymbol{p}}\cdot\boldsymbol{J}\,v(p) = \frac{1}{2}\,v(p) \tag{10.314}$$

with eigenvalue 1/2. A particle whose spin is parallel to its momentum is said to have **positive helicity** or to be **right handed**. Nearly massless antineutrinos are nearly right handed. □

The Majorana mass term

$$\mathcal{L}_M(x) = -\tfrac{1}{2} m\, v^\dagger(x)\, \sigma_2\, v^*(x) - \tfrac{1}{2} m^*\, v^\mathsf{T}(x)\, \sigma_2\, v(x) \tag{10.315}$$

like (10.301) is Lorentz covariant. The action density of a right-handed field of mass m is the sum $\mathcal{L} = \mathcal{L}_r + \mathcal{L}_M$ of the kinetic one (10.310) and this Majorana mass term (10.315). The resulting equations of motion

$$\begin{aligned} 0 &= i\,(\partial_0 + \nabla \cdot \boldsymbol{\sigma})\, v - m\sigma_2 v^* \\ 0 &= \left(\partial_0^2 - \nabla^2 + |m|^2\right) v \end{aligned} \tag{10.316}$$

show that the field v represents particles of mass $|m|$.

10.39 The Dirac Representation of the Lorentz Group

Dirac's representation of $SO(3,1)$ is the direct sum $D^{(1/2,0)} \oplus D^{(0,1/2)}$ of $D^{(1/2,0)}$ and $D^{(0,1/2)}$. Its generators are the 4×4 matrices

$$\boldsymbol{J} = \frac{1}{2}\begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix} \quad \text{and} \quad \boldsymbol{K} = \frac{i}{2}\begin{pmatrix} -\boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}. \tag{10.317}$$

Dirac's representation uses the **Clifford algebra** of the gamma matrices γ^a which satisfy the anticommutation relation

$$\{\gamma^a, \gamma^b\} \equiv \gamma^a\, \gamma^b + \gamma^b\, \gamma^a = 2\eta^{ab} I \tag{10.318}$$

in which η is the 4×4 diagonal matrix (10.256) with $\eta^{00} = -1$ and $\eta^{jj} = 1$ for $j = 1$, 2, and 3, and I is the 4×4 identity matrix.

Remarkably, the generators of the Lorentz group

$$J^{ij} = \epsilon_{ijk} J_k \quad \text{and} \quad J^{0j} = K_j \tag{10.319}$$

may be represented as commutators of gamma matrices

$$J^{ab} = -\frac{i}{4}\,[\gamma^a, \gamma^b]. \tag{10.320}$$

They transform the gamma matrices as a 4-vector

$$[J^{ab}, \gamma^c] = -i\gamma^a\, \eta^{bc} + i\gamma^b\, \eta^{ac} \tag{10.321}$$

(exercise 10.42) and satisfy the commutation relations

$$i[J^{ab}, J^{cd}] = \eta^{bc}\, J^{ad} - \eta^{ac}\, J^{bd} - \eta^{da}\, J^{cb} + \eta^{db}\, J^{ca} \tag{10.322}$$

of the Lorentz group (Weinberg, 1995, p. 213–217) (exercise 10.43).

The gamma matrices γ^a are not unique; if S is any 4×4 matrix with an

inverse, then the matrices $\gamma'^a \equiv S\gamma^a S^{-1}$ also satisfy the definition (10.318). The choice

$$\gamma^0 = -i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \boldsymbol{\gamma} = -i\begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix} \tag{10.323}$$

makes $\boldsymbol{J}$ and $\boldsymbol{K}$ block diagonal (10.317) and lets us assemble a left-handed spinor u and a right-handed spinor v neatly into a 4-component spinor

$$\psi = \begin{pmatrix} u \\ v \end{pmatrix}. \tag{10.324}$$

Dirac's action density for a 4-spinor is

$$\mathcal{L} = -\overline{\psi}\,(\gamma^a\partial_a + m)\,\psi \equiv -\overline{\psi}\,(\not{\partial} + m)\,\psi \tag{10.325}$$

in which

$$\overline{\psi} \equiv i\psi^\dagger\gamma^0 = \psi^\dagger\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} v^\dagger & u^\dagger \end{pmatrix}. \tag{10.326}$$

The kinetic part is the sum of the left-handed $\mathcal{L}_\ell$ and right-handed $\mathcal{L}_r$ action densities (10.293 & 10.310)

$$-\overline{\psi}\,\gamma^a\partial_a\psi = iu^\dagger\,(\partial_0 I - \boldsymbol{\nabla}\cdot\boldsymbol{\sigma})\,u + i\,v^\dagger\,(\partial_0 I + \boldsymbol{\nabla}\cdot\boldsymbol{\sigma})\,v. \tag{10.327}$$

If u is a left-handed spinor transforming as (10.292), then the spinor

$$v = \sigma_2\,u^* \equiv \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} u_1^\dagger \\ u_2^\dagger \end{pmatrix} \tag{10.328}$$

transforms as a right-handed spinor (10.309), that is (exercise 10.44)

$$e^{\boldsymbol{z}^*\cdot\boldsymbol{\sigma}/2}\sigma_2\,u^* = \sigma_2\left(e^{-\boldsymbol{z}\cdot\boldsymbol{\sigma}/2}u\right)^*. \tag{10.329}$$

Similarly, if v is right handed, then $u = -\sigma_2\,v^*$ is left handed.

The simplest 4-spinor is the Majorana spinor

$$\psi_M = \begin{pmatrix} u \\ \sigma_2 u^* \end{pmatrix} = \begin{pmatrix} -\sigma_2 v^* \\ v \end{pmatrix} = -i\gamma^2\psi_M^* \tag{10.330}$$

whose particles are the same as its antiparticles.

If two Majorana spinors $\psi_M^{(1)}$ and $\psi_M^{(2)}$ have the same mass, then one may combine them into a Dirac spinor

$$\psi_D = \frac{1}{\sqrt{2}}\left(\psi_M^{(1)} + i\psi_M^{(2)}\right) = \frac{1}{\sqrt{2}}\begin{pmatrix} u^{(1)} + iu^{(2)} \\ v^{(1)} + iv^{(2)} \end{pmatrix} = \begin{pmatrix} u_D \\ v_D \end{pmatrix}. \tag{10.331}$$

The **Dirac mass term**

$$-m\,\overline{\psi}_D\psi_D = -m\left(v_D^\dagger u_D + u_D^\dagger v_D\right) \tag{10.332}$$

conserves charge, and since $\exp(\boldsymbol{z}^*\cdot\sigma/2)^\dagger\exp(-\boldsymbol{z}\cdot\sigma/2) = I$ it also is Lorentz invariant. For a Majorana field, it reduces to

$$\begin{aligned} -\tfrac{1}{2}m\overline{\psi}_M\psi_M &= -\tfrac{1}{2}m\left(v^\dagger u + u^\dagger v\right) = -\tfrac{1}{2}m\left(u^\dagger\sigma_2 u^* + u^\mathsf{T}\sigma_2 u\right) \\ &= -\tfrac{1}{2}m\left(v^\dagger\sigma_2 v^* + v^\mathsf{T}\sigma_2 v\right) \end{aligned} \tag{10.333}$$

a Majorana mass term (10.301 or 10.315).

10.40 The Poincaré Group

The elements of the Poincaré group are products of Lorentz transformations and translations in space and time. The Lie algebra of the Poincaré group therefore includes the generators $\boldsymbol{J}$ and $\boldsymbol{K}$ of the Lorentz group as well as the hamiltonian H and the momentum operator $\boldsymbol{P}$ which respectively generate translations in time and space.

Suppose $T(y)$ is a translation that takes a 4-vector x to $x+y$ and $T(z)$ is a translation that takes a 4-vector x to $x+z$. Then $T(z)T(y)$ and $T(y)T(z)$ both take x to $x+y+z$. So if a translation $T(y) = T(t,\boldsymbol{y})$ is represented by a unitary operator $U(t,\boldsymbol{y}) = \exp(iHt - i\boldsymbol{P}\cdot\boldsymbol{y})$, then the hamiltonian H and the momentum operator $\boldsymbol{P}$ commute with each other

$$[H, P^j] = 0 \quad\text{and}\quad [P^i, P^j] = 0. \tag{10.334}$$

We can figure out the commutation relations of H and $\boldsymbol{P}$ with the angular-momentum $\boldsymbol{J}$ and boost $\boldsymbol{K}$ operators by realizing that $P^a = (H,\boldsymbol{P})$ is a 4-vector. Let

$$U(\boldsymbol{\theta},\boldsymbol{\lambda}) = e^{-i\boldsymbol{\theta}\cdot\boldsymbol{J} - i\boldsymbol{\lambda}\cdot\boldsymbol{K}} \tag{10.335}$$

be the (infinite-dimensional) unitary operator that represents (in Hilbert space) the infinitesimal Lorentz transformation

$$L = I + \boldsymbol{\theta}\cdot\boldsymbol{R} + \boldsymbol{\lambda}\cdot\boldsymbol{B} \tag{10.336}$$

where $\boldsymbol{R}$ and $\boldsymbol{B}$ are the six 4×4 matrices (10.265 & 10.266). Then because P is a 4-vector under Lorentz transformations, we have

$$U^{-1}(\boldsymbol{\theta},\boldsymbol{\lambda})PU(\boldsymbol{\theta},\boldsymbol{\lambda}) = e^{+i\boldsymbol{\theta}\cdot\boldsymbol{J}+i\boldsymbol{\lambda}\cdot\boldsymbol{K}}Pe^{-i\boldsymbol{\theta}\cdot\boldsymbol{J}-i\boldsymbol{\lambda}\cdot\boldsymbol{K}} = (I + \boldsymbol{\theta}\cdot\boldsymbol{R} + \boldsymbol{\lambda}\cdot\boldsymbol{B})\,P \tag{10.337}$$

or using (10.308)

$$(I + i\boldsymbol{\theta}\cdot\boldsymbol{J} + i\boldsymbol{\lambda}\cdot\boldsymbol{K})\, H\, (I - i\boldsymbol{\theta}\cdot\boldsymbol{J} - i\boldsymbol{\lambda}\cdot\boldsymbol{K}) = H + \boldsymbol{\lambda}\cdot\boldsymbol{P} \tag{10.338}$$
$$(I + i\boldsymbol{\theta}\cdot\boldsymbol{J} + i\boldsymbol{\lambda}\cdot\boldsymbol{K})\, \boldsymbol{P}\, (I - i\boldsymbol{\theta}\cdot\boldsymbol{J} - i\boldsymbol{\lambda}\cdot\boldsymbol{K}) = \boldsymbol{P} + H\boldsymbol{\lambda} + \boldsymbol{\theta}\wedge\boldsymbol{P}.$$

Thus one finds (exercise 10.44) that H is invariant under rotations, while $\boldsymbol{P}$ transforms as a 3-vector

$$[J_i, H] = 0 \quad \text{and} \quad [J_i, P_j] = i\epsilon_{ijk}P_k \tag{10.339}$$

and that

$$[K_i, H] = -iP_i \quad \text{and} \quad [K_i, P_j] = -i\delta_{ij}H. \tag{10.340}$$

By combining these equations with (10.322), one may write (exercise 10.46) the Lie algebra of the Poincaré group as

$$\begin{aligned} i[J^{ab}, J^{cd}] &= \eta^{bc}\, J^{ad} - \eta^{ac}\, J^{bd} - \eta^{da}\, J^{cb} + \eta^{db}\, J^{ca} \\ i[P^a, J^{bc}] &= \eta^{ab}P^c - \eta^{ac}P^b \\ [P^a, P^b] &= 0. \end{aligned} \tag{10.341}$$

Further reading

The books *Group theory in a Nutshell for Physicists* (Zee, 2016), *Lie Algebras in Particle Physics* (Georgi, 1999), *Unitary Symmetry and Elementary Particles* (Lichtenberg, 1978), and *Group theory and Its Application to the Quantum Mechanics of Atomic Spectra* (Wigner, 1964) are excellent. For applications to molecular physics, see *Chemical Applications of Group theory* (Cotton, 1990); for applications to condensed-matter physics, see *Group theory and Quantum Mechanics* (Tinkham, 2003).

Exercises

10.1 Show that all $n \times n$ (real) orthogonal matrices O leave invariant the quadratic form $x_1^2 + x_2^2 + \ldots + x_n^2$, that is, that if $x' = Ox$, then $x'^2 = x^2$.

10.2 Show that the set of all $n \times n$ orthogonal matrices forms a group.

10.3 Show that all $n \times n$ unitary matrices U leave invariant the quadratic form $|x_1|^2 + |x_2|^2 + \ldots + |x_n|^2$, that is, that if $x' = Ux$, then $|x'|^2 = |x|^2$.

10.4 Show that the set of all $n \times n$ unitary matrices forms a group.

10.5 Show that the set of all $n \times n$ unitary matrices with unit determinant forms a group.

10.6 Show that the matrix $D^{(j)}_{m'm}(g) = \langle j, m'|U(g)|j, m\rangle$ is unitary because the rotation operator $U(g)$ is unitary $\langle j, m'|U^\dagger(g)U(g)|j, m\rangle = \delta_{m'm}$.

10.7 Invent a group of order 3 and compute its multiplication table. For extra credit, prove that the group is unique.

10.8 Show that the relation (10.23) between two equivalent representations is an isomorphism.

10.9 Suppose that D_1 and D_2 are equivalent, finite-dimensional, irreducible representations of a group G so that $D_2(g) = SD_1(g)S^{-1}$ for all $g \in G$. What can you say about a matrix A that satisfies $D_2(g)\,A = A\,D_1(g)$ for all $g \in G$?

10.10 Find all components of the matrix $\exp(i\alpha A)$ in which

$$A = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}. \tag{10.342}$$

10.11 If $[A, B] = B$, find $e^{i\alpha A} B e^{-i\alpha A}$. Hint: what are the α-derivatives of this expression?

10.12 Show that the direct-product matrix (10.34) of two representations D and D' is a representation.

10.13 Find a 4×4 matrix S that relates the direct-product representation $D^{(1/2,1/2)}$ to the direct sum $D^{(1)} \oplus D^{(0)}$.

10.14 Find the generators in the adjoint representation of the group with structure constants $f_{abc} = \epsilon_{abc}$ where a, b, c run from 1 to 3. *Hint:* The answer is three 3×3 matrices t_a, often written as L_a.

10.15 Show that the generators (10.94) satisfy the commutation relations (10.97).

10.16 Show that the demonstrated equation (10.103) implies the commutation relation (10.104).

10.17 Use the Cayley-Hamilton theorem (1.290) to show that the 3×3 matrix (10.100) that represents a right-handed rotation of θ radians about the axis $\boldsymbol{\theta}$ is given by (10.101).

10.18 Verify the mixed Jacobi identity (10.162).

10.19 For the group $SU(3)$, find the structure constants f_{123} and f_{231}.

10.20 Show that every 2×2 unitary matrix of unit determinant is a quaternion of unit norm.

10.21 Show that the quaternions as defined by (10.211) are closed under addition and multiplication and that the product xq is a quaternion if x is real and q is a quaternion.

10.22 Show that the square of the matrix (10.199) is $J^2 = -I$, where I is the $2n \times 2n$ identity matrix. Then by setting $R = \exp(\epsilon t)$ with $0 < \epsilon \ll 1$, show that $t^\intercal J = -Jt$ and $JtJ = t^\intercal$.

10.23 Show that $JtJ = t^\intercal$ implies that t is given by (10.204).

10.24 Show that the one-sided derivative $f'(q)$ (10.221) of the quaternionic function $f(q) = q^2$ depends upon the direction along which $q' \to 0$.

10.25 Show that the generators (10.225) of $Sp(2n)$ obey commutation relations of the form (10.226) for some real structure constants f_{abc} and a suitably extended set of matrices A, A', ... and S_k, S'_k, ...

10.26 Show that for $0 < \epsilon \ll 1$, the real $2n \times 2n$ matrix $T = \exp(\epsilon J S)$ in which S is symmetric satisfies $T^{\mathsf{T}} J T = J$ (at least up to terms of order ϵ^2) and so is in $Sp(2n, R)$.

10.27 Show that the matrix T of (10.207) is in $Sp(2, R)$.

10.28 Use the parametrization (10.251) of the group $SU(2)$, show that the parameters $\boldsymbol{a}(\boldsymbol{c}, \boldsymbol{b})$ that describe the product $g(\boldsymbol{a}(\boldsymbol{c}, \boldsymbol{b})) = g(\boldsymbol{c})\, g(\boldsymbol{b})$ are those of (10.253).

10.29 Use formulas (10.253) and (10.246) to show that the left-invariant measure for $SU(2)$ is given by (10.254).

10.30 In tensor notation, which is explained in chapter 12, the condition (10.263) that $I + \omega$ be an infinitesimal Lorentz transformation reads $\left(\omega^{\mathsf{T}}\right)_b{}^a = \omega^a{}_b = -\eta_{bc}\,\omega^c{}_d\,\eta^{da}$ in which sums over c and d from 0 to 3 are understood. In this notation, the matrix η_{ef} lowers indices and η^{gh} raises them, so that $\omega_b{}^a = -\omega_{bd}\,\eta^{da}$. (Both η_{ef} and η^{gh} are numerically equal to the matrix η displayed in equation (10.256).) Multiply both sides of the condition (10.263) by $\eta_{ae} = \eta_{ea}$ and use the relation $\eta^{da}\,\eta_{ae} = \eta^d{}_e \equiv \delta^d{}_e$ to show that the matrix ω_{ab} with both indices lowered (or raised) is antisymmetric, that is,

$$\omega_{ba} = -\omega_{ab} \quad \text{and} \quad \omega^{ba} = -\omega^{ab}. \tag{10.343}$$

10.31 Show that the six matrices (10.265) and (10.266) satisfy the $SO(3, 1)$ condition (10.263).

10.32 Show that the six generators $\boldsymbol{J}$ and $\boldsymbol{K}$ obey the commutations relations (10.268–10.270).

10.33 Show that if $\boldsymbol{J}$ and $\boldsymbol{K}$ satisfy the commutation relations (10.268–10.270) of the Lie algebra of the Lorentz group, then so do $\boldsymbol{J}$ and $-\boldsymbol{K}$.

10.34 Show that if the six generators $\boldsymbol{J}$ and $\boldsymbol{K}$ obey the commutation relations (10.268–10.270), then the six generators $\boldsymbol{J}^+$ and $\boldsymbol{J}^-$ obey the commutation relations (10.277).

10.35 Relate the parameter α in the definition (10.288) of the standard boost $B(p)$ to the 4-vector p and the mass m.

10.36 Derive the formulas for $D^{(1/2,0)}(\boldsymbol{0}, \alpha\,\hat{\boldsymbol{p}})$ given in equation (10.289).

10.37 Derive the formulas for $D^{(0,1/2)}(\boldsymbol{0}, \alpha\,\hat{\boldsymbol{p}})$ given in equation (10.307).

10.38 For infinitesimal complex $\boldsymbol{z}$, derive the 4-vector properties (10.290 & 10.308) of $(-I, \boldsymbol{\sigma})$ under $D^{(1/2,0)}$ and of $(I, \boldsymbol{\sigma})$ under $D^{(0,1/2)}$.

10.39 Show that under the unitary Lorentz transformation (10.292), the action density (10.293) is Lorentz covariant (10.294).

10.40 Show that under the unitary Lorentz transformation (10.309), the action density (10.310) is Lorentz covariant (10.311).

10.41 Show that under the unitary Lorentz transformations (10.292 & 10.309), the Majorana mass terms (10.301 & 10.315) are Lorentz covariant.

10.42 Show that the definitions of the gamma matrices (10.318) and of the generators (10.320) imply that the gamma matrices transform as a 4-vector under Lorentz transformations (10.321).

10.43 Show that (10.320) and (10.321) imply that the generators J^{ab} satisfy the commutation relations (10.322) of the Lorentz group.

10.44 Show that the spinor $v = \sigma_2 u^*$ defined by (10.328) is right handed (10.309) if u is left handed (10.292).

10.45 Use (10.338) to get (10.339 & 10.340).

10.46 Derive (10.341) from (10.322, 10.334, 10.339, & 10.340).