Spinors Made Simple

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Creation and annihilation operators inherit the transformation properties of states of particles. Spinors, which are the coefficients of the creation and annihilation operators in the Fourier expansions of fields, are defined so that the fields transform according to representations of the Poincaré group. This note is intended to explain to students how and why the Dirac spinors v, which are the coefficients of the creation operators, are so different from the Dirac spinors u, which are the coefficients of the annihilation operators.

I. SPIN-ONE-HALF PARTICLES

A spin-one-half particle of kind n and 4-momentum $p = (\boldsymbol{p}, p^0)$ with $p^2 = -m^2$ and spin $s = \pm 1/2$ in the z direction is represented by a state $|\boldsymbol{p}, s, n\rangle$. For fixed kind and momentum, these states form a two-dimensional space spanned by $|\boldsymbol{p}, 1/2, n\rangle$ and $|\boldsymbol{p}, -1/2, n\rangle$.

If the momentum is in the z direction, then the state $|p\hat{\boldsymbol{z}}, s, n\rangle$ is an eigenstate of the z component J_z of the angular momentum operator $\boldsymbol{J}$ with eigenvalue s

$$J_z |p\hat{\boldsymbol{z}}, s, n\rangle = s|p\hat{\boldsymbol{z}}, s, n\rangle. \tag{1}$$

So under a rotation by angle θ about the z axis, the state $|p\hat{\boldsymbol{z}}, s, n\rangle$ goes as

$$e^{-i\theta J_z} |p\hat{\boldsymbol{z}}, s, n\rangle = e^{-is\theta} |p\hat{\boldsymbol{z}}, s, n\rangle. \tag{2}$$

The creation operator $a^\dagger(\boldsymbol{p}, s, n)$ adds such a particle to a state. It turns the vacuum state $|0\rangle$ into the state $|\boldsymbol{p}, s, n\rangle$

$$a^\dagger(\boldsymbol{p}, s, n)|0\rangle = |\boldsymbol{p}, s, n\rangle. \tag{3}$$

It turns a state of several particles $|\boldsymbol{p}_1, s_1, n_1, \ldots\rangle$ into a state

$$a^\dagger(\boldsymbol{p}, s, n)|\boldsymbol{p}_1, s_1, n_1; \ldots\rangle = |\boldsymbol{p}, s, n; \boldsymbol{p}_1, s_1, n_1; \ldots\rangle \tag{4}$$

that represents those particles plus a particle of 4-momentum $p = (\boldsymbol{p}, p^0)$, spin s in the z direction, and kind n.

The antiparticle of a particle of kind n is of kind n_c. For antiparticles, the particle equations (1–4) apply with $n \to n_c$:

$$J_z|p\hat{\boldsymbol{z}}, s, n_c\rangle = s|p\hat{\boldsymbol{z}}, s, n_c\rangle \tag{5}$$

$$e^{-i\theta J_z}|p\hat{\boldsymbol{z}}, s, n_c\rangle = e^{-is\theta}|p\hat{\boldsymbol{z}}, s, n_c\rangle \tag{6}$$

$$a^\dagger(\boldsymbol{p}, s, n_c)|0\rangle = |\boldsymbol{p}, s, n_c\rangle \tag{7}$$

$$a^\dagger(\boldsymbol{p}, s, n_c)|\boldsymbol{p}_1, s_1, n_1; \ldots\rangle = |\boldsymbol{p}, s, n_c; \boldsymbol{p}_1, s_1, n_1; \ldots\rangle. \tag{8}$$

Under a rotation by angle $\boldsymbol{\theta}$, the creation operator $a^\dagger(\boldsymbol{p}, s, n)$ goes as

$$\begin{aligned} U(R(\boldsymbol{\theta}))a^\dagger(\boldsymbol{p}, s, n)U^{-1}(R(\boldsymbol{\theta})) &= e^{-i\boldsymbol{\theta}\cdot\boldsymbol{J}}a^\dagger(\boldsymbol{p}, s, n)e^{i\boldsymbol{\theta}\cdot\boldsymbol{J}} \\ &= D(R(\boldsymbol{\theta}))_{s's}\, a^\dagger(R(\boldsymbol{\theta})\boldsymbol{p}, s', n) \end{aligned} \tag{9}$$

in which

$$D(R(\boldsymbol{\theta}))_{ss'} = \left[e^{-i\boldsymbol{\theta}\cdot\boldsymbol{\sigma}/2}\right]_{ss'} = \delta_{ss'}\, \cos(\theta/2) - i\boldsymbol{\theta}\cdot(\boldsymbol{\sigma})_{ss'}\, \sin(\theta/2) \tag{10}$$

is the 2×2 representation of the rotation group. The annihilation operator goes as the adjoint equation (9)

$$\begin{aligned} U(R(\boldsymbol{\theta}))a(\boldsymbol{p}, s, n)U^{-1}(R(\boldsymbol{\theta})) &= e^{-i\boldsymbol{\theta}\cdot\boldsymbol{J}}a(\boldsymbol{p}, s, n)e^{i\boldsymbol{\theta}\cdot\boldsymbol{J}} \\ &= D^*(R(\boldsymbol{\theta}))_{s's}\, a(R(\boldsymbol{\theta})\boldsymbol{p}, s', n) \end{aligned} \tag{11}$$

For a rotation by angle θ about the z axis, the matrix $D(R(\theta\hat{\boldsymbol{z}}))$ is diagonal, and so the creation operator $a^\dagger(\boldsymbol{p}, s, n)$ goes as

$$\begin{aligned} U(R(\theta\hat{\boldsymbol{z}}))a^\dagger(\boldsymbol{p}, s, n)U^{-1}(R(\theta\hat{\boldsymbol{z}})) &= e^{-i\theta J_z}a^\dagger(R(\theta\hat{\boldsymbol{z}})\boldsymbol{p}, s, n)e^{i\theta J_z} \\ &= e^{-is\theta}a^\dagger(R(\theta\hat{\boldsymbol{z}})\boldsymbol{p}, s, n) \end{aligned} \tag{12}$$

which makes sense since $a^\dagger(R(\theta\hat{\boldsymbol{z}})\boldsymbol{p}, s, n)$ adds s units of angular momentum to a state. Taking the adjoint of this equation, we see that under a rotation by angle θ about the z axis, the annihilation operator $a(\boldsymbol{p}, s, n)$ goes as

$$\begin{aligned} U(R(\theta\hat{\boldsymbol{z}}))a(\boldsymbol{p}, s, n)U^{-1}(R(\theta\hat{\boldsymbol{z}})) &= e^{-i\theta J_z}a(R(\theta\hat{\boldsymbol{z}})\boldsymbol{p}, s, n)e^{i\theta J_z} \\ &= e^{is\theta}a(R(\theta\hat{\boldsymbol{z}})\boldsymbol{p}, s, n) \end{aligned} \tag{13}$$

which makes sense since $a(R(\theta\hat{\boldsymbol{z}})\boldsymbol{p}, s, n)$ subtracts s units of angular momentum from a state. Creation and annihilation operators transform differently under rotations. That's why the spinors u and v must be different.

II. THE SPINORS OF DIRAC FIELDS

Under a rotation $R(\theta\hat{\boldsymbol{z}})$ about the $\hat{\boldsymbol{z}}$ axis by angle θ, a Dirac field

$$\psi_a(x) = \sum_s \int \frac{d^3p}{(2\pi)^{3/2}} \; u_a(\boldsymbol{p}, s, n)\, e^{ip\cdot x}\, a(\boldsymbol{p}, s, n) + v_a(\boldsymbol{p}, s, n_c)\, e^{-ip\cdot x}\, a^\dagger(\boldsymbol{p}, s, n_c) \tag{14}$$

transforms as its creation and annihilation operators transform (12 and 13)

$$\begin{aligned} U(R(\theta\hat{\boldsymbol{z}}))\psi_a(x)U^{-1}(R(\theta\hat{\boldsymbol{z}})) &= e^{-i\theta J_z}\psi_a(x)e^{i\theta J_z} \\ &= \sum_s \int \frac{d^3p}{(2\pi)^{3/2}} \Big[u_a(\boldsymbol{p}, s, n)e^{ip\cdot x}e^{-i\theta J_z}a(\boldsymbol{p}, s, n)e^{i\theta J_z} \\ &\quad + v_a(\boldsymbol{p}, s, n_c)e^{-ip\cdot x}e^{-i\theta J_z}a^\dagger(\boldsymbol{p}, s, n_c)e^{i\theta J_z}\Big] \\ &= \sum_s \int \frac{d^3p}{(2\pi)^{3/2}} \Big[u_a(\boldsymbol{p}, s, n)e^{ip\cdot x}e^{is\theta}a(R(\theta\hat{\boldsymbol{z}})\boldsymbol{p}, s, n) \\ &\quad + v_a(\boldsymbol{p}, s, n_c)e^{-ip\cdot x}e^{-is\theta}a^\dagger(R(\theta\hat{\boldsymbol{z}})\boldsymbol{p}, s, n_c)\Big]. \end{aligned} \tag{15}$$

Since $R\boldsymbol{p}\cdot R\boldsymbol{x} = \boldsymbol{p}\cdot\boldsymbol{x}$, and $d^3R\boldsymbol{p} = d^3\boldsymbol{p}$, we can write this as

$$\begin{aligned} U(R(\theta\hat{\boldsymbol{z}}))\psi_a(x)U^{-1}(R(\theta\hat{\boldsymbol{z}})) = \sum_s \int \frac{d^3p}{(2\pi)^{3/2}} &\Big[u_a(R^{-1}(\theta\hat{\boldsymbol{z}})\boldsymbol{p}, s)e^{i\boldsymbol{p}\cdot R\boldsymbol{x}}e^{is\theta}a(\boldsymbol{p}, s) \\ &+ v_a(R^{-1}(\theta\hat{\boldsymbol{z}})\boldsymbol{p}, s)e^{-i\boldsymbol{p}\cdot R\boldsymbol{x}}e^{-is\theta}a_c^\dagger(\boldsymbol{p}, s)\Big]. \end{aligned} \tag{16}$$

On the other hand, under a rotation $R(\theta\hat{\boldsymbol{z}})$ about the $\hat{\boldsymbol{z}}$ axis by angle θ, a Dirac field goes as

$$e^{-i\theta J_z}\psi_a(x)e^{i\theta J_z} = D(R^{-1}(\theta\hat{\boldsymbol{z}}))_{ab}\psi_b(R(\theta\hat{\boldsymbol{z}})x) \tag{17}$$

in which D is the $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ representation of the Lorentz group. For rotations about the $\hat{\boldsymbol{z}}$ axis $D^{(1/2,0)\oplus(0,1/2)}(R^{-1})$ is diagonal

$$D(R^{-1}) = \begin{pmatrix} e^{i\theta\sigma_3/2} & 0 \\ 0 & e^{i\theta\sigma_3/2} \end{pmatrix} = \begin{pmatrix} e^{i\theta/2} & 0 & 0 & 0 \\ 0 & e^{-i\theta/2} & 0 & 0 \\ 0 & 0 & e^{i\theta/2} & 0 \\ 0 & 0 & 0 & e^{-i\theta/2} \end{pmatrix} \tag{18}$$

because the two 2×2 representations of the Lorentz group are the same for rotations, $D^{(1/2,0)}(R) = D^{(0,1/2)}(R)$. Thus under a rotation $R(\theta\hat{\boldsymbol{z}})$ by angle θ about the $\hat{\boldsymbol{z}}$ axis, a Dirac field goes as

$$
\begin{aligned}
U(R(\theta\hat{\boldsymbol{z}}))\psi_a(x^0,\boldsymbol{x})U^{-1}(R(\theta\hat{\boldsymbol{z}})) &= e^{-i\theta J_z}\psi_a(x^0,\boldsymbol{x})e^{i\theta J_z} \\
&= \begin{cases} e^{i\theta/2}\,\psi_a(x^0,R(\theta\hat{\boldsymbol{z}})\boldsymbol{x}) & \text{if } a=1,3 \\ e^{-i\theta/2}\,\psi_a(x^0,R(\theta\hat{\boldsymbol{z}})\boldsymbol{x}) & \text{if } a=2,4 \end{cases} .
\end{aligned} \tag{19}
$$

If we now compare this equation with (16), then we see that for momentum in the $\hat{\boldsymbol{z}}$ direction, the $u_a(p\hat{\boldsymbol{z}},s)$ spinors must satisfy

$$
e^{is\theta}\,u_a(p\hat{\boldsymbol{z}},s) = \begin{cases} e^{i\theta/2}\,u_a(p\hat{\boldsymbol{z}},s) & \text{if } a=1,3 \\ e^{-i\theta/2}\,u_a(p\hat{\boldsymbol{z}},s) & \text{if } a=2,4 \end{cases} \tag{20}
$$

while the $v_a(p\hat{\boldsymbol{z}},s)$ spinors must satisfy

$$
e^{-is\theta}\,v_a(p\hat{\boldsymbol{z}},s) = \begin{cases} e^{i\theta/2}\,v_a(p\hat{\boldsymbol{z}},s) & \text{if } a=1,3 \\ e^{-i\theta/2}\,v_a(p\hat{\boldsymbol{z}},s) & \text{if } a=2,4 \end{cases} . \tag{21}
$$

Thus for momentum in the $\hat{\boldsymbol{z}}$ direction, the $u_a(p\hat{\boldsymbol{z}},\frac{1}{2})$ spinors can have nonzero components only for $a=1$ and 3, while the $u_a(p\hat{\boldsymbol{z}},-\frac{1}{2})$ spinors can have nonzero components only for $a=2$ and 4. This is what one expects. But we also see that the $v_a(p\hat{\boldsymbol{z}},\frac{1}{2})$ spinors can have nonzero components only for $a=2$ and 4, while the $v_a(p\hat{\boldsymbol{z}},-\frac{1}{2})$ spinors can have nonzero components only for $a=1$ and 3. This surprises many physicists, but it is stated correctly in books by Steven Weinberg [1, Chap. 5] and by Peskin and Schroeder [2, pp. 803–804] and in various articles [3, 4].

Weinberg's zero-momentum spinors are

$$
u(\mathbf{0},\tfrac{1}{2}) = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\1\\0\end{bmatrix} \quad \text{and} \quad u(\mathbf{0},-\tfrac{1}{2}) = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\0\\1\end{bmatrix} \tag{22}
$$

and

$$
v(\mathbf{0},\tfrac{1}{2}) = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\0\\-1\end{bmatrix} \quad \text{and} \quad v(\mathbf{0},-\tfrac{1}{2}) = \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\0\\1\\0\end{bmatrix} . \tag{23}
$$

The spinors for momentum $\boldsymbol{p}$ then are [3]

$$u(\boldsymbol{p},s) = \frac{m - i\not{p}}{\sqrt{2p^0(p^0+m)}}\, u(\boldsymbol{0},s) \quad \text{and} \quad v(\boldsymbol{p},s) = \frac{m + i\not{p}}{\sqrt{2p^0(p^0+m)}}\, v(\boldsymbol{0},s) \tag{24}$$

or more explicitly

$$u(\boldsymbol{p},\tfrac{1}{2}) = \frac{1}{2\sqrt{p^0(p^0+m)}} \begin{pmatrix} m + p^0 - p_3 \\ -p_1 - ip_2 \\ m + p^0 + p_3 \\ p_1 + ip_2 \end{pmatrix} \tag{25}$$

$$u(\boldsymbol{p},-\tfrac{1}{2}) = \frac{1}{2\sqrt{p^0(p^0+m)}} \begin{pmatrix} -p_1 + ip_2 \\ m + p^0 + p_3 \\ p_1 - ip_2 \\ m + p^0 - p_3 \end{pmatrix} \tag{26}$$

$$v(\boldsymbol{p},\tfrac{1}{2}) = \frac{1}{2\sqrt{p^0(p^0+m)}} \begin{pmatrix} -p_1 + ip_2 \\ m + p^0 + p_3 \\ -p_1 + ip_2 \\ -m - p^0 + p_3 \end{pmatrix} \tag{27}$$

$$v(\boldsymbol{p},-\tfrac{1}{2}) = \frac{1}{2\sqrt{p^0(p^0+m)}} \begin{pmatrix} -m - p^0 + p_3 \\ p_1 + ip_2 \\ m + p^0 + p_3 \\ p_1 + ip_2 \end{pmatrix} . \tag{28}$$

III. CROSSCHECKS

If we compare the two transformation laws (16) and (19) for how a Dirac field goes under a rotation $R(\theta\hat{\boldsymbol{z}})$ about the z axis, then we see that for momentum $\boldsymbol{p}$ the spinors must obey the rules

$$e^{is\theta}\, u_a(R^{-1}(\theta\hat{\boldsymbol{z}})\boldsymbol{p},s) = \begin{cases} e^{i\theta/2}\, u_a(\boldsymbol{p},s) & \text{if } a = 1,3 \\ e^{-i\theta/2}\, u_a(\boldsymbol{p},s) & \text{if } a = 2,4 \end{cases} \tag{29}$$

and

$$e^{-is\theta}\, v_a(R^{-1}(\theta\hat{\boldsymbol{z}})\boldsymbol{p},s) = \begin{cases} e^{i\theta/2}\, v_a(p\hat{\boldsymbol{z}},s) & \text{if } a = 1,3 \\ e^{-i\theta/2}\, v_a(p\hat{\boldsymbol{z}},s) & \text{if } a = 2,4 \end{cases} . \tag{30}$$

As a check, we set $s = \frac{1}{2}$ and $a = 2$ in (29) and find

$$-p_1' - ip_2' = e^{-i\theta}(-p_1 - ip_2)$$

where the primes mean $\boldsymbol{p}' = R^{-1}(\theta\hat{\boldsymbol{z}})\boldsymbol{p}$. That is,

$$p_1' = \cos\theta p_1 + \sin\theta p_2 \quad \text{and} \quad p_2' = \cos\theta p_2 - \sin\theta p_1$$

which is a left-handed rotation, $R^{-1}(\theta\hat{\boldsymbol{z}})$, about the z axis.

As a final crosscheck, let's examine a state of one antiparticle at rest. The state $a^\dagger(0, s, n_c)|0\rangle$ is (summed over a and s')

$$\begin{aligned}
\frac{1}{(2\pi)^{3/2}} \int d^3x\, \bar{v}_a(0,s)\psi_a(x)|0\rangle &= \bar{v}_a(0,s) \int \frac{d^3x d^3p}{(2\pi)^3}\, v(\boldsymbol{p}, s')\, e^{-ip\cdot x}\, a^\dagger(\boldsymbol{p}, s', n_c)|0\rangle \\
&= \bar{v}_a(0,s) \int d^3p\, \delta^3(\boldsymbol{p}) v(\boldsymbol{p}, s')\, a^\dagger(\boldsymbol{p}, s', n_c)|0\rangle \\
&= \bar{v}_a(0,s) v(0, s')\, a^\dagger(0, s', n_c)|0\rangle \qquad (31) \\
&= \delta_{ss'}\, a^\dagger(0, s', n_c)|0\rangle = a^\dagger(0, s, n_c)|0\rangle.
\end{aligned}$$

So

$$|0, s, n_c\rangle = \frac{1}{(2\pi)^{3/2}} \sum_a \int d^3x\, \bar{v}_a(0,s)\psi_a(x)|0\rangle. \qquad (32)$$

To determine its spin, we act on it with the operator $e^{-i\theta J_3}$ that rotates states about the z axis by angle θ in a right-handed way.

$$\begin{aligned}
e^{-i\theta J_3}|0, s, n_c\rangle &= e^{-i\theta J_3} \frac{1}{(2\pi)^{3/2}} \sum_a \int d^3x\, \bar{v}_a(0,s)\psi_a(x)|0\rangle \qquad (33) \\
&= \frac{1}{(2\pi)^{3/2}} \sum_a \int d^3x\, \bar{v}_a(0,s) e^{-i\theta J_3}\psi_a(x) e^{i\theta J_3} e^{-i\theta J_3}|0\rangle.
\end{aligned}$$

Since the vacuum is invariant, this is

$$\begin{aligned}
e^{-i\theta J_3}|0, s, n_c\rangle &= \frac{1}{(2\pi)^{3/2}} \sum_a \int d^3x\, \bar{v}_a(0,s) e^{-i\theta J_3}\psi_a(x) e^{i\theta J_3}|0\rangle \\
&= \frac{1}{(2\pi)^{3/2}} \sum_a \int d^3x\, \bar{v}_a(0,s) D(R^{-1})_{ab}\psi_b(Rx)|0\rangle. \qquad (34)
\end{aligned}$$

Since the jacobian of a rotation is unity, we have

$$e^{-i\theta J_3}|0, s, n_c\rangle = \frac{1}{(2\pi)^{3/2}} \sum_a \int d^3x\, \bar{v}_a(0,s) D(R^{-1})_{ab}\psi_b(x)|0\rangle \qquad (35)$$

in which

$$D(R^{-1})_{ab} = \begin{pmatrix} e^{i\theta/2} & 0 & 0 & 0 \\ 0 & e^{-i\theta/2} & 0 & 0 \\ 0 & 0 & e^{i\theta/2} & 0 \\ 0 & 0 & 0 & e^{-i\theta/2} \end{pmatrix}. \tag{36}$$

So this is

$$e^{-i\theta J_3}|0, s, n_c\rangle = \frac{1}{(2\pi)^{3/2}} \sum_a \int d^3x \, \bar{v}_a(0, s) e^{i\sigma(a)\theta/2} \psi_a(x)|0\rangle \tag{37}$$

in which $\sigma(a) = 1$ for $a = 1$ & 3, and $\sigma(a) = -1$ for $a = 2$ & 4. For $s = \pm 1/2$, the spinors $v(0, s)$ are

$$v(\mathbf{0}, \tfrac{1}{2}) = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} \quad \text{and} \quad v(\mathbf{0}, -\tfrac{1}{2}) = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}. \tag{38}$$

For $s = 1/2$, slots $a = 2$ & 4 of the spinor $v(\mathbf{0}, s)$ are nonzero, while slots $a = 1$ & 3 are zero. Thus

$$e^{-i\theta J_3}|0, 1/2, n_c\rangle = e^{-i\theta/2} \frac{1}{(2\pi)^{3/2}} \sum_a \int d^3x \, \bar{v}_a(0, s) \psi_a(x)|0\rangle \tag{39}$$

which means that the state has spin 1/2 in the z direction. For $s = -1/2$, slots $a = 1$ & 3 of the spinor $v(\mathbf{0}, s)$ are nonzero, while slots $a = 2$ & 4 are zero. Thus

$$e^{-i\theta J_3}|0, 1/2, n_c\rangle = e^{i\theta/2} \frac{1}{(2\pi)^{3/2}} \sum_a \int d^3x \, \bar{v}_a(0, s) \psi_a(x)|0\rangle \tag{40}$$

which means that the state has spin $-1/2$ in the z direction.

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[1] S. Weinberg, *The Quantum Theory of Fields*, Vol. I (Cambridge University Press, Cambridge, UK, 1995).

[2] M. E. Peskin and D. V. Schroeder, *An Introduction to Quantum Field Theory*, Advanced book program (Westview Press Reading (Mass.), Boulder (Colo.), 1995) reprinted in 1997.

[3] P. Cahill and K. Cahill, Learning about spin-one-half fields, *Eur. J. Phys.* **27**, 29 (2006).

[4] H. K. Dreiner, H. E. Haber, and S. P. Martin, Two-component spinor techniques and Feynman rules for quantum field theory and supersymmetry, Phys. Rept. **494**, 1 (2010), arXiv:0812.1594 [hep-ph].