## 10.15 Generators

To study continuous groups, we will use calculus and algebra, and we will focus on the simplest part of the group—the elements $g(\alpha)$ for $\alpha \approx 0$ which are near the identity $e = g(0)$ for which all $\alpha_a = 0$. Each element $g(\alpha)$ of the group is represented by a matrix $D(\alpha) \equiv D(g(\alpha))$ in the $D$ representation of the group and by another matrix $D'(\alpha) \equiv D'(g(\alpha))$ in any other $D'$ representation of the group. Every representation respects the multiplication law of the group. So if $g(\beta)g(\alpha) = g(\gamma)$, then the matrices of the $D$ representation must satisfy $D(\beta)D(\alpha) = D(\gamma)$, and those of any other representation $D'$ must satisfy $D'(\beta)D'(\alpha) = D'(\gamma)$.

A **generator** $t_a$ of a representation $D$ is the partial derivative of the matrix $D(\alpha)$ with respect to the component $\alpha_a$ of $\alpha$ evaluated at $\alpha = 0$

$$t_a = -i \, \frac{\partial D(\alpha)}{\partial \alpha_a}\bigg|_{\alpha=0} . \tag{10.57}$$

When all the parameters $\alpha_a$ are infinitesimal, $|\alpha_a| \ll 1$, the matrix $D(\alpha)$ is very close to the identity matrix $I$

$$D(\alpha) \simeq I + i \sum_a \alpha_a \, t_a. \tag{10.58}$$

Replacing $\alpha$ by $\alpha/n$, we get a relation that becomes exact as $n \to \infty$

$$D\left(\frac{\alpha}{n}\right) = I + i \sum_a \frac{\alpha_a}{n} \, t_a. \tag{10.59}$$

The $n$th power of this equation is the matrix $D(\alpha)$ that represents the group element $g(\alpha)$ in the **exponential parametrization**

$$D(\alpha) = D\left(\frac{\alpha}{n}\right)^n = \lim_{n\to\infty} \left(I + i \sum_a \frac{\alpha_a}{n} \, t_a\right)^n = \exp\left(i \sum_a \alpha_a \, t_a\right). \tag{10.60}$$

The $i$'s appear in these equations so that when the generators $t_a$ are hermitian matrices, $(t_a)^\dagger = t_a$, and the $\alpha$'s are real, the matrices $D(\alpha)$ are unitary

$$D^{-1}(\alpha) = \exp\left(-i \sum_a \alpha_a \, t_a\right) = D^\dagger(\alpha) = \exp\left(-i \sum_a \alpha_a \, t_a\right). \tag{10.61}$$

Compact groups have finite-dimensional, unitary representations with hermitian generators.

## 10.16 Lie algebra

If $t_a$ and $t_b$ are any two generators of a representation $D$, then the matrices

$$D(\alpha) = e^{i\epsilon t_a} \quad \text{and} \quad D(\beta) = e^{i\epsilon t_b} \tag{10.62}$$

represent the group elements $g(\alpha)$ and $g(\beta)$ with infinitesimal exponential parameters $\alpha_i = \epsilon\,\delta_{ia}$ and $\beta_i = \epsilon\,\delta_{ib}$. The inverses of these group elements $g^{-1}(\alpha) = g(-\alpha)$ and $g^{-1}(\beta) = g(-\beta)$ are represented by the matrices $D(-\alpha) = e^{-i\epsilon t_a}$ and $D(-\beta) = e^{-i\epsilon t_b}$. The multiplication law of the group determines the parameters $\gamma(\alpha,\beta)$ of the product

$$g(\beta)\,g(\alpha)\,g(-\beta)\,g(-\alpha) = g(\gamma(\alpha,\beta)). \tag{10.63}$$

The matrices of any two representations $D$ with generators $t_a$ and $D'$ with generators $t'_a$ obey the same multiplication law

$$\begin{aligned} D(\beta)\,D(\alpha)\,D(-\beta)\,D(-\alpha) &= D(\gamma(\alpha,\beta)) \\ D'(\beta)\,D'(\alpha)\,D'(-\beta)\,D'(-\alpha) &= D'(\gamma(\alpha,\beta)) \end{aligned} \tag{10.64}$$

with the same infinitesimal exponential parameters $\alpha, \beta$, and $\gamma(\alpha,\beta)$. To order $\epsilon^2$, the product of the four $D$'s is

$$\begin{aligned} e^{i\epsilon t_b}\,e^{i\epsilon t_a}\,e^{-i\epsilon t_b}\,e^{-i\epsilon t_a} &\approx (1 + i\epsilon\,t_b - \frac{\epsilon^2}{2}\,t_b^2)(1 + i\epsilon\,t_a - \frac{\epsilon^2}{2}\,t_a^2) \\ &\quad \times (1 - i\epsilon\,t_b - \frac{\epsilon^2}{2}\,t_b^2)(1 - i\epsilon\,t_a - \frac{\epsilon^2}{2}\,t_a^2) \\ &\approx 1 + \epsilon^2(t_a\,t_b - t_b\,t_a) = 1 + \epsilon^2[t_a, t_b]. \end{aligned} \tag{10.65}$$

The other representation gives the same result but with primes

$$e^{i\epsilon t'_b}\,e^{i\epsilon t'_a}\,e^{-i\epsilon t'_b}\,e^{-i\epsilon t'_a} \approx 1 + \epsilon^2[t'_a, t'_b]. \tag{10.66}$$

The products (10.65 & 10.66) represent the same group element $g(\gamma(\alpha,\beta))$, so they have the same infinitesimal parameters $\gamma(\alpha,\beta)$ and therefore are the same linear combinations of their respective generators $t_c$ and $t'_c$

$$\begin{aligned} D(\gamma(\alpha,\beta)) &\approx 1 + \epsilon^2[t_a, t_b] = 1 + i\epsilon^2 \sum_{c=1}^{n} f_{ab}^c\,t_c \\ D'(\gamma(\alpha,\beta)) &\approx 1 + \epsilon^2[t'_a, t'_b] = 1 + i\epsilon^2 \sum_{c=1}^{n} f_{ab}^c\,t'_c. \end{aligned} \tag{10.67}$$

which in turn imply the **Lie algebra** formulas

$$[t_a, t_b] = \sum_{c=1}^{n} f_{ab}^c\,t_c \quad \text{and} \quad [t'_a, t'_b] = \sum_{c=1}^{n} f_{ab}^c\,t'_c. \tag{10.68}$$

**The commutator of any two generators is a linear combination of the generators. The coefficients $f^c{}_{ab}$ are the structure constants of the group. They are the same for all representations of the group.**

Unless the parameters $\alpha_a$ are redundant, the generators are linearly independent. They span a vector space, and any linear combination may be called a generator. By using the Gram-Schmidt procedure (section 1.10), we may make the generators $t_a$ orthogonal with respect to the inner product (1.91)

$$(t_a, t_b) = \text{Tr}\left(t_a^\dagger t_b\right) = k\,\delta_{ab} \tag{10.69}$$

in which $k$ is a nonnegative normalization constant that depends upon the representation. We can't normalize the generators, making $k$ unity, because the structure constants $f^c_{ab}$ are the same in all representations.

In what follows, I will often omit the summation symbol $\sum$ when an index is repeated. In this notation, the structure-constant formulas (10.68) are

$$[t_a, t_b] = f^c_{ab}\, t_c \quad \text{and} \quad [t'_a, t'_b] = f^c_{ab}\, t'_c. \tag{10.70}$$

This summation convention avoids unnecessary summation symbols.

By multiplying both sides of the first of the two Lie algebra formulas (10.68) by $t_d^\dagger$ and using the orthogonality (10.69) of the generators, we find

$$\text{Tr}\left([t_a, t_b]\, t_d^\dagger\right) = i\, f^c{}_{ab}\, \text{Tr}\left(t_c\, t_d^\dagger\right) = i\, f^c{}_{ab}\, k\, \delta_{cd} = ik\, f^d{}_{ab} \tag{10.71}$$

which implies that the structure constant $f^c{}_{ab}$ is the trace

$$f^c{}_{ab} = -\frac{i}{k}\, \text{Tr}\left([t_a, t_b]\, t_c^\dagger\right). \tag{10.72}$$

Because of the antisymmetry of the commutator $[t_a, t_b]$, structure constants are **antisymmetric in their lower indices**

$$f^c{}_{ab} = -f^c{}_{ba}. \tag{10.73}$$

From any $n \times n$ matrix $A$, one may make a hermitian matrix $A + A^\dagger$ and an antihermitian one $A - A^\dagger$. Thus, one may separate the $n_G$ generators into a set that are hermitian $t_a^{(h)}$ and a set that are antihermitian $t_a^{(ah)}$. The exponential of any imaginary linear combination of $n \times n$ hermitian generators $D(\alpha) = \exp\left(i\alpha_a\, t_a^{(h)}\right)$ is an $n \times n$ unitary matrix since

$$D^\dagger(\alpha) = \exp\left(-i\alpha_a\, t_a^{\dagger(h)}\right) = \exp\left(-i\alpha_a\, t_a^{(h)}\right) = D^{-1}(\alpha). \tag{10.74}$$

A group with only hermitian generators is **compact** and has finite-dimensional unitary representations.

On the other hand, the exponential of any imaginary linear combination of antihermitian generators $D(\alpha) = \exp\left(i\alpha_a\, t_a^{(ah)}\right)$ is a real exponential of their hermitian counterparts $i\, t_a^{(ah)}$ whose squared norm

$$\|D(\alpha)\|^2 = \text{Tr}\left[D(\alpha)^\dagger D(\alpha)\right] = \text{Tr}\left[\exp\left(2\alpha_a\, it_a^{(ah)}\right)\right] \tag{10.75}$$

grows exponentially and without limit as the parameters $\alpha_a \to \pm\infty$. A group with some antihermitian generators is **noncompact** and does not have finite-dimensional unitary representations. (The unitary representations of the translations and of the Lorentz and Poincaré groups are infinite dimensional.)

Compact Lie groups have hermitian generators, and so the structure-constant formula (10.72) reduces in this case to

$$f^c{}_{ab} = (-i/k)\text{Tr}\left([t_a, t_b]\, t_c^\dagger\right) = (-i/k)\text{Tr}\left([t_a, t_b]\, t_c\right). \tag{10.76}$$

Now, since the trace is cyclic, we have

$$\begin{aligned} f^b{}_{ac} &= (-i/k)\text{Tr}\left([t_a, t_c]\, t_b\right) = (-i/k)\text{Tr}\left(t_a t_c t_b - t_c t_a t_b\right) \\ &= (-i/k)\text{Tr}\left(t_b t_a t_c - t_a t_b t_c\right) \\ &= (-i/k)\text{Tr}\left([t_b, t_a]\, t_c\right) = f^c{}_{ba} = -f^c{}_{ab}. \end{aligned} \tag{10.77}$$

Interchanging $a$ and $b$, we get

$$f^a{}_{bc} = f^c{}_{ab} = -f^c{}_{ba}. \tag{10.78}$$

Finally, interchanging $b$ and $c$ in (10.77) gives

$$f^c{}_{ab} = f^b{}_{ca} = -f^b{}_{ac}. \tag{10.79}$$

Combining (10.77, 10.78, & 10.79), we see that **the structure constants of a compact Lie group are totally antisymmetric**

$$f^b{}_{ac} = -f^b{}_{ca} = f^c{}_{ba} = -f^c{}_{ab} = -f^a{}_{bc} = f^a{}_{cb}. \tag{10.80}$$

Because of this antisymmetry, it is usual to lower the upper index

$$f^c{}_{ab} = f_{cab} = f_{abc} \tag{10.81}$$

and write the antisymmetry of the structure constants of compact Lie groups as

$$f_{acb} = -f_{cab} = f_{bac} = -f_{abc} = -f_{bca} = f_{cba}. \tag{10.82}$$

For compact Lie groups, the generators are hermitian, and so the **structure constants $f_{abc}$ are real**, as we may see by taking the complex conjugate of the formula (10.76) for $f_{abc}$

$$f^*_{abc} = (i/k)\text{Tr}\,(t_c\,[t_b, t_a]) = (-i/k)\text{Tr}\,([t_a, t_b]\,t_c) = f_{abc}. \tag{10.83}$$

It follows from (10.68 & 10.81–10.83) that **the commutator of any two generators of a Lie group is a linear combination**

$$[t_a, t_b] = i\, f^c_{ab}\, t_c \tag{10.84}$$

**of its generators $t_c$, and that the structure constants $f_{abc} \equiv f^c_{ab}$ are real and totally antisymmetric if the group is compact**.

**Example 10.19** (Gauge Transformation) The action density of a Yang-Mills theory is unchanged when a spacetime dependent unitary matrix $U(x)$ changes a vector $\psi(x)$ of matter fields to $\psi'(x) = U(x)\psi(x)$. Terms like $\psi^\dagger\psi$ are invariant because $\psi^\dagger(x)U^\dagger(x)U(x)\psi(x) = \psi^\dagger(x)\psi(x)$, but how can kinetic terms like $\partial_i\psi^\dagger\,\partial^i\psi$ be made invariant? Yang and Mills introduced matrices $A_i$ of gauge fields, replaced ordinary derivatives $\partial_i$ by **covariant derivatives** $D_i \equiv \partial_i + A_i$, and required that $D'_i\psi' = UD_i\psi$ or that

$$\left(\partial_i + A'_i\right) U\psi = \left(\partial_i U + U\partial_i + A'_i U\right)\psi = U\left(\partial_i + A_i\right)\psi. \tag{10.85}$$

Their nonabelian gauge transformation is

$$A'_i(x) = U(x)A_i(x)U^\dagger(x) - \left(\partial_i U(x)\right)U^\dagger(x). \tag{10.86}$$

One can write the unitary matrix as $U(x) = \exp(-ig\,\theta_a(x)\,t_a)$ in which $g$ is a coupling constant, the functions $\theta_a(x)$ parametrize the gauge transformation, and the generators $t_a$ belong to the representation that acts on the vector $\psi(x)$ of matter fields. □

## 10.17 The Rotation Group

The rotations and reflections in three-dimensional space form a compact group $O(3)$ whose elements $R$ are $3 \times 3$ real matrices that leave invariant the dot product of any two three vectors

$$(R\boldsymbol{x}) \cdot (R\boldsymbol{y}) = \boldsymbol{x}^\mathsf{T} R^\mathsf{T} R\,\boldsymbol{y} = \boldsymbol{x}^\mathsf{T} I\boldsymbol{y} = \boldsymbol{x} \cdot \boldsymbol{y}. \tag{10.87}$$

These matrices therefore are orthogonal (1.181)

$$R^{\mathsf{T}} R = I. \tag{10.88}$$

Taking the determinant of both sides and using the transpose (1.209) and product (1.222) rules, we have

$$(\det R)^2 = 1 \tag{10.89}$$

whence $\det R = \pm 1$. The group $O(3)$ contains reflections as well as rotations and is disjoint. The subgroup with $\det R = 1$ is the group $SO(3)$. An $SO(3)$ element near the identity $R = I + \omega$ must satisfy

$$(I + \omega)^{\mathsf{T}} (I + \omega) = I. \tag{10.90}$$

Neglecting the tiny quadratic term, we find that the infinitesimal matrix $\omega$ is antisymmetric

$$\omega^{\mathsf{T}} = -\omega. \tag{10.91}$$

One complete set of real $3 \times 3$ antisymmetric matrices is

$$\omega_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \omega_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \omega_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{10.92}$$

which we may write as

$$[\omega_b]_{ac} = \epsilon_{abc} \tag{10.93}$$

in which $\epsilon_{abc}$ is the **Levi-Civita symbol** which is totally antisymmetric with $\epsilon_{123} = 1$ (Tullio Levi-Civita 1873–1941). The $\omega_b$ are antihermitian, but we make them hermitian by multiplying by $i$

$$t_b = i\,\omega_b \quad \text{so that} \quad [t_b]_{ac} = i\epsilon_{abc} \tag{10.94}$$

and $R = I - i\theta_b\, t_b$.

The three hermitian generators $t_a$ satisfy (exercise 10.15) the commutation relations

$$[t_a, t_b] = i\, f_{abc}\, t_c \tag{10.95}$$

in which the structure constants are given by the Levi-Civita symbol $\epsilon_{abc}$

$$f_{abc} = \epsilon_{abc} \tag{10.96}$$

so that

$$[t_a, t_b] = i\, \epsilon_{abc}\, t_c. \tag{10.97}$$

They are the generators of the **defining representation** of $SO(3)$ (and also of the **adjoint representation** of $SU(2)$ (section 10.24)).

Physicists usually scale the generators by $\hbar$ and define the angular-momentum generator $L_a$ as

$$L_a = \hbar\, t_a \tag{10.98}$$

so that the eigenvalues of the angular-momentum operators are the physical values of the angular momenta. With $\hbar$, the commutation relations are

$$[L_a, L_b] = i\,\hbar\,\epsilon_{abc}\,L_c. \tag{10.99}$$

The matrix that represents a right-handed rotation (of an object) by an angle $\theta = |\boldsymbol{\theta}|$ about an axis $\boldsymbol{\theta}$ is

$$D(\boldsymbol{\theta}) = e^{-i\boldsymbol{\theta}\cdot\boldsymbol{t}} = e^{-i\boldsymbol{\theta}\cdot\boldsymbol{L}/\hbar}. \tag{10.100}$$

By using the fact (1.290) that a matrix obeys its characteristic equation, one may show (exercise 10.17) that the $3\times 3$ matrix $D(\boldsymbol{\theta})$ that represents a right-handed rotation of $\theta$ radians about the axis $\boldsymbol{\theta}$ is the matrix (10.7) whose $i,j$th entry is

$$D_{ij}(\boldsymbol{\theta}) = \cos\theta\,\delta_{ij} - \sin\theta\,\epsilon_{ijk}\,\theta_k/\theta + (1-\cos\theta)\,\theta_i\theta_j/\theta^2 \tag{10.101}$$

in which a sum over $k = 1,\ 2,\ 3$ is understood.

**Example 10.20** (Demonstration of commutation relations) Take a big sphere with a distinguished point and orient the sphere so that the point lies in the $y$-direction from the center of the sphere. Now rotate the sphere by a small angle, say 15 degrees or $\epsilon = \pi/12$, right-handedly about the $x$-axis, then right-handedly about the $y$-axis by the same angle, then left-handedly about the $x$-axis and then left-handedly about the $y$-axis. Using the approximations (10.65 & 10.67) for the product of these four rotation matrices and the definitions (10.98) of the generators and of their structure constants (10.99), we have $\hbar t_a = L_1 = L_x$, $\hbar t_b = L_2 = L_y$, $\hbar f_{abc} t_c = \epsilon_{12c} L_c = L_3 = L_z$, and

$$e^{i\epsilon L_y/\hbar}\, e^{i\epsilon L_x/\hbar}\, e^{-i\epsilon L_y/\hbar}\, e^{-i\epsilon L_x/\hbar} \approx 1 + \frac{\epsilon^2}{\hbar^2}[L_x, L_y] = 1 + i\frac{\epsilon^2}{\hbar}L_z \approx e^{i\epsilon^2 L_z/\hbar} \tag{10.102}$$

which is a left-handed rotation about the (vertical) $z$-axis. The magnitude of that rotation should be about $\epsilon^2 = (\pi/12)^2 \approx 0.069$ or about 3.9 degrees. Photographs of an actual demonstration are displayed in Fig. 10.1.

The demonstrated equation (10.102) shows (exercise 10.16) that the generators $L_x$ and $L_y$ satisfy the commutation relation

$$[L_x, L_y] = i\hbar L_z \tag{10.103}$$

of the rotation group. □

## 10.18 Rotations and reflections in $2n$ dimensions

The orthogonal group $O(2n)$ of rotations and reflections in $2n$ dimensions is the group of all real $2n \times 2n$ matrices $O$ whose transposes $O^{\mathsf{T}}$ are their inverses

$$O^{\mathsf{T}}\, O = O\, O^{\mathsf{T}} = I \tag{10.104}$$

in which $I$ is the $2n \times 2n$ identity matrix. These orthogonal matrices leave unchanged the distances from the origin of points in $2n$ dimensions. Those with unit determinant, $\det O = 1$, constitute the subgroup $SO(2n)$ of rotations in $2n$ dimensions.

A symmetric sum $\{A, B\} = AB + BA$ is called an **anticommutator**. Complex fermionic variables $\psi_i$ obey the anticommutation relations

$$\{\psi_i, \psi_k^\dagger\} = \hbar\, \delta_{ik}, \quad \{\psi_i, \psi_k\} = 0, \quad \text{and} \quad \{\psi_i^\dagger, \psi_k^\dagger\} = 0. \tag{10.105}$$

Their real $x_i$ and imaginary $y_i$ parts

$$x_i = \frac{1}{\sqrt{2}}(\psi_i + \psi_i^\dagger) \quad \text{and} \quad y_i = \frac{1}{i\sqrt{2}}(\psi_i - \psi_i^\dagger) \tag{10.106}$$

obey the anticommutation relations

$$\{x_i, x_k\} = \hbar\, \delta_{ik}, \quad \{y_i, y_k\} = \hbar\, \delta_{ik}, \quad \text{and} \quad \{x_i, y_k\} = 0. \tag{10.107}$$

More simply, the anticommutation relations of these $2n$ hermitian variables $v = (x_1, \dots, x_n, y_1, \dots, y_n)$ are

$$\{v_i, v_k\} = \hbar\, \delta_{ik}. \tag{10.108}$$

If the real linear transformation $v_i' = L_{i1}\, v_1 + L_{i2}\, v_2 + \cdots + L_{i2n}\, v_{2n}$ preserves these anticommutation relations, then the matrix $L$ must satisfy

$$\hbar\, \delta_{ik} = \{v_i', v_k'\} = L_{ij}\, L_{k\ell}\{v_j, v_\ell\} = L_{ij}\, L_{k\ell}\, \hbar\, \delta_{j\ell} = \hbar L_{ij}\, L_{kj} \tag{10.109}$$

which is the statement that it is orthogonal, $LL^{\mathsf{T}} = I$. Thus the group $O(2n)$ is the largest group of linear transformations that preserve the anticommutation relations of the $2n$ hermitian real and imaginary parts $v_i$ of $n$ complex fermionic variables $\psi_i$.

Physical Demonstration of the Commutation Relations

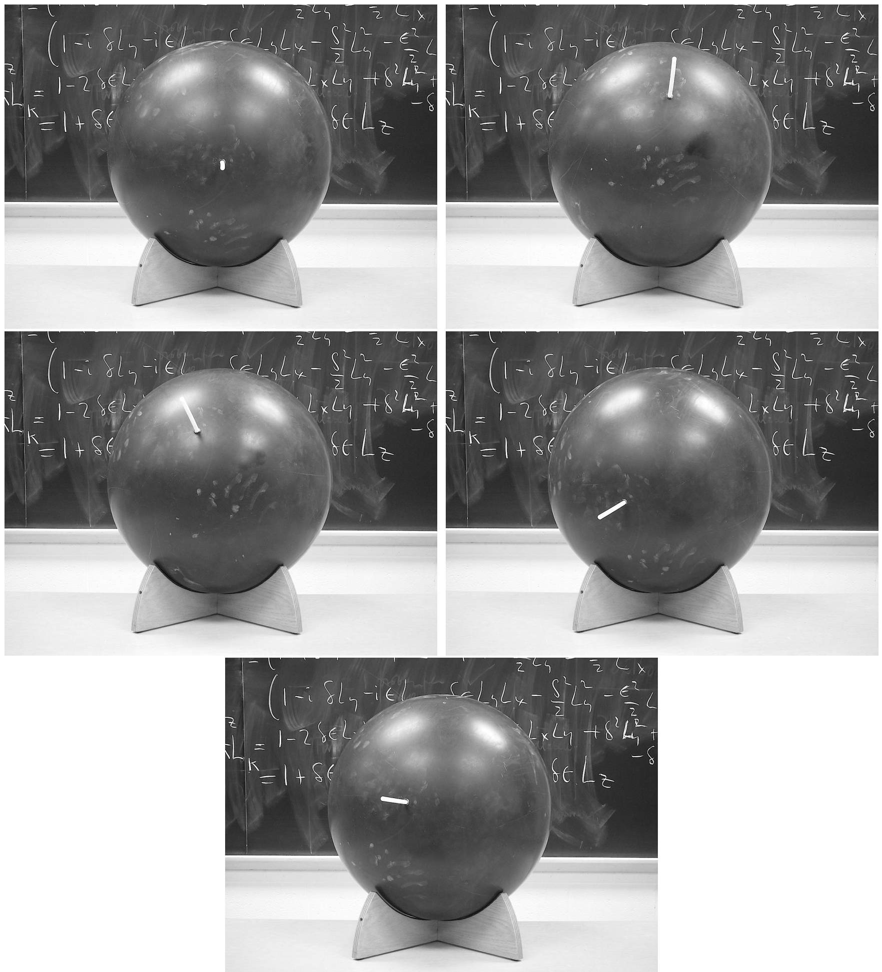

Figure 10.1 Demonstration of equation (10.102) and the commutation relation (10.103). Upper left: black ball with a white stick pointing in the $y$-direction; the $x$-axis is to the reader's left, the $z$-axis is vertical. Upper right: ball after a small right-handed rotation about the $x$-axis. Center left: ball after that rotation is followed by a small right-handed rotation about the $y$-axis. Center right: ball after these rotations are followed by a small left-handed rotation about the $x$-axis. Bottom: ball after these rotations are followed by a small left-handed rotation about the $y$-axis. The net effect is approximately a small left-handed rotation about the $z$-axis.

## 10.19 The defining representation of $SU(2)$

The smallest positive value of angular momentum is $\hbar/2$. The spin-one-half angular momentum operators are represented by three $2 \times 2$ matrices

$$S_a = \frac{\hbar}{2}\, \sigma_a \tag{10.110}$$

in which the $\sigma_a$ are the **Pauli matrices**

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \text{and} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{10.111}$$

which obey the multiplication law

$$\sigma_i\, \sigma_j = \delta_{ij} + i\epsilon_{ijk}\, \sigma_k \tag{10.112}$$

summed over $k$ from 1 to 3. Since the symbol $\epsilon_{ijk}$ is totally antisymmetric in $i, j$, and $k$, the Pauli matrices obey the commutation and anticommutation relations

$$\begin{aligned} [\sigma_i, \sigma_j] &\equiv \sigma_i \sigma_j - \sigma_j \sigma_i = 2i\epsilon_{ijk}\, \sigma_k \\ \{\sigma_i, \sigma_j\} &\equiv \sigma_i\, \sigma_j + \sigma_j\, \sigma_i = 2\delta_{ij}. \end{aligned} \tag{10.113}$$

The Pauli matrices divided by 2 satisfy the commutation relations (10.97) of the rotation group

$$\left[\tfrac{1}{2}\sigma_a, \tfrac{1}{2}\sigma_b\right] = i\, \epsilon_{abc}\, \tfrac{1}{2}\sigma_c \tag{10.114}$$

and generate the elements of the group $SU(2)$

$$\exp\left(i\ \boldsymbol{\theta} \cdot \frac{\boldsymbol{\sigma}}{2}\right) = I\ \cos\frac{\theta}{2} + i\, \hat{\boldsymbol{\theta}} \cdot \boldsymbol{\sigma}\ \sin\frac{\theta}{2} \tag{10.115}$$

in which $I$ is the $2{\times}2$ identity matrix, $\theta = \sqrt{\boldsymbol{\theta}^2}$ and $\hat{\boldsymbol{\theta}} = \boldsymbol{\theta}/\theta$.

It follows from (10.114) that the spin operators (10.110) satisfy

$$[S_a, S_b] = i\, \hbar\, \epsilon_{abc}\, S_c. \tag{10.116}$$

## 10.20 The Lie Algebra and Representations of *SU(2)*

The three generators of $SU(2)$ in its $2 \times 2$ defining representation are the Pauli matrices divided by 2, $t_a = \sigma_a/2$. The structure constants of $SU(2)$ are $f_{abc} = \epsilon_{abc}$ which is totally antisymmetric with $\epsilon_{123} = 1$

$$[t_a, t_b] = i f_{abc} t_c = \left[\tfrac{1}{2}\sigma_a, \tfrac{1}{2}\sigma_b\right] = i\epsilon_{abc}\, \tfrac{1}{2}\sigma_c. \tag{10.117}$$

For every half-integer

$$j = \frac{n}{2} \quad \text{for} \quad n = 0, 1, 2, 3, \dots \tag{10.118}$$

there is an irreducible representation of $SU(2)$

$$D^{(j)}(\boldsymbol{\theta}) = e^{-i\boldsymbol{\theta}\cdot\boldsymbol{J}^{(j)}} \tag{10.119}$$

in which the three generators $t_a^{(j)} \equiv J_a^{(j)}$ are $(2j+1) \times (2j+1)$ square hermitian matrices. In a basis in which $J_3^{(j)}$ is diagonal, the matrix elements of the complex linear combinations $J_\pm^{(j)} \equiv J_1^{(j)} \pm i J_2^{(j)}$ are

$$\left[J_1^{(j)} \pm i J_2^{(j)}\right]_{s',s} = \delta_{s',s\pm1}\sqrt{(j \mp s)(j \pm s + 1)} \tag{10.120}$$

where $s$ and $s'$ run from $-j$ to $j$ in integer steps and those of $J_3^{(j)}$ are

$$\left[J_3^{(j)}\right]_{s',s} = s\,\delta_{s',s}. \tag{10.121}$$

Borrowing a trick from section 10.25, one may show that the commutator of the square $\boldsymbol{J}^{(j)} \cdot \boldsymbol{J}^{(j)}$ of the angular momentum matrix commutes with every generator $J_a^{(j)}$. Thus $\boldsymbol{J}^{(j)2}$ commutes with $D^{(j)}(\boldsymbol{\theta})$ for every element of the group. Part 2 of Schur's lemma (section 10.7) then implies that $\boldsymbol{J}^{(j)2}$ must be a multiple of the $(2j+1) \times (2j+1)$ identity matrix. The coefficient turns out to be $j(j+1)$

$$\boldsymbol{J}^{(j)} \cdot \boldsymbol{J}^{(j)} = j(j+1)\,I. \tag{10.122}$$

Combinations of generators that are a multiple of the identity are called **Casimir operators**.

**Example 10.21** (Spin 2) For $j = 2$, the spin-two matrices $J_+^{(2)}$ and $J_3^{(2)}$ are

$$J_+^{(2)} = \begin{pmatrix} 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{6} & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad J_3^{(2)} = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{pmatrix} \tag{10.123}$$

and $J_- = \left(J_+^{(2)}\right)^\dagger$. □

The tensor product of any two irreducible representations $D^{(j)}$ and $D^{(k)}$

of $SU(2)$ is equivalent to the direct sum of all the irreducible representations $D^{(\ell)}$ for $|j-k| \le \ell \le j+k$

$$D^{(j)} \otimes D^{(k)} = \bigoplus_{\ell=|j-k|}^{j+k} D^{(\ell)} \tag{10.124}$$

each $D^{(\ell)}$ occurring once.

**Example 10.22** (Addition theorem) The spherical harmonics $Y_{\ell m}(\theta,\phi) = \langle \theta,\phi|\ell,m\rangle$ of section 8.13 transform under the $(2\ell+1)$-dimensional representation $D^{\ell}$ of the rotation group. If a rotation $R$ takes $\theta,\phi$ into the vector $\theta',\phi'$, so that $|\theta',\phi'\rangle = U(R)|\theta,\phi\rangle$, then summing over $m'$ from $-\ell$ to $\ell$, we get

$$\begin{aligned} Y^*_{\ell,m}(\theta',\phi') &= \langle \ell,m|\theta',\phi'\rangle = \langle \ell,m|U(R)|\theta,\phi\rangle \\ &= \langle \ell,m|U(R)|\ell,m'\rangle\langle \ell,m'|\theta,\phi\rangle = D^{\ell}(R)_{m,m'} Y^*_{\ell,m'}(\theta,\phi). \end{aligned}$$

Suppose now that a rotation $R$ maps $|\theta_1,\phi_1\rangle$ and $|\theta_2,\phi_2\rangle$ into $|\theta_1',\phi_1'\rangle = U(R)\,|\theta_1,\phi_1\rangle$ and $|\theta_2',\phi_2'\rangle = U(R)\,|\theta_2,\phi_2\rangle$. Then summing over the repeated indices $m$, $m'$, and $m''$ from $-\ell$ to $\ell$, we find

$$Y_{\ell,m}(\theta_1',\phi_1')Y^*_{\ell,m}(\theta_2',\phi_2') = D^{\ell}(R)^*_{m,m'}Y_{\ell,m'}(\theta_1,\phi_1)D^{\ell}(R)_{m,m''}Y^*_{\ell,m''}(\theta_2,\phi_2).$$

In this equation, the matrix element $D^{\ell}(R)^*_{m,m'}$ is

$$D^{\ell}(R)^*_{m,m'} = \langle \ell,m|U(R)|\ell,m'\rangle^* = \langle \ell,m'|U^\dagger(R)|\ell,m\rangle = D^{\ell}(R^{-1})_{m',m}.$$

Thus since $D^{\ell}$ is a representation of the rotation group, the product of the two $D^{\ell}$'s in (10.22) is

$$\begin{aligned} D^{\ell}(R)^*_{m,m'}D^{\ell}(R)_{m,m''} &= D^{\ell}(R^{-1})_{m',m}D^{\ell}(R)_{m,m''} \\ &= D^{\ell}(R^{-1}R)_{m',m''} = D^{\ell}(I)_{m',m''} = \delta_{m',m''}. \end{aligned}$$

So as long as the same rotation $R$ maps $\theta_1,\phi_1$ into $\theta_1',\phi_1'$ and $\theta_2,\phi_2$ into $\theta_2',\phi_2'$, then we have

$$\sum_{m=-\ell}^{\ell} Y_{\ell,m}(\theta_1',\phi_1')Y^*_{\ell,m}(\theta_2',\phi_2') = \sum_{m=-\ell}^{\ell} Y_{\ell,m}(\theta_1,\phi_1)Y^*_{\ell,m}(\theta_2,\phi_2).$$

We choose the rotation $R$ as the product of a rotation that maps the unit vector $\hat{n}(\theta_2,\phi_2)$ into $\hat{n}(\theta_2',\phi_2') = \hat{\boldsymbol{z}} = (0,0,1)$ and a rotation about the $z$ axis that maps $\hat{n}(\theta_1,\phi_1)$ into $\hat{n}(\theta_1',\phi_1') = (\sin\theta, 0, \cos\theta)$ in the $x$-$z$ plane where it makes an angle $\theta$ with $\hat{n}(\theta_2',\phi_2') = \hat{\boldsymbol{z}}$. We then have $Y^*_{\ell,m}(\theta_2',\phi_2') = Y^*_{\ell,m}(0,0)$ and $Y_{\ell,m}(\theta_1',\phi_1') = Y_{\ell,m}(\theta,0)$ in which $\theta$ is the angle between the unit vectors $\hat{n}(\theta_1', \phi_1')$ and $\hat{n}(\theta_2', \phi_2')$, which is the same as the angle between the unit vectors $\hat{n}(\theta_1, \phi_1)$ and $\hat{n}(\theta_2, \phi_2)$. The vanishing (8.108) at $\theta = 0$ of the associated Legendre functions $P_{\ell,m}$ for $m \neq 0$ and the definitions (8.4, 8.101, & 8.112–8.114) say that $Y^*_{\ell,m}(0,0) = \sqrt{(2\ell+1)/4\pi}\,\delta_{m,0}$, and that $Y_{\ell,0}(\theta, 0) = \sqrt{(2\ell+1)/4\pi}\,P_\ell(\cos\theta)$. Thus, our identity (10.22) gives us the for the spherical harmonics **addition theorem** (8.123)

$$P_\ell(\cos\theta) = \frac{2\ell+1}{4\pi}\sum_{m=-\ell}^{\ell} Y_{\ell,m}(\theta_1,\phi_1) Y^*_{\ell,m}(\theta_2,\phi_2).$$

□

## 10.21 How a field transforms under a rotation

Under a rotation $R$, a field $\psi_s(x)$ that transforms under the $D^{(j)}$ representation of $SU(2)$ responds as

$$U(R)\,\psi_s(x)\,U^{-1}(R) = D^{(j)}_{s,s'}(R^{-1})\,\psi_{s'}(Rx). \tag{10.125}$$

**Example 10.23** (Spin and Statistics) Suppose $|a, m\rangle$ and $|b, m\rangle$ are any eigenstates of the rotation operator $J_3$ with eigenvalue $m$ (in units with $\hbar = c = 1$). If $u$ and $v$ are any two space-like points, then in some Lorentz frame they have spacetime coordinates $u = (t, x, 0, 0)$ and $v = (t, -x, 0, 0)$. Let $U$ be the unitary operator that represents a right-handed rotation by $\pi$ about the 3-axis or $z$-axis of this Lorentz frame. Then

$$U|a,m\rangle = e^{-im\pi}|a,m\rangle \quad \text{and} \quad \langle b,m|U^{-1} = \langle b,m|e^{im\pi}. \tag{10.126}$$

And by (10.125), $U$ transforms a field $\psi$ of spin $j$ with $\boldsymbol{x} \equiv (x, 0, 0)$ to

$$U(R)\,\psi_s(t,\boldsymbol{x})\,U^{-1}(R) = D^{(j)}_{ss'}(R^{-1})\,\psi_{s'}(t,-\boldsymbol{x}) = e^{i\pi s}\psi_s(t,-\boldsymbol{x}). \tag{10.127}$$

Thus by inserting the identity operator in the form $I = U^{-1}U$ and using both (10.126) and (10.127), we find, since the phase factors $\exp(-im\pi)$ and $\exp(im\pi)$ cancel,

$$\begin{aligned}\langle b,m|\psi_s(t,\boldsymbol{x})\,\psi_s(t,-\boldsymbol{x})|a,m\rangle &= \langle b,m|U\psi_s(t,\boldsymbol{x})U^{-1}U\psi_s(t,-\boldsymbol{x})U^{-1}|a,m\rangle \\ &= e^{2i\pi s}\,\langle b,m|\psi_s(t,-\boldsymbol{x})\psi_s(t,\boldsymbol{x})|a,m\rangle.\end{aligned} \tag{10.128}$$

Now if $j$ is an integer, then so is $s$, and the phase factor $\exp(2i\pi s) = 1$ is unity. In this case, we find that the mean value of the equal-time commutator vanishes

$$\langle b, m|[\psi_s(t,\boldsymbol{x}), \psi_s(t,-\boldsymbol{x})]|a,m\rangle = 0 \tag{10.129}$$

which suggests that fields of integral spin commute at space-like separations. They represent bosons. On the other hand, if $j$ is half an odd integer, that is, $j = (2n+1)/2$, where $n$ is an integer, then the phase factor $\exp(2i\pi s) = -1$ is minus one. In this case, the mean value of the equal-time anticommutator vanishes

$$\langle b, m|\{\psi_s(t,\boldsymbol{x}), \psi_s(t,-\boldsymbol{x})\}|a,m\rangle = 0 \tag{10.130}$$

which suggests that fields of half-odd-integral spin anticommute at space-like separations. They represent fermions. This argument shows that the behavior of fields under rotations is related to their equal-time commutation or anticommutation relations

$$\psi_s(t,\boldsymbol{x})\psi_{s'}(t,\boldsymbol{x'}) + (-1)^{2j}\psi_{s'}(t,\boldsymbol{x'})\psi_s(t,\boldsymbol{x}) = 0 \tag{10.131}$$

and their statistics. □

## 10.22 The addition of two spin-one-half systems

The spin operators (10.110)

$$S_a = \frac{\hbar}{2}\sigma_a \tag{10.132}$$

obey the commutation relation (10.116)

$$[S_a, S_b] = i\,\hbar\,\epsilon_{abc}\,S_c. \tag{10.133}$$

The raising and lowering operators

$$S_\pm = S_1 \pm iS_2 \tag{10.134}$$

have simple commutators with $S_3$

$$[S_3, S_\pm] = \pm\,\hbar\,S_\pm. \tag{10.135}$$

This relation implies that if the state $|\frac{1}{2}, m\rangle$ is an eigenstate of $S_3$ with eigenvalue $\hbar m$, then the states $S_\pm|\frac{1}{2}, m\rangle$ either vanish or are eigenstates of $S_3$ with eigenvalues $\hbar(m \pm 1)$

$$S_3S_\pm|\tfrac{1}{2}, m\rangle = S_\pm S_3|\tfrac{1}{2}, m\rangle \pm \hbar S_\pm|\tfrac{1}{2}, m\rangle = \hbar(m\pm 1)S_\pm|\tfrac{1}{2}, m\rangle. \tag{10.136}$$

Thus the raising and lowering operators raise and lower the eigenvalues of $S_3$. The eigenvalues of $S_3 = \hbar\sigma_3/2$ are $\pm\hbar/2$. So with the usual sign and normalization conventions

$$S_+|-\rangle = \hbar|+\rangle \quad \text{and} \quad S_-|+\rangle = \hbar|-\rangle \tag{10.137}$$

while

$$S_+|+\rangle = 0 \quad \text{and} \quad S_-|-\rangle = 0. \tag{10.138}$$

The square of the total spin operator is simply related to the raising and lowering operators and to $S_3$

$$\boldsymbol{S}^2 = S_1^2 + S_2^2 + S_3^2 = \frac{1}{2}S_+S_- + \frac{1}{2}S_-S_+ + S_3^2. \tag{10.139}$$

But the squares of the Pauli matrices are unity, and so $S_a^2 = (\hbar/2)^2$ for all three values of $a$. Thus

$$\boldsymbol{S}^2 = \frac{3}{4}\,\hbar^2 \tag{10.140}$$

is a Casimir operator (10.122) for a spin one-half system.

Consider two spin operators $\boldsymbol{S}^{(1)}$ and $\boldsymbol{S}^{(2)}$ as defined by (10.110) acting on two spin-one-half systems. Let the tensor-product states

$$|\pm,\pm\rangle = |\pm\rangle_1|\pm\rangle_2 = |\pm\rangle_1 \otimes |\pm\rangle_2 \tag{10.141}$$

be eigenstates of $S_3^{(1)}$ and $S_3^{(2)}$ so that

$$\begin{aligned} S_3^{(1)}|+,\pm\rangle &= \frac{\hbar}{2}\,|+,\pm\rangle \quad \text{and} \quad S_3^{(2)}|\pm,+\rangle = \frac{\hbar}{2}\,|\pm,+\rangle \\ S_3^{(1)}|-,\pm\rangle &= -\,\frac{\hbar}{2}\,|-,\pm\rangle \quad \text{and} \quad S_3^{(2)}|\pm,-\rangle = -\,\frac{\hbar}{2}\,|\pm,-\rangle \end{aligned} \tag{10.142}$$

The total spin of the system is the sum of the two spins $\boldsymbol{S} = \boldsymbol{S}^{(1)} + \boldsymbol{S}^{(2)}$, so

$$\boldsymbol{S}^2 = \left(\boldsymbol{S}^{(1)} + \boldsymbol{S}^{(2)}\right)^2 \quad \text{and} \quad S_3 = S_3^{(1)} + S_3^{(2)}. \tag{10.143}$$

The state $|+,+\rangle$ is an eigenstate of $S_3$ with eigenvalue $\hbar$

$$S_3|+,+\rangle = S_3^{(1)}|+,+\rangle + S_3^{(2)}|+,+\rangle = \frac{\hbar}{2}|+,+\rangle + \frac{\hbar}{2}|+,+\rangle = \hbar|+,+\rangle. \tag{10.144}$$

So the state of angular momentum $\hbar$ in the 3-direction is $|1,1\rangle = |+,+\rangle$. Similarly, the state $|-,-\rangle$ is an eigenstate of $S_3$ with eigenvalue $-\hbar$

$$S_3|-,-\rangle = S_3^{(1)}|-,-\rangle + S_3^{(2)}|-,-\rangle = -\,\frac{\hbar}{2}|-,-\rangle - \frac{\hbar}{2}|-,-\rangle = -\,\hbar|-,-\rangle \tag{10.145}$$

and so the state of angular momentum $\hbar$ in the negative 3-direction is $|1,-1\rangle = |-,-\rangle$. The states $|+,-\rangle$ and $|-,+\rangle$ are eigenstates of $S_3$ with eigenvalue 0

$$\begin{aligned} S_3|+,-\rangle &= S_3^{(1)}|+,-\rangle + S_3^{(2)}|+,-\rangle = \frac{\hbar}{2}|+,-\rangle - \frac{\hbar}{2}|+,-\rangle = 0 \\ S_3|-,+\rangle &= S_3^{(1)}|-,+\rangle + S_3^{(2)}|-,+\rangle = -\frac{\hbar}{2}|-,+\rangle + \frac{\hbar}{2}|-,+\rangle = 0. \end{aligned} \tag{10.146}$$

To see which states are eigenstates of $\boldsymbol{S}^2$, we use the lowering operator for the combined system $S_- = S_-^{(1)} + S_-^{(2)}$ and the rules (10.120, 10.137, & 10.138) to lower the state $|1,1\rangle$

$$S_-|+,+\rangle = \left(S_-^{(1)} + S_-^{(2)}\right)|+,+\rangle = \hbar\left(|-,+\rangle + |+,-\rangle\right) = \hbar\sqrt{2}\,|1,0\rangle.$$

Thus the state $|1,0\rangle$ is

$$|1,0\rangle = \frac{1}{\sqrt{2}}\left(|+,-\rangle + |-,+\rangle\right). \tag{10.147}$$

The orthogonal and normalized combination of $|+,-\rangle$ and $|-,+\rangle$ must be the state of spin zero

$$|0,0\rangle = \frac{1}{\sqrt{2}}\left(|+,-\rangle - |-,+\rangle\right) \tag{10.148}$$

with the usual sign convention.

To check that the states $|1,0\rangle$ and $|0,0\rangle$ really are eigenstates of $\boldsymbol{S}^2$, we use (10.139 & 10.140) to write $\boldsymbol{S}^2$ as

$$\begin{aligned} \boldsymbol{S}^2 &= \left(\boldsymbol{S}^{(1)} + \boldsymbol{S}^{(2)}\right)^2 = \frac{3}{2}\hbar^2 + 2\boldsymbol{S}^{(1)}\cdot\boldsymbol{S}^{(2)} \\ &= \frac{3}{2}\hbar^2 + S_+^{(1)}S_-^{(2)} + S_-^{(1)}S_+^{(2)} + 2S_3^{(1)}S_3^{(2)}. \end{aligned} \tag{10.149}$$

Now the sum $S_+^{(1)}S_-^{(2)} + S_-^{(1)}S_+^{(2)}$ merely interchanges the states $|+,-\rangle$ and $|-,+\rangle$ and multiplies them by $\hbar^2$, so

$$\begin{aligned} \boldsymbol{S}^2|1,0\rangle &= \frac{3}{2}\hbar^2|1,0\rangle + \hbar^2|1,0\rangle - \frac{2}{4}\hbar^2|1,0\rangle \\ &= 2\hbar^2|1,0\rangle = s(s+1)\hbar^2|1,0\rangle \end{aligned} \tag{10.150}$$

which confirms that $s = 1$. Because of the relative minus sign in formula (10.148) for the state $|0,0\rangle$, we have

$$\begin{aligned} \boldsymbol{S}^2|0,0\rangle &= \frac{3}{2}\hbar^2|0,0\rangle - \hbar^2|1,0\rangle - \frac{1}{2}\hbar^2|1,0\rangle \\ &= 0\hbar^2|1,0\rangle = s(s+1)\hbar^2|1,0\rangle \end{aligned} \tag{10.151}$$

which confirms that $s = 0$.

**Example 10.24** (Two equivalent representations of $SU(2)$) The identity

$$\left[\exp\left(i\boldsymbol{\theta}\cdot\frac{\boldsymbol{\sigma}}{2}\right)\right]^* = \sigma_2 \exp\left(i\boldsymbol{\theta}\cdot\frac{\boldsymbol{\sigma}}{2}\right)\sigma_2 \tag{10.152}$$

shows that the defining representation of $SU(2)$ (section 10.19) and its complex conjugate are equivalent (10.8) representations. To prove this identity, we expand the exponential on the right-hand side in powers of its argument

$$\sigma_2 \exp\left(i\boldsymbol{\theta}\cdot\frac{\boldsymbol{\sigma}}{2}\right)\sigma_2 = \sigma_2\left[\sum_{n=0}^{\infty}\frac{1}{n!}\left(i\boldsymbol{\theta}\cdot\frac{\boldsymbol{\sigma}}{2}\right)^n\right]\sigma_2 \tag{10.153}$$

and use the fact that $\sigma_2$ is its own inverse to get

$$\sigma_2 \exp\left(i\boldsymbol{\theta}\cdot\frac{\boldsymbol{\sigma}}{2}\right)\sigma_2 = \sum_{n=0}^{\infty}\frac{1}{n!}\left[\sigma_2\left(i\boldsymbol{\theta}\cdot\frac{\boldsymbol{\sigma}}{2}\right)\sigma_2\right]^n. \tag{10.154}$$

Since the Pauli matrices obey the anticommutation relation (10.113), and since both $\sigma_1$ and $\sigma_3$ are real, while $\sigma_2$ is imaginary, we can write the $2\times 2$ matrix within the square brackets as

$$\sigma_2\left(i\boldsymbol{\theta}\cdot\frac{\boldsymbol{\sigma}}{2}\right)\sigma_2 = -i\theta_1\frac{\sigma_1}{2} + i\theta_2\frac{\sigma_2}{2} - i\theta_3\frac{\sigma_3}{2} = \left(i\boldsymbol{\theta}\cdot\frac{\boldsymbol{\sigma}}{2}\right)^* \tag{10.155}$$

which implies the identity (10.152)

$$\sigma_2 \exp\left(i\boldsymbol{\theta}\cdot\frac{\boldsymbol{\sigma}}{2}\right)\sigma_2 = \sum_{n=0}^{\infty}\frac{1}{n!}\left[\left(i\boldsymbol{\theta}\cdot\frac{\boldsymbol{\sigma}}{2}\right)^*\right]^n = \left[\exp\left(i\boldsymbol{\theta}\cdot\frac{\boldsymbol{\sigma}}{2}\right)\right]^*. \tag{10.156}$$

□

## 10.23 The Jacobi Identity

Any three square matrices $A$, $B$, and $C$ satisfy the commutator-product rule

$$\begin{aligned}[A, BC] &= ABC - BCA = ABC - BAC + BAC - BCA \\ &= [A, B]C + B[A, C].\end{aligned} \tag{10.157}$$

Interchanging $B$ and $C$ gives

$$[A, CB] = [A, C]B + C[A, B]. \tag{10.158}$$

Subtracting the second equation from the first, we get the Jacobi identity

$$[A, [B, C]] = [[A, B], C] + [B, [A, C]] \tag{10.159}$$

and its equivalent cyclic form

$$[A,[B,C]]+[B,[C,A]]+[C,[A,B]]=0. \tag{10.160}$$

Another Jacobi identity uses the anticommutator $\{A,B\}\equiv AB+BA$

$$\{[A,B],C\}+\{[A,C],B\}+[\{B,C\},A]=0. \tag{10.161}$$

## 10.24 The Adjoint Representation

Any three generators $t_a$, $t_b$, and $t_c$ satisfy the Jacobi identity (10.160)

$$[t_a,[t_b,t_c]]+[t_b,[t_c,t_a]]+[t_c,[t_a,t_b]]=0. \tag{10.162}$$

By using the structure-constant formula (10.84), we may express each of these double commutators as a linear combination of the generators

$$\begin{aligned}
[t_a,[t_b,t_c]] &= [t_a, i f^d_{bc} t_d] = \; - f^d_{bc} f^e_{ad} t_e \\
[t_b,[t_c,t_a]]] &= [t_b, i f^d_{ca} t_d] = \; - f^d_{ca} f^e_{bd} t_e \\
[t_c,[t_a,t_b]] &= [t_c, i f^d_{ab} t_d] = \; - f^d_{ab} f^e_{cd} t_e .
\end{aligned} \tag{10.163}$$

So the Jacobi identity (10.162) implies that

$$\left(f^d_{bc} f^e_{ad} + f^d_{ca} f^e_{bd} + f^d_{ab} f^e_{cd}\right) t_e = 0 \tag{10.164}$$

or since the generators are linearly independent

$$f^d_{bc} f^e_{ad} + f^d_{ca} f^e_{bd} + f^d_{ab} f^e_{cd} = 0. \tag{10.165}$$

If we define a set of matrices $T_a$ by

$$(T_b)_{ac} = i f^c_{ab} \tag{10.166}$$

then, since the structure constants are antisymmetric in their lower indices, we may write the three terms in the preceding equation (10.165) as

$$f^d_{bc} f^e_{ad} = f^d_{cb} f^e_{da} = (-T_b T_a)_{ce} \tag{10.167}$$

$$f^d_{ca} f^e_{bd} = - f^d_{ca} f^e_{db} = (T_a T_b)_{ce} \tag{10.168}$$

and

$$f^d_{ab} f^e_{cd} = -i f^d_{ab} (T_d)_{ce} \tag{10.169}$$

or in matrix notation

$$[T_a, T_b] = i f^c_{ab}\, T_c . \tag{10.170}$$

So the matrices $T_a$, which we made out of the structure constants by the rule $(T_b)_{ac} = if^c_{ab}$ (10.166), obey the same algebra (10.68) as do the generators $t_a$. They are the **generators in the adjoint representation** of the Lie algebra. If the Lie algebra has $N$ generators $t_a$, then the $N$ generators $T_a$ in the adjoint representation are $N \times N$ matrices.

## 10.25 Casimir operators

For any compact Lie algebra, the sum of the squares of all the generators

$$C = \sum_{a=1}^{N} t_a t_a \equiv t_a t_a \tag{10.171}$$

commutes with every generator $t_b$

$$\begin{aligned}[C, t_b] &= [t_a t_a, t_b] = [t_a, t_b] t_a + t_a [t_a, t_b] \\ &= i f_{abc} t_c t_a + t_a i f_{abc} t_c = i \left( f_{abc} + f_{cba} \right) t_c t_a = 0\end{aligned} \tag{10.172}$$

because of the total antisymmetry (10.82) of the structure constants. This sum, called a **Casimir operator**, commutes with every matrix

$$[C, D(\alpha)] = [C, \exp(i\alpha_a t_a)] = 0 \tag{10.173}$$

of the representation generated by the $t_a$'s. Thus by part 2 of Schur's lemma (section 10.7), it must be a multiple of the identity matrix

$$C = t_a t_a = cI. \tag{10.174}$$

The constant $c$ depends upon the representation $D(\alpha)$ and is called the **quadratic Casimir**

$$C_2(D) = \mathrm{Tr}\left(t_a^2\right) / \mathrm{Tr} I\ . \tag{10.175}$$

**Example 10.25** (Quadratic Casimirs of $SU(2)$) The quadratic Casimir $C(\mathbf{2})$ of the defining representation of $SU(2)$ with generators $t_a = \sigma_a/2$ (10.111) is

$$C_2(\mathbf{2}) = \mathrm{Tr}\left( \sum_{a=1}^{3} \left(\tfrac{1}{2}\sigma_a\right)^2 \right) / \mathrm{Tr}\,(I) = \frac{3 \cdot 2 \cdot \left(\frac{1}{2}\right)^2}{2} = \frac{3}{4}. \tag{10.176}$$

That of the adjoint representation (10.94) is

$$C_2(\mathbf{3}) = \mathrm{Tr}\left( \sum_{b=1}^{3} t_b^2 \right) / \mathrm{Tr}\,(I) = \sum_{a,b,c=1}^{3} \frac{i\epsilon_{abc} i\epsilon_{cba}}{3} = 2. \tag{10.177}$$

□

The generators of some noncompact groups come in pairs $t_a$ and $it_a$, and so the sum of the squares of these generators vanishes, $C = t_a t_a - t_a t_a = 0$.

## 10.26 Tensor operators for the rotation group

Suppose $A^{(j)}_m$ is a set of $2j+1$ operators whose commutation relations with the generators $J_i$ of rotations are

$$[J_i, A^{(j)}_m] = A^{(j)}_s (J^{(j)}_i)_{sm} \tag{10.178}$$

in which the sum over $s$ runs from $-j$ to $j$. Then $A^{(j)}$ is said to be a **spin-$j$ tensor operator** for the group $SU(2)$.

**Example 10.26** (A Spin-One Tensor Operator) For instance, if $j = 1$, then $(J^{(1)}_i)_{sm} = i\hbar\epsilon_{sim}$, and so a spin-1 tensor operator of $SU(2)$ is a vector $A^{(1)}_m$ that transforms as

$$[J_i, A^{(1)}_m] = A^{(1)}_s \, i\,\hbar\,\epsilon_{sim} = i\,\hbar\,\epsilon_{ims}\,A^{(1)}_s \tag{10.179}$$

under rotations. □

Let's rewrite the definition (10.178) as

$$J_i A^{(j)}_m = A^{(j)}_s (J^{(j)}_i)_{sm} + A^{(j)}_m J_i \tag{10.180}$$

and specialize to the case $i = 3$ so that $(J^{(j)}_3)_{sm}$ is diagonal, $(J^{(j)}_3)_{sm} = \hbar m \delta_{sm}$

$$J_3 A^{(j)}_m = A^{(j)}_s (J^{(j)}_3)_{sm} + A^{(j)}_m J_3 = A^{(j)}_s \hbar m \delta_{sm} + A^{(j)}_m J_3 = A^{(j)}_m \left(\hbar m + J_3\right) \tag{10.181}$$

Thus if the state $|j, s, E\rangle$ is an eigenstate of $J_3$ with eigenvalue $\hbar s$, then the state $A^{(j)}_m |j, s, E\rangle$ is an eigenstate of $J_3$ with eigenvalue $\hbar(m+s)$

$$J_3 A^{(j)}_m |j, s, E\rangle = A^{(j)}_m \left(\hbar m + J_3\right)|j, s, E\rangle = \hbar\,(m+s)\,A^{(j)}_m |j, s, E\rangle. \tag{10.182}$$

The $J_3$ eigenvalues of the tensor operator $A^{(j)}_m$ and the state $|j, s, E\rangle$ add.

## 10.27 Simple and semisimple Lie algebras

An **invariant subalgebra** is a set of generators $t^{(i)}_a$ whose commutator with every generator $t_b$ of the group is a linear combination of the generators $t^{(i)}_c$ of the invariant subalgebra

$$[t^{(i)}_a, t_b] = i f_{abc} t^{(i)}_c. \tag{10.183}$$

The whole algebra and the null algebra are trivial invariant subalgebras.

An algebra with no nontrivial invariant subalgebras is a **simple** algebra. A simple algebra generates a **simple group**. An algebra that has no nontrivial abelian invariant subalgebras is a **semisimple** algebra. A semisimple algebra generates a **semisimple group**.

**Example 10.27** (Some Simple Lie Groups) The groups of unitary matrices of unit determinant $SU(2)$, $SU(3)$, ... are simple. So are the groups of orthogonal matrices of unit determinant $SO(n)$ (except $SO(4)$, which is semisimple) and the groups of symplectic matrices $Sp(2n)$ (section 10.31). □

**Example 10.28** (Unification and Grand Unification) The symmetry group of the **standard model of particle physics** is a **direct product** of an $SU(3)$ group that acts on colored fields, an $SU(2)$ group that acts on **left-handed** quark and lepton fields, and a $U(1)$ group that acts on fields that carry hypercharge. Each of these three groups, is an invariant subgroup of the full symmetry group $SU(3)_c \otimes SU(2)_\ell \otimes U(1)_Y$, and the last one is abelian. Thus the symmetry group of the standard model is neither simple nor semisimple. In theories of **grand unification**, the strong and electroweak interactions unify at very high energies and are described by a simple group which makes all its charges simple multiples of each other. Georgi and Glashow suggested the group $SU(5)$ in 1976 (Howard Georgi, 1947–; Sheldon Glashow, 1932–). Others have proposed $SO(10)$ and even bigger groups. □

## 10.28 $SU(3)$

The Gell-Mann matrices are

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$
$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \text{and} \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \tag{10.184}$$

The generators $t_a$ of the $3 \times 3$ defining representation of $SU(3)$ are these Gell-Mann matrices divided by 2

$$t_a = \lambda_a/2 \tag{10.185}$$

(Murray Gell-Mann, 1929–).

The eight generators $t_a$ are orthogonal with $k = 1/2$

$$\mathrm{Tr}\,(t_a t_b) = \frac{1}{2}\delta_{ab} \tag{10.186}$$

and satisfy the commutation relation

$$[t_a, t_b] = i f_{abc}\, t_c. \tag{10.187}$$

The trace formula (10.72) gives us the ***SU*(3) structure constants** as

$$f_{abc} = \,-\,2i\,\mathrm{Tr}\left([t_a, t_b]t_c\right). \tag{10.188}$$

They are real and totally antisymmetric with $f_{123} = 1$, $f_{458} = f_{678} = \sqrt{3}/2$, and $f_{147} = -f_{156} = f_{246} = f_{257} = f_{345} = -f_{367} = 1/2$.

While no two generators of $SU(2)$ commute, two generators of $SU(3)$ do. In the representation (10.184,10.185), $t_3$ and $t_8$ are diagonal and so commute

$$[t_3, t_8] = 0. \tag{10.189}$$

They generate the **Cartan subalgebra** (section 10.30) of $SU(3)$.

## 10.29 $SU(3)$ and quarks

The generators defined by Eqs.(10.185 & 10.184) give us the $3 \times 3$ representation

$$D(\alpha) = \exp\left(i\alpha_a t_a\right) \tag{10.190}$$

in which the sum $a = 1, 2, \ldots 8$ is over the eight generators $t_a$. This representation acts on complex 3-vectors and is called the **3**.

Note that if

$$D(\alpha_1)D(\alpha_2) = D(\alpha_3) \tag{10.191}$$

then the complex conjugates of these matrices obey the same multiplication rule

$$D^*(\alpha_1)D^*(\alpha_2) = D^*(\alpha_3) \tag{10.192}$$

and so form another representation of $SU(3)$. It turns out that (unlike in $SU(2)$) this representation is inequivalent to the **3**; it is the $\overline{\mathbf{3}}$.

There are three quarks with masses less than about 100 MeV/c$^2$—the u, d, and s quarks. The other three quarks c, b, and t are more massive; $m_c = 1.28$ GeV, $m_b = 4.18$ GeV, and $m_t = 173.1$ GeV. Nobody knows why. Gell-Mann and Zweig suggested that the low-energy strong interactions were approximately invariant under unitary transformations of the three light quarks, which they represented by a **3**, and of the three light antiquarks, which they represented by a $\overline{\mathbf{3}}$. They imagined that the eight light pseudoscalar mesons, that is, the three pions $\pi^-$, $\pi^0$, $\pi^+$, the neutral $\eta$, and the four kaons $K^0$, $K^+$, $K^- \overline{K}^0$, were composed of a quark and an antiquark. So they should transform as the tensor product

$$\mathbf{3} \otimes \overline{\mathbf{3}} = \mathbf{8} \oplus \mathbf{1}. \tag{10.193}$$

They put the eight pseudoscalar mesons into an **8**.

They imagined that the eight light baryons — the two nucleons $N$ and $P$, the three sigmas $\Sigma^-$, $\Sigma^0$, $\Sigma^+$, the neutral lambda $\Lambda$, and the two cascades $\Xi^-$ and $\Xi^0$ were each made of three quarks. They should transform as the tensor product

$$\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = \mathbf{10} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{1}. \tag{10.194}$$

They put the eight light baryons into one of these **8**'s. When they were writing these papers, there were nine spin-3/2 resonances with masses somewhat heavier than 1200 MeV/c$^2$ — four $\Delta$'s, three $\Sigma^*$'s, and two $\Xi^*$'s. They put these into the **10** and predicted the tenth and its mass. In 1964, a tenth spin-3/2 resonance, the $\Omega^-$, was found with a mass close to their prediction of 1680 MeV/c$^2$, and by 1973 an MIT-SLAC team had discovered quarks inside protons and neutrons. (George Zweig, 1937–)

## 10.30 Cartan Subalgebra

In any Lie group, the maximum set of mutually commuting generators $H_a$ generate the **Cartan subalgebra**

$$[H_a, H_b] = 0 \tag{10.195}$$

which is an abelian subalgebra. The number of generators in the Cartan subalgebra is the **rank** of the Lie algebra. The Cartan generators $H_a$ can be simultaneously diagonalized, and their eigenvalues or diagonal elements are the **weights**

$$H_a|\mu, x, D\rangle = \mu_a|\mu, x, D\rangle \tag{10.196}$$

in which $D$ labels the representation and $x$ whatever other variables are needed to specify the state. The vector $\mu$ is the **weight vector**. The **roots** are the weights of the adjoint representation.

## 10.31 The Symplectic Group *Sp(2n)*

The real symplectic group $Sp(2n, \mathbb{R})$ is the group of linear transformations that preserve the canonical commutation relations of quantum mechanics

$$[q_i, p_k] = i\hbar\delta_{ik}, \quad [q_i, q_k] = 0, \quad \text{and} \quad [p_i, p_k] = 0 \tag{10.197}$$

for $i, k = 1, \ldots, n$. In terms of the $2n$ vector $v = (q_1, \ldots, q_n, p_1, \ldots, p_n)$ of quantum variables, these commutation relations are $[v_i, v_k] = i\hbar J_{ik}$ where $J$ is the $2n \times 2n$ real matrix

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \tag{10.198}$$

in which $I$ is the $n \times n$ identity matrix. The real linear transformation

$$v'_i = \sum_{\ell=1}^{2n} R_{i\ell}\, v_\ell \tag{10.199}$$

will preserve the quantum-mechanical commutation relations (10.197) if

$$[v'_i, v'_k] = \left[\sum_{\ell=1}^{2n} R_{i\ell}\, v_\ell, \sum_{m=1}^{2n} R_{km}\, v_m\right] = i\hbar \sum_{\ell,k=1}^{2n} R_{i\ell}\, J_{\ell m}\, R_{km} = i\hbar J_{ik} \tag{10.200}$$

which in matrix notation is just the condition

$$R\, J\, R^{\mathsf{T}} = J \tag{10.201}$$

that the matrix $R$ be in the real symplectic group $Sp(2n, \mathbb{R})$. The transpose and product rules (1.209 & 1.219) for determinants imply that $\det(R) = \pm 1$, but the condition (10.201) itself implies that $\det(R) = 1$ (Zee, 2016, p. 281).

In terms of the matrix $J$ and the hamiltonian $H(v) = H(q, p)$, Hamilton's equations have the symplectic form

$$\dot{q}_i = \frac{\partial H(q,p)}{\partial p_i} \quad \text{and} \quad \dot{p}_i = -\frac{\partial H(q,p)}{\partial q_i} \quad \text{are} \quad \frac{dv_i}{dt} = \sum_{\ell=1}^{2n} J_{i\ell}\, \frac{\partial H(v)}{\partial v_\ell}. \tag{10.202}$$

A matrix $R = e^t$ obeys the defining condition (10.201) if $t\, J = -J\, t^{\mathsf{T}}$ (exercise 10.22) or equivalently if $JtJ = t^{\mathsf{T}}$. It follows (exercise 10.23) that the generator $t$ must be

$$t = \begin{pmatrix} b & s_1 \\ s_2 & -b^{\mathsf{T}} \end{pmatrix} \tag{10.203}$$

in which the matrices $b, s_1, s_2$ are real, and both $s_1$ and $s_2$ are symmetric.

The group $Sp(2n, \mathbb{R})$ is noncompact.

**Example 10.29** (Squeezed states) A coherent state $|\alpha\rangle$ is an eigenstate of the annihilation operator $a = (\lambda q + ip/\lambda)/\sqrt{2\hbar}$ with a complex eigenvalue $\alpha$ (2.146), $a|\alpha\rangle = \alpha|\alpha\rangle$. In a coherent state with $\lambda = \sqrt{m\omega}$, the variances are $(\Delta q)^2 = \langle\alpha|(q - \bar{q})^2|\alpha\rangle = \hbar/(2m\omega)$ and $(\Delta p)^2 = \langle\alpha|(p - \bar{p})^2|\alpha\rangle = \hbar m\omega/2$. Thus coherent states have minimum uncertainty, $\Delta q\, \Delta p = \hbar/2$.

A squeezed state $|\alpha\rangle'$ is an eigenstate of $a' = (\lambda q' + ip'/\lambda)/\sqrt{2\hbar}$ in which $q'$ and $p'$ are related to $q$ and $p$ by an $Sp(2)$ transformation

$$\begin{pmatrix} q' \\ p' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix} \text{ with inverse } \begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} q' \\ p' \end{pmatrix} \tag{10.204}$$

in which $ad - bc = 1$. The standard deviations of the variables $q$ and $p$ in the squeezed state $|\alpha'\rangle$ are

$$\Delta q = \sqrt{\frac{\hbar}{2}} \sqrt{\frac{d^2}{m\omega} + m\omega\, b^2} \quad \text{and} \quad \Delta p = \sqrt{\frac{\hbar}{2}} \sqrt{\frac{c^2}{m\omega} + m\omega\, a^2}. \tag{10.205}$$

Thus by making $b$ and $d$ tiny, one can reduce the uncertainty $\Delta q$ by any factor, but then $\Delta p$ will increase by the same factor since the determinant of the $Sp(2)$ transformation must remain equal to unity, $ad - bc = 1$. □

**Example 10.30** ($Sp(2, \mathbb{R})$) The matrices (exercise 10.27)

$$T = \pm \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix} \tag{10.206}$$

are elements of the noncompact symplectic group $Sp(2, \mathbb{R})$. □

A dynamical map $\mathcal{M}$ that takes a $2n$ vector $v = (q_1, \ldots, q_n, p_1, \ldots, p_n)$ from $v(t_1)$ to $v(t_2)$ has a jacobian (section 1.21)

$$M_{ab} = \frac{\partial z_a(t_2)}{\partial z_b(t_1)} \tag{10.207}$$

in $Sp(2n, \mathbb{R})$ if and only if its dynamics are hamiltonian (10.202, section 17.1) (Carl Jacobi 1804–1851, William Hamilton 1805–1865).

The complex symplectic group $Sp(2n, \mathbb{C})$ consists of all $2n \times 2n$ complex matrices $C$ that satisfy the condition

$$C\, J\, C^{\mathsf{T}} = J. \tag{10.208}$$

The group $Sp(2n, \mathbb{C})$ also is noncompact.

The unitary symplectic group $USp(2n)$ consists of all $2n \times 2n$ complex unitary matrices $U$ that satisfy the condition

$$U\,J\,U^{\mathsf{T}} = J. \tag{10.209}$$

It is compact.

## 10.32 Quaternions

If $z$ and $w$ are any two complex numbers, then the $2 \times 2$ matrix

$$q = \begin{pmatrix} z & w \\ -w^* & z^* \end{pmatrix} \tag{10.210}$$

is a quaternion. The quaternions are closed under addition and multiplication and under multiplication by a real number (exercise 10.21), but not under multiplication by an arbitrary complex number. The squared norm of $q$ is its determinant

$$\|q\|^2 = |z|^2 + |w|^2 = \det q. \tag{10.211}$$

The matrix products $q^\dagger q$ and $q\,q^\dagger$ are the squared norm $\|q\|^2$ multiplied by the $2 \times 2$ identity matrix

$$q^\dagger q = q\,q^\dagger = \|q\|^2\, I \tag{10.212}$$

The $2 \times 2$ matrix

$$i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{10.213}$$

provides another expression for $\|q\|^2$ in terms of $q$ and its transpose $q^{\mathsf{T}}$

$$q^{\mathsf{T}} i\sigma_2\, q = \|q\|^2\, i\sigma_2. \tag{10.214}$$

Clearly $\|q\| = 0$ implies $q = 0$. The norm of a product of quaternions is the product of their norms

$$\|q_1 q_2\| = \sqrt{\det(q_1 q_2)} = \sqrt{\det q_1 \det q_2} = \|q_1\| \|q_2\|. \tag{10.215}$$

The quaternions therefore form an **associative division algebra** (over the real numbers); the only others are the real numbers and the complex numbers; the **octonions** are a nonassociative division algebra.

One may use the Pauli matrices to define for any real 4-vector $x$ a quaternion $q(x)$ as

$$\begin{aligned} q(x) &= x_0 + i\,\sigma_k\, x_k = x_0 + i\,\boldsymbol{\sigma} \cdot \boldsymbol{x} \\ &= \begin{pmatrix} x_0 + i\,x_3 & x_2 + i\,x_1 \\ -\,x_2 + i\,x_1 & x_0 - i\,x_3 \end{pmatrix} \end{aligned} \tag{10.216}$$

with squared norm

$$\|q(x)\|^2 = x_0^2 + x_1^2 + x_2^2 + x_3^2. \tag{10.217}$$

The product rule (10.112) for the Pauli matrices tells us that the product of two quaternions is

$$\begin{aligned} q(x)\, q(y) = &\ (x_0 + i\,\boldsymbol{\sigma}\cdot\boldsymbol{x})(y_0 + i\,\boldsymbol{\sigma}\cdot\boldsymbol{y}) \\ = &\ x_0 y_0 + i\,\boldsymbol{\sigma}\cdot(y_0\,\boldsymbol{x} + x_0\,\boldsymbol{y}) - i\,(\boldsymbol{x}\times\boldsymbol{y})\cdot\boldsymbol{\sigma} - \boldsymbol{x}\cdot\boldsymbol{y} \end{aligned} \tag{10.218}$$

so their commutator is

$$[q(x), q(y)] = -2\,i\,(\boldsymbol{x}\times\boldsymbol{y})\cdot\boldsymbol{\sigma}. \tag{10.219}$$

**Example 10.31** (Lack of Analyticity) One may define a function $f(q)$ of a quaternionic variable and then ask what functions are analytic in the sense that the (one-sided) derivative

$$f'(q) = \lim_{q'\to 0} \left[f(q+q') - f(q)\right] q'^{-1} \tag{10.220}$$

exists and is independent of the direction through which $q' \to 0$. This space of functions is extremely limited and does not even include the function $f(q) = q^2$ (exercise 10.24). □

## 10.33 Quaternions and symplectic groups

This section is optional on a first reading.

One may regard the unitary symplectic group $USp(2n)$ as made of $2n\times 2n$ matrices $W$ that map $n$-tuples $q$ of quaternions into $n$-tuples $q' = Wq$ of quaternions with the same value of the quadratic quaternionic form

$$\|q'\|^2 = \|q'_1\|^2 + \|q'_2\|^2 + \ldots + \|q'_n\|^2 = \|q_1\|^2 + \|q_2\|^2 + \ldots + \|q_n\|^2 = \|q\|^2. \tag{10.221}$$

By (10.212), the quadratic form $\|q'\|^2$ times the $2\times 2$ identity matrix $I$ is equal to the hermitian form $q'^\dagger q'$

$$\|q'\|^2\, I = q'^\dagger q' = q_1'^\dagger q_1' + \ldots + q_n'^\dagger q_n' = q^\dagger W^\dagger W q \tag{10.222}$$

and so any matrix $W$ that is both a $2n\times 2n$ unitary matrix and an $n\times n$ matrix of quaternions keeps $\|q'\|^2 = \|q\|^2$

$$\|q'\|^2\, I = q^\dagger W^\dagger W q = q^\dagger q = \|q\|^2\, I. \tag{10.223}$$

The group $USp(2n)$ thus consists of all $2n\times 2n$ unitary matrices that also are $n \times n$ matrices of quaternions. (This last requirement is needed so that $q' = Wq$ is an $n$-tuple of quaternions.)

The generators $t_a$ of the symplectic group $USp(2n)$ are $2n \times 2n$ direct-product matrices of the form

$$I \otimes A, \quad \sigma_1 \otimes S_1, \quad \sigma_2 \otimes S_2, \quad \text{and} \quad \sigma_3 \otimes S_3 \tag{10.224}$$

in which $I$ is the $2 \times 2$ identity matrix, the three $\sigma_i$'s are the Pauli matrices, $A$ is an imaginary $n \times n$ anti-symmetric matrix, and the $S_i$ are $n \times n$ real symmetric matrices. These generators $t_a$ close under commutation

$$[t_a, t_b] = i f_{abc} t_c. \tag{10.225}$$

Any imaginary linear combination $i\alpha_a t_a$ of these generators is not only a $2n \times 2n$ antihermitian matrix but also an $n \times n$ matrix of quaternions. Thus the matrices

$$D(\alpha) = e^{i\alpha_a t_a} \tag{10.226}$$

are both unitary $2n \times 2n$ matrices and $n \times n$ quaternionic matrices and so are elements of the group $Sp(2n)$.

**Example 10.32** ($USp(2) \cong SU(2)$) There is no $1 \times 1$ anti-symmetric matrix, and there is only one $1 \times 1$ symmetric matrix. So the generators $t_a$ of the group $Sp(2)$ are the Pauli matrices $t_a = \sigma_a$, and $Sp(2) = SU(2)$. The elements $g(\alpha)$ of $SU(2)$ are quaternions of unit norm (exercise 10.20), and so the product $g(\alpha)q$ is a quaternion

$$\|g(\alpha)q\|^2 = \det(g(\alpha)q) = \det(g(\alpha)) \det q = \det q = \|q\|^2 \tag{10.227}$$

with the same squared norm. □

**Example 10.33** ($SO(4) \cong SU(2) \otimes SU(2)$) If $g$ and $h$ are any two elements of the group $SU(2)$, then the squared norm (10.217) of the quaternion $q(x) = x_0 + i\boldsymbol{\sigma} \cdot \boldsymbol{x}$ is invariant under the transformation $q(x') = g\, q(x)\, h^{-1}$, that is, $x_0'^2 + x_1'^2 + x_2'^2 + x_3'^2 = x_0^2 + x_1^2 + x_2^2 + x_3^2$. So $x \to x'$ is an $SO(4)$ rotation of the 4-vector $x$. The Lie algebra of $SO(4)$ thus contains two commuting invariant $SU(2)$ subalgebras and so is semisimple. □

**Example 10.34** ($USp(4) \cong SO(5)$) Apart from scale factors, there are three real symmetric $2 \times 2$ matrices $S_1 = \sigma_1$, $S_2 = I$, and $S_3 = \sigma_3$ and one imaginary anti-symmetric $2 \times 2$ matrix $A = \sigma_2$. So there are 10 generators of $USp(4) = SO(5)$

$$t_1 = I \otimes \sigma_2 = \begin{pmatrix} 0 & -iI \\ iI & 0 \end{pmatrix}, \quad t_{k1} = \sigma_k \otimes \sigma_1 = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix}$$
$$t_{k2} = \sigma_k \otimes I = \begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix}, \quad t_{k3} = \sigma_k \otimes \sigma_3 = \begin{pmatrix} \sigma_k & 0 \\ 0 & -\sigma_k \end{pmatrix} \tag{10.228}$$

where $k$ runs from 1 to 3. □

Another way of looking at $USp(2n)$ is to use (10.214) to write the quadratic form $\|q\|^2$ as

$$\|q\|^2 \mathcal{J} = q^{\mathsf{T}} \mathcal{J} q \tag{10.229}$$

in which the $2n \times 2n$ matrix $\mathcal{J}$ has $n$ copies of $i\sigma_2$ on its $2 \times 2$ diagonal

$$\mathcal{J} = \begin{pmatrix} i\sigma_2 & 0 & 0 & 0 & \dots & 0 \\ 0 & i\sigma_2 & 0 & 0 & \dots & 0 \\ 0 & 0 & i\sigma_2 & 0 & \dots & 0 \\ 0 & 0 & 0 & i\sigma_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & i\sigma_2 \end{pmatrix} \tag{10.230}$$

and is the matrix $J$ (10.198) in a different basis. Thus any $n \times n$ matrix of quaternions $W$ that satisfies

$$W^{\mathsf{T}} \mathcal{J} W = \mathcal{J} \tag{10.231}$$

also satisfies

$$\|Wq\|^2 \mathcal{J} = q^{\mathsf{T}} W^{\mathsf{T}} \mathcal{J} W q = q^{\mathsf{T}} \mathcal{J} q = \|q\|^2 \mathcal{J} \tag{10.232}$$

and so leaves invariant the quadratic form (10.221). The group $USp(2n)$ therefore consists of all $2n \times 2n$ matrices $W$ that satisfy (10.231) and that also are $n \times n$ matrices of quaternions.

## 10.34 Compact Simple Lie Groups

Élie Cartan (1869–1951) showed that all compact, simple Lie groups fall into four infinite classes and five discrete cases. For $n = 1, 2, \dots$, his four classes are

- $A_n = SU(n+1)$ which are $(n+1) \times (n+1)$ unitary matrices with unit determinant,

- $B_n = SO(2n+1)$ which are $(2n+1) \times (2n+1)$ orthogonal matrices with unit determinant,
- $C_n = Sp(2n)$ which are $2n \times 2n$ symplectic matrices, and
- $D_n = SO(2n)$ which are $2n \times 2n$ orthogonal matrices with unit determinant.

The five discrete cases are the **exceptional groups** $G_2$, $F_4$, $E_6$, $E_7$, and $E_8$.

The exceptional groups are associated with the **octonians**

$$a + b_\alpha i_\alpha \tag{10.233}$$

where the $\alpha$-sum runs from 1 to 7; the eight numbers $a$ and $b_\alpha$ are real; and the seven $i_\alpha$'s obey the multiplication law

$$i_\alpha \, i_\beta = -\delta_{\alpha\beta} + g_{\alpha\beta\gamma} \, i_\gamma \tag{10.234}$$

in which $g_{\alpha\beta\gamma}$ is totally antisymmetric with

$$g_{123} = g_{247} = g_{451} = g_{562} = g_{634} = g_{375} = g_{716} = 1. \tag{10.235}$$

Like the quaternions and the complex numbers, the octonians form a **division algebra** with an absolute value

$$|a + b_\alpha i_\alpha| = \left(a^2 + b_\alpha^2\right)^{1/2} \tag{10.236}$$

that satisfies

$$|AB| = |A||B| \tag{10.237}$$

but they lack associativity.

The group $G_2$ is the subgroup of $SO(7)$ that leaves the $g_{\alpha\beta\gamma}$'s of (10.234) invariant.

## 10.35 Group Integration

Suppose we need to integrate some function $f(g)$ over a group. Naturally, we want to do so in a way that gives equal weight to every element of the group. In particular, if $g'$ is any group element, we want the integral of the shifted function $f(g'g)$ to be the same as the integral of $f(g)$

$$\int f(g) \, dg = \int f(g'g) \, dg. \tag{10.238}$$

Such a measure $dg$ is said to be **left invariant** (Creutz, 1983, chap. 8).

Let's use the letters $a = a_1, \dots, a_n$, $b = b_1, \dots, b_n$, and so forth to label the elements $g(a)$, $g(b)$, so that an integral over the group is

$$\int f(g)\, dg = \int f(g(a))\, m(a)\, d^n a \tag{10.239}$$

in which $m(a)$ is the left-invariant measure and the integration is over the $n$-space of $a$'s that label all the elements of the group.

To find the left-invariant measure $m(a)$, we use the multiplication law of the group

$$g(a(c,b)) \equiv g(c)\, g(b) \tag{10.240}$$

and impose the requirement (10.238) of left invariance with $g' \equiv g(c)$

$$\int f(g(b))\, m(b)\, d^n b = \int f(g(c)g(b))\, m(b)\, d^n b = \int f(g(a(c,b)))\, m(b)\, d^n b. \tag{10.241}$$

We change variables from $b$ to $a = a(c,b)$ by using the jacobian $\det(\partial b/\partial a)$ which gives us $d^n b = \det(\partial b/\partial a)\, d^n a$

$$\int f(g(b))\, m(b)\, d^n b = \int f(g(a))\, \det(\partial b/\partial a)\, m(b)\, d^n a. \tag{10.242}$$

Replacing $b$ by $a = a(c,b)$ on the left-hand side of this equation, we find

$$m(a) = \det(\partial b/\partial a)\, m(b) \tag{10.243}$$

or since $\det(\partial b/\partial a) = 1/\det(\partial a(c,b)/\partial b)$

$$m(a(c,b)) = m(b)/\det(\partial a(c,b)/\partial b). \tag{10.244}$$

So if we let $g(b) \to g(0) = e$, the identity element of the group, and set $m(e) = 1$, then we find for the measure

$$m(a) = m(c) = m(a(c,b))|_{b=0} = 1/\det(\partial a(c,b)/\partial b)|_{b=0}\,. \tag{10.245}$$

**Example 10.35** (The Invariant Measure for SU(2)) A general element of the group $SU(2)$ is given by (10.115) as

$$\exp\left(i\, \boldsymbol{\theta} \cdot \frac{\boldsymbol{\sigma}}{2}\right) = I\, \cos\frac{\theta}{2} + i\, \hat{\boldsymbol{\theta}} \cdot \boldsymbol{\sigma}\, \sin\frac{\theta}{2}. \tag{10.246}$$

Setting $a_0 = \cos(\theta/2)$ and $\boldsymbol{a} = \hat{\boldsymbol{\theta}}\, \sin(\theta/2)$, we have

$$g(a) = a_0 + i\, \boldsymbol{a} \cdot \boldsymbol{\sigma} \tag{10.247}$$

in which $a^2 \equiv a_0^2 + \boldsymbol{a} \cdot \boldsymbol{a} = 1$. Thus, the parameter space for $SU(2)$ is the unit sphere $S_3$ in four dimensions. Its invariant measure is

$$\int \delta(1-a^2)\, d^4a = \int \delta(1-a_0^2-\boldsymbol{a}^2)\, d^4a = \int (1-\boldsymbol{a}^2)^{-1/2}\, d^3a \tag{10.248}$$

or

$$m(\boldsymbol{a}) = (1-\boldsymbol{a}^2)^{-1/2} = \frac{1}{|\cos(\theta/2)|}. \tag{10.249}$$

We also can write the arbitrary element (10.247) of $SU(2)$ as

$$g(\boldsymbol{a}) = \pm\sqrt{1-\boldsymbol{a}^2} + i\,\boldsymbol{a}\cdot\boldsymbol{\sigma} \tag{10.250}$$

and the group-multiplication law (10.240) as

$$\sqrt{1-\boldsymbol{a}^2} + i\,\boldsymbol{a}\cdot\boldsymbol{\sigma} = \left(\sqrt{1-\boldsymbol{c}^2} + i\,\boldsymbol{c}\cdot\boldsymbol{\sigma}\right)\left(\sqrt{1-\boldsymbol{b}^2} + i\,\boldsymbol{b}\cdot\boldsymbol{\sigma}\right). \tag{10.251}$$

Thus, by multiplying both sides of this equation by $\sigma_i$ and taking the trace, we find (exercise 10.28) that the parameters $\boldsymbol{a}(\boldsymbol{c},\boldsymbol{b})$ that describe the product $g(\boldsymbol{c})\,g(\boldsymbol{b})$ are

$$\boldsymbol{a}(\boldsymbol{c},\boldsymbol{b}) = \sqrt{1-\boldsymbol{c}^2}\,\boldsymbol{b} + \sqrt{1-\boldsymbol{b}^2}\,\boldsymbol{c} - \boldsymbol{c}\times\boldsymbol{b}. \tag{10.252}$$

To compute the jacobian of our formula (10.245) for the invariant measure, we differentiate this expression (10.252) at $\boldsymbol{b}=\boldsymbol{0}$ and so find (exercise 10.29)

$$m(\boldsymbol{a}) = 1/\det(\partial a(c,b)/\partial b)|_{b=0} = (1-\boldsymbol{a}^2)^{-1/2} \tag{10.253}$$

as the left-invariant measure in agreement with (10.249). □

## 10.36 The Lorentz group

The Lorentz group $O(3,1)$ is the set of all linear transformations $L$ that leave invariant the Minkowski inner product

$$xy \equiv \boldsymbol{x}\cdot\boldsymbol{y} - x^0y^0 = x^{\mathsf{T}}\eta y \tag{10.254}$$

in which $\eta$ is the diagonal matrix

$$\eta = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{10.255}$$

So $L$ is in $O(3,1)$ if for all 4-vectors $x$ and $y$

$$(Lx)^{\mathsf{T}}\,\eta L\,y = x^{\mathsf{T}}L^{\mathsf{T}}\,\eta\,Ly = x^{\mathsf{T}}\eta\,y. \tag{10.256}$$

Since $x$ and $y$ are arbitrary, this condition amounts to

$$L^{\mathsf{T}}\eta\, L = \eta. \tag{10.257}$$

Taking the determinant of both sides and using the transpose (1.209) and product (1.222) rules, we have

$$(\det L)^2 = 1. \tag{10.258}$$

So $\det L = \pm 1$, and every Lorentz transformation $L$ has an inverse. Multiplying (10.257) by $\eta$, we get

$$\eta L^{\mathsf{T}}\eta L = \eta^2 = I \tag{10.259}$$

which identifies $L^{-1}$ as

$$L^{-1} = \eta L^{\mathsf{T}}\eta. \tag{10.260}$$

The subgroup of $O(3,1)$ with $\det L = 1$ is the proper Lorentz group $SO(3,1)$. The subgroup of $SO(3,1)$ that leaves invariant the sign of the time component of timelike vectors is the **proper orthochronous Lorentz group** $SO^+(3,1)$.

To find the Lie algebra of $SO^+(3,1)$, we take a Lorentz matrix $L = I + \omega$ that differs from the identity matrix $I$ by a tiny matrix $\omega$ and require $L$ to obey the condition (10.257) for membership in the Lorentz group

$$\left(I + \omega^{\mathsf{T}}\right)\eta\,(I + \omega) = \eta + \omega^{\mathsf{T}}\eta + \eta\,\omega + \omega^{\mathsf{T}}\omega = \eta. \tag{10.261}$$

Neglecting $\omega^{\mathsf{T}}\omega$, we have $\omega^{\mathsf{T}}\eta = -\eta\,\omega$ or since $\eta^2 = I$

$$\omega^{\mathsf{T}} = -\,\eta\,\omega\,\eta. \tag{10.262}$$

This equation says (exercise 10.31) that under transposition the time-time and space-space elements of $\omega$ change sign, while the time-space and space-time elements do not. That is, the tiny matrix $\omega$ is for infinitesimal $\boldsymbol{\theta}$ and $\boldsymbol{\lambda}$ a linear combination

$$\omega = \boldsymbol{\theta}\cdot\boldsymbol{R} + \boldsymbol{\lambda}\cdot\boldsymbol{B} \tag{10.263}$$

of three antisymmetric space-space matrices

$$R_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad R_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad R_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{10.264}$$

and of three symmetric time-space matrices

$$B_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad B_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad B_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{10.265}$$

all of which satisfy condition (10.262). The three $R_\ell$ are $4 \times 4$ versions of the rotation generators (10.92); the three $B_\ell$ generate Lorentz boosts.

If we write $L = I + \omega$ as

$$L = I - i\theta_\ell \, iR_\ell - i\lambda_\ell \, iB_\ell \equiv I - i\theta_\ell J_\ell - i\lambda_\ell K_\ell \tag{10.266}$$

then the three matrices $J_\ell = iR_\ell$ are imaginary and antisymmetric, and therefore hermitian. But the three matrices $K_\ell = iB_\ell$ are imaginary and symmetric, and so are antihermitian. The $4\times4$ matrix $L = \exp(i\theta_\ell J_\ell - i\lambda_\ell K_\ell)$ is **not unitary** because the Lorentz group is **not compact**.

One may verify (exercise 10.32) that the six generators $J_\ell$ and $K_\ell$ satisfy three sets of commutation relations:

$$[J_i, J_j] = i\epsilon_{ijk} J_k \tag{10.267}$$

$$[J_i, K_j] = i\epsilon_{ijk} K_k \tag{10.268}$$

$$[K_i, K_j] = -\, i\epsilon_{ijk} J_k. \tag{10.269}$$

The first (10.267) says that the three $J_\ell$ generate the rotation group $SO(3)$; the second (10.268) says that the three boost generators transform as a 3-vector under $SO(3)$; and the third (10.269) implies that four canceling infinitesimal boosts can amount to a rotation. These three sets of commutation relations form the Lie algebra of the Lorentz group $SO(3,1)$. Incidentally, one may show (exercise 10.33) that if $\boldsymbol{J}$ and $\boldsymbol{K}$ satisfy these commutation relations (10.267–10.269), then so do

$$\boldsymbol{J} \quad \text{and} \quad -\boldsymbol{K}. \tag{10.270}$$

The infinitesimal Lorentz transformation (10.266) is the $4 \times 4$ matrix

$$L = I + \omega = I + \theta_\ell R_\ell + \lambda_\ell B_\ell = \begin{pmatrix} 1 & \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1 & 1 & -\theta_3 & \theta_2 \\ \lambda_2 & \theta_3 & 1 & -\theta_1 \\ \lambda_3 & -\theta_2 & \theta_1 & 1 \end{pmatrix}. \tag{10.271}$$

It moves any 4-vector $x$ to $x' = L\,x$ or in components $x'^a = L^a{}_b\, x^b$

$$
\begin{aligned}
x'^0 &= x^0 + \lambda_1 x^1 + \lambda_2 x^2 + \lambda_3 x^3 \\
x'^1 &= \lambda_1 x^0 + x^1 - \theta_3 x^2 + \theta_2 x^3 \\
x'^2 &= \lambda_2 x^0 + \theta_3 x^1 + x^2 - \theta_1 x^3 \\
x'^3 &= \lambda_3 x^0 - \theta_2 x^1 + \theta_1 x^2 + x^3.
\end{aligned} \tag{10.272}
$$

More succinctly with $t = x^0$, this is

$$
\begin{aligned}
t' &= t + \boldsymbol{\lambda} \cdot \boldsymbol{x} \\
\boldsymbol{x}' &= \boldsymbol{x} + t\boldsymbol{\lambda} + \boldsymbol{\theta} \wedge \boldsymbol{x}
\end{aligned} \tag{10.273}
$$

in which $\wedge \equiv \times$ means cross-product.

For arbitrary real $\boldsymbol{\theta}$ and $\boldsymbol{\lambda}$, the matrices

$$
L = e^{-i\boldsymbol{\theta}\cdot\boldsymbol{J} - i\boldsymbol{\lambda}\cdot\boldsymbol{K}} \tag{10.274}
$$

form the subgroup of $O(3,1)$ that is connected to the identity matrix $I$. The matrices of this subgroup have unit determinant and preserve the sign of the time of time-like vectors, that is, if $x^2 < 0$, and $y = Lx$, then $y^0 x^0 > 0$. This is the proper orthochronous Lorentz group $SO^+(3,1)$. The rest of the (homogeneous) Lorentz group can be obtained from it by space $\mathcal{P}$, time $\mathcal{T}$, and spacetime $\mathcal{PT}$ reflections.

The task of finding all the finite-dimensional irreducible representations of the proper orthochronous homogeneous Lorentz group becomes vastly simpler when we write the commutation relations (10.267–10.269) in terms of the hermitian matrices

$$
J^\pm_\ell = \frac{1}{2}\left(J_\ell \pm i K_\ell\right) \tag{10.275}
$$

which generate two independent rotation groups

$$
\begin{aligned}
[J^+_i, J^+_j] &= i\epsilon_{ijk} J^+_k \\
[J^-_i, J^-_j] &= i\epsilon_{ijk} J^-_k \\
[J^+_i, J^-_j] &= 0.
\end{aligned} \tag{10.276}
$$

Thus the Lie algebra of the Lorentz group is equivalent to two copies of the Lie algebra (10.117) of $SU(2)$.

The hermitian generators of the rotation subgroup $SU(2)$ are by (10.275)

$$
\boldsymbol{J} = \boldsymbol{J}^+ + \boldsymbol{J}^-. \tag{10.277}
$$

The antihermitian generators of the boosts are (also by 10.275)

$$
\boldsymbol{K} = -i\left(\boldsymbol{J}^+ - \boldsymbol{J}^-\right). \tag{10.278}
$$

Since $\boldsymbol{J}^+$ and $\boldsymbol{J}^-$ commute, the finite-dimensional irreducible representations of the Lorentz group are the direct products

$$\begin{aligned} D(j,j')(\boldsymbol{\theta},\boldsymbol{\lambda}) = e^{-i\boldsymbol{\theta}\cdot\boldsymbol{J}-i\boldsymbol{\lambda}\cdot\boldsymbol{K}} &= e^{(-i\boldsymbol{\theta}-\boldsymbol{\lambda})\cdot\boldsymbol{J}^+ + (-i\boldsymbol{\theta}+\boldsymbol{\lambda})\cdot\boldsymbol{J}^-} \\ &= e^{(-i\boldsymbol{\theta}-\boldsymbol{\lambda})\cdot\boldsymbol{J}^+}\, e^{(-i\boldsymbol{\theta}+\boldsymbol{\lambda})\cdot\boldsymbol{J}^-} \end{aligned} \tag{10.279}$$

of the nonunitary representations

$$D^{(j,0)}(\boldsymbol{\theta},\boldsymbol{\lambda}) = e^{(-i\boldsymbol{\theta}-\boldsymbol{\lambda})\cdot\boldsymbol{J}^+} \quad \text{and} \quad D^{(0,j')}(\boldsymbol{\theta},\boldsymbol{\lambda}) = e^{(-i\boldsymbol{\theta}+\boldsymbol{\lambda})\cdot\boldsymbol{J}^-} \tag{10.280}$$

generated by the three $(2j+1)\times(2j+1)$ matrices $J_\ell^+$ and by the three $(2j'+1)\times(2j'+1)$ matrices $J_\ell^-$.

Under a Lorentz transformation $L$, a field $\psi^{(j,j')}_{m,m'}(x)$ that transforms under the $D^{(j,j')}$ representation of the Lorentz group responds as

$$U(L)\,\psi^{(j,j')}_{m,m'}(x)\,U^{-1}(L) = D^{(j,0)}_{mm''}(L^{-1})\,D^{(0,j')}_{m'm'''}(L^{-1})\,\psi^{(j,j')}_{m'',m'''}(Lx). \tag{10.281}$$

The representation $D^{(j,j')}$ describes objects of the spins $s$ that can arise from the direct product of spin-$j$ with spin-$j'$ (Weinberg, 1995, p. 231)

$$s = j+j',\, j+j'-1, \ldots, |j-j'|. \tag{10.282}$$

For instance, $D^{(0,0)}$ describes a spinless field or particle, while $D^{(1/2,0)}$ and $D^{(0,1/2)}$ respectively describe left-handed and right-handed spin-1/2 fields or particles. The representation $D^{(1/2,1/2)}$ describes objects of spin 1 and spin 0—the spatial and time components of a 4-vector. The interchange of $\boldsymbol{J}^+$ and $\boldsymbol{J}^-$ replaces the generators $\boldsymbol{J}$ and $\boldsymbol{K}$ with $\boldsymbol{J}$ and $-\boldsymbol{K}$, a substitution that we know (10.270) is legitimate.

## 10.37 Two-dimensional left-handed representation of the Lorentz group

The generators of the representation $D^{(1/2,0)}$ with $j = 1/2$ and $j' = 0$ are given by (10.277 & 10.278) with $\boldsymbol{J}^+ = \boldsymbol{\sigma}/2$ and $\boldsymbol{J}^- = 0$. They are

$$\boldsymbol{J} = \frac{1}{2}\boldsymbol{\sigma} \quad \text{and} \quad \boldsymbol{K} = -i\frac{1}{2}\boldsymbol{\sigma}. \tag{10.283}$$

The $2\times2$ matrix $D^{(1/2,0)}$ that represents the Lorentz transformation (10.274)

$$L = e^{-i\boldsymbol{\theta}\cdot\boldsymbol{J}-i\boldsymbol{\lambda}\cdot\boldsymbol{K}} \tag{10.284}$$

is

$$D^{(1/2,0)}(\boldsymbol{\theta},\boldsymbol{\lambda}) = \exp\left(-i\boldsymbol{\theta}\cdot\boldsymbol{\sigma}/2 - \boldsymbol{\lambda}\cdot\boldsymbol{\sigma}/2\right). \tag{10.285}$$

And so the generic $D^{(1/2,0)}$ matrix is

$$D^{(1/2,0)}(\boldsymbol{\theta}, \boldsymbol{\lambda}) = e^{-\boldsymbol{z}\cdot\boldsymbol{\sigma}/2} \tag{10.286}$$

with $\boldsymbol{\lambda} = \mathrm{Re}\boldsymbol{z}$ and $\boldsymbol{\theta} = \mathrm{Im}\boldsymbol{z}$. It is nonunitary and of unit determinant; it is a member of the group $SL(2,C)$ of complex unimodular $2\times 2$ matrices. The (covering) group $SL(2,C)$ relates to the Lorentz group $SO(3,1)$ as $SU(2)$ relates to the rotation group $SO(3)$.

**Example 10.36** (The standard left-handed boost) For a particle of mass $m > 0$, the standard boost that takes the 4-vector $k = (m, \mathbf{0})$ to $p = (p^0, \boldsymbol{p})$, where $p^0 = \sqrt{m^2 + \boldsymbol{p}^2}$ is a boost in the $\hat{\boldsymbol{p}}$ direction. It is the $4 \times 4$ matrix

$$B(p) = R(\hat{\boldsymbol{p}})\, B_3(p^0)\, R^{-1}(\hat{\boldsymbol{p}}) = \exp\left(\alpha\, \hat{\boldsymbol{p}} \cdot \boldsymbol{B}\right) \tag{10.287}$$

in which $\cosh\alpha = p^0/m$ and $\sinh\alpha = |\boldsymbol{p}|/m$, as one may show by expanding the exponential (exercise 10.35). This standard boost is represented by $D^{(1/2,0)}(\mathbf{0}, \boldsymbol{\lambda})$, the $2 \times 2$ matrix (10.284), with $\boldsymbol{\lambda} = \alpha\, \hat{\boldsymbol{p}}$. The power-series expansion of this matrix is (exercise 10.36)

$$\begin{aligned} D^{(1/2,0)}(\mathbf{0}, \alpha\, \hat{\boldsymbol{p}}) &= e^{-\alpha\hat{\boldsymbol{p}}\cdot\boldsymbol{\sigma}/2} = I\, \cosh(\alpha/2) - \hat{\boldsymbol{p}} \cdot \boldsymbol{\sigma}\, \sinh(\alpha/2) \\ &= I\, \sqrt{(p^0 + m)/(2m)} - \hat{\boldsymbol{p}} \cdot \boldsymbol{\sigma}\, \sqrt{(p^0 - m)/(2m)} \\ &= \frac{(p^0 + m)I - \boldsymbol{p} \cdot \boldsymbol{\sigma}}{\sqrt{2m(p^0 + m)}} \end{aligned} \tag{10.288}$$

in which $I$ is the $2 \times 2$ identity matrix. □

Under $D^{(1/2,0)}$, the vector $(-I, \boldsymbol{\sigma})$ transforms like a 4-vector. For tiny $\boldsymbol{\theta}$ and $\boldsymbol{\lambda}$, one may show (exercise 10.38) that the vector $(-I, \boldsymbol{\sigma})$ transforms as

$$\begin{aligned} D^{\dagger(1/2,0)}(\boldsymbol{\theta}, \boldsymbol{\lambda})(-I)D^{(1/2,0)}(\boldsymbol{\theta}, \boldsymbol{\lambda}) &= -I + \boldsymbol{\lambda} \cdot \boldsymbol{\sigma} \\ D^{\dagger(1/2,0)}(\boldsymbol{\theta}, \boldsymbol{\lambda})\, \boldsymbol{\sigma}\, D^{(1/2,0)}(\boldsymbol{\theta}, \boldsymbol{\lambda}) &= \boldsymbol{\sigma} + (-I)\boldsymbol{\lambda} + \boldsymbol{\theta} \wedge \boldsymbol{\sigma} \end{aligned} \tag{10.289}$$

which is how the 4-vector $(t, \boldsymbol{x})$ transforms (10.273). Under a finite Lorentz transformation $L$, the 4-vector $S^a \equiv (-I, \boldsymbol{\sigma})$ goes to

$$D^{\dagger(1/2,0)}(L)\, S^a\, D^{(1/2,0)}(L) = L^a{}_b S^b. \tag{10.290}$$

A massless field $u(x)$ that responds to a unitary Lorentz transformation $U(L)$ like

$$U(L)\, u(x)\, U^{-1}(L) = D^{(1/2,0)}(L^{-1})\, u(Lx) \tag{10.291}$$

is called a **left-handed Weyl spinor**. Its action density

$$\mathcal{L}_\ell(x) = i\, u^\dagger(x)\, (\partial_0 I - \boldsymbol{\nabla} \cdot \boldsymbol{\sigma})\, u(x) \tag{10.292}$$

is Lorentz covariant, that is

$$U(L)\,\mathcal{L}_\ell(x)\,U^{-1}(L) = \mathcal{L}_\ell(Lx). \tag{10.293}$$

**Example 10.37** (Why $\mathcal{L}_\ell$ is Lorentz covariant) We first note that the derivatives $\partial'_b$ in $\mathcal{L}_\ell(Lx)$ are with respect to $x' = Lx$. Since the inverse matrix $L^{-1}$ takes $x'$ back to $x = L^{-1}x'$ or in tensor notation $x^a = L^{-1a}{}_b\, x'^b$, the derivative $\partial'_b$ is

$$\partial'_b = \frac{\partial}{\partial x'^b} = \frac{\partial x^a}{\partial x'^b}\frac{\partial}{\partial x^a} = L^{-1a}{}_b\,\frac{\partial}{\partial x^a} = \partial_a\, L^{-1a}{}_b. \tag{10.294}$$

Now using the abbreviation $\partial_0 I - \boldsymbol{\nabla}\cdot\boldsymbol{\sigma} \equiv -\,\partial_a S^a$ and the transformation laws (10.290 & 10.291), we have

$$\begin{aligned} U(L)\,\mathcal{L}_\ell(x)\,U^{-1}(L) &= i\,u^\dagger(Lx) D^{(1/2,0)\dagger}(L^{-1})(-\,\partial_a S^a) D^{(1/2,0)}(L^{-1})\,u(Lx) \\ &= i\,u^\dagger(Lx)(-\,\partial_a L^{-1a}{}_b S^b)\,u(Lx) \\ &= i\,u^\dagger(Lx)(-\,\partial'_b S^b)\,u(Lx) = \mathcal{L}_\ell(Lx) \end{aligned} \tag{10.295}$$

which shows that $\mathcal{L}_\ell$ is Lorentz covariant. □

Incidentally, the rule (10.294) ensures, among other things, that the divergence $\partial_a V^a$ is invariant

$$(\partial_a V^a)' = \partial'_a V'^a = \partial_b\, L^{-1b}{}_a\, L^a{}_c V^c = \partial_b\, \delta^b{}_c\, V^c = \partial_b\, V^b. \tag{10.296}$$

**Example 10.38** (Why $u$ is left handed) The spacetime integral $S$ of the action density $\mathcal{L}_\ell$ is stationary when $u(x)$ satisfies the wave equation

$$(\partial_0 I - \boldsymbol{\nabla}\cdot\boldsymbol{\sigma})\,u(x) = 0 \tag{10.297}$$

or in momentum space

$$(E + \boldsymbol{p}\cdot\boldsymbol{\sigma})\,u(p) = 0. \tag{10.298}$$

Multiplying from the left by $(E - \boldsymbol{p}\cdot\boldsymbol{\sigma})$, we see that the energy of a particle created or annihilated by the field $u$ is the same as its momentum, $E = |\boldsymbol{p}|$. The particles of the field $u$ are massless because the action density $\mathcal{L}_\ell$ has no mass term. The spin of the particle is represented by the matrix $\boldsymbol{J} = \boldsymbol{\sigma}/2$, so the momentum-space relation (10.298) says that $u(p)$ is an eigenvector of $\hat{\boldsymbol{p}}\cdot\boldsymbol{J}$ with eigenvalue $-\,1/2$

$$\hat{\boldsymbol{p}}\cdot\boldsymbol{J}\,u(p) = -\,\frac{1}{2}\,u(p). \tag{10.299}$$

A particle whose spin is opposite to its momentum is said to have **negative helicity** or to be **left handed**. Nearly massless neutrinos are nearly left handed. □

One may add to this action density the **Majorana mass term**

$$\mathcal{L}_M(x) = -\tfrac{1}{2} m\, u^\dagger(x)\, \sigma_2\, u^*(x) - \tfrac{1}{2} m^*\, u^\mathsf{T}(x)\, \sigma_2\, u(x) \tag{10.300}$$

which is Lorentz covariant because the matrices $\sigma_1$ and $\sigma_3$ anti-commute with $\sigma_2$ which is antisymmetric (exercise 10.41). This term would vanish if $u_1 u_2$ were equal to $u_2 u_1$. Since charge is conserved, only neutral fields like neutrinos can have Majorana mass terms. The action density of a left-handed field of mass $m$ is the sum $\mathcal{L} = \mathcal{L}_\ell + \mathcal{L}_M$ of the kinetic one (10.292) and the Majorana mass term (10.300). The resulting equations of motion

$$\begin{aligned} 0 &= i\,(\partial_0 - \nabla \cdot \boldsymbol{\sigma})\, u - m \sigma_2 u^* \\ 0 &= \left(\partial_0^2 - \nabla^2 + |m|^2\right) u \end{aligned} \tag{10.301}$$

show that the field $u$ represents particles of mass $|m|$.

## 10.38 Two-dimensional right-handed representation of the Lorentz group

The generators of the representation $D^{(0,1/2)}$ with $j = 0$ and $j' = 1/2$ are given by (10.277 & 10.278) with $\boldsymbol{J}^+ = 0$ and $\boldsymbol{J}^- = \boldsymbol{\sigma}/2$; they are

$$\boldsymbol{J} = \frac{1}{2}\boldsymbol{\sigma} \quad \text{and} \quad \boldsymbol{K} = i\frac{1}{2}\boldsymbol{\sigma}. \tag{10.302}$$

Thus $2 \times 2$ matrix $D^{(0,1/2)}(\boldsymbol{\theta}, \boldsymbol{\lambda})$ that represents the Lorentz transformation (10.274)

$$L = e^{-i\boldsymbol{\theta}\cdot\boldsymbol{J} - i\boldsymbol{\lambda}\cdot\boldsymbol{K}} \tag{10.303}$$

is

$$D^{(0,1/2)}(\boldsymbol{\theta}, \boldsymbol{\lambda}) = \exp\left(-i\boldsymbol{\theta} \cdot \boldsymbol{\sigma}/2 + \boldsymbol{\lambda} \cdot \boldsymbol{\sigma}/2\right) = D^{(1/2,0)}(\boldsymbol{\theta}, -\boldsymbol{\lambda}) \tag{10.304}$$

which differs from $D^{(1/2,0)}(\boldsymbol{\theta}, \boldsymbol{\lambda})$ only by the sign of $\boldsymbol{\lambda}$. The generic $D^{(0,1/2)}$ matrix is the complex unimodular $2 \times 2$ matrix

$$D^{(0,1/2)}(\boldsymbol{\theta}, \boldsymbol{\lambda}) = e^{\boldsymbol{z}^* \cdot \boldsymbol{\sigma}/2} \tag{10.305}$$

with $\boldsymbol{\lambda} = \mathrm{Re}\boldsymbol{z}$ and $\boldsymbol{\theta} = \mathrm{Im}\boldsymbol{z}$.

**Example 10.39** (The standard right-handed boost ) For a particle of mass $m > 0$, the "standard" boost (10.287) that transforms $k = (m, \boldsymbol{0})$ to $p = (p^0, \boldsymbol{p})$ is the $4 \times 4$ matrix $B(p) = \exp\left(\alpha\, \hat{\boldsymbol{p}} \cdot \boldsymbol{B}\right)$ in which $\cosh \alpha = p^0/m$

and $\sinh\alpha = |\boldsymbol{p}|/m$. This Lorentz transformation with $\boldsymbol{\theta} = \boldsymbol{0}$ and $\boldsymbol{\lambda} = \alpha\,\hat{\boldsymbol{p}}$ is represented by the matrix (exercise 10.37)

$$\begin{aligned} D^{(0,1/2)}(\boldsymbol{0}, \alpha\,\hat{\boldsymbol{p}}) &= e^{\alpha\hat{\boldsymbol{p}}\cdot\boldsymbol{\sigma}/2} = I\,\cosh(\alpha/2) + \hat{\boldsymbol{p}}\cdot\boldsymbol{\sigma}\,\sinh(\alpha/2) \\ &= I\,\sqrt{(p^0+m)/(2m)} + \hat{\boldsymbol{p}}\cdot\boldsymbol{\sigma}\,\sqrt{(p^0-m)/(2m)} \\ &= \frac{p^0 + m + \boldsymbol{p}\cdot\boldsymbol{\sigma}}{\sqrt{2m(p^0+m)}} \end{aligned} \tag{10.306}$$

in the third line of which the $2 \times 2$ identity matrix $I$ is suppressed. □

Under $D^{(0,1/2)}$, the vector $(I, \boldsymbol{\sigma})$ transforms as a 4-vector; for tiny $\boldsymbol{z}$

$$\begin{aligned} D^{\dagger(0,1/2)}(\boldsymbol{\theta},\boldsymbol{\lambda})\,I\,D^{(0,1/2)}(\boldsymbol{\theta},\boldsymbol{\lambda}) &= I + \boldsymbol{\lambda}\cdot\boldsymbol{\sigma} \\ D^{\dagger(0,1/2)}(\boldsymbol{\theta},\boldsymbol{\lambda})\,\boldsymbol{\sigma}\,D^{(0,1/2)}(\boldsymbol{\theta},\boldsymbol{\lambda}) &= \boldsymbol{\sigma} + I\boldsymbol{\lambda} + \boldsymbol{\theta}\wedge\boldsymbol{\sigma} \end{aligned} \tag{10.307}$$

as in (10.273).

A massless field $v(x)$ that responds to a unitary Lorentz transformation $U(L)$ as

$$U(L)\,v(x)\,U^{-1}(L) = D^{(0,1/2)}(L^{-1})\,v(Lx) \tag{10.308}$$

is called a **right-handed Weyl spinor**. One may show (exercise 10.40) that the action density

$$\mathcal{L}_r(x) = i\,v^\dagger(x)\,(\partial_0 I + \boldsymbol{\nabla}\cdot\boldsymbol{\sigma})\,v(x) \tag{10.309}$$

is Lorentz covariant

$$U(L)\,\mathcal{L}_r(x)\,U^{-1}(L) = \mathcal{L}_r(Lx). \tag{10.310}$$

**Example 10.40** (Why $v$ is right handed) An argument like that of example (10.38) shows that the field $v(x)$ satisfies the wave equation

$$(\partial_0 I + \boldsymbol{\nabla}\cdot\boldsymbol{\sigma})\,v(x) = 0 \tag{10.311}$$

or in momentum space

$$(E - \boldsymbol{p}\cdot\boldsymbol{\sigma})\,v(p) = 0. \tag{10.312}$$

Thus, $E = |\boldsymbol{p}|$, and $v(p)$ is an eigenvector of $\hat{\boldsymbol{p}}\cdot\boldsymbol{J}$

$$\hat{\boldsymbol{p}}\cdot\boldsymbol{J}\,v(p) = \frac{1}{2}\,v(p) \tag{10.313}$$

with eigenvalue 1/2. A particle whose spin is parallel to its momentum is said to have **positive helicity** or to be **right handed**. Nearly massless antineutrinos are nearly right handed. □

The Majorana mass term

$$\mathcal{L}_M(x) = -\tfrac{1}{2} m\, v^\dagger(x)\, \sigma_2\, v^*(x) - \tfrac{1}{2} m^*\, v^{\mathsf{T}}(x)\, \sigma_2\, v(x) \tag{10.314}$$

like (10.300) is Lorentz covariant. The action density of a right-handed field of mass $m$ is the sum $\mathcal{L} = \mathcal{L}_r + \mathcal{L}_M$ of the kinetic one (10.309) and this Majorana mass term (10.314). The resulting equations of motion

$$\begin{aligned} 0 &= i\,(\partial_0 + \nabla \cdot \boldsymbol{\sigma})\, v - m\sigma_2 v^* \\ 0 &= \left(\partial_0^2 - \nabla^2 + |m|^2\right) v \end{aligned} \tag{10.315}$$

show that the field $v$ represents particles of mass $|m|$.

## 10.39 The Dirac Representation of the Lorentz Group

Dirac's representation of $SO(3,1)$ is the direct sum $D^{(1/2,0)} \oplus D^{(0,1/2)}$ of $D^{(1/2,0)}$ and $D^{(0,1/2)}$. Its generators are the $4 \times 4$ matrices

$$\boldsymbol{J} = \frac{1}{2}\begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix} \quad \text{and} \quad \boldsymbol{K} = \frac{i}{2}\begin{pmatrix} -\boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}. \tag{10.316}$$

Dirac's representation uses the **Clifford algebra** of the gamma matrices $\gamma^a$ which satisfy the anticommutation relation

$$\{\gamma^a, \gamma^b\} \equiv \gamma^a\, \gamma^b + \gamma^b\, \gamma^a = 2\eta^{ab} I \tag{10.317}$$

in which $\eta$ is the $4 \times 4$ diagonal matrix (10.255) with $\eta^{00} = -1$ and $\eta^{jj} = 1$ for $j = 1$, 2, and 3, and $I$ is the $4 \times 4$ identity matrix.

Remarkably, the generators of the Lorentz group

$$J^{ij} = \epsilon_{ijk} J_k \quad \text{and} \quad J^{0j} = K_j \tag{10.318}$$

may be represented as commutators of gamma matrices

$$J^{ab} = -\frac{i}{4}\,[\gamma^a, \gamma^b]. \tag{10.319}$$

They transform the gamma matrices as a 4-vector

$$[J^{ab}, \gamma^c] = -i\gamma^a\, \eta^{bc} + i\gamma^b\, \eta^{ac} \tag{10.320}$$

(exercise 10.42) and satisfy the commutation relations

$$i[J^{ab}, J^{cd}] = \eta^{bc}\, J^{ad} - \eta^{ac}\, J^{bd} - \eta^{da}\, J^{cb} + \eta^{db}\, J^{ca} \tag{10.321}$$

of the Lorentz group (Weinberg, 1995, p. 213–217) (exercise 10.43).

The gamma matrices $\gamma^a$ are not unique; if $S$ is any $4 \times 4$ matrix with an inverse, then the matrices $\gamma'^a \equiv S\gamma^a S^{-1}$ also satisfy the definition (10.317). The choice

$$\gamma^0 = -i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \boldsymbol{\gamma} = -i \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix} \tag{10.322}$$

makes $\boldsymbol{J}$ and $\boldsymbol{K}$ block diagonal (10.316) and lets us assemble a left-handed spinor $u$ and a right-handed spinor $v$ neatly into a 4-component spinor

$$\psi = \begin{pmatrix} u \\ v \end{pmatrix}. \tag{10.323}$$

Dirac's action density for a 4-spinor is

$$\mathcal{L} = -\overline{\psi}\,(\gamma^a \partial_a + m)\,\psi \equiv -\overline{\psi}\,(\partial\!\!\!/ + m)\,\psi \tag{10.324}$$

in which

$$\overline{\psi} \equiv i\psi^\dagger \gamma^0 = \psi^\dagger \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} v^\dagger & u^\dagger \end{pmatrix}. \tag{10.325}$$

The kinetic part is the sum of the left-handed $\mathcal{L}_\ell$ and right-handed $\mathcal{L}_r$ action densities (10.292 & 10.309)

$$-\overline{\psi}\,\gamma^a \partial_a \psi = iu^\dagger\,(\partial_0 I - \boldsymbol{\nabla}\cdot\boldsymbol{\sigma})\,u + i\,v^\dagger\,(\partial_0 I + \boldsymbol{\nabla}\cdot\boldsymbol{\sigma})\,v. \tag{10.326}$$

If $u$ is a left-handed spinor transforming as (10.291), then the spinor

$$v = \sigma_2\, u^* \equiv \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} u_1^\dagger \\ u_2^\dagger \end{pmatrix} \tag{10.327}$$

transforms as a right-handed spinor (10.308), that is (exercise 10.44)

$$e^{\boldsymbol{z}^*\cdot\boldsymbol{\sigma}/2}\sigma_2\, u^* = \sigma_2 \left(e^{-\boldsymbol{z}\cdot\boldsymbol{\sigma}/2} u\right)^*. \tag{10.328}$$

Similarly, if $v$ is right handed, then $u = -\sigma_2\, v^*$ is left handed.

The simplest 4-spinor is the Majorana spinor

$$\psi_M = \begin{pmatrix} u \\ \sigma_2 u^* \end{pmatrix} = \begin{pmatrix} -\sigma_2 v^* \\ v \end{pmatrix} = -i\gamma^2 \psi_M^* \tag{10.329}$$

whose particles are the same as its antiparticles.

If two Majorana spinors $\psi_M^{(1)}$ and $\psi_M^{(2)}$ have the same mass, then one may combine them into a Dirac spinor

$$\psi_D = \frac{1}{\sqrt{2}}\left(\psi_M^{(1)} + i\psi_M^{(2)}\right) = \frac{1}{\sqrt{2}} \begin{pmatrix} u^{(1)} + iu^{(2)} \\ v^{(1)} + iv^{(2)} \end{pmatrix} = \begin{pmatrix} u_D \\ v_D \end{pmatrix}. \tag{10.330}$$

The **Dirac mass term**

$$-m\,\overline{\psi}_D\psi_D = -m\left(v_D^\dagger u_D + u_D^\dagger v_D\right) \tag{10.331}$$

conserves charge, and since $\exp(\boldsymbol{z}^*\cdot\sigma/2)^\dagger\exp(-\boldsymbol{z}\cdot\sigma/2) = I$ it also is Lorentz invariant. For a Majorana field, it reduces to

$$\begin{aligned}-\tfrac{1}{2}m\overline{\psi}_M\psi_M &= -\tfrac{1}{2}m\left(v^\dagger u + u^\dagger v\right) = -\tfrac{1}{2}m\left(u^\dagger\sigma_2 u^* + u^\mathsf{T}\sigma_2 u\right)\\ &= -\tfrac{1}{2}m\left(v^\dagger\sigma_2 v^* + v^\mathsf{T}\sigma_2 v\right)\end{aligned} \tag{10.332}$$

a Majorana mass term (10.300 or 10.314).

## 10.40 The Poincaré Group

The elements of the Poincaré group are products of Lorentz transformations and translations in space and time. The Lie algebra of the Poincaré group therefore includes the generators $\boldsymbol{J}$ and $\boldsymbol{K}$ of the Lorentz group as well as the hamiltonian $H$ and the momentum operator $\boldsymbol{P}$ which respectively generate translations in time and space.

Suppose $T(y)$ is a translation that takes a 4-vector $x$ to $x+y$ and $T(z)$ is a translation that takes a 4-vector $x$ to $x+z$. Then $T(z)T(y)$ and $T(y)T(z)$ both take $x$ to $x+y+z$. So if a translation $T(y) = T(t,\boldsymbol{y})$ is represented by a unitary operator $U(t,\boldsymbol{y}) = \exp(iHt - i\boldsymbol{P}\cdot\boldsymbol{y})$, then the hamiltonian $H$ and the momentum operator $\boldsymbol{P}$ commute with each other

$$[H, P^j] = 0 \quad \text{and} \quad [P^i, P^j] = 0. \tag{10.333}$$

We can figure out the commutation relations of $H$ and $\boldsymbol{P}$ with the angular-momentum $\boldsymbol{J}$ and boost $\boldsymbol{K}$ operators by realizing that $P^a = (H, \boldsymbol{P})$ is a 4-vector. Let

$$U(\boldsymbol{\theta},\boldsymbol{\lambda}) = e^{-i\boldsymbol{\theta}\cdot\boldsymbol{J} - i\boldsymbol{\lambda}\cdot\boldsymbol{K}} \tag{10.334}$$

be the (infinite-dimensional) unitary operator that represents (in Hilbert space) the infinitesimal Lorentz transformation

$$L = I + \boldsymbol{\theta}\cdot\boldsymbol{R} + \boldsymbol{\lambda}\cdot\boldsymbol{B} \tag{10.335}$$

where $\boldsymbol{R}$ and $\boldsymbol{B}$ are the six $4\times 4$ matrices (10.264 & 10.265). Then because $P$ is a 4-vector under Lorentz transformations, we have

$$U^{-1}(\boldsymbol{\theta},\boldsymbol{\lambda})PU(\boldsymbol{\theta},\boldsymbol{\lambda}) = e^{+i\boldsymbol{\theta}\cdot\boldsymbol{J}+i\boldsymbol{\lambda}\cdot\boldsymbol{K}}Pe^{-i\boldsymbol{\theta}\cdot\boldsymbol{J}-i\boldsymbol{\lambda}\cdot\boldsymbol{K}} = \left(I + \boldsymbol{\theta}\cdot\boldsymbol{R} + \boldsymbol{\lambda}\cdot\boldsymbol{B}\right)P \tag{10.336}$$

or using (10.307)

$$(I + i\boldsymbol{\theta} \cdot \boldsymbol{J} + i\boldsymbol{\lambda} \cdot \boldsymbol{K})\, H\, (I - i\boldsymbol{\theta} \cdot \boldsymbol{J} - i\boldsymbol{\lambda} \cdot \boldsymbol{K}) = H + \boldsymbol{\lambda} \cdot \boldsymbol{P} \qquad (10.337)$$
$$(I + i\boldsymbol{\theta} \cdot \boldsymbol{J} + i\boldsymbol{\lambda} \cdot \boldsymbol{K})\, \boldsymbol{P}\, (I - i\boldsymbol{\theta} \cdot \boldsymbol{J} - i\boldsymbol{\lambda} \cdot \boldsymbol{K}) = \boldsymbol{P} + H\boldsymbol{\lambda} + \boldsymbol{\theta} \wedge \boldsymbol{P}.$$

Thus one finds (exercise 10.44) that $H$ is invariant under rotations, while $\boldsymbol{P}$ transforms as a 3-vector

$$[J_i, H] = 0 \quad \text{and} \quad [J_i, P_j] = i\epsilon_{ijk} P_k \qquad (10.338)$$

and that

$$[K_i, H] = -iP_i \quad \text{and} \quad [K_i, P_j] = -i\delta_{ij} H. \qquad (10.339)$$

By combining these equations with (10.321), one may write (exercise 10.46) the Lie algebra of the Poincaré group as

$$\begin{aligned} i[J^{ab}, J^{cd}] &= \eta^{bc}\, J^{ad} - \eta^{ac}\, J^{bd} - \eta^{da}\, J^{cb} + \eta^{db}\, J^{ca} \\ i[P^a, J^{bc}] &= \eta^{ab} P^c - \eta^{ac} P^b \\ [P^a, P^b] &= 0. \end{aligned} \qquad (10.340)$$

## Further reading

The books *Group theory in a Nutshell for Physicists* (Zee, 2016), *Lie Algebras in Particle Physics* (Georgi, 1999), *Unitary Symmetry and Elementary Particles* (Lichtenberg, 1978), and *Group theory and Its Application to the Quantum Mechanics of Atomic Spectra* (Wigner, 1964) are excellent. For applications to molecular physics, see *Chemical Applications of Group theory* (Cotton, 1990); for applications to condensed-matter physics, see *Group theory and Quantum Mechanics* (Tinkham, 2003).

## Exercises

10.1 Show that all $n \times n$ (real) orthogonal matrices $O$ leave invariant the quadratic form $x_1^2 + x_2^2 + \ldots + x_n^2$, that is, that if $x' = Ox$, then $x'^2 = x^2$.

10.2 Show that the set of all $n \times n$ orthogonal matrices forms a group.

10.3 Show that all $n \times n$ unitary matrices $U$ leave invariant the quadratic form $|x_1|^2 + |x_2|^2 + \ldots + |x_n|^2$, that is, that if $x' = Ux$, then $|x'|^2 = |x|^2$.

10.4 Show that the set of all $n \times n$ unitary matrices forms a group.

10.5 Show that the set of all $n \times n$ unitary matrices with unit determinant forms a group.

10.6 Show that the matrix $D^{(j)}_{m'm}(g) = \langle j, m'|U(g)|j, m\rangle$ is unitary because the rotation operator $U(g)$ is unitary $\langle j, m'|U^\dagger(g)U(g)|j, m\rangle = \delta_{m'm}$.