which might arise from an interaction such as

$$L = \tfrac{1}{2} m_P^2 \, A_i A^i - 4\sqrt{\pi} A_i \partial^i \phi. \tag{9.30}$$

The sign of the mass term is wrong, however.

9.3 Quantization of Fields in Curved Space

Some good references for these ideas are:

Einstein's Gravity in a Nutshell by Zee
Quantum Fields in Curved Space by Birrell and Davies,
Conformal Field Theory by Di Francesco, Mathieu, and Sénéchal

Let's focus on scalar fields for simplicity. We usually expand a scalar field in flat space as Fourier might have (1.56)

$$\phi(x) = \int \frac{d^3p}{\sqrt{(2\pi)^3 2p^0}} \left[a(p)\, e^{ip\cdot x} + a^\dagger(p)\, e^{-ip\cdot x} \right]. \tag{9.31}$$

The field ϕ obeys the Klein-Gordon equation

$$(\nabla^2 - \partial_0^2 - m^2)\, \phi(x) \equiv (\Box - m^2)\, \phi(x) = 0 \tag{9.32}$$

because the flat-space modes, which have $p^2 = -m^2$,

$$f_p(x) = e^{ip\cdot x} \tag{9.33}$$

do

$$(\nabla^2 - \partial_0^2 - m^2)\, f_p(x) \equiv (\Box - m^2)\, f_p(x) = 0. \tag{9.34}$$

In terms of these functions $f_p(x)$, the field is

$$\phi(x) = \int \frac{d^3p}{\sqrt{(2\pi)^3 2p^0}} \left[a(p)\, f_p(x) + a^\dagger(p)\, f_p^*(x) \right]. \tag{9.35}$$

In terms of a discrete set of modes f_n, it is

$$\phi(x) = \sum_n \left[a_n(p)\, f_n(x) + a_n^\dagger\, f_n^*(x) \right]. \tag{9.36}$$

The action for a scalar field in a space described by the metric g_{ik} is

$$S = -\frac{1}{2} \int \sqrt{g}\, d^4x \left[g^{ik} \phi_{,i} \phi_{,k} + \left(m^2 + \xi R \right) \phi^2 \right] \tag{9.37}$$

in which R is the scalar curvature, ξ is a constant, commas denote derivatives as in $\phi_{,k} = \partial_k \phi$, g^{ij} is the inverse of the metric g_{ij}, and g is the absolute value of the determinant of the metric g_{ij}. If the spacetime metric is g_{ij}, then instead of (9.32), the field ϕ obeys the covariant Klein-Gordon equation

$$\partial_i \left[\sqrt{g}\, g^{ij}\, \partial_j\, \phi(x) \right] - \left(m^2 + \xi R\right) \sqrt{g}\, \phi(x) = 0. \tag{9.38}$$

However, to simplify what follows, we will now set $\xi = 0$.

To quantize in the new coordinates or in the gravitational field of the metric g_{ij}, we need solutions $f'_n(x)$ of the curved-space equation (9.38)

$$\partial_i \left[\sqrt{g}\, g^{ij}\, \partial_j\, f'_n(x) \right] - m^2 \sqrt{g}\, f'_n(x) = 0 \tag{9.39}$$

which we label with primes to distinguish them from the flat-space solutions (9.33). We use these solutions to expand the field in terms of curved-space annihilation and creation operators, which we also label with primes

$$\phi(x) = \sum_n \left[a'_n(p)\, f'_n(x) + a'^{\dagger}_n\, f'^{*}_n(x) \right]. \tag{9.40}$$

The flat-space modes obey the orthonormality relations

$$\begin{aligned} (f_p, f_q) &= i \int d^3x \Big[f^*_p(x)\, \partial_t f_q(x) - \big(\partial_t f^*_p(x)\big) f_q(x) \Big] \\ &= i \int d^3x \Big[e^{-ipx}\, (-iq^0) e^{iqx} - ip^0 e^{-ipx}\, e^{iqx} \Big] \\ &= 2p^0 (2\pi)^3 \delta^3(\boldsymbol{p} - \boldsymbol{q}), \end{aligned} \tag{9.41}$$

$$(f^*_p, f^*_q) = \, - 2p^0 (2\pi)^3 \delta^3(\boldsymbol{p} - \boldsymbol{q}) \quad \text{and} \quad (f_p, f^*_q) = 0. \tag{9.42}$$

In terms of discrete modes, the flat-space scalar product is

$$(f_n, f_m) = i \int d^3x \Big[f^*_n(x)\, \partial_t f_m(x) - \big(\partial_t f^*_n(x)\big) f_m(x) \Big] = \delta_{nm} \tag{9.43}$$

and its orthonormality relations are

$$(f_n, f_m) = \delta_{nm}, \quad (f^*_n, f^*_m) = \, - \delta_{nm} \quad \text{and} \quad (f_n, f^*_m) = 0. \tag{9.44}$$

The scalar product (9.43) is a special case of more general curved-space scalar product

$$(f, g) = i \int_S \sqrt{g_S}\; d^3S\, v^a \Big[f^*(x)\, \partial_a g(x) - \big(\partial_a f^*(x)\big) g(x) \Big] \tag{9.45}$$

in which the integral is over a spacelike surface S with a future-pointing timelike vector v^a, and g_S is the absolute value of the spatial part of the metric g_{ik}. This more general scalar product (9.45) is hermitian

$$(f, g)^* = (g, f). \tag{9.46}$$

It also satisfies the rule

$$(f, g)^* = -(f^*, g^*). \tag{9.47}$$

One may use Gauss's theorem to show (Hawking and Ellis, 1973, section 2.8) that this inner product is independent of S and v. The curved-space modes $f_n(x)$ obey orthonormality relations

$$(f'_n, f'_m) = \delta_{nm}, \quad (f'^*_n, f'^*_m) = -\delta_{nm}, \quad \text{and} \quad (f'_n, f'^*_m) = 0 \tag{9.48}$$

like those (9.44) of the flat-space modes.

The flat-space modes $f_n(x) = e^{ip_n x}$ naturally describe particles of momentum p_n in flat Minkowski space. In curved space, however, there are in general no equally natural modes. We can consider other complete sets of modes $f''_n(x)$ that are solutions of the curved-space Klein-Gordon equation (9.39) and obey the orthonormality relations (9.48). We can use any of these complete sets of mode functions to expand a scalar field $\phi(x)$

$$\begin{aligned} \phi(x) &= \sum_n a_n f_n(x) + a_n^\dagger f_n^*(x) \\ \phi(x) &= \sum_n a'_n f'_n(x) + a'^\dagger_n f'^*_n(x) \\ \phi(x) &= \sum_n a''_n f''_n(x) + a''^\dagger_n f''^*_n(x). \end{aligned} \tag{9.49}$$

The curved-space orthonormality relations (9.48) imply that

$$\begin{aligned} (f_\ell, \phi) &= \sum_n a_n (f_\ell, f_n) + a_n^\dagger (f_\ell, f_n^*) = a_\ell \\ (f'_\ell, \phi) &= \sum_n a'_n (f'_\ell, f'_n) + a'^\dagger_n (f'_\ell, f'^*_n) = a'_\ell \\ (f''_\ell, \phi) &= \sum_n a''_n (f''_\ell, f''_n) + a''^\dagger_n (f''_\ell, f''^*_n) = a''_\ell \end{aligned} \tag{9.50}$$

and that

$$\begin{aligned} (f_\ell^*, \phi) &= \sum_n a_n (f_\ell^*, f_n) + a_n^\dagger (f_\ell^*, f_n^*) = -a_\ell^\dagger \\ (f'^*_\ell, \phi) &= \sum_n a'_n (f'^*_\ell, f'_n) + a'^\dagger_n (f'^*_\ell, f'^*_n) = -a'^\dagger_\ell \\ (f''^*_\ell, \phi) &= \sum_n a''_n (f''^*_\ell, f''_n) + a''^\dagger_n (f''^*_\ell, f''^*_n) = -a''^\dagger_\ell. \end{aligned} \tag{9.51}$$

The completeness of the mode functions lets us expand them in terms of each other. Suppressing the spacetime argument x, we have

$$f'_j = \sum_i \left(\alpha_{ji} f_i + \beta_{ji}\, f_i^*\right). \tag{9.52}$$

The curved-space orthonormality relations (9.48) let us identify these Bogoliubov coefficients

$$\begin{aligned}(f_\ell, f'_j) &= \sum_i \Big[\alpha_{ji}(f_\ell, f_i) + \beta_{ji}\,(f_\ell, f_i^*)\Big] = \alpha_{j\ell}\\ (f_\ell^*, f'_j) &= \sum_i \Big[\alpha_{ji}(f_\ell^*, f_i) + \beta_{ji}\,(f_\ell^*, f_i^*)\Big] = -\beta_{j\ell}.\end{aligned} \tag{9.53}$$

To find the inverse relations, we use the completeness of the mode functions f'_i to expand the f_j's

$$f_j = \sum_i \left(c_{ji} f'_i + d_{ji}\, f_i'^*\right) \tag{9.54}$$

and then use the orthonormality relations (9.48) to form the inner products

$$\begin{aligned}(f'_\ell, f_j) &= \sum_i \Big[c_{ji}(f'_\ell, f'_i) + d_{ji}\,(f'_\ell, f_i'^*)\Big] = c_{j\ell}\\ (f_\ell'^*, f_j) &= \sum_i \Big[c_{ji}(f_\ell'^*, f'_i) + d_{ji}\,(f_\ell'^*, f_i'^*)\Big] = -d_{j\ell}.\end{aligned} \tag{9.55}$$

The hermiticity (9.46) of the scalar product tells us that the $c_{j\ell}$'s are related to the α's

$$c_{j\ell} = (f_j, f'_\ell)^* = \alpha^*_{\ell j}. \tag{9.56}$$

The hermiticity (9.46) of the scalar product and the rule (9.47) show that

$$d_{j\ell} = -\,(f_\ell'^*, f_j) = -\,(f_j, f_\ell'^*)^* = (f_j^*, f'_\ell) = -\beta_{\ell j}. \tag{9.57}$$

So the inverse relation (9.54) is

$$f_j = \sum_i \left(\alpha^*_{ij} f'_i - \beta_{ij}\, f_i'^*\right). \tag{9.58}$$

The formulas (9.52 & 9.58) that relate the mode functions of different metrics are known as Bogoliubov transformations.

The vacuum state for a given metric is the state that is mapped to zero by all the annihilation operators. Our formulas (9.51) for the annihilation and creation operators

$$\begin{aligned}a_\ell &= (f_\ell, \phi) \quad \text{and} \quad a'_\ell = (f'_\ell, \phi)\\ a_\ell^\dagger &= -\,(f_\ell^*, \phi) \quad \text{and} \quad a_\ell'^\dagger = -\,(f_\ell'^*, \phi)\end{aligned} \tag{9.59}$$

let us express the annihilation and creation operators for one metric in terms of those for a different metric. Thus, using our formula (9.52) for the f''s in terms of the f's, we find

$$a'_j = (f'_j, \phi) = \sum_i \Big[\alpha_{ji}\,(f_i, \phi) + \beta_{ji}\,(f_i^*, \phi)\Big] = \sum_i \Big(\alpha_{ji}\,a_i - \beta_{ji}\,a_i^\dagger\Big). \quad (9.60)$$

Our formula (9.58) for the f's in terms of the f''s gives us the inverse relation

$$a_j = (f_j, \phi) = \sum_i \Big[\alpha_{ij}^*\,(f'_i, \phi) - \beta_{ij}\,(f_i'^*, \phi)\Big] = \sum_i \Big(\alpha_{ij}^*\,a'_i + \beta_{ij}\,a_i'^\dagger\Big). \quad (9.61)$$

The annihilation operators a'_j define the vacuum state $|0\rangle'$ by the rules

$$a'_j\,|0\rangle' = 0 \quad (9.62)$$

for all modes j. Thus our formula (9.61) for a_j says that

$$a_j\,|0\rangle' = \sum_i \Big(\alpha_{ij}^*\,a'_i + \beta_{ij}\,a_i'^\dagger\Big)|0\rangle' = \sum_i \beta_{ij}\,a_i'^\dagger|0\rangle' \quad (9.63)$$

The adjoint of this equation is

$${}'\langle 0|a_j^\dagger = {}'\langle 0| \sum_i a'_i \beta_{ij}^*. \quad (9.64)$$

Thus the mean value of the number operator $a_j^\dagger a_j$ in the $|0\rangle'$ vacuum is

$${}'\langle 0|a_j^\dagger a_j\,|0\rangle' = {}'\langle 0| \sum_i a'_i \beta_{ij}^* \sum_k \beta_{kj}\,a_k'^\dagger|0\rangle'. \quad (9.65)$$

The commutation relations

$$[a'_i, a_k'^\dagger] = \delta_{ik} \quad (9.66)$$

and the definition $a'_j\,|0\rangle' = 0$ (9.62) of the vacuum $|0\rangle'$ imply that the average number (9.65) of particles of mode j in the (normalized) vacuum $|0\rangle'$ is

$${}'\langle 0|a_j^\dagger a_j\,|0\rangle' = {}'\langle 0| \sum_{ik} \beta_{ij}^* \beta_{kj}\,\delta_{ik}|0\rangle' = \sum_i \beta_{ij}^* \beta_{ij}. \quad (9.67)$$

It follows that the vacuum of one metric contains particles of the other metric unless the Bogoliubov matrix

$$\beta_{j\ell} = \, -\,(f_\ell^*, f'_j) \quad (9.68)$$

vanishes. The value of $\beta_{j\ell}$ in the Minkowski-space scalar product (9.41) is

$$\beta_{j\ell} = \, -\,(f_\ell^*, f'_j) = -i \int d^3x \Big[f_\ell(x)\,\partial_t f'_j(x) - \big(\partial_t f_\ell(x)\big)\,f'_j(x)\Big]. \quad (9.69)$$

This integral for $\beta_{j\ell}$ is nonzero, for example, when the functions f_ℓ and f'_j have different frequencies but are not spatially orthogonal.

An example due to Rindler. Let us consider the two metrics

$$ds^2 = dx^2 + dy^2 + dz^2 - dt^2 = dr^2 + dy^2 + dz^2 - r^2 du^2 \tag{9.70}$$

in which y and z are the same, r plays the role of x and u that of time. The first metric has $g_{ik} = \eta_{ik}$ and $g = |\det(\eta)| = 1$. The second metric has

$$g_{ik} = \begin{pmatrix} -r^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad g = r^2. \tag{9.71}$$

The inverse metric is

$$g^{ik} = \begin{pmatrix} -r^{-2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{9.72}$$

The equation of motion is

$$\partial_i \left[\sqrt{g} g^{ik} \partial_k f \right] = m^2 \sqrt{g} f. \tag{9.73}$$

So we must solve

$$\partial_u(-r^{-1}\partial_u f) + \partial_r(r\partial_r f) + \partial_y(r\partial_y f) + \partial_z(r\partial_z f) = m^2 r f. \tag{9.74}$$

Now u, r, y, z are independent coordinates, so this equation of motion is

$$-r^{-1}\partial_u^2 f + \partial_r(r\partial_r f) + r\partial_y^2 f + r\partial_z^2 f = m^2 r f \tag{9.75}$$

or

$$-r^{-2}\partial_u^2 f + r^{-1}\partial_r f + \partial_r^2 f + \partial_y^2 f + \partial_z^2 f = m^2 f. \tag{9.76}$$

Let's make f a plane wave in the y and z directions

$$f(u, r, y, z) = f(u, r) e^{iyp_y + izP_z}. \tag{9.77}$$

If we also set

$$M^2 = m^2 + P_y^2 + P_z^2, \tag{9.78}$$

then we must solve

$$-r^{-2}\partial_u^2 f + r^{-1}\partial_r f + \partial_r^2 f = M^2 f \tag{9.79}$$

where $f = f(u, r)$. We now make the Daniel-Middleton transformation, separating the dependence of f upon r and u

$$f(u, r) = a(u)b(r). \tag{9.80}$$

Our differential equation (9.79) reduces to

$$-r^{-2}\ddot{a}b + r^{-1}ab' + ab'' = M^2 ab \tag{9.81}$$

in which dots denote ∂_u and primes ∂_r. Dividing by a, we get

$$-\frac{\ddot{a}}{a}\frac{b}{r^2} + \frac{b'}{r} + b'' - M^2 b = 0. \tag{9.82}$$

As $a(u)$, we choose

$$a(u) = e^{i\omega u} \quad \text{and} \quad \frac{\ddot{a}}{a} = -\omega^2. \tag{9.83}$$

Then our equation (9.82) for $b(r)$ is

$$b'' + \frac{b'}{r} - \left(M^2 - \frac{\omega^2}{r^2}\right)b = 0 \tag{9.84}$$

or equivalently

$$r^2 b'' + rb' - \left(M^2 r^2 - \omega^2\right)b = 0. \tag{9.85}$$

Yi's solution is a modified Bessel function $I_\nu(z)$ for imaginary ν.

Bogoliubov's β matrix is

$$\beta_{j\ell} = -(f_\ell^*, f_j'). \tag{9.86}$$

The scalar product (9.45) uses the metric of one, not both, of the solutions. If we choose the flat-space metric, then we need to write the solution $f_j'(u, r, y, z)$ in terms of the flat-space coordinates t, x, y, z. The mean occupation number (9.67) is the sum

$$\sum_j {}'\langle 0|a_j^\dagger a_j|0\rangle' = \sum_{ij} |\beta_{ij}|^2 = \sum_{ij} |(f_j^*, f_i')|^2. \tag{9.87}$$

9.4 Accelerated coordinate systems

We recall the Lorentz transformations

$$\begin{aligned} t' &= \gamma(t - vx), \quad x' = \gamma(x - vt), \quad y' = y, \quad \text{and} \quad z' = z \\ t &= \gamma(t' + vx'), \quad x = \gamma(x' + vt'), \quad y = y', \quad \text{and} \quad z = z' \end{aligned} \tag{9.88}$$

in which $\gamma = 1/\sqrt{1-v^2}$. The velocities (in the x direction) are

$$u = \frac{dx}{dt} \quad \text{and} \quad u' = \frac{dx'}{dt'} = \frac{\gamma(dx - vdt)}{\gamma(dt - vdx)} = \frac{(u-v)}{(1-vu)}. \tag{9.89}$$

So the accelerations (in the x direction) are

$$\begin{aligned} a &= \frac{du}{dt} \\ a' &= \frac{du'}{dt'} = d\Big[\frac{(u-v)}{(1-vu)}\Big] \Big/ \gamma(dt - vdx) \\ &= \Big[\frac{du}{(1-vu)} + \frac{(u-v)vdu}{(1-vu)^2}\Big] \Big/ \gamma(dt - vdx) \\ &= \Big[\frac{du(1-vu)}{(1-vu)^2} + \frac{(u-v)vdu}{(1-vu)^2}\Big] \Big/ \gamma(dt - vdx) \\ &= \frac{du(1-v^2)}{\gamma(1-vu)^2(dt - vdx)} \\ &= \frac{(1-v^2)a}{\gamma(1-vu)^2(1-vu)} \\ &= \frac{(1-v^2)a}{\gamma(1-vu)^3} = \frac{a}{\gamma^3(1-vu)^3}. \end{aligned} \tag{9.90}$$

Now we let the acceleration a' be a constant. That is, the acceleration in the (instantaneous) rest frame of the frame moving instantaneously at $u = v$ in the laboratory frame is a constant, $a' = \alpha$. In this case, since $u = v$, we get an equation (Rindler, 2006, sec. 3.7)

$$\begin{aligned} a' = \alpha &= \frac{a}{\gamma^3(1-vu)^3} = \frac{a}{\gamma^3(1-v^2)^3} = \frac{a}{(1-v^2)^{3/2}} \\ &= \frac{1}{(1-v^2)^{3/2}}\frac{du}{dt} = \frac{1}{(1-u^2)^{3/2}}\frac{du}{dt} = \frac{d}{dt}\left(\frac{u}{\sqrt{1-u^2}}\right) \end{aligned} \tag{9.91}$$

that we can integrate

$$\frac{u}{\sqrt{1-u^2}} = \alpha\,(t - t_0). \tag{9.92}$$

Squaring and solving for u, we find

$$u = \frac{dx}{dt} = \frac{\alpha(t-t_0)}{\sqrt{1+\alpha^2(t-t_0)^2}} \tag{9.93}$$

which we can integrate to

$$x = \alpha^{-1}\int dt\,\frac{\alpha^2(t-t_0)}{\sqrt{1+\alpha^2(t-t_0)^2}} = \frac{\sqrt{1+\alpha^2(t-t_0)^2} - 1}{\alpha} + x_0. \tag{9.94}$$

The proper time of the accelerating frame is $\tau = t'$. An interval dt' of proper time is one at which $dx' = 0$. So $dt' = \sqrt{1-v^2}dt$ or $d\tau = \sqrt{1-u^2}dt$ where $u(t)$ is the velocity (9.93). Integrating the equation

$$dt' = d\tau = \sqrt{1-u^2}dt = \sqrt{1 - \frac{\alpha^2(t-t_0)^2}{1+\alpha^2(t-t_0)^2}}\, dt = \frac{dt}{\sqrt{1+\alpha^2(t-t_0)^2}}, \tag{9.95}$$

we get

$$\alpha(t'-t'_0) = \alpha(\tau-\tau_0) = \int \frac{\alpha\, dt}{\sqrt{1+\alpha^2(t-t_0)^2}} = \operatorname{arcsinh}(\alpha(t-t_0)). \tag{9.96}$$

So

$$\alpha(t-t_0) = \sinh(\alpha(\tau-\tau_0)). \tag{9.97}$$

The formula (9.94) for x now gives

$$\alpha(x-x_0) = \cosh(\alpha(\tau-\tau_0)). \tag{9.98}$$

9.5 Scalar field in an accelerating frame

For simplicity, we'll work with a real scalar field (1.54)

$$\phi(x) = \int \frac{d^3p}{\sqrt{(2\pi)^3 2p^0}} \left[a(p)\, e^{ip\cdot x} + a^\dagger(p)\, e^{-ip\cdot x}\right]. \tag{9.99}$$

If the field is quantized in a box of volume V, then the expansion of the field is

$$\phi(x) = \sum_k \frac{1}{\sqrt{2\omega_k V}} \left[a(k)\, e^{ik\cdot x} + a^\dagger(k)\, e^{-ik\cdot x}\right]. \tag{9.100}$$

For the scalar field (9.99), the zero-temperature correlation function is the mean value in the vacuum

$$\begin{aligned}\langle 0|\phi(\boldsymbol{x},t)\phi(\boldsymbol{x}',t')|0\rangle &= \langle 0|\int \frac{d^3p}{\sqrt{(2\pi)^3 2p^0}}\left[a(p)\,e^{ip\cdot x}+a^\dagger(p)\,e^{-ip\cdot x}\right]\\ &\quad\times\int\frac{d^3p'}{\sqrt{(2\pi)^3 2p'^0}}\left[a(p')\,e^{ip'\cdot x'}+a^\dagger(p')\,e^{-ip'\cdot x'}\right]|0\rangle\\ &= \langle 0|\int\frac{d^3p d^3p'}{(2\pi)^3\sqrt{2p^0 2p'^0}}\,a(p)a^\dagger(p')\,e^{ip\cdot x-ip'\cdot x'}|0\rangle\\ &= \int\frac{d^3p d^3p'}{(2\pi)^3\sqrt{2p^0 2p'^0}}\,\delta^3(\boldsymbol{p}-\boldsymbol{p}')e^{ip\cdot x-ip'\cdot x'}\\ &= \int\frac{d^3p}{(2\pi)^3 2p^0}\,e^{ip\cdot(x-x')}. \end{aligned} \tag{9.101}$$

This integral is simpler for massless fields. Setting $p=|\boldsymbol{p}|$ and $r=|\boldsymbol{x}-\boldsymbol{x}'|$, we add a small imaginary part to the exponential and find

$$\begin{aligned}\langle 0|\phi(\boldsymbol{x},t)\phi(\boldsymbol{x}',t')|0\rangle &= \int\frac{d^3p}{(2\pi)^3 2p}\,e^{ipr\cos\theta-ip(t-t'-i\epsilon)}\\ &= \int\frac{pdp\,d\cos\theta}{(2\pi)^2 2}\,e^{ipr\cos\theta-ip(t-t'-i\epsilon)}\\ &= \int\frac{pdp}{(2\pi)^2 2}\left(\frac{e^{ipr}-e^{-ipr}}{ipr}\right)\,e^{-ip(t-t'-i\epsilon)}\\ &= \int_0^\infty\frac{dp}{(2\pi)^2 2ir}\left(e^{ipr}-e^{-ipr}\right)\,e^{-ip(t-t'-i\epsilon)}\\ &= \frac{1}{(2\pi)^2 2ir}\left(-\frac{1}{i(r-(t-t'))}-\frac{1}{i(r+(t-t'))}\right)\\ &= \frac{1}{(2\pi)^2 2r}\left(\frac{1}{r-(t-t')}+\frac{1}{r+(t-t')}\right)\\ &= \frac{1}{(2\pi)^2[(\boldsymbol{x}-\boldsymbol{x}')^2-(t-t')^2]}. \end{aligned} \tag{9.102}$$

This is the zero-temperature correlation function.

In the instantaneous rest frame of an accelerated observer moving in the x direction with coordinates (9.97 & 9.98), this two-point function is

$$\begin{aligned}\langle 0|\phi(\boldsymbol{x},t)\phi(\boldsymbol{x}',t')|0\rangle &= \frac{\alpha^2/(2\pi)^2}{\left[\cosh(\alpha\tau)-\cosh(\alpha\tau')\right]^2-\left[\sinh(\alpha\tau)-\sinh(\alpha\tau')\right]^2}\\ &= -\frac{\alpha^2}{(4\pi)^2\sinh^2[\alpha(\tau-\tau')/2]} \end{aligned} \tag{9.103}$$

in which the identities

$$\cosh(\alpha\tau)\cosh(\alpha\tau') - \sinh(\alpha\tau)\sinh(\alpha\tau') = \cosh(\alpha(\tau - \tau')) \qquad (9.104)$$

and

$$\cosh(\alpha(\tau - \tau')) - 1 = 2\sinh(\alpha(\tau - \tau')/2) \qquad (9.105)$$

as well as $\cosh^2 \alpha\tau - \sinh^2 \alpha\tau = 1$ were used.

Now let's compute the same correlation function at a finite inverse temperature $\beta = 1/(k_B T)$

$$\langle\phi(\mathbf{0},\tau)\phi(\mathbf{0},\tau')\rangle_\beta = \mathrm{Tr}\big[\phi(\mathbf{0},0)\phi(\mathbf{0},t)e^{-\beta H}\big] \Big/ \mathrm{Tr}\big(e^{-\beta H}\big) \ . \qquad (9.106)$$

The mean value of the number operator for a given momentum k is

$$\langle a^\dagger(k)\, a(k)\rangle_\beta = n_k = \mathrm{Tr}\big[a^\dagger(k)\, a(k) e^{-\beta H}\big] \Big/ \mathrm{Tr}\big(e^{-\beta H}\big) \qquad (9.107)$$

in which

$$H_0 = \sum_k \omega_k \big(a^\dagger(k)\, a(k) + \tfrac{1}{2}\big). \qquad (9.108)$$

The trace over all momenta with $k' \neq k$ is unity, and we are left with

$$\begin{aligned}\langle a^\dagger\, a\rangle_\beta = n_k &= \mathrm{Tr}\big[a^\dagger\, a e^{-\beta\omega_k a^\dagger a}\big] \Big/ \mathrm{Tr}\big(e^{-\beta\omega_k a^\dagger a}\big) \\ &= -\,\frac{1}{\omega_k \mathrm{Tr}\big(e^{-\beta\omega_k a^\dagger a}\big)} \frac{\partial}{\partial\beta} \mathrm{Tr}\big(e^{-\beta\omega_k a^\dagger a}\big)\end{aligned} \qquad (9.109)$$

in which $a^\dagger a \equiv a^\dagger(k)\, a(k)$ and the 1/2 terms have cancelled. The trace is

$$\mathrm{Tr}\big(e^{-\beta\omega_k a^\dagger a}\big) = \sum_n e^{-\beta n \omega_k} = \frac{1}{1 - e^{-\beta\omega_k}}. \qquad (9.110)$$

So we have

$$\langle a^\dagger\, a\rangle_\beta = -\,\frac{\big(1 - e^{-\beta\omega_k}\big)}{\omega_k} \frac{\big(-\omega_k e^{-\beta\omega_k}\big)}{\big(1 - e^{-\beta\omega_k}\big)^2} = \frac{1}{e^{\beta\omega_k} - 1}. \qquad (9.111)$$

In the trace (9.106), only terms that don't change the number of quanta in each mode contribute. So the mean value of the correlation function at inverse temperature $\beta = 1/(k_B T)$ is

$$\begin{aligned}\langle\phi(\mathbf{0},\tau)\phi(\mathbf{0},\tau')\rangle_\beta = \sum_k \frac{1}{2kV} \Big\{ &\Big(e^{\hbar\omega_k/k_B T} - 1\Big)^{-1} e^{i\omega_k(\tau-\tau')} \\ &+ \Big[\Big(e^{\hbar\omega_k/k_B T} - 1\Big)^{-1} + 1\Big] e^{-i\omega_k(\tau-\tau')}\Big\}.\end{aligned} \qquad (9.112)$$

In the continuum limit, this τ, τ' correlation function is two integrals. For massless particles, the simpler one requires some regularization because it involves zero-point energies

$$
\begin{aligned}
A_\beta &= \sum_k \frac{1}{2kV}\, e^{-ik(\tau-\tau')} = \int \frac{d^3k}{(2\pi)^3 2k}\, e^{-ik(\tau-\tau')} = \int \frac{k^2 dk}{(2\pi)^2 k} e^{-ik(\tau-\tau')} \\
&= \int_0^\infty \frac{k dk}{(2\pi)^2} e^{-ik(\tau-\tau')}.
\end{aligned}
\tag{9.113}
$$

We send $\tau - \tau' \to \tau - \tau' + i\epsilon$ and find

$$
\begin{aligned}
A_\beta &= \int_0^\infty \frac{k dk}{(2\pi)^2} e^{-ik(\tau-\tau'+i\epsilon)} = i\frac{d}{d\tau}\int_0^\infty \frac{dk}{(2\pi)^2} e^{-ik(\tau-\tau'+i\epsilon)} \\
&= i\frac{d}{d\tau}\int_0^\infty \frac{dk}{(2\pi)^2} e^{-ik(\tau-\tau'+i\epsilon)} = \frac{1}{(2\pi)^2}\frac{d}{d\tau}\frac{1}{(\tau-\tau')} \\
&= -\frac{1}{(2\pi)^2}\frac{1}{(\tau-\tau')^2}
\end{aligned}
\tag{9.114}
$$

as $\epsilon \to 0$.

The second integral is

$$
B_\beta = \int_0^\infty \frac{k dk}{(2\pi)^2} \left(e^{\beta k} - 1\right)^{-1} 2\cos(k(\tau - \tau')) \tag{9.115}
$$

which Mathematica says is

$$
B_\beta = \frac{1}{(2\pi)^2(\tau-\tau')^2} - \frac{\operatorname{csch}^2(\pi(\tau-\tau')/\beta)}{4\beta^2}. \tag{9.116}
$$

Thus the finite-temperature correlation function is

$$
\langle \phi(\mathbf{0},\tau)\phi(\mathbf{0},\tau')\rangle_\beta = A_\beta + B_\beta = -\frac{1}{4\beta^2\sinh^2(\pi(\tau-\tau')/\beta)}. \tag{9.117}
$$

Equating this formula to the zero–temperature correlation function (9.103) in the accelerating frame, we find

$$
\frac{1}{4\beta^2\sinh^2(\pi(\tau-\tau')/\beta)} = \frac{\alpha^2}{(4\pi)^2\sinh^2[\alpha(\tau-\tau')/2]} \tag{9.118}
$$

which says redundantly

$$
k_B T = \frac{\alpha}{2\pi} \quad \text{and} \quad \pi k_B T = \frac{\alpha}{2}. \tag{9.119}
$$

So a detector accelerating uniformly with acceleration α in the vacuum feels a nonzero temperature

$$T = \frac{\hbar \alpha}{2\pi c k_B}. \tag{9.120}$$

This result (Davies, 1975) is equivalent to the finding (Hawking, 1974) that a gravitational field of local acceleration g makes empty space radiate at a temperature

$$T = \frac{\hbar g}{2\pi c k_B}. \tag{9.121}$$

Black holes are not black.

9.6 Maximally symmetric spaces

The spheres S^2 and S^3 and the hyperboloids H^2 and H^3 are maximally symmetric spaces. A transformation $x \to x'$ is an **isometry** if $g'_{ik}(x') = g_{ik}(x')$ in which case the distances $g_{ik}(x) dx^i dx^k = g'_{ik}(x') dx'^i dx'^k = g_{ik}(x') dx'^i dx'^k$ are the same. To see what this symmetry condition means, we consider the infinitesimal transformation $x'^\ell = x^\ell + \epsilon y^\ell(x)$ under which to lowest order $g_{ik}(x') = g_{ik}(x) + g_{ik,\ell} \epsilon y^\ell$ and $dx'^i = dx^i + \epsilon y^i_{,j} dx^j$. The symmetry condition requires

$$g_{ik}(x) dx^i dx^k = (g_{ik}(x) + g_{ik,\ell} \epsilon y^\ell)(dx^i + \epsilon y^i_{,j} dx^j)(dx^k + \epsilon y^k_{,m} dx^m) \tag{9.122}$$

or

$$0 = g_{ik,\ell}\, y^\ell + g_{im}\, y^m_{,k} + g_{jk}\, y^j_{,i}. \tag{9.123}$$

The vector field $y^i(x)$ must satisfy this condition if $x'^i = x^i + \epsilon y^i(x)$ is to be a symmetry of the metric $g_{ik}(x)$. Since the covariant derivative of the metric tensor vanishes, $g_{ik;\ell} = 0$, we may write the condition on the symmetry vector $y^\ell(x)$ as

$$0 = y_{i;k} + y_{k;i}. \tag{9.124}$$