10

Field Theory on a Lattice

10.1 Scalar Fields

To represent a hermitian field $\phi(x)$, we put a real number $\phi(i,j,k,\ell)$ at each vertex of the lattice and set

$$x = a(i,j,k,\ell) \tag{10.1}$$

where a is the lattice spacing. The derivative $\partial_i \phi(x)$ is approximated as

$$\partial_i \phi(x) \approx \frac{\phi(x+\hat{\imath}) - \phi(x)}{a} \tag{10.2}$$

in which x is the discrete position 4-vector and $\hat{\imath}$ is a unit 4-vector pointing in the i direction. So the euclidian action is the sum over all lattice sites of

$$\begin{aligned} S_e &= \sum_x \frac{1}{2} \left(\partial_i \phi(x)\right)^2 a^4 + \frac{1}{2} m^2 \phi^2(x)\, a^4 \\ &= \sum_x \frac{1}{2} \Big(\phi(x+\hat{\imath}) - \phi(x)\Big)^2 a^2 + \frac{1}{2} m_0^2 \, \phi^2(x)\, a^4 \\ &= -\frac{1}{2} \sum_{xi} \phi(x+\hat{\imath})\phi(x)\, a^2 + \frac{1}{2}\left(8 + m_0^2\right) \phi^2(x)\, a^4 \end{aligned} \tag{10.3}$$

if the self interaction happens to be quartic.

$$S_e = \frac{1}{2} \left(\partial_i \phi(x)\right)^2 a^4 + \frac{1}{2} m_0^2 \, \phi^2(x)\, a^4 + \frac{\lambda}{4} \phi^4(x)\, a^4 \tag{10.4}$$

if the self interaction happens to be quartic.

10.2 Pure Gauge Theory

The gauge-covariant derivative is defined in terms of the generators t_a of a compact Lie algebra

$$[t_a, t_b] = i f_{abc} t_c \tag{10.5}$$

and a gauge-field matrix $A_i = igA_i^b t_b$ as

$$D_i = \partial_i - A_i = \partial_i - igA_i^b t_b \tag{10.6}$$

summed over all the generators, and g is a coupling constant. Since the group is compact, we may raise and lower group indexes without worrying about factors or minus signs.

The Faraday matrix is

$$F_{ij} = [D_i, D_j] = [I\partial_i - A_i(x), I\partial_j - A_j(x)] = -\partial_i A_j + \partial_j A_i + [A_i, A_j] \tag{10.7}$$

in matrix notation. With more indices exposed, it is

$$\begin{aligned}(F_{ij})_{cd} &= (-\partial_i A_j + \partial_j A_i + [A_i, A_j])_{cd} \\ &= -igt^b_{cd}\left(\partial_i A^b_j - \partial_j A^b_i\right) + \left([igt^b A^b_i, igt^e A^e_j]\right)_{cd}.\end{aligned} \tag{10.8}$$

Summing over repeated indices, we get

$$\begin{aligned}(F_{ij})_{cd} &= -igt^b_{cd}\left(\partial_i A^b_j - \partial_j A^b_i\right) - g^2 A^b_i A^e_j \left([t^b, t^e]\right)_{cd} \\ &= -igt^b_{cd}\left(\partial_i A^b_j - \partial_j A^b_i\right) - g^2 A^b_i A^e_j i f_{bef} t^f_{cd} \\ &= -igt^b_{cd}\left(\partial_i A^b_j - \partial_j A^b_i\right) - ig^2 A^b_i A^e_j f_{bef} t^f_{cd} \\ &= -igt^b_{cd}\left(\partial_i A^b_j - \partial_j A^b_i\right) - ig^2 A^f_i A^e_j f_{feb} t^b_{cd} \\ &= -igt^b_{cd}\left(\partial_i A^b_j - \partial_j A^b_i + gA^f_i A^e_j f_{feb}\right) \\ &= -igt^b_{cd}\left(\partial_i A^b_j - \partial_j A^b_i + g f_{bfe}\, A^f_i A^e_j\right) = -igt^b_{cd}\, F^b_{ij}\end{aligned} \tag{10.9}$$

where

$$F^b_{ij} = \partial_i A^b_j - \partial_j A^b_i + g f_{bfe}\, A^f_i A^e_j \tag{10.10}$$

is the Faraday tensor.

The action density of this tensor is

$$L_F = -\frac{1}{4} F^b_{ij} F^{ij}_b. \tag{10.11}$$

The trace of the square of the Faraday matrix is

$$\begin{aligned}\mathrm{Tr}\left[F_{ij}F^{ij}\right] &= \mathrm{Tr}\left[igt^b\,F^b_{ij}\,igt^c\,F^{ij}_c\right]\\ &= -\,g^2F^b_{ij}\,F^{ij}_c\,\mathrm{Tr}(t^bt^c) = -g^2F^b_{ij}\,F^{ij}_c\,k\delta_{bc}\\ &= -\,kg^2\,F^b_{ij}\,F^{ij}_b.\end{aligned} \tag{10.12}$$

So the Faraday action density is

$$L_F = \,-\,\frac{1}{4}F^b_{ij}F^{ij}_b = \frac{1}{4kg^2}\,\mathrm{Tr}\left[F_{ij}F^{ij}\right] = \frac{1}{2g^2}\,\mathrm{Tr}\left[F_{ij}F^{ij}\right]. \tag{10.13}$$

The theory described by this action density, without scalar or spinor fields, is called **pure** gauge theory.

The purpose of a gauge field is to make the gauge-invariant theories. So if a field ψ_a is transformed by the group element $g(x)$

$$\psi'_a(x) = g(x)_{ab}\psi_b(x), \tag{10.14}$$

then we want the covariant derivative of the field to go as

$$\begin{aligned}(D_i\psi(x))' &= \left(\partial_i - A'_i(x)\right)g(x)\psi(x)\\ &= g(x)D_i\psi(x) = g(x)\left[\left(\partial_i - A_i(x)\right)\psi(x)\right].\end{aligned} \tag{10.15}$$

So the gauge field must go as

$$\partial_i g - A'_i g = \,-\,gA_i \tag{10.16}$$

or as

$$A'_i(x) = g(x)A_i(x)g^{-1}(x) + (\partial_i g(x))g^{-1}(x). \tag{10.17}$$

So if $g(x) = \exp(-i\theta^a(x)t^a)$, then a gauge transforms as

$$A'_i(x) = e^{-i\theta^a(x)t^a}\,A_i(x)\,e^{i\theta^a(x)t^a} + (\partial_i e^{-i\theta^a(x)t^a})\,e^{i\theta^a(x)t^a}. \tag{10.18}$$

How does an exponential of a path-ordered, very short line integral of gauge fields go? We will evaluate how the path-ordered exponential in which g_0 is a coupling constant

$$\begin{aligned}P\exp\left(g_0A_i(x)dx^i\right)' &= P\exp\left(\left[g(x)g_0A_i(x)g^{-1}(x) + (\partial_i g(x))g^{-1}(x)\right]dx^i\right)\\ &= P\exp\left(\left[g(x)g_0A_i(x)g^{-1}(x) + \partial_i\log g(x)\right]dx^i\right)\end{aligned} \tag{10.19}$$

changes under the gauge transformation (10.18) in the limitt $dx^i \to 0$. We find

$$\begin{aligned} P\exp\left(g_0 A_i(x)dx^i\right)' &= P\Big[g(x)e^{g_0 A_i(x)dx^i}g^{-1}(x) \\ &\quad\times e^{\log g(x+dx^i/2)-\log g(x)+\log g(x)-\log g(x-dx^i/2)}\Big] \\ &= P\Big[g(x)e^{g_0 A_i(x)dx^i}g^{-1}(x) \\ &\quad\times g(x+dx^i/2)g^{-1}(x)g(x)g^{-1}(x-dx^i/2)\Big] \\ &= g(x+dx^i/2)e^{g_0 A_i(x)dx^i}g^{-1}(x-dx^i/2). \end{aligned} \tag{10.20}$$

Putting together a chain of such infinitesimal links, we get

$$P\exp\left(\int_y^x g_0 A_i(x')dx'^i\right)' = g(x)P\exp\left(\int_y^x g_0 A_i(x')dx'^i\right)g^{-1}(y). \tag{10.21}$$

In particular, this means that the trace of a closed loop is gauge invariant

$$\begin{aligned} \left[\mathrm{Tr}P\exp\left(\oint g_0 A_i(x)dx^i\right)\right]' &= \mathrm{Tr}g(x)P\exp\left(\oint g_0 A_i(x)dx^i\right)g^{-1}(x) \\ &= \mathrm{Tr}P\exp\left(\oint g_0 A_i(x)dx^i\right). \end{aligned} \tag{10.22}$$

10.3 Pure Gauge Theory on a Lattice

Wilson's lattice gauge theory is inspired by these last two equations. Another source of inspiration is the approximation for a loop of tiny area $dx\wedge dy$ which in the joint limit $dx\to 0$ and $dy\to 0$ is

$$\begin{aligned} W = P\exp\left(\oint g_0 A_i(x)dx^i\right) &= \exp\Big[g_0\Big(A_{y,x}-A_{x,y}+g_0[A_x,A_y]\Big)dxdy\Big] \\ &= \exp\Big(-g_0 F_{xy}\,dxdy\Big). \end{aligned} \tag{10.23}$$

To derive this formula, we will ignore the bare coupling constant g_0 for the moment and apply the Baker-Campbell-Hausdorf identity

$$e^A\,e^B = \exp\left(A+B+\frac{1}{2}[A,B]+\frac{1}{12}[A,[A,B]]+\frac{1}{12}[B,[B,A]]+\dots\right). \tag{10.24}$$

to the product

$$W = \exp\Big(A_x\,dx\Big)\exp\Big(A_y\,dy + A_{y,x}\,dxdy\Big) \times \exp\Big(-A_x\,dx - A_{x,y}\,dxdy\Big)\exp\Big(-A_y\,dy\Big). \tag{10.25}$$

We get

$$W = \exp\Big(A_x\,dx + A_y\,dy + A_{y,x}\,dxdy + \frac{1}{2}[A_x\,dx, (A_y + A_{y,x}\,dx)dy]\Big) \times \exp\Big(-A_x\,dx - A_y\,dy - A_{x,y}\,dxdy + \frac{1}{2}[(A_x + A_{x,y}\,dy)dx, A_y\,dy]\Big). \tag{10.26}$$

Applying again the BCH identity, we get

$$W = \exp\Big[\Big(A_{y,x} - A_{x,y} + [A_x, A_y]\Big)dxdy\Big] = \exp\Big(F_{xy}\,dxdy\Big), \tag{10.27}$$

an identity that is the basis of lattice gauge theory.

Restoring g_0, we divide the trace of W by the dimension n of the matrices t_a and subtract from unity

$$1 - \frac{1}{n}\mathrm{Tr}\left[P\exp\left(\oint g_0 A_i(x)dx^i\right)\right] = 1 - \frac{1}{n}\mathrm{Tr}\Big[\exp\Big(g_0F_{xy}\,dxdy\Big)\Big] = -\frac{1}{n}\mathrm{Tr}\Big[g_0F_{xy}\,dxdy + \frac{1}{2}\Big(g_0F_{xy}\,dxdy\Big)^2\Big]. \tag{10.28}$$

The generators of $SU(n)$, $SO(n)$, and $Sp(2n)$ are traceless, so the first term vanishes, and we get

$$1 - \frac{1}{n}\mathrm{Tr}\left[P\exp\left(\oint g_0 A_i(x)dx^i\right)\right] = -\frac{1}{2n}\mathrm{Tr}\Big(g_0F_{xy}\,dxdy\Big)^2. \tag{10.29}$$

Recalling the more explicit form (10.9) of the Faraday matrix, we have

$$1 - \frac{1}{n}\mathrm{Tr}(W) = \frac{g_0^2}{2n}\mathrm{Tr}\Big[t_aF^a_{xy}\,t_bF^b_{xy}\,(dxdy)^2\Big] = \frac{kg_0^2}{2n}\left(F^a_{xy}dxdy\right)^2 \tag{10.30}$$

in which k is the constant of the normalization $\mathrm{Tr}(t_at_b) = k\delta_{ab}$. For $SU(2)$ with $t_a = \sigma_a/2$ and for $SU(3)$ with $t_a = \lambda_a/2$, this constant is $k = 1/2$.

The Wilson action is a sum over all the smallest squares of the lattice, called the plaquettes, of the quantity

$$S_\Box = \frac{n}{2kg_0^2}\left[1 - \frac{1}{n}\mathrm{Tr}(W)\right] = \frac{1}{4}\left(F^a_{ij}\right)^2 a^4 \tag{10.31}$$

in which a is the lattice spacing. The full Wilson action is the sum of this quantity over all the elementary squares of the lattice and over $i, j = 1, 2, 3, 4$.