and $a_1, a_1, \ldots, a_n$ are the eigenvalues of the matrix $A$. This definition makes sense if $f(A)$ is a series in powers of $A$ because then

$$f(A) = \sum_{k=0}^{\infty} c_k A^k = \sum_{k=0}^{\infty} c_k \left( SA^{(d)} S^{-1} \right)^k. \quad (1.305)$$

So since $S^{-1} S = I$, we have $(SA^{(d)} S^{-1})^k = S (A^{(d)})^k S^{-1}$ and thus

$$f(A) = S \left[ \sum_{k=0}^{\infty} c_k \left( A^{(d)} \right)^k \right] S^{-1} = S f(A^{(d)}) S^{-1} \quad (1.306)$$

which is (1.303).

**Example 1.44** (Momentum operators generate spatial translations) The position operator $x$ and the momentum operator $p$ obey the commutation relation $[x, p] = xp - px = i\hbar$. Thus the $a$-derivative $\dot{x}(a)$ of the operator $x(a) = e^{iap/\hbar} x e^{-iap/\hbar}$ is unity

$$\dot{x}(a) = e^{iap/\hbar} (-i[x, p]) e^{-iap/\hbar} = e^{iap/\hbar} \hbar e^{-iap/\hbar} = 1. \quad (1.307)$$

Since $x(0) = x$, we see that the unitary transformation $U(a) = e^{iap/\hbar}$ moves $x$ to $x + a$

$$e^{iap/\hbar} x e^{-iap/\hbar} = x(a) = x(0) + \int_0^a \dot{x}(a') da' = x + a. \quad (1.308)$$

**Example 1.45** (Glauber’s identity) The commutator of the annihilation operator $a$ and the creation operator $a^\dagger$ for a given mode is the number 1

$$[a, a^\dagger] = a a^\dagger - a^\dagger a = 1. \quad (1.309)$$

Thus $a$ and $a^\dagger$ commute with their commutator $[a, a^\dagger] = 1$ just as $x$ and $p$ commute with their commutator $[x, p] = i\hbar$.

Suppose that $A$ and $B$ are any two operators that commute with their commutator $[A, B] = AB - BA$

$$[A, [A, B]] = [B, [A, B]] = 0. \quad (1.310)$$

As in the $[x, p]$ example (1.44), we define $A_B(t) = e^{-tB} A e^{tB}$ and note that because $[B, [A, B]] = 0$, its $t$-derivative is simply

$$\dot{A}_B(t) = e^{-tB} [A, B] e^{tB} = [A, B]. \quad (1.311)$$

Since $A_B(0) = A$, an integration gives

$$A_B(t) = A + \int_0^t \dot{A}(t) dt = A + \int_0^t [A, B] dt = A + t [A, B]. \quad (1.312)$$
Multiplication from the left by $e^{tB}$ now gives $e^{tB} A_B(t)$ as
\[ e^{tB} A_B(t) = A e^{tB} = e^{tB} (A + t [A, B]). \] (1.313)

Now we define
\[ G(t) = e^{tA} e^{tB} e^{-t(A+B)} \] (1.314)
and use our formula (1.313) to compute its $t$-derivative as
\[ \dot{G}(t) = e^{tA} (A e^{tB} + e^{tB} B - e^{tB} (A + B)) e^{-t(A+B)} = e^{tA} t [A, B] e^{(t(A+B))} = t [A, B] G(t) = t G(t) [A, B]. \] (1.315)

Since $\dot{G}(t), G(t),$ and $[A, B]$ all commute with each other, we can integrate this operator equation
\[ \frac{d}{dt} \log G(t) = \frac{\dot{G}(t)}{G(t)} = t [A, B] \] (1.316)
from 0 to 1 and get since $G(0) = 1$
\[ \log G(1) - \log G(0) = \log G(1) = \frac{1}{2} [A, B]. \] (1.317)
Thus $G(1) = e^{[A,B]/2}$ and so
\[ e^{A} e^{B} e^{-(A+B)} = e^{\frac{1}{2}[A,B]} \quad \text{or} \quad e^{A} e^{B} = e^{A+B+\frac{1}{2}[A,B]} \] (1.318)
which is Glauber’s identity. \hfill \Box

**Example 1.46** (Chemical reactions) The chemical reactions $[A] \xrightarrow{a} [B]$, $[B] \xrightarrow{b} [A]$, and $[B] \xrightarrow{c} [C]$ make the concentrations $[A] \equiv A$, $[B] \equiv B$, and $[C] \equiv C$ of three kinds of molecules vary with time as
\[ \dot{A} = -aA + bB, \quad \dot{B} = aA - (b + c)B \quad \text{and} \quad \dot{C} = cB. \] (1.319)

We can group these concentrations into a 3-vector $V = (A, B, C)$ and write the three equations (1.319) as $\dot{V} = K V$ in which $K$ is the matrix
\[ K = \begin{pmatrix} -a & b & 0 \\ a & -b - c & 0 \\ 0 & c & 0 \end{pmatrix}. \] (1.320)

The solution to the differential equation $\dot{V} = K V$ is $V(t) = e^{Kt} V(0)$.

The eigenvalues of the matrix $K$ are the roots of the cubic equation $\det(K - \lambda I) = 0$. One root vanishes, and the other two are the roots of the quadratic equation $\lambda^2 + (a + b + c)\lambda + ac = 0$. Their sum is the trace