10

Field Theory on a Lattice

10.1 Pure Gauge Theory

The gauge-covariant derivative is defined in terms of the generators t_a of a compact Lie algebra

$$[t_a, t_b] = i f_{abc} t_c \tag{10.1}$$

as

$$D_i = \partial_i + A_i = \partial_i + i g A_i^b \, t_b \tag{10.2}$$

summed over all the generators, and g is a coupling constant. Since the group is compact, we may raise and lower group indexes without worrying about factors or minus signs.

The Faraday matrix is

$$F_{ij} = [D_i, D_j] = [I\partial_i + A_i(x), I\partial_j + A_j(x)] = \partial_i A_j - \partial_j A_i + [A_i, A_j] \tag{10.3}$$

in matrix notation. With more indices exposed, it is

$$\begin{aligned} (F_{ij})_{cd} &= (\partial_i A_j - \partial_j A_i + [A_i, A_j])_{cd} \\ &= igt^b_{cd} \left(\partial_i A^b_j - \partial_j A^b_i \right) + \left([igt^b A^b_i, igt^e A^e_j] \right)_{cd} . \end{aligned} \tag{10.4}$$

Summing over repeated indices, we get

$$
\begin{aligned}
(F_{ij})_{cd} &= igt^b_{cd}\left(\partial_i A^b_j - \partial_j A^b_i\right) - g^2 A^b_i A^e_j \left([t^b, t^e]\right)_{cd} \\
&= igt^b_{cd}\left(\partial_i A^b_j - \partial_j A^b_i\right) - g^2 A^b_i A^e_j i f_{bef} t^f_{cd} \\
&= igt^b_{cd}\left(\partial_i A^b_j - \partial_j A^b_i\right) - ig^2 A^b_i A^e_j f_{bef} t^f_{cd} \\
&= igt^b_{cd}\left(\partial_i A^b_j - \partial_j A^b_i\right) - ig^2 A^f_i A^e_j f_{feb} t^b_{cd} \\
&= igt^b_{cd}\left(\partial_i A^b_j - \partial_j A^b_i - gA^f_i A^e_j f_{feb}\right) \\
&= igt^b_{cd}\left(\partial_i A^b_j - \partial_j A^b_i - gf_{bfe}\, A^f_i A^e_j\right) = igt^b_{cd}\, F^b_{ij}
\end{aligned}
\tag{10.5}
$$

where

$$
F^b_{ij} = \partial_i A^b_j - \partial_j A^b_i - gf_{bfe}\, A^f_i A^e_j \tag{10.6}
$$

is the Faraday tensor.

The action density of this tensor is

$$
L_F = -\frac{1}{4} F^b_{ij} F^{ij}_b . \tag{10.7}
$$

The trace of the square of the Faraday matrix is

$$
\begin{aligned}
\mathrm{Tr}\left[F_{ij} F^{ij}\right] &= \mathrm{Tr}\left[igt^b\, F^b_{ij}\, igt^c\, F^{ij}_c\right] \\
&= -g^2 F^b_{ij}\, F^{ij}_c\, \mathrm{Tr}(t^b t^c) = -g^2 F^b_{ij}\, F^{ij}_c\, k\delta_{bc} \\
&= -kg^2\, F^b_{ij}\, F^{ij}_b .
\end{aligned}
\tag{10.8}
$$

So the Faraday action density is

$$
L_F = -\frac{1}{4} F^b_{ij} F^{ij}_b = \frac{1}{4kg^2}\, \mathrm{Tr}\left[F_{ij} F^{ij}\right] = \frac{1}{2g^2}\, \mathrm{Tr}\left[F_{ij} F^{ij}\right] . \tag{10.9}
$$

The theory described by this action density, without scalar or spinor fields, is called **pure** gauge theory.

Under a gauge transformation, gauge fields go as

$$
\begin{aligned}
A'_i(x) &= g(x) A_i(x) g^{-1}(x) - (\partial_i g(x)) g^{-1}(x) \\
&= e^{-i\theta^a(x) t^a}\, A_i(x)\, e^{i\theta^a(x) t^a} - (\partial_i e^{-i\theta^a(x) t^a}) e^{i\theta^a(x) t^a} .
\end{aligned}
\tag{10.10}
$$

How does an exponential of a path-ordered very short line integral of gauge fields go? We will evaluate how the path-ordered exponential

$$\begin{aligned} P\exp\left(-A_i(x)dx^i\right)' &= P\exp\left(-\left[g(x)A_i(x)g^{-1}(x) - (\partial_i g(x))g^{-1}(x)\right]dx^i\right) \\ &= P\exp\left(-\left[g(x)A_i(x)g^{-1}(x) - \partial_i \log g(x)\right]dx^i\right). \end{aligned} \tag{10.11}$$

changes under the gauge transformation (10.10) in the limitt $dx^i \to 0$. We find

$$\begin{aligned} P\exp\left(-A_i(x)dx^i\right)' &= P\Big[g(x)e^{-A_i(x)dx^i}g^{-1}(x) \\ &\quad \times e^{\log g(x+dx^i/2) - \log g(x) + \log g(x) - \log g(x-dx^i/2)}\Big] \\ &= P\Big[g(x)e^{-A_i(x)dx^i}g^{-1}(x) \\ &\quad \times g(x+dx^i/2)g^{-1}(x)g(x)g^{-1}(x-dx^i/2)\Big] \\ &= g(x+dx^i/2)e^{-A_i(x)dx^i}g^{-1}(x-dx^i/2). \end{aligned} \tag{10.12}$$

Putting together a chain of such infinitesimal links, we get

$$P\exp\left(-\int_y^x A_i(x')dx'^i\right)' = g(x)P\exp\left(-\int_y^x A_i(x')dx'^i\right)g^{-1}(y). \tag{10.13}$$

In particular, this means that a closed loop is gauge invariant

$$P\exp\left(-\oint A_i(x)dx^i\right)' = P\exp\left(-\oint A_i(x)dx^i\right). \tag{10.14}$$

10.2 Pure Gauge Theory on a Lattice

Wilson's compactification of gauge theory is inspired in part by these last two equations. Another source of inspiration is the approximation for a loop of tiny area $dx \wedge dy$ which in the joint limit $dx \to 0$ and $dy \to 0$ is

$$\begin{aligned} W = P\exp\left(-\oint A_i(x)dx^i\right) &= \exp\Big[-\Big(A_{y,x} - A_{x,y} + [A_x, A_y]\Big)dxdy\Big] \\ &= \exp\Big(-F_{xy}\,dxdy\Big). \end{aligned} \tag{10.15}$$

To derive this formula, we will apply the Baker-Campbell-Hausdorf identity

$$e^A e^B = \exp\left(A + B + \frac{1}{2}[A,B] + \frac{1}{12}[A,[A,B]] + \frac{1}{12}[B,[B,A]] + \dots\right). \tag{10.16}$$

to the product

$$W = \exp\Big(-A_x\,dx\Big)\exp\Big(-A_y\,dy - A_{y,x}\,dxdy\Big)$$
$$\times\exp\Big(A_x\,dx + A_{x,y}\,dxdy\Big)\exp\Big(A_y\,dy\Big). \tag{10.17}$$

We get

$$W = \exp\Big(-A_x\,dx - A_y\,dy - A_{y,x}\,dxdy - \frac{1}{2}[A_x\,dx, (A_y + A_{y,x}\,dx)dy]\Big)$$
$$\times\exp\Big(A_x\,dx + A_y\,dy + A_{x,y}\,dxdy - \frac{1}{2}[(A_x + A_{x,y}\,dy)dx, A_y\,dy]\Big). \tag{10.18}$$

Applying again the BCH identity, we get

$$W = \exp\Big[-\Big(A_{y,x} - A_{x,y} + [A_x, A_y]\Big)dxdy\Big] = \exp\Big(-F_{xy}\,dxdy\Big) \tag{10.19}$$

which is the desired identity which is the basis of lattice gauge theory.

After subtracting the identity matrix, we take the trace

$$\mathrm{Tr}\left[P\exp\left(-\oint A_i(x)dx^i\right) - 1\right] = \mathrm{Tr}\Big[\exp\Big(-F_{xy}\,dxdy\Big) - 1\Big]$$
$$= \mathrm{Tr}\Big[-F_{xy}\,dxdy + \frac{1}{2}\Big(F_{xy}\,dxdy\Big)^2\Big]. \tag{10.20}$$

The generators of $SU(n)$, $SO(n)$, and $Sp(2n)$ are traceless, so the first term vanishes

$$W = \mathrm{Tr}\left[P\exp\left(-\oint A_i(x)dx^i\right) - 1\right] = \frac{1}{2}\mathrm{Tr}\Big(F_{xy}\,dxdy\Big)^2. \tag{10.21}$$