and a_1, a_1, $\dots a_n$ are the eigenvalues of the matrix A. This definition makes sense if $f(A)$ is a series in powers of A because then

$$f(A) = \sum_{k=0}^{\infty} c_k A^k = \sum_{k=0}^{\infty} c_k \left(S A^{(d)} S^{-1} \right)^k . \tag{1.305}$$

So since $S^{-1} S = I$, we have $\left(S A^{(d)} S^{-1} \right)^k = S \left(A^{(d)} \right)^k S^{-1}$ and thus

$$f(A) = S \left[\sum_{k=0}^{\infty} c_k \left(A^{(d)} \right)^k \right] S^{-1} = S f(A^{(d)}) S^{-1} \tag{1.306}$$

which is (1.303).

Example 1.44 (Momentum operators generate spatial translations) The position operator x and the momentum operator p obey the commutation relation $[x, p] = xp - px = i\hbar$. Thus the a-derivative $\dot{x}(a)$ of the operator $x(a) = e^{iap/\hbar} \, x \, e^{-iap/\hbar}$ is unity

$$\dot{x}(a) = e^{iap/\hbar} \left(-i[x, p] \right) e^{-iap/\hbar} = e^{iap/\hbar} \, \hbar \, e^{-iap/\hbar} = 1. \tag{1.307}$$

Since $x(0) = x$, we see that the unitary transformation $U(a) = e^{iap/\hbar}$ moves x to $x + a$

$$e^{iap/\hbar} \, x \, e^{-iap/\hbar} = x(a) = x(0) + \int_0^a \dot{x}(a') \, da' = x + a. \tag{1.308}$$

□

Example 1.45 (Glauber's identity) The commutator of the annihilation operator a and the creation operator $a^\dagger$ for a given mode is the number 1

$$[a, a^\dagger] = a \, a^\dagger - a^\dagger \, a = 1. \tag{1.309}$$

Thus a and $a^\dagger$ commute with their commutator $[a, a^\dagger] = 1$ just as x and p commute with their commutator $[x, p] = i\hbar$.

Suppose that A and B are any two operators that commute with their commutator $[A, B] = AB - BA$

$$[A, [A, B]] = [B, [A, B]] = 0. \tag{1.310}$$

As in the $[x, p]$ example (1.44), we define $A_B(t) = e^{-tB} \, A \, e^{tB}$ and note that because $[B, [A, B]] = 0$, its t-derivative is simply

$$\dot{A}_B(t) = e^{-tB} \, [A, B] \, e^{tB} = [A, B]. \tag{1.311}$$

Since $A_B(0) = A$, an integration gives

$$A_B(t) = A + \int_0^t \dot{A}(t) \, dt = A + \int_0^t [A, B] \, dt = A + t \, [A, B]. \tag{1.312}$$

Multiplication from the left by e^{tB} now gives $e^{tB}\, A_B(t)$ as

$$e^{tB}\, A_B(t) = A\, e^{tB} = e^{tB}\, (A + t\, [A, B])\,. \tag{1.313}$$

Now we define

$$G(t) = e^{tA}\, e^{tB}\, e^{-t(A+B)} \tag{1.314}$$

and use our formula (1.313) to compute its t-derivative as

$$\begin{aligned} \dot{G}(t) &= e^{tA}\left(A\, e^{tB} + e^{tB}\, B - \, e^{tB}(A+B)\right) e^{-t(A+B)} \\ &= e^{tA}\left(e^{tB}\,(A + t\,[A,B]) + e^{tB}\, B - \, e^{tB}(A+B)\right) e^{-t(A+B)} \\ &= e^{tA}\, e^{tB}\, t\,[A,B]\, e^{t(A+B)} = t\,[A,B]\, G(t) = t\, G(t)\,[A,B]. \end{aligned} \tag{1.315}$$

Since $\dot{G}(t)$, $G(t)$, and $[A, B]$ all commute with each other, we can integrate this operator equation

$$\frac{d}{dt} \log G(t) = \frac{\dot{G}(t)}{G(t)} = t\,[A, B] \tag{1.316}$$

from 0 to 1 and get since $G(0) = 1$

$$\log G(1) - \log G(0) = \log G(1) = \frac{1}{2}\,[A, B]. \tag{1.317}$$

Thus $G(1) = e^{[A,B]/2}$ and so

$$e^A\, e^B\, e^{-(A+B)} = e^{\frac{1}{2}[A,B]} \quad \text{or} \quad e^A\, e^B = e^{A+B+\frac{1}{2}[A,B]} \tag{1.318}$$

which is Glauber's identity. □

Example 1.46 (Chemical reactions) The chemical reactions [A] $\xrightarrow{a}$ [B], [B] $\xrightarrow{b}$ [A], and [B] $\xrightarrow{c}$ [C] make the concentrations $[A] \equiv A$, $[B] \equiv B$, and $[C] \equiv C$ of three kinds of molecules vary with time as

$$\dot{A} = \, -aA + bB, \quad \dot{B} = aA - (b+c)B \quad \text{and} \quad \dot{C} = cB. \tag{1.319}$$

We can group these concentrations into a 3-vector $V = (A, B, C)$ and write the three equations (1.319) as $\dot{V} = K\, V$ in which K is the matrix

$$K = \begin{pmatrix} -a & b & 0 \\ a & -b-c & 0 \\ 0 & c & 0 \end{pmatrix}. \tag{1.320}$$

The solution to the differential equation $\dot{V} = K\, V$ is $V(t) = e^{Kt}\, V(0)$.

The eigenvalues of the matrix K are the roots of the cubic equation $\det(K - \lambda I) = 0$. One root vanishes, and the other two are the roots of the quadratic equation $\lambda^2 + (a + b + c)\lambda + ac = 0$. Their sum is the trace